Tutorial - 0
Group Submission - Room 4

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Ans.1.a

Let's denote the maximum number of regions that can be formed using *n* lines is l_n

- Our formula -

We claim that the maximum number of regions that can be formed when using *n* lines is

$$res_n = \frac{n * (n+1)}{2} + 1$$

- Firstly, we claim that we can achieve this value for any *n*, i.e. formally,

$$res_n \leq l_n$$

proof of this result:

For proving this result, we first prove the following subclaim

- subclaim: We can always find *n* lines which are non-parallel and such that not more than two lines intersect at a point.

proof of subclaim:
Proof by induction:

base case: n = 1

Clearly a single line trivially satisfies the claim

inductive step: Assume that the result holds for n - 1 lines, we would now prove for n lines

Now, we have a set of n - 1 lines which are not pairwise parallel, nor do their intersection points have intersection of more than 2 lines. Hence, for choosing a line which is not parallel to any of these n - 1 lines, we choose a slope which does not equal any of the slopes of these n - 1 lines. Now, that we have picked a non-parallel line, there are two possibilities,

a) The line does not intersect any intersection point => hence we have our result for n lines
b) The line intersections atleast one previous intersection point, then we "perturb this line's slope" by a small value ϵsuch that it does not intersect now, and also it's slope does not equal any other lines' slope.

Hence proved.

Now, we use this subclaim for our proof of lower bound, we prove using induction:

base case: n = 1, clearly any line divides the plane into 2 parts, hence res_1can be achieved.
inductive step: assume that res_{n-1} can be achieved using n - 1 lines, now we we show that res_n can be achieved with n lines

with n - 1 lines we can achieve $res_{n-1} = \frac{n*(n-1)}{2} + 1$
now, we choose a line which does not intersect any of the previous n - 1 lines at more than 2 points, and also is not parallel to any of our n - 1 lines - this can be done since we have proven it in the subclaim.

Now, clearly, this line divides n of the previous regions into 2 parts each, hence n more regions are added, so we have achieved

$$res_{n-1} + n = \frac{n*(n-1)}{2} + 1 + n = res_n$$

- Proof of Upper bound -

Let's assume that there are (n-1) lines existing in a way to maximise the number of regions, and we have to put n^{th} line in the 2D area to maximise the number of regions. Since the maximum number of intersections this new line can make is (n-1), so it can pass through at max n regions and divide each of them into 2 regions, thus creating n extra regions. So at max n regions can be added by this n^{th}line.

If we claim that n^{th}line can pass through more than n regions, say (n + i) where i >= 1. Then for creating (n + i) regions, this new line will have to intersect (n + i - 1) lines. Since (n + i - 1) >= n, and we were only having (n-1) lines previously, our assumption was incorrect that a line can pass through more than n regions.

- Proof of our formula -

Using the proof of subclaim and the upper bound we can clearly say that nth line can be inserted in a way to make exactly n intersections.

Ans. 1.b