Foundations of Data Science & Machine Learning

Tutotial 07

April 16, 2021

Question 1. Let P be the transition probability matrix of a random walk on a finite directed graph G on the vertex set $[n]$. Prove formally (using induction on k) that the (i,j)-th entry of the matrix P^k is equal to the probability that the random walk is in node j in step k given that it was in node i in step 0.

Question 2. Design a random walk on a finite graph G such that G has two different stationary distributions. Try to find the smallest possible G.

The next question is to correct an error that I made in Lecture 15. The same was pointed out to me Ahmed.

Error 1. I mentioned in Lecture 15 that the eigenvector corresponding to the eigenvalue 1 of the transition probability matrix P of a random walk on a finite graph G is unique (up to scaling) if and *only if* G is strongly connected. The *only if* part is wrong.

Question 3. Design a random walk on a finite graph G which is not strongly connected such that its transition probability matrix P has a unique eigenvector (up to scaling) for the eigenvalue 1.

A more serious error (in a proof) that was pointed out by Ahmed is the following.

Error 2. I mentioned in Lecture 15 that the eigenvalues of P and P^T are the same and have the same multiplicities. But this is true only for the algebraic multiplicity. Hence we cannot assume, using this fact alone that the stationary distribution of a random walk on a finite strongly connected graph is unique.

The result is still true, but my ingenious proof is wrong. The correct proofs seem more involved and we will have to skip them for now. See for example

1. Ding, Jiu, and Noah H. Rhee. "On the equality of algebraic and geometric multiplicities of matrix eigenvalues." Applied mathematics letters 24.12 (2011): 2211-2215. `https://core.ac.uk/download/pdf/82820022.pdf`

2. Lecture notes by Aleksandar Nikolov
   `http://www.cs.toronto.edu/~anikolov/CSC473W20/Lectures/MarkovChains.pdf`