Foundations of Data Science & Machine Learning

Tutotial 04

March 19, 2021

Definition 1. Let X be a subset of $\mathbb{R}^n$. A function $f : X \to \{+1, -1\}$ is called *linearly separable* if there exists a vector $a \in \mathbb{R}^n$ such that for every $x \in X$, $\langle a, x \rangle \geq 1$ if and only if $f(x) = +1$. A 2-partition $(G, X \setminus G)$ of X is called *linear* if there exists a vector $a \in \mathbb{R}^n$ such that for every $x \in G$, $\langle a, x \rangle \geq 1$ and for every $x \in X \setminus G$, $\langle a, x \rangle < 1$.

Remark. The demand that $b = 1$ for a hyperplane defined by $\langle a, x \rangle = b$ restricts the choice of hyperplanes by half. In fact, we are now restricted to those hyperplanes where origin is on its negative side.

Question 1. Let X be a set of N points in $\mathbb{R}^2$. Prove that the number of linearly separable functions $f : X \to \{+1, -1\}$ is at most $\binom{N+1}{2} + 1$.

It is easy to see that a function $f : X \to \{+1, -1\}$ is linearly separable if and only if the partition $(f^{-1}(+1), f^{-1}(-1))$ of X is linear. Hence Question 1 is equivalent to the following claim.

Claim. *The number of linear partitions of a set X of N points in $\mathbb{R}^2$ is at most $\binom{N+1}{2} + 1$.*