# CS5014 Foundations of Data Science & Machine Learning

Quiz 01 — Solutions

March 10, 2021

**Question 1.** A set $S \subset \mathbb{R}^n$ is *convex* if for any two points $x, y \in S$ and any $\lambda \in [0,1]$, the point $z = \lambda x + (1-\lambda)y$ lies in $S$. Show that if $S \subset \mathbb{R}^n$ is convex, then

(a) for any 3 points $x_1, x_2, x_3 \in S$ and any $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, the point $z = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$ lies in $S$.

(b) for any $k$ points $x_1, \ldots, x_k \in S$, $k \geq 2$, and any $\lambda_1, \ldots, \lambda_k \in [0,1]$ such that $\sum_{i=1}^k \lambda_i = 1$, the point $z = \sum_{i=1}^k \lambda_i x_i$ lies in $S$.

**1.(a)** We have to show $z = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$ lies in $S$. If $\lambda_3 = 1$, then $\lambda_1 = \lambda_2 = 0$ and hence $z = x_3 \in S$. Otherwise $\lambda_1 + \lambda_2 > 0$, and

$$\begin{aligned}
z &= \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \\
&= \lambda_{12}\left(\frac{\lambda_1}{\lambda_{12}} x_1 + \frac{\lambda_2}{\lambda_{12}} x_2\right) + \lambda_3 x_3, && \text{where } \lambda_{12} = \lambda_1 + \lambda_2 \\
&= \lambda_{12}\left(\lambda x_1 + (1-\lambda)x_2\right) + \lambda_3 x_3, && \text{where } \lambda = \lambda_1/\lambda_{12} \\
&= \lambda_{12} w + \lambda_3, && \text{where } w = \lambda x_1 + (1-\lambda)x_2 \in S \text{ by definition of convexity} \\
&= (1-\lambda_3)w + \lambda_3 x_3, && \text{since } \lambda_{12} + \lambda_3 = 1 \\
&\in S && \text{by definition of convexity.}
\end{aligned}$$

**1.(b)** We will use induction on $k$. Base case, $k = 2$, follows directly from definition. Now suppose the claim is true for $k-1$ points and consider $z = \sum_{i=1}^k \lambda_i x_i$. If $\lambda_k = 1$, then $z = x_k \in S$. Otherwise, for $i \in [k-1]$, let $\lambda'_i = \lambda_i / (\sum_{i=1}^{k-1} \lambda_i)$. Since $\lambda'_1, \ldots, \lambda'_{k-1}$ are coefficients in $[0,1]$ which add up to 1, by induction hypothesis (on $k-1$ vertices), we can conclude that $w = \sum_{i=1}^{k-1} \lambda'_i x_i$ is in $S$. Hence $z = (\sum_{i=1}^{k-1} \lambda_i) w + \lambda_k x_k$ is in $S$ by the definition of a convex set. Note that the two coefficients above are also in $[0,1]$ and add up to 1.

**Question 2.** Let $X \subset \mathbb{R}^n$ be the set of (corner) vertices of the hypercube $\{0,1\}^n$. That is, $X = \{(x_1, \ldots, x_n) : x_i = 0 \text{ or } 1\}$. We say that a point $x = (x_1, \ldots, x_n) \in X$ is good if the last 1 in $x$ is at an even position. That is, $\max\{i : x_i = 1\} = 0 \pmod 2$. The remaining points of $X$ are bad. Are the good and bad points defined above linearly separable. Prove your answer.

**2.** They are linearly separable by a hyperplane defined by $\langle a, x \rangle = b$, where
$a = -(1, -2, 4, -8, \ldots, (-2)^{n-1})$ and $b = -0.5$.

**Question 3.** Find a vector space $V$ and an embedding function $\phi : \mathbb{R}^2 \to V$ such that the resulting kernel $K$ on $\mathbb{R}^2 \times \mathbb{R}^2$ is the function $K(x,y) = (1 + \langle x, y \rangle)^2$.

**3.** Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

$$\begin{aligned}
K(x, y) &= (1 + \langle x, y \rangle)^2 \\
&= (1 + x_1 y_1 + x_2 y_2)^2 \\
&= (1 + 2x_1 y_1 + 2x_2 y_2 + 2x_1 y_1 x_2 y_2 + x_1^2 y_1^2 + x_2^2 y_2^2 \\
&= \left\langle (1, \sqrt{2}x_1, \sqrt{2}x_2, \sqrt{2}x_1 x_2, x_1^2, x_2^2), (1, \sqrt{2}y_1, \sqrt{2}y_2, \sqrt{2}y_1 y_2, y_1^2, y_2^2) \right\rangle \\
&= \langle \phi(x), \phi(y) \rangle \,,
\end{aligned}$$

where $\phi$ is an embedding from $\mathbb{R}^2$ into $\mathbb{R}^6$ given by

$$\phi(x) = (1, \sqrt{2}x_1, \sqrt{2}x_2, \sqrt{2}x_1 x_2, x_1^2, x_2^2).$$