CS5014 Foundations of Data Science & Machine Learning

Quiz 04

May 10, 2021 — 8.30 - 9.30 AM

Instructions

1. This is a **zoom proctored** exam. Please adjust your seating so that **your face, hands, answer book and the mobile phone that you will use to scan the sheets are always in the webcam view**. Do not leave your seat or talk to anyone during the exam.
2. Write your answer on plain paper with your **name and roll number on the first sheet.**
3. This is a **closed book** exam. Do not refer to any books, notes, the Internet or any other person during the exam.
4. You can take **maximum 5 minutes after the exam to scan** the sheets into a **single PDF** file and upload to Moodle. Submissions made after 9:35 AM will be evaluated only if there is a genuine reason for the delay.

Questions

Question 1. Let H be the 3-dimensional lattice $[n]^3$ with self loops at all the boundary vertices. All the non-loop edges have weight 1 while the self-loop at a boundary vertex v has weight $6 - d_v$, where d_v is the degree of v. Hence the total weight of all edges incident to any vertex is 6. Find the normalised conductance $\phi(H)$ of H.

Solution. We will ignore rounding errors in this analysis throughout. If you choose S to be the left half vertices of the lattice, i.e., $S = [n/2] \times [n] \times [n]$, we get $E(S, \bar{S}) = n^2$, $w(S) = 6|S| = 3n^3$, and hence $\phi(S, \bar{S}) = n^2/(3n^3) = 1/3n$. Hence $\phi(H) \leq 1/3n$.

Showing that $\phi(H) \geq 1/3n$ rigorously is a bit more tricky. One can do some hand waving and say that any S which maximises $\phi(S, \bar{S})$ will be a rectangular box sticking to a corner of H and then analyse three cases based on how many dimensions of the box are full.

A more rigorous argument can be made as follows. Let S be a subset of $V(H)$ which maximises $\phi(S, \bar{S})$ and size *less than* $n^3/2$. Then there is no vertex v outside S such that it has a total edge weight of 3 or more into $S \cup \{v\}$. Otherwise $S \cup \{v\}$ has a larger ϕ-value than S since the weight of escaping edges did not increase while the weight of the set itself increased by one. Geometrically this means that there are no "concave corners" in S. The same argument can be extended to show that there is no pair of adjacent vertices u and v outside of S such that the total edge weight between $\{u, v\}$ and $S \cup \{u, v\}$ is at least 6. Otherwise $S \cup \{u, v\}$ is better than S. This geometrically means that there are no "concave edges" in S. Putting it together we get that S is convex and being a subset of the lattice, it has to be a rectangular box. It is an easy argument that this rectangular box has to touch a corner of H if it is maximising ϕ. Further, because of the self loops at the boundary of the lattice, a pair of boundary vertices $\{u, v\}$ of the lattice next to an edge of S has a total edge weight of at least 6 between $\{u, v\}$ and $S \cup \{u, v\}$. Hence S has size exactly $n^3/2$. Once the size of S is $n^3/2$ we cannot do the fiddling around by adding more vertices. But we can still remove a vertex from a convex corner of S and add it to a concave corner, remove a line from a convex edge and add it to a concave edge etc. Thus we can always end up with a ϕ-maximising S of size $n^{/}2$ which is the left half vertices of H. Thus our upper bound from the previous paragraph is tight.

Question 2. Let C_4 be the cycle on 4-vertices $\{0, 1, 2, 3\}$. Find the hitting time $h(0, 1)$, for the standard random walk on C_4.
Hint. Search for symmetries.
Warning. It is recommended that you do this question at the end.

Solution. By symmetry $h(i, i+1)$ should be the same for all $i \in \{0, \ldots, 3\}$, where addition is modulo 4. Call this common value x. Symmetry again tells you that $\forall i,\ h(i, i-1) = x$. Similarly $h(i, i+2)$ should be the same for all $i \in \{0, \ldots, 3\}$, where addition is modulo 4. Call this common value y. Now consider

$$\begin{aligned} h(0,2) &= \frac{1}{2}(1 + h(1,2)) + \frac{1}{2}(1 + h(3,2)) \\ y &= \frac{1}{2}(1 + x) + \frac{1}{2}(1 + x) \\ y &= 1 + x \end{aligned}$$

$$\begin{aligned} h(1,2) &= \frac{1}{2}1 + \frac{1}{2}(1 + h(0,2)) \\ x &= \frac{1}{2}1 + \frac{1}{2}(1 + y) \\ x &= 1 + y/2 \end{aligned}$$

Solving the above two equations gives $x = 3$ and $y = 4$. Hence our answer $h(0, 1) = x = 3$.

Question 3. Let A be a real symmetric square matrix. Find a relation between

1. Singular vectors and eigenvectors of A.

2. Singular values and eigenvalues of A.

Prove your answer.

Solution. Since A is real symmetric, it is always diagonalisable and has n eigenvalues $\lambda_1, \ldots, \lambda_n$ and n orthonormal eigenvectors $e_1, \ldots, e_n$ corresponding to them. Proving this is a typical linear algebra exercise:

For any vector $x = x_1e_1 + \cdots + x_ne_n$ in $\mathbb{R}^n$ $y = Ax = \lambda_1x_1e_1 + \cdots + \lambda_nx_ne_n$. Hence one of the unit vectors which maximises $\|Ax\|$ is e_1. That is, e_1 qualifies to be a first singular vector of A with singular value $\sigma_1 = |\lambda_1|$. Any vector x orthogonal to e_1 can be represented as $x = x_2e_2 + \cdots x_ne_n$ and it is easy to see that one of the vectors maximising $\|Ax\|$ from this set is e_2. Hence e_2 qualifies to be a second singular vector with $\sigma_2 = |\lambda_2|$.

Hence $\forall i \in [n]$, $\sigma_i = |\lambda_i|$. The set of eigenvectors $e_1, \ldots, e_n$ of A qualify to be singular vectors of A also. But remember that neither the set of eigenvectors nor the set of singular vectors is unique. Hence we can only say that any set of orthonormal eigenvectors of A will also qualify to be singular vectors of A.