Lab 2 - Mathematical Expressions

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Cauchy Integral Theorem

If $\int(x)$ is analytic in some simply connected region $\mathbf{R}$, then

$$\oint_y f(z)dz = 0 \tag{1}$$

for any closed contour γ completely contained in $\mathbf{R}$. Writing z as

$$z \equiv x + iy \tag{2}$$

and $\int(z)$ as

$$f(z) \equiv u + iv \tag{3}$$

then gives

$$\oint_y f(z)dz = \int_y (u+iv)(dx+idy) \tag{4}$$

$$= \int_y udx - vdy + i\int_y vdx - udy \tag{5}$$

From Green's theorem,

$$\int_y f(x,y)dy - g(x,y)dy = -\iint\left(\frac{\delta g}{\delta x} + \frac{\delta f}{\delta y}\right)dxdy \tag{6}$$

$$\int_y f(x,y)dy + g(x,y)dy = -\iint\left(\frac{\delta g}{\delta x} - \frac{\delta f}{\delta y}\right)dxdy, \tag{7}$$

so ($\diamond$) becomes

$$\oint_y f(z)dz = -\iint\left(\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y}\right)dxdy + i\iint\left(\frac{\delta u}{\delta x} - \frac{\delta v}{\delta y}\right)dxdy. \tag{8}$$

But the Cauchy-Riemann equations require that

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \tag{9}$$

$$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}, \tag{10}$$

so

$$\oint_y f(z)dz = 0, \tag{11}$$

Q.E.D.
For a multiply connected region,

$$\oint_{c_1} f(z)dz = \oint_{c_2} f(z)dz. \tag{12}$$

SEE ALSO:
Argument Principle, Cauchy Integral Formula, Contour Integral, Morera's Theorem, Residue Theorem

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