# von Karman plate equation

$$\Delta^2 w = [w,F] - p\Delta w + f$$
$$\Delta^2 F = -\frac{1}{2}[w,w] + g$$

with $[u,v] := u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy} = \mathrm{cof}(\nabla^2 u) : \nabla^2 v$.

Weak form

$$\int_\Omega \nabla^2 w : \nabla^2 \delta w \, dx = \int_\Omega [w,F]\delta w + p\nabla w \cdot \nabla \delta w + f\delta w \, dx$$
$$\int_\Omega \nabla^2 F : \nabla^2 \delta F \, dx = \int_\Omega -\frac{1}{2}[w,w]\delta F + g\delta F \, dx$$

Mixed form, $\sigma := \nabla^2 w, \tau := \nabla^2 F$

$$\int_\Omega \sigma : \delta\sigma \, dx + \langle \mathrm{div}(\delta\sigma), \nabla w \rangle = 0$$
$$\langle \mathrm{div}(\sigma) : \nabla \delta w \rangle = -\int_\Omega \mathrm{cof}(\sigma) : \tau \, \delta w + p\nabla w \cdot \nabla \delta w + f\delta w \, dx$$
$$\langle \mathrm{div}(\tau), \nabla \delta F \rangle = \int_\Omega \frac{1}{2}\mathrm{cof}(\sigma) : \sigma \, \delta F - g\delta F \, dx$$
$$\int_\Omega \tau : \delta\tau \, dx + \langle \mathrm{div}(\delta\tau), \nabla F \rangle = 0$$

In [ ]:
```
from ngsolve import *
from ngsolve.webgui import Draw
from netgen.occ import *
```

In [ ]:
```
def SolveKarman(mesh, force, p, dirichlet, g=CF(0), order=3):
    Q = H1(mesh, order=order, dirichlet=dirichlet)
    V = HDivDiv(mesh, order=order-1)
    X = Q*Q*V*V
    (w,F,sigma,tau), (dw,dF,dsigma,dtau) = X.TnT()

    n = specialcf.normal(2)

    def tang(u): return u-(u*n)*n

    B = BilinearForm(X, symmetric=True)
    B += (InnerProduct (sigma, dsigma) + div(sigma)*grad(dw) \
          + div(dsigma)*grad(w) - 1e-10*w*dw )*dx \
          + (-(sigma*n) * tang(grad(dw)) - (dsigma*n)*tang(grad(w)))*dx(element_boundary=True)
    B += (InnerProduct (tau, dtau) + div(tau)*grad(dF) \
          + div(dtau)*grad(F) - 1e-10*F*dF )*dx \
          + (-(tau*n) * tang(grad(dF)) - (dtau*n)*tang(grad(F)))*dx(element_boundary=True)
    B += (p*Grad(w)*Grad(dw)+InnerProduct(Cof(sigma),tau)*dw-0.5*InnerProduct(Cof(sigma),sigma)*dF + force*dw + g*d
F)*dx

    gfsol = GridFunction(X)

    solvers.Newton(B,gfsol,inverse="sparsecholesky")

    return gfsol
```

## Example with known solution

Adapt second equation to $\Delta^2 F = -\frac{1}{2}[w,w] + g$ for generating right-hand side.

$w = x^2(1-x)^2 y^2(1-y)^2$, $F = \sin(\pi x)^2 \sin(\pi y)^2$

In [ ]:
```
rect = Rectangle(1,1).Face()
mesh = Mesh(OCCGeometry(rect,dim=2).GenerateMesh(maxh=0.05))

wex = 100*x**2*(1-x)**2*y**2*(1-y)**2
Fex =sin(pi*x)**2*sin(pi*y)**2

p=0
gradwex = CF( (wex.Diff(x),wex.Diff(y)) )
hessewex = CF( (wex.Diff(x).Diff(x),wex.Diff(x).Diff(y),
                wex.Diff(y).Diff(x),wex.Diff(y).Diff(y)), dims=(2,2) )

gradFex = CF( (Fex.Diff(x),Fex.Diff(y)) )
hesseFex = CF( (Fex.Diff(x).Diff(x),Fex.Diff(x).Diff(y),
                Fex.Diff(y).Diff(x),Fex.Diff(y).Diff(y)), dims=(2,2) )
bilaplacewex = wex.Diff(x).Diff(x).Diff(x).Diff(x)+2*wex.Diff(x).Diff(x).Diff(y).Diff(y)+wex.Diff(y).Diff(y).Diff
(y).Diff(y)
bilaplaceFex = Fex.Diff(x).Diff(x).Diff(x).Diff(x)+2*Fex.Diff(x).Diff(x).Diff(y).Diff(y)+Fex.Diff(y).Diff(y).Diff
(y).Diff(y)

g = bilaplaceFex + 0.5*InnerProduct(Cof(hessewex),hessewex)
f = bilaplacewex - InnerProduct(Cof(hessewex),hesseFex) + p*Trace(hessewex)


with TaskManager():
    gfsol = SolveKarman(mesh, p=p, force=f, g=g, dirichlet=".*", order=1)

gfw,gfF,gfsigma,gftau = gfsol.components
Draw(gfw, mesh, name="u", deformation=True)
Draw(gfF, mesh, name="v", deformation=True)

errw = sqrt(Integrate((gfw-wex)**2+(Grad(gfw)-gradwex)**2,mesh))
errF = sqrt(Integrate((gfF-Fex)**2+(Grad(gfF)-gradFex)**2,mesh))

print("errw = ", errw, ", errF = ", errF)
```