Phase reduction and optimal control theory

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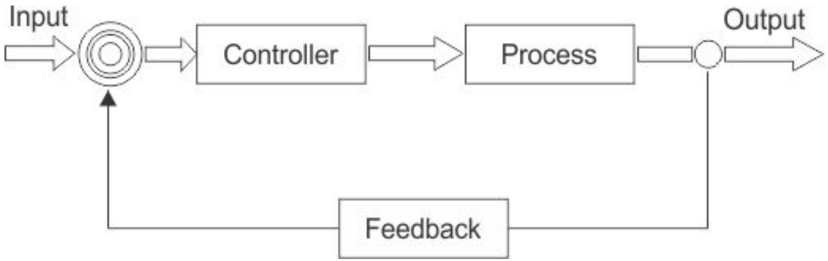
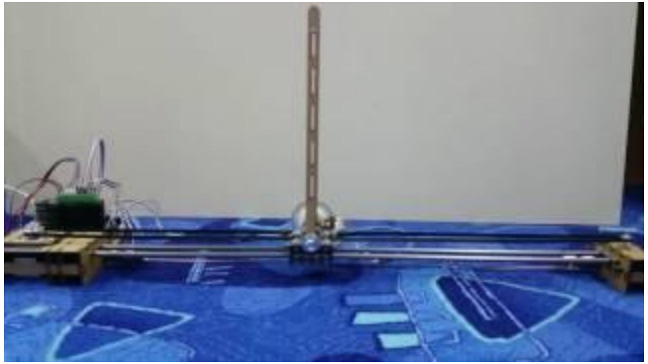


$$\Delta m = K_p \Delta e \qquad \text{Proportional action}$$

$$\Delta m = \frac{1}{\tau_i}\int e\,dt \qquad \text{Integral action}$$

$$\Delta m = \tau_d \frac{de}{dt} \qquad \text{Derivative action}$$



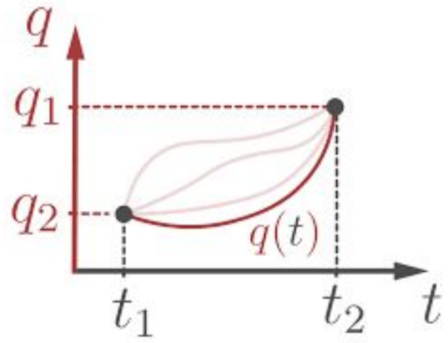
Limitations:

- Linear and time-invariant system dynamics
- Limited tuning/adaptability
- …

Control theory

$$S[q] = \int_{t_1}^{t_2} \mathrm{d}t\, L(t, q, \dot{q})$$

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0$$



- **Solution of constrained minimisation:**

$$\min_{\boldsymbol{x}} \; J(\boldsymbol{x})$$

$$s.t. \; \boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$$

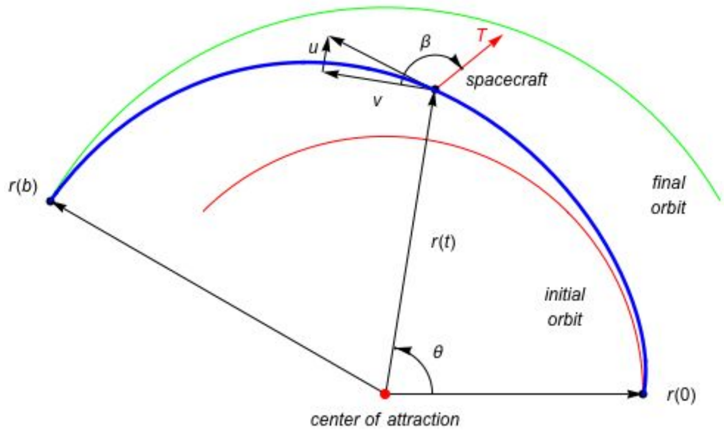
- Build *Lagrangian function*

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = J(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{f}(x)$$

Optimal control theory

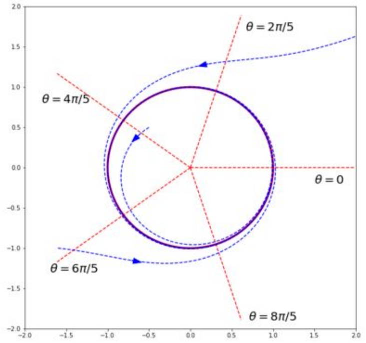
Key features:

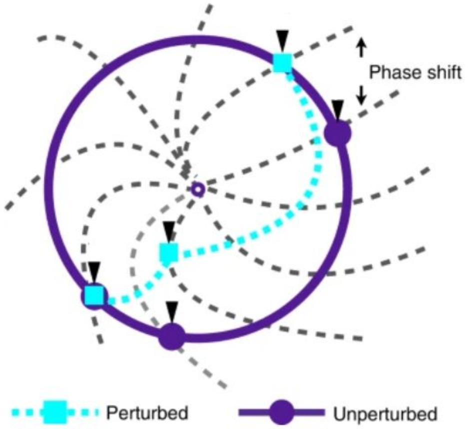
- Mathematical framework.
- Controllers can handle complex dynamics and disturbances without any feedback loop.
- It allows for the incorporation of constraints on the system.
- Trade-off: achieving a desired system behavior and minimizing the control effort.



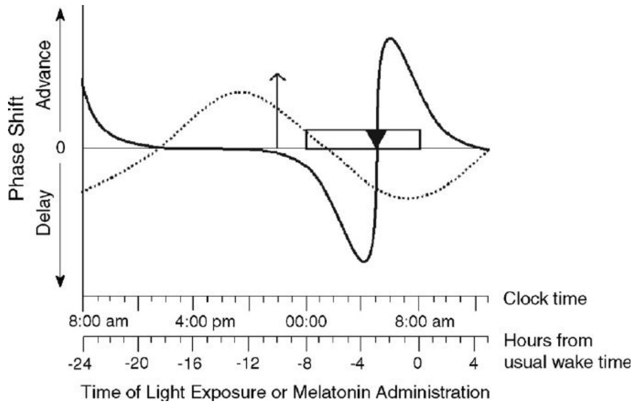
Optimal control theory

Non-linear oscillator in N dimensions





PRC



System control?

Phase reduction

$$\dot{v} = \frac{-I_L(v) - I_{\mathrm{Na}}(v,h) - I_K(v,h) - I_T(v,r) + I_{\mathrm{b}}}{C_m}$$

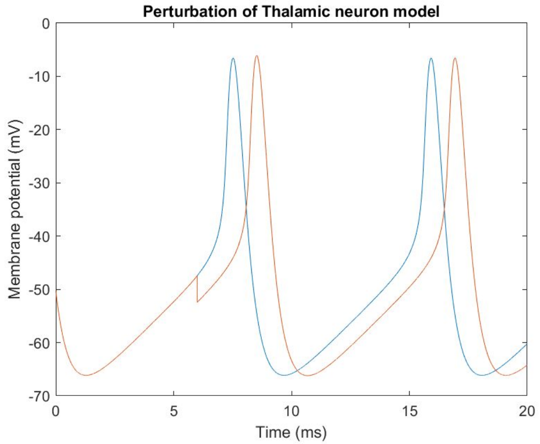
$$\Delta\Psi(\theta) = \frac{\Psi'(\theta) - \Psi(\theta)}{T_{unp}} 2\pi$$

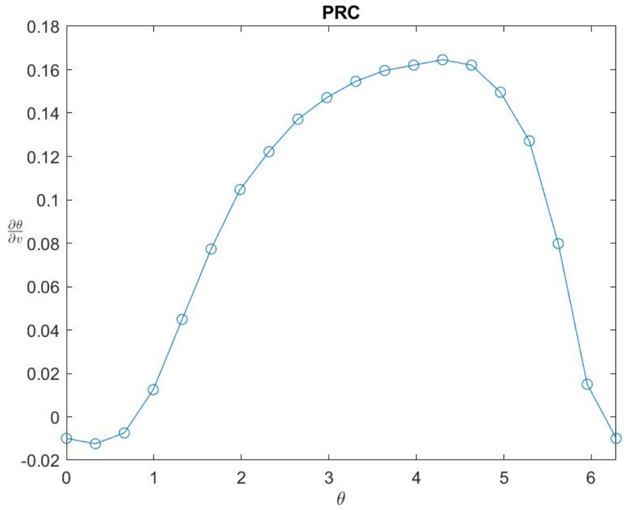
$$\dot{h} = \frac{h_\infty(v) - h}{\tau_h(v)},$$

$$\dot{r} = \frac{r_\infty(v) - r}{\tau_r(v)}.$$

$$PRC(\theta) = \frac{\Delta\Psi(\theta)}{|\delta|}$$

Control objective:
Change the T of the system.





Case study: Problem overview

Phase reduction + control

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + Z(\theta)u(t)$$

Notation

$$\dot{} = \frac{d}{dt}$$

$$' = \frac{d}{d\theta}$$

Cost function and Lagrangian

$$\mathcal{G}[u(t)] = \int_0^{T_1} [u(t)]^2 \mathrm{d}t$$

$$C[u(t)] = \int_0^{T_1} \underbrace{\left\{[u(t)]^2 + \lambda(t)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} - \omega - Z(\theta)u(t)\right)\right\}}_{\mathcal{L}} \mathrm{d}t$$

Constraints

$$\theta(0) = 0, \quad \theta(T_1) = 2\pi$$

Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right), \quad \frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\lambda}}\right), \frac{\partial \mathcal{L}}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)$$

$$u(t) = \frac{\lambda(t)Z(\theta(t))}{2},$$

POSTPROCESS

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + Z(\theta)u(t) = \omega + \frac{\lambda(t)[Z(\theta)]^2}{2},$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -\lambda(t)Z'(\theta)u(t) = -\frac{[\lambda(t)]^2 Z(\theta)Z'(\theta)}{2}$$

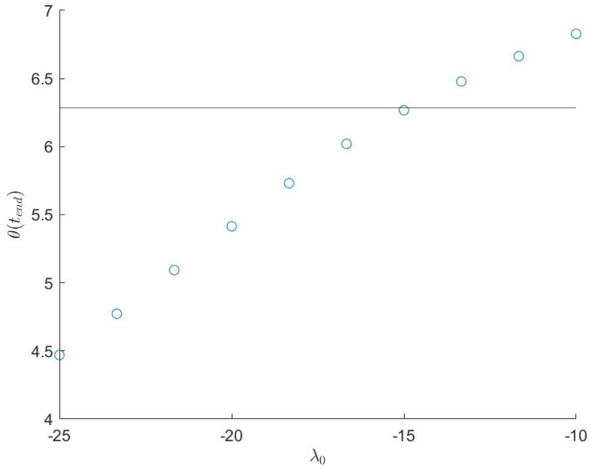
Case study: Main equations

Our variables: $\begin{cases} t \\ \lambda(t) \\ \theta(t) \end{cases}$

$$\theta(T_{des}) - 2\pi = \phi(\lambda_0)$$

Constraints: $\begin{aligned} \theta(0) &= 0, \\ \theta(T_d) &= 2\pi. \end{aligned}$

$$\lambda_0^{n+1} = \lambda_0^n - \frac{\phi(\lambda_0^n)}{\phi'(\lambda_0^n)}$$



$$\begin{cases} \dfrac{d\dot\theta}{d\lambda_0} = \dfrac{d\dot\theta}{d\lambda}\dfrac{d\lambda}{dt}\dfrac{dt}{d\lambda_0} + \dfrac{d\dot\theta}{d\theta}\dfrac{d\theta}{d\lambda_0} + \dfrac{d\dot\theta}{d\lambda}\dfrac{d\lambda}{d\lambda_0} \\ \dfrac{d\dot\lambda}{d\lambda_0} = \dfrac{d\dot\lambda}{d\lambda}\dfrac{d\lambda}{dt}\dfrac{dt}{d\lambda_0} + \dfrac{d\dot\lambda}{d\theta}\dfrac{d\theta}{d\lambda_0} + \dfrac{d\dot\lambda}{d\lambda}\dfrac{d\lambda}{d\lambda_0} \\ \dfrac{d\theta}{d\lambda_0}(0) = 0 \\ \dfrac{d\lambda}{d\lambda_0}(0) = 1 \end{cases}$$

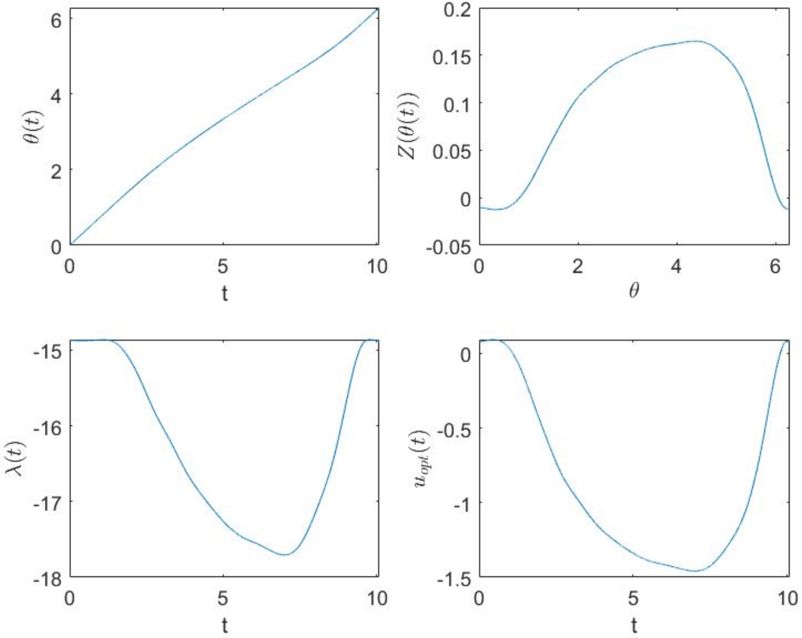
$$x = \frac{d\theta}{d\lambda_0}$$

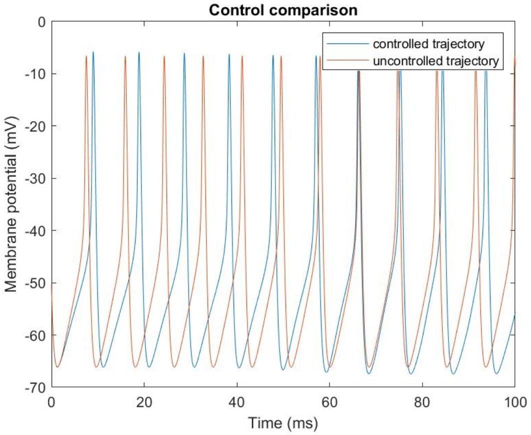
$$x(T_{des}) = \frac{d\theta}{d\lambda_0}(T_{des}) = \frac{d}{d\lambda_0}(\theta(T_{des}) - 2\pi) = \phi'(\lambda_0)$$

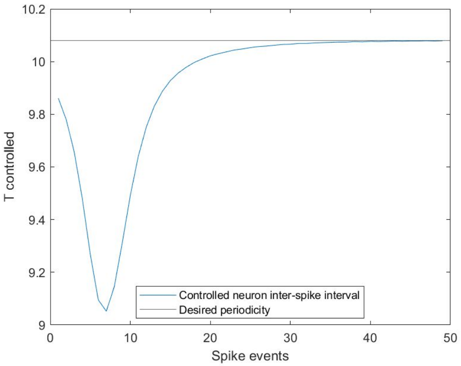
Case study: Shooting method with newton solver

Case study: Solution

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + Z(\theta)u(t)$$







Case study: Recap

$$\dot{v} = \frac{-I_L(v) - I_{\mathrm{Na}}(v,h) - I_K(v,h) - I_T(v,r) + I_{\mathrm{b}}}{C_m} + u(t),$$

$$\dot{h} = \frac{h_\infty(v) - h}{\tau_h(v)},$$

$$\dot{r} = \frac{r_\infty(v) - r}{\tau_r(v)}.$$

$$\boxed{\int_0^{T_1} \text{Euler-Lagrange equations}}$$

$$\downarrow$$

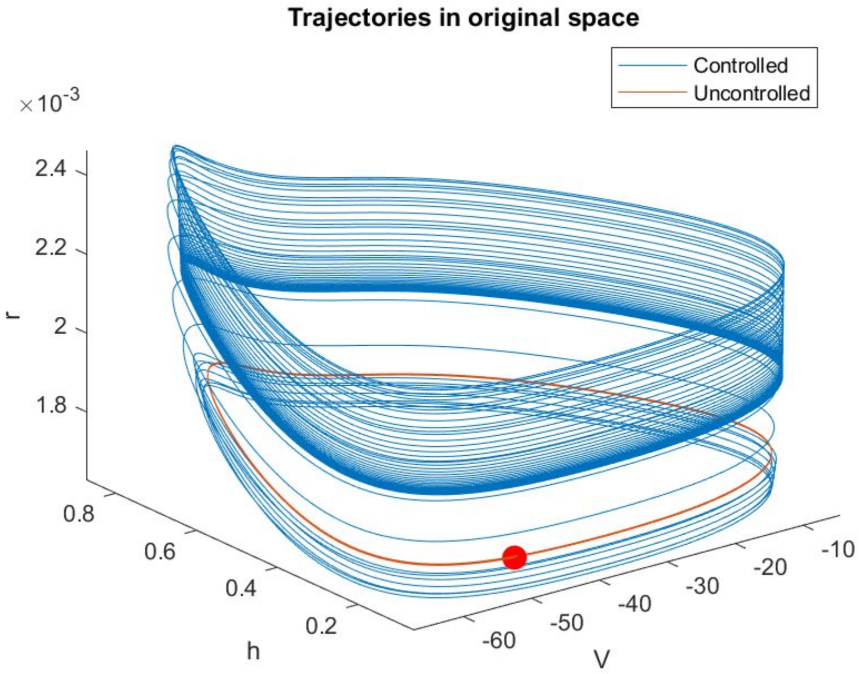
$$\boxed{\theta(t), \lambda(t)}$$

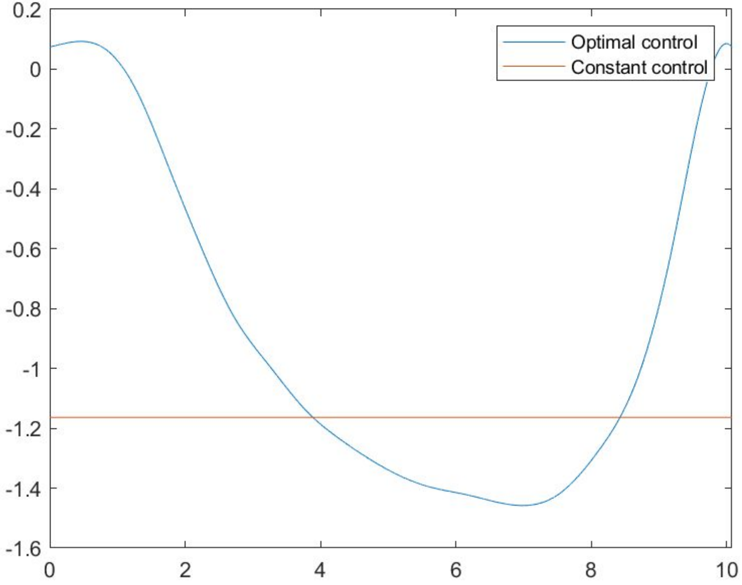
$$\downarrow$$

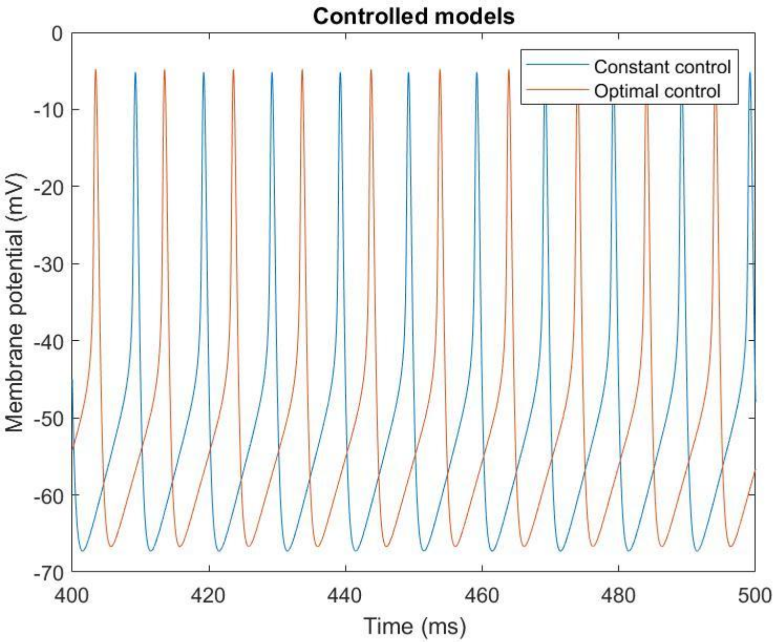
$$\boxed{\mathbf{U}(t) = \begin{cases} \left[\frac{\lambda(t) Z(\theta(t))}{2},\ 0, \ldots, 0\right]^T & 0 \le t \le T_1 \\ [0,\ 0, \ldots, 0]^T & t > T_1 \end{cases}}$$

$$\downarrow$$

$$\boxed{\mathbf{x}(t) = \int_0^t [\mathbf{F}(\mathbf{x}) + \mathbf{U}(t)]\, dt}$$







Cost function

$$\mathcal{G}[u(t)] = \int_0^{T_1} [u(t)]^2 \mathrm{d}t$$

Total control energy consumed:

- Constant control: 13.6374
- Optimal control: 10.7155

Case study: Final comparison

1. Monga, B., Wilson, D., Matchen, T., & Moehlis, J. (2019). Phase reduction and phase-based optimal control for biological systems: a tutorial. *Biological cybernetics*, *113*(1-2), 11-46.

References