Probability 2

Exercise sheet nb. 10

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Due until: 26th November at 5 p.m.

Exercises marked with * are easier after the lecture on Thursday.

Exercise 1 (3 points*). Consider the Markov chain $(X_n)_{n\geq 0}$ on $S := \{1,2\}$ starting at 1 with transition matrix

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

1. Compute the distribution of the first return time $T_1 := \inf_{n\geq 1}\{X_n = 1\}$ of $(X_n)_{n\geq 0}$ at state 1.
2. Find a stationary probability measure μ on S. Compare $\mu(1)^{-1}$ and $\mathbb{E}(T_1)$.

Exercise 2 (3 points*). Fix $p \in (0,1)$, set $q = 1 - p$ and consider the Markov chain $\{X_n\}_{n\geq 0}$ with state space $\mathbb{Z}_{\geq 0}$ and $Q_{i,j} = \mathbb{P}[X_{n+1} = j | X_n = i]$ transition matrix given by

$$Q_{i,j} = \begin{cases} p, & \text{if } j - 1 = i \geq 1, \\ q, & \text{if } j + 1 = i \geq 1, \\ 1, & \text{if } i = 0,\ j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

1. Show that the measure μ on $\mathbb{Z}_{\geq 0}$ defined by $\mu(i) = \left(\frac{p}{q}\right)^i$ for $i \geq 1$ and $\mu(0) = p$, is stationary.
2. Show that for $p < 0.5$, all states are recurrent.

Exercise 3 (5 points). Consider the same Markov chain $\{X_n\}_{n\geq 0}$ from Exercise 2. This exercise is independent from Exercise 2. Define the hitting times for $j \geq 0$ as

$$H_j := \inf\{n > 0 :\ X_n = j\}\,,$$

and let ϕ be the generating function of H_0, for the Markov chain starting at 1, i.e. $\phi(s) := \mathbb{E}_1[s^{H_0}]$ for $s \in [0,1)$.

1. Show that $\mathbb{E}_2[s^{H_0}] = \phi(s)^2$. Conclude that

$$\mathbb{E}_1[s^{H_0}\mathbb{1}[X_1 = 2]] = ps\phi(s)^2.$$

   Hint: working under $\mathbb{P}_2$ (i.e. with the Markov chain starting at 2), define $\tilde{H}_0 = H_0 - H_1$ and show that $\tilde{H}_0$ is independent of H_1 and that both, under $\mathbb{P}_2$, have the distributions of H_0 under $\mathbb{P}_1$.

2. Show that ϕ satisfies $ps\phi^2(s) + qs = \phi(s)$. Show that $\phi(s) = \frac{1-\sqrt{1-4pqs^2}}{2ps}$, where $s \in [0, 1)$.

   Hint: The quadratic equation has two solutions, so assume the fact that ϕ is continuous on $[0, 1)$ to decide which solution must be chosen.

3. Compute $\mathbb{P}_1[H_0 = 3]$ from $\phi(s)$ directly.

   Hint: Recall that $\phi(s) = \sum_{k\geq 0} \mathbb{P}_1[H_0 = k]s^k$.

4. Show that $\mathbb{P}_1[H_0 < \infty] = \lim_{s\to 1^-} \phi(s)$ and compute it.

   Hint: Note that $1 - 4pq = (1 - 2p)^2$.

5. Show that $\mathbb{E}_1[H_0] = \lim_{s\to 1^-} \phi'(s)$ and compute it for $p < 0.5$. (Remark: For $p \geq 0.5$, the value of $\mathbb{E}[H_0]$ is infinite: why?)