Homework Assignment 7

Hopf algebras - Spring Semester 2018

Exercise 1

Let G be a group.

- Let $N \leq G$ be a subgroup. Show that $k[G]$ is free as a left and right $k[N]$-module.
- If $N \trianglelefteq G$ is a normal subgroup, let $p : G \mapsto G/N$ denote the natural projection and $\pi : k[G] \to k[G/N]$ the induced algebra morphism. Recall that $k[N]^+$ denotes the augmentation ideal, that is the kernel of the counit ϵ. Let $k[G]^{co\, k[G/N]}$ be the space of $k[N]$-coinvariant elements of $k[G]$ with respect to the right $k[G/N]$ - comodule algebra structure given by $(\mathrm{id} \otimes \pi)\Delta$ (see last exercise sheet).

  Show that $\ker \pi = k[G](k[N])^+$ and $k[G]^{\mathrm{co}\, k[G/N]} = k[N]$.

Exercise 2

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a} \subset \mathfrak{g}$ a Lie subalgebra.

- Show that the map $U(\iota) : U(\mathfrak{a}) \to U(\mathfrak{g})$ induced by the inclusion $\iota : \mathfrak{a} \to \mathfrak{g}$ is injective. Show as well that $U(\mathfrak{g})$ is free as a left and right $U(\mathfrak{a})$-module.
- Suppose that $\mathfrak{a}$ is a Lie ideal. Let $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ be the canonical map $\pi : U(\mathfrak{g}) \to U(\mathfrak{g}/\mathfrak{a})$ the induced algebra homomorphism. Let $U(\mathfrak{a})^+$ be the augmentation ideal and let $U(\mathfrak{g})^{\mathrm{co}\, U(\mathfrak{g}/\mathfrak{a})}$ be the space of $U(\mathfrak{g}/\mathfrak{a})$-coinvariant elements of $U(\mathfrak{g})$ with respect to the right $U(\mathfrak{g}/\mathfrak{a})$ - comodule algebra structure given by $(\mathrm{id} \otimes \pi)\Delta$ (see last exercise sheet).

  Show that $\ker \pi = U(\mathfrak{g})U(\mathfrak{a})^+$ and describe $U(\mathfrak{g})^{\mathrm{co}\, U(\mathfrak{g}/\mathfrak{a})}$.

Exercise 3

Let $\mathfrak{g}$ be a Lie algebra, I a set, and $x : I \to \mathfrak{g}$ an injective map. We say that $\mathfrak{g}$ is freely generated by I if for every Lie algebra $\mathfrak{h}$ and any map $f : I \to \mathfrak{h}$ there is a unique Lie algebra homomorphism $\bar{f} : \mathfrak{g} \to \mathfrak{h}$ such that the following diagram commutes:

$$\begin{tikzcd} I \arrow[r, "x"] \arrow[dr, "f"'] & \mathfrak{g} \arrow[d, "\exists!\,\tilde{f}"] \\ & \mathfrak{h} \end{tikzcd}$$

Show that for any set I there is a sub Lie algebra $\mathfrak{g}_I \subset k < x_i \mid i \in I >^-$ that is freely generated by I. Show that $U(\mathfrak{g}_I) \simeq k < x_i \mid i \in I >$.