# From presheaves to Hopf algebras

## Permutation Patterns 2019, Zurich

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Slides can be found at
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# Counting occurences of a pattern

Permutation $\pi$ on a set $S$ (a pair of orders on $S$).
$I$ subset of $S$ - set of columns of the square configuration of $\pi$.
The **restriction to** $I$ can be defined $\pi|_I$ and is a permutation in $I$.
We can count occurences!
We write

$$\mathbf{p}_{12}(132) = 2,\ \mathbf{p}_{123}(123456) = 20,\ \mathbf{p}_{2413}(762341895) = 0\,.$$

# Permutation pattern algebra

Set of permutations in $[n]$ for $n \geq 0$ - $\mathcal{G}(\texttt{Per})$

Pattern function $p_\pi$ are in the space of functions $\mathcal{F}(\mathcal{G}(\texttt{Per}), \mathbb{R})$

The linear span of all pattern functions - $\mathcal{A}(\texttt{Per})$ - Is an algebra (closed for multiplication).

**Adding another ingredient**

$$\pi \oplus \tau = \begin{array}{|c|c|} \hline & \tau \\ \hline \pi & \\ \hline \end{array} \qquad \pi \ominus \tau = \begin{array}{|c|c|} \hline \pi & \\ \hline & \tau \\ \hline \end{array}$$

By *magic properties* of dualization, these give coproducts on $\mathcal{A}(\texttt{Per})$, for instance:

$$\Delta \, \mathbf{p}_\pi = \sum_{\pi = \tau_1 \oplus \tau_2} \mathbf{p}_{\tau_1} \otimes \mathbf{p}_{\tau_2} \, ,$$

so that we have a Hopf algebra

$$\mathbf{p}_\pi(\sigma_1 \oplus \sigma_2) = \Delta \, \mathbf{p}_\pi(\sigma_1 \otimes \sigma_2) \, .$$

# Permutation pattern algebra

### Proposition (Linear independence)

*The set* $\{\mathbf{p}_\pi \mid \pi \in \uplus_{n\geq 0} S_n\}$ *is linearly independent* - *Triangularity argument*

### Proposition (Product formula)

*Let* $\binom{\sigma}{\pi, \tau}$ *count the number of covers of* $\sigma$ *with permutations* $\pi$, $\tau$.

$$\mathbf{p}_\pi \cdot \mathbf{p}_\tau = \sum_\sigma \binom{\sigma}{\pi, \tau} \mathbf{p}_\sigma \,,$$

*where* $\sigma$ *runs over equivalence classes of pairs of orders.*

### Theorem (Vargas, 2014)

*The Hopf algebra* $\mathcal{A}(\texttt{Per})$ *is free comutative.* *what is free?*

# Outline of the talk

# Pattern algebra

What do we need to have a pattern Hopf algebra?

1. Assignment $S \mapsto h[S] = \{\text{structures over } S\}$ + notion of *relabelling*.
2. For any inclusion $V \hookrightarrow W$, a restriction map $h[W] \to h[V]$.
3. An associative *monoid* operation $*$ with unit, in $\mathcal{G}(\mathtt{h})$ that is compatible with restrictions.
4. A unique element of size zero.

$1 + 2 =$ *combinatorial presheaf* $\to$ Algebra.
$1 + 2 + 3 =$ *monoid in combinatorial presheaves*.
$1 + 2 + 4 =$ *connected presheaf*.
$1 + 2 + 3 + 4 \to$ Hopf algebra.

# Category theory formulation

Observation: The product structure on $\mathcal{A}(\mathtt{h})$ depends only on the **combinatorial presheaf structure**, and **not** on the **monoid structure.**
The same product structure may be compatible with several coproducts.

Examples: the presheaves of **marked graphs** or **permutations.**

We have a functor $\mathcal{A}$ that sends

$$\mathcal{A} : \mathtt{CPSh} \to \mathrm{GAlg}_{\mathbb{R}} \,,$$

and restricts $\mathcal{A} : \mathrm{Mon}(\mathtt{CPSh}) \to \mathrm{GHopf}_{\mathbb{R}}$.

# A presheaf on graphs

- For each set $V$ we are given the set $\mathtt{G}[V]$ of graphs with vertex set $V$, and for any bijection $\phi : V \to W$ we are given a relabelling of graphs $\mathtt{G}[W] \to \mathtt{G}[V]$.
- Induced subgraphs $\to$ restrictions.
- The disjoint union of graphs is an associative monoid structure. It is also **commutative**.
- The empty graph fortunately exists!

### Theorem (P - 2019+)

*If $\mathtt{h}$ is a connected commutative presheaf, then $\mathcal{A}(\mathtt{h})$ is free. The free generators are the indecomposable objects with respect to the commutative product.*

# Simple example

The presheaf of sets.
For each $n \geq 0$, we have a unique element $*_n$ of size $n$ up to isomorphism.

$$\mathbf{p}_{*_n}(*_m) = \binom{m}{n} \qquad \binom{*_d}{*_a, *_b} = \binom{d}{a}\binom{a}{a+b-d}.$$

So

$$\mathbf{p}_{*_a}\,\mathbf{p}_{*_b}(*_c) = \sum_{d \geq 0} \binom{d}{a}\binom{a}{a+b-d}\,\mathbf{p}_{*_d}(*_c)$$

**Product rule** - Disjoint union: $*_n *_m = *_{n+m}$.

$$\Delta\,\mathbf{p}_{*_a} = \sum_{k=0}^{a} \mathbf{p}_{*_k} \otimes \mathbf{p}_{*_{a-k}}, \qquad \mathcal{A}(\mathtt{Set}) = \mathbb{R}\langle \mathbf{p}_{*_1} \rangle$$

# Connected commutative combinatorial presheafs

Proof (by example):
Graphs, with a disjoint union, form a commutative presheaf. Every graph has a unique factorisation into **indecomposables** $\mathcal{I}$.

$$\mathcal{A}(\mathtt{G}) \text{ is free commutative} \Leftrightarrow \{\prod_{l\in L} \mathbf{p}_l \,|L \subseteq \mathcal{I} \text{ multiset } \} \text{ is lin. ind.}$$

$$\Leftrightarrow_{\substack{\text{triangularity}\\ \text{argument}}} \prod_{l\in L} \mathbf{p}_l = \mathbf{p}_\alpha + \sum_{\beta\leq\alpha} c_\beta \, \mathbf{p}_\beta \text{ for some order } \leq \, .$$

where $\alpha = \biguplus_{l\in L} l$.

Highly important: **We have a unique factorisation theorem on graphs**.
To remember: A unique factorisation theorem up to commutativity of factors $\Rightarrow$ an order to use the triangularity argument.

# Unique factorisation theorem on permutations

Vargas used the $\oplus$ product on permutations to obtain a unique factorisation theorem on permutations.

$$\pi = \tau_1 \oplus \cdots \oplus \tau_k = \begin{array}{|c|c|c|} \hline & & \tau_k \\ \hline & \iddots & \\ \hline \tau_1 & & \\ \hline \end{array}$$

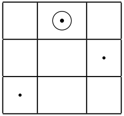
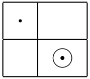
- The factorisation is **not** unique up to order of factors.

Enlarge the set $\mathcal{I}$ to $\mathcal{L}$ with **Lyndon permutations**, by adding some decomposable elements. Choose between $\pi_1 \oplus \pi_2$ and $\pi_2 \oplus \pi_1$, and between more factors.

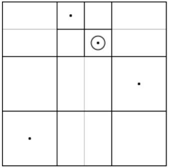
**Lyndon words** - used for the freeness of the shuffle algebra on $\mathbb{K}\langle A \rangle$.

# The inflation product - Marked permutations

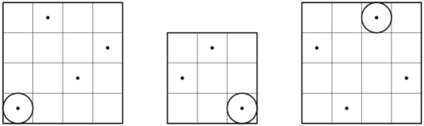
Presheaf of marked permutations - use the inflation product.

$\pi =$  , $\sigma_1 =$ 

Inflation of $\pi * \sigma$ is



Examples of indecomposable marked permutations:



# Unique factorisation theorem on marked permutations

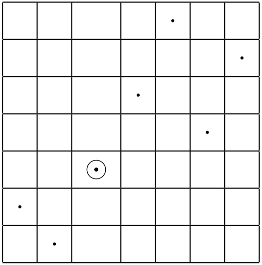
- The factorisation is **not** unique up to order of factors.
- The order of the factors does matter **only to some extent.**

The inflation map is a morphism of monoids $* : \mathcal{W}(\mathcal{I}) \to \mathcal{A}(\texttt{MPer})$.
If $\tau_1, \tau_2$ $\oplus$-indecomposable.

$$(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) * (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1 \,.$$

For $\tau_1 = 2413$ and $\tau_2 = 21$ we have

$$(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = 21 \oplus \bar{1} \oplus 2413 =$$



# Unique factorisation theorem on marked permutations

Monoid morphism $* : \mathcal{W}(\mathcal{I}) \to \mathcal{A}(\texttt{MPer})$

$\oplus$- relations : $(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) * (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1$ .

**Theorem (P - 2019+)**

*The equivalence relation* $\ker *$ *is spanned by relations as the one above and their* $\ominus$ *equivalent.*

# Further questions - *Permutons*

*Permutons* - Notion of patterns of a permutation $\pi$ can be extended to a permuton $P$: $\mathbf{p}_\pi(P) = \mathbb{P}[$ n i.i.d. points with law P form pattern $\pi]$.

**Conjecture**

*Let $\mathcal{L}_q = \{\mathbf{p}_l \,|\, l$ is a Lyndon permutation with size $\leq q\}$ be the set of free generators of $\mathcal{A}(\mathtt{Per})$. The image of the map*

$$\prod_{l \in \mathcal{L}_q} \mathbf{p}_l : \{\textit{Permutons}\} \to \mathbb{R}^{\#\mathcal{L}_q}\,,$$

*is full dimensional.*

Partial results for the map $\prod_{l \in \mathcal{I}} \mathbf{p}_l$ by Kenyon, Krall et al.

# Further questions - *Algebra*

- *Character Theory:* simple characters can be constructed. All these have ”compact support”:

$$\zeta_a(\mathbf{p}_b) = \mathbf{p}_b(a)\,,$$

  and all its convolutions. Can we describe all characters? Are these all ”compactly supported characters” of a free pattern algebra?
- *Freeness:* Are pattern algebras free in general? Other examples include set compositions, polyominoes, etc.
- *Other monoidal products:* The ”marked” operations come from the *operad* monoidal structure on permutations. *Hadamard product*? *Heisenberg product*?

# Biblio


- Aguiar, M., & Mahajan, S. A. (2010). Monoidal functors, species and Hopf algebras (Vol. 29). *Providence, RI: American Mathematical Society*.
- Vargas, Y. (2014). Hopf algebra of permutation pattern functions. In Discrete Mathematics and Theoretical Computer Science (pp. 839-850). *Discrete Mathematics and Theoretical Computer Science*.
- Kenyon, R., Kral, D., Radin, C., & Winkler, P. (2015). Permutations with fixed pattern densities. *arXiv preprint* arXiv:1506.02340.

# Thank you