Probability 2

Exercise sheet nb. 4

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Due until: 15th October at 5 p.m.

Exercises marked with * should be easier after attending the lecture on Thursday.

Exercise 1 (3 points). Let $(X_n)_{n\geq 1}$ be a martingale, and suppose that each X_n is in L^2. We say that $(X_n)_{n\geq 0}$ is *bounded in* L^2 if $\sup_{n\geq 1}\mathbb{E}[X_n^2] < \infty$.

1. Show that if $(X_n)_{n\geq 1}$ is bounded in L^2, then $\mathbb{E}[X_n^2]$ converges.
2. Show that if $p \geq q$, then
$$\mathbb{E}[X_p X_q] = \mathbb{E}[X_q^2]\,.$$
3. Conclude that if $(X_n)_{n\geq 1}$ is bounded in L^2, then the sequence $(X_n)_{n\geq 1}$ converges in L^2 (Hint: show that it is a Cauchy sequence, what do you know about Cauchy sequences in L^2?).

Exercise 2 (4 points). Let $p \in (0, 0.5)$ and $\{Y_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with
$$\mathbb{P}[Y_1 = 1] = p,\ \mathbb{P}[Y_1 = -1] = 1-p.$$
Let $\{X_n\}_{n\geq 0}$ be the random process defined by $X_n = \sum_{k=1}^n Y_k$, and consider the associated random process $Z_n = \left(\frac{1-p}{p}\right)^{X_n}$. Recall that in exercise 3 of sheet 3, we showed that $\{Z_n\}_{n\geq 0}$ is a martingale w.r.t. the filtration $\mathcal{F}_n = \sigma(Y_1,\dots,Y_n)$.

1. Show that
$$\mathbb{P}\Big(\sup_{n\geq 0} X_n \geq k\Big) \leq \left(\frac{p}{1-p}\right)^k.$$
Deduce that $\sup_{n\geq 0} X_n < \infty$ almost surely, and that
$$\mathbb{E}\Big(\sup_{n\geq 0} X_n\Big) \leq \frac{p}{1-2p}.$$

2. Recall that the strong law of large numbers asserts that $\frac{X_n}{n}$ tends almost surely to $\mathbb{E}(X_1)$. Use this to give another proof that $\sup\limits_{n \geq 1} X_n < \infty$ almost surely.

Exercise 3 (3 points). Let $(M_n)_{n \geq 1}$ be a bounded martingale with $M_0 = 0$ a.s. that it, it exists some constant A (independent of n) such that $M_n \leq A$ a.s. Consider $X_n = \exp(M_n)$.

1. Show that X_n is a submartingale.
2. Use Doob's decomposition of X_n to show that $\{\mathbb{E}(\exp(M_n))\}_{n \geq 1}$ is non-decreasing.
3. * Show that X_n converges a.s.