Probability 2

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Due until: 24th September at 17h

Exercises marked with * should be easier after attending the lecture on Thursday.

Exercise 1 (2 points). Let X be a $\mathrm{Binom}(n, p)$ distributed r.v. and let Y follow a $\mathrm{Ber}\left(\frac{X}{n}\right)$ distribution.

(a) Describe the law of Y;

(b) Describe the r.v. $\mathbb{E}[X|Y]$ as a function of Y.

Exercise 2 (2 points). Let X and Y be independent uniform random variables in $\{1, \dots, 6\}$ and let $S = X + Y$. Compute $\mathbb{E}(X|X)$, $\mathbb{E}(X|Y)$, $\mathbb{E}(X|S)$, $\mathbb{E}(Y|X)$ and the remaining 5 conditional expectations.

(Hint: for $\mathbb{E}(X|S)$, $\mathbb{E}(Y|S)$, it might help to first prove that they are equal to each other and consider the sum.)

Exercise 3 (4 points *). For a real-valued random variable Y, we characterized all random variables X that are $\sigma(Y)$-measurable in Lemma 2.12. In this exercise we will prove one of the implications in this lemma[1]. Let X be a nonnegative or integrable function $\Omega \to \mathbb{R}$, measurable with respect to the σ-algebra

$$\sigma(Y) := \{A \subseteq \Omega : \, A = Y^{-1}(B) \text{ for some Borel subset } B \subseteq \mathbb{R}\}.$$

The goal is to show that $X = f(Y)$ for some measurable function $f : \mathbb{R} \to \mathbb{R}$.

(a) Prove the statement when X is a step function, i.e. a finite sum $\sum_i a_i \mathbf{1}_{A_i}$; (Hint: we can choose A_i that are $\sigma(Y)$ measurable)

(b) Prove the statement when X is nonnegative (recall that nonnegative measurable functions can be written as pointwise limits of measurable step functions X_n);

(c) Prove the statement when X is in L^1 (write $X = X^+ - X^-$ with X^+ and X^- nonnegative).

[1] This is in fact the only non-trivial implication in the lemma.

Exercise 4 (2 points *). If $\mathcal{H}$ is a real Hilbert space, and $\mathcal{L} \subseteq \mathcal{H}$ a subspace, the projection operator $\Pi : \mathcal{H} \to \mathcal{L}$ is defined as

$$\Pi x := \min \arg_{y \in \mathcal{L}} \{||x - y||\} .$$

Show the following properties of Π:

(a) $\Pi x = x$ for $x \in \mathcal{L}$;

(b) $\Pi^2 = \Pi$, that is, Π is idempotent;

(c) Πx is the unique element in $\mathcal{L}$ such that for any $y \in \mathcal{L}$

$$\langle y, \Pi x \rangle = \langle y, x \rangle ; \tag{CP}$$

(d) $\Pi x = 0$ for $x \in \mathcal{L}^{\perp}$.