Chromatic problems in polytope Hopf algebras

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The chromatic symmetric function on graphs

A *colouring* on a graph G is a map $f : V(G) \to \mathbb{N}$.
It is *proper* if $f(v_1) \neq f(v_2)$ when $\{v_1, v_2\} \in E(G)$.

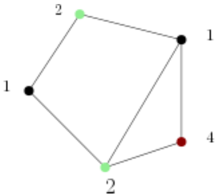


Figure: A proper colouring f^* of a graph

Set $x_f = \prod_v x_{f(v)}$. We have $x_{f^*} = x_1^2 x_2^2 x_4$ in the figure.

The *chromatic symmetric function* (CF) is $\Psi_{\mathbf{G}}(G) = \sum_{f \text{ proper}} x_f$.

CF on graphs - The kernel problem

Question (The kernel problem on graphs)

Describe all linear relations of the form

$$\sum_i a_i \Psi_{\mathbf{G}}(G_i) = 0\,.$$

Let $\mathbf{G}$ = the linear span of all graphs.
Equivalent to find kernel of the linear extension of $\Psi_{\mathbf{G}} : \mathbf{G} \to QSym$.

Outline

Symmetric functions

A weak composition of n is an infinite list $\alpha = (\alpha_1, \dots,)$ of non-negative integers that sum up to n.
Write $x^\alpha = \prod_i x_i^{\alpha_i}$.
Example: $\beta = (3, 1, 2, 1, 0, 0, \cdots)$ weakly composes 7.
We have $x^\beta = x_1^3 x_2 x_3^2 x_4$.
A homogeneous symmetric function of degree n is a sum of the form

$$f = \sum_\alpha a_\alpha x^\alpha \,,$$

where the sum runs over weak compositions of n, and reordering $\alpha \to \beta$ preserves the coefficient $a_\alpha = a_\beta$ (i.e. changing $x_i \leftrightarrow x_j$ does not change the sum).

Symmetric functions

The graded ring of *symmetric functions* $Sym = \oplus_{n\geq 0} Sym_n$ is the span of all homogeneous symmetric functions.

Monomial basis of Sym_n is $m_\lambda = \sum_{\lambda(\alpha)=\lambda} x^\alpha$, where the sum runs over weak compositions that, after reordering, generate the partition λ.

The *chromatic symmetric function* on a graph is a symmetric function.

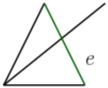
Proposition (Monomial formula for graphs)

$$\Psi_{\mathbf{G}}(G) = \sum_{\pi} \mathrm{aut}_{\lambda(\pi)}\, m_{\lambda(\pi)}\,,$$

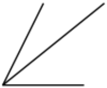
where the sum runs over all stable set partitions.

Graphs terminology

The edge deletion of a graph: $H \setminus \{e\}$.

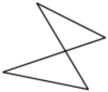


H



$H \setminus \{e\}$

The edge addition of a graph: $G + \{e\}$.



G

$G + \{e\}$

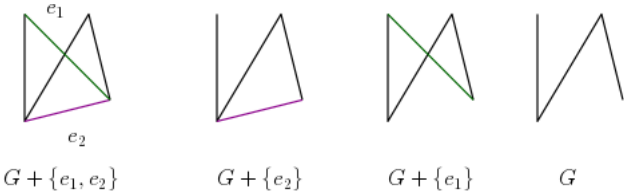
Modular relations

$$\Psi_{\mathbf{G}}(G) = \sum_{f \text{ proper on } G} x_f \,.$$

Proposition (Modular relations - Guay-Paquet, Orellana, Scott, 2013)

Let G be a graph that contains an edge e_3 and does not contain e_1, e_2 such that the edges $\{e_1, e_2, e_3\}$ form a triangle. Then,

$$\Psi_{\mathbf{G}}(G) - \Psi_{\mathbf{G}}(G + \{e_1\}) - \Psi_{\mathbf{G}}(G + \{e_2\}) + \Psi_{\mathbf{G}}(G + \{e_1, e_2\}) = 0\,.$$



The kernel problem

For G_1, G_2 isomorphic graphs, we have $G_1 - G_2 \in \ker \Psi_{\mathbf{G}}$. These are called *isomorphism relation*.

Theorem (RP-2017)

The kernel of $\Psi_{\mathbf{G}}$ is generated by modular relations and isomorphism relations.

Let $\mathcal{M} = \langle \text{ modular relations}, \text{ isomorphism relations } \rangle \subseteq \mathbf{G}$.
Goal: $\ker \Psi_{\mathbf{G}} = \mathcal{M}$.

Idea of proof - Rewriting graph combinations

$$\Psi_{\mathbf{G}}(G) - \Psi_{\mathbf{G}}(G + \{e_1\}) - \Psi_{\mathbf{G}}(G + \{e_2\}) + \Psi_{\mathbf{G}}(G + \{e_1, e_2\}) = 0\,.$$

- Take $z = \sum_i G_i a_i \in \mathbf{G}/\mathcal{M}$ in the kernel of $\tilde{\Psi}_{\mathbf{G}} : \mathbf{G}/\mathcal{M} \to Sym$.
  Goal: show that $z = 0$.
- Some of the G_i can be rewritten as graphs with more edges (through modular relation). We call them *extendible*.
- The badly behaved graphs $\{H_1, H_2, \cdots\}$ are not a lot, and $\{\Psi_{\mathbf{G}}(H_1), \Psi_{\mathbf{G}}(H_2), \cdots\}$ is linearly independent.
- Linear algebra magic. Cash in the theorem.

Idea of proof - Rewriting graph combinations

$$\Psi_{\mathbf{G}}(G) - \Psi_{\mathbf{G}}(G + \{e_1\}) - \Psi_{\mathbf{G}}(G + \{e_2\}) + \Psi_{\mathbf{G}}(G + \{e_1, e_2\}) = 0\,.$$

Proposition (Non-extendible graphs)

A graph is non-extendible if and only if any connected component G^c, the complement graph of G, is a complete graph.

Consequence: Up to isomorphism, we can identify naturally a partition λ with a non-extendible graph K^c_λ in such a way $\lambda = \lambda(G^c)$.
Possible to show: the set $\{\Psi_{\mathbf{G}}(K^c_\lambda)\}_\lambda$ is linearly independent.

$$z = \sum_\lambda K^c_\lambda a_\lambda \in \ker \Psi_{\mathbf{G}}\,,$$

Idea of proof - Rewriting graph combinations

So

$$z = \sum_{\lambda} K_{\lambda}^{c} a_{\lambda} \in \ker \Psi_{\mathbf{G}} \, ,$$

Apply $\Psi_{\mathbf{G}}$ to get

$$0 = \sum_{\lambda} \Psi_{\mathbf{G}}(K_{\lambda}^{c}) a_{\lambda} \Rightarrow a_{\lambda} = 0 \, .$$

So $z = 0$, as desired.

Quasisymmetric functions

A homogeneous *quasisymmetric* function of degree n is a sum of the form

$$f = \sum_{\alpha} a_\alpha x^\alpha \,,$$

where the sum runs over weak compositions of n, and the coefficients respect $a_\alpha = a_\beta$ whenever β is obtained from α by changing the order **of the zeroes.**

Monomial basis of $QSym_n$:

$$M_\alpha = \sum_{\alpha(\beta)=\alpha} x^\beta \,,$$

where the sum runs over weak compositions that, after deleting zeroes, generate the (strong) composition α.

CF on matroids

Let $M = (I, \mathcal{B})$ be a matroid, for I finite set and $\mathcal{B} \subseteq \mathcal{P}(I)$ a set of bases.
A colouring f of M is a map $f : I \to \mathbb{N}$. It is called M-*generic* if

$$B \mapsto \sum_{b \in B} f(b)$$

has a minimum in a unique basis $B \in \mathcal{B}$.
The chromatic quasisymmetric function on matroids is then defined as

$$\Psi_{\mathbf{Mat}}(M) = \sum_{f \text{ is M-generic}} x_f \, .$$

CF on posets

For a poset, a colouring $f : P \to \mathbb{N}$ is called *non-decreasing* if $a \leq b \Rightarrow f(a) \leq f(b)$.
The chromatic quasisymmetric function on posets is then defined as

$$\Psi_{\mathbf{Pos}}(P) = \sum_{f \text{ non-decreasing}} x_f \, .$$

Theorem (Féray, 2014)

The kernel of $\Psi_{\mathbf{Pos}}$ is generated by the cyclic inclusion exclusion relations and isomorphism relations.

(Graded) Hopf algebras

Given a field $\mathbb{K}$, a graded Hopf algebra is a linear space $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}_n$ with graded operations μ and Δ.

- Operation $\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is a multiplication and says how to merge two objects together.
- Operation $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a comultiplication and says how to split an object into two.

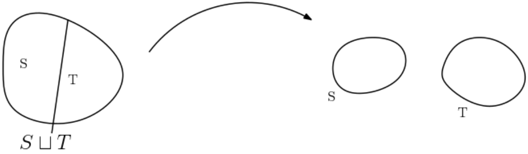


Figure: The coproduct determines how objects decompose

- Some extra conditions for compatibility and an antipode $s : \mathcal{H} \to \mathcal{H}$.

(Graded) Hopf algebras

Examples:

- The one dimensional vector space $\mathbb{K}$.
- Sym and $QSym$.
- The vector space spanned freely by graphs $\mathbf{G}$.

Hopf algebra structure on graphs:
The multiplication $\mu(G_1, G_2)$ is a graph with vertices $V(G_1) \sqcup V(G_2)$, and edges $E(G_1) \sqcup E(G_2)$, with some relabelling.
Graph comultiplication ΔG is a linear combination of graphs

$$\Delta G = \sum_{S \sqcup T = V(G)} G|_S \otimes G|_T \, .$$

CF in combinatorial Hopf algebras

Character: a linear map $\eta : \mathcal{H} \to \mathbb{K}$, preserves multiplication and unit.

On graphs: $\eta(G) = \mathbb{1}[G \text{ has no edges}]$.
On $QSym$: $\eta_0(M_\alpha) = \mathbb{1}[\exists_{n\geq 0} \alpha = (n)]$.

Theorem (Aguiar, Bergeron and Sottile, 2006)

For a combinatorial Hopf algebra $(\mathcal{H}, \eta)$ there is a unique Hopf algebra morphism $\Psi_\mathcal{H}$ that makes the diagram commute:

$$\begin{tikzcd}
\mathcal{H} \arrow[rr, "\Psi_\mathcal{H}"] \arrow[dr, "\eta"'] & & QSym \arrow[dl, "\eta_0"] \\
& k &
\end{tikzcd}$$

CF in combinatorial Hopf algebras

For a composition α of size l, η_α is the composition:

$$\mathcal{H} \xrightarrow{\Delta^{(l-1)}} \mathcal{H}^{\otimes l} \xrightarrow{\pi_\alpha} \mathcal{H}^{\otimes l} \xrightarrow{\eta^{\otimes l}} \mathbb{K}^{\otimes l} \cong \mathbb{K}\,.$$

For $a \in \mathcal{H}_n$, the unique Hopf algebra morphism is

$$\Psi_{\mathcal{H}}(a) = \sum_\alpha \eta_\alpha(a) M_\alpha\,,$$

where the sum runs over compositions of n.

CF in combinatorial Hopf algebras

For the graph Hopf algebra $\mathbf{G}$, if we choose the character $\eta(G) = \mathbb{1}[G$ has no edges$]$, we obtain $\Psi_{\mathbf{G}}$.

For the poset Hopf algebra $\mathbf{Pos}$, if we choose the character $\eta(P) = \mathbb{1}[P$ is an anti-chain$]$, we obtain $\Psi_{\mathbf{Pos}}$.

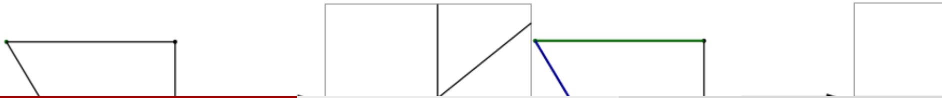
For the matroid Hopf algebra $\mathbf{Mat}$, if we choose the character $\eta(M) = \mathbb{1}[M$ has a unique basis$]$, we obtain $\Psi_{\mathbf{Mat}}$.

Polytopes and fans

A polytope is a bounded set of the form $\mathfrak{q} = \{x \in \mathbb{R}^n | Ax \leq b\}$.

Functional $f : \{1, \cdots, n\} \to \mathbb{R}$. Colouring $f : \{1, \cdots, n\} \to \mathbb{N}$.
$\to$ Linear optimisation problem $\min_{(x_i)_{i=1}^n} \sum_{i=1}^n f(i) x_i$
$\to$ Solution $\mathfrak{q}_f$ is called a *face*.

This partitions the colourings into *cones*, for each face. This partition is called the *normal fan* of a polytope.



The permutahedron and its generalisations

The n order permutahedron is $\mathfrak{per} = \mathrm{conv}\{(\sigma(1), \ldots, \sigma(n)) | \sigma \in S_n\}$.
Is $(n-1)$-dimentional.

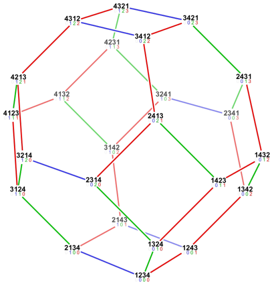


Figure: The 4-permutahedron[1]

[1] https://en.wikipedia.org/wiki/Permutohedron

The permutahedron and its generalisations

For a cone C of $\mathfrak{per}$,

$$\sum_{f \in C} x_f$$

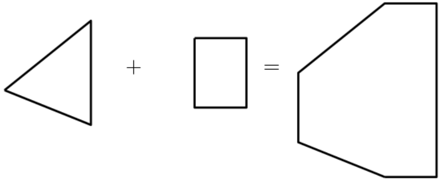
is a quasisymmetric function.

A *generalised permutahedra* $\mathfrak{q}$ is a polytope which fan coarsens the normal fan of the permutahedron (i.e. results from merging cones from the n-permutahedron).Define the CF:

$$\Psi_{\mathbf{GP}}(\mathfrak{q}) = \sum_{\mathfrak{q}_f = \text{ point}} x_f \,.$$

Minkowsky sum

$$A +_M B = \{a + b | a \in A, b \in B\} .$$



$C := A -_M B$ if $A = C +_M B$.
C may not exist but if exists it is **unique** (only for polytopes).

Minkowsky sum

Examples of generalised permutahedra:
The J-simplex, for $J \subseteq \{1, \cdots, n\}$: $\mathfrak{s}_J = \operatorname{conv}\{e_j | j \in J\}$ and its dilations.
The permutahedron

$$\mathfrak{per} = \operatorname{conv}\{(\sigma(1), \dots, \sigma(n)) | \sigma \in S_n\} .$$

The permutahedron is also given as

$$\mathfrak{per} = {}^{M}\sum_{i \leq j} \mathfrak{s}_{\{i,j\}} .$$

Generalised permutahedra and nestohedra

A *generalised permutahedron* is a polytope $\mathfrak{q}$ of the form

$$\mathfrak{q} = \left({}^{M}\!\!\sum_{\substack{J \neq \emptyset \\ a_J > 0}} a_J \mathfrak{s}_J \right) -_M \left({}^{M}\!\!\sum_{\substack{J \neq \emptyset \\ a_J < 0}} |a_J| \mathfrak{s}_J \right) ,$$

A *nestohedron* is only the positive part:

$$\mathfrak{q} = {}^{M}\!\!\sum_{\substack{J \neq \emptyset \\ a_J > 0}} a_J \mathfrak{s}_J \, .$$

Zonotopes and other embedings

Given a graph G, its zonotope is defined as

$$Z(G) = {}^{M}\sum_{e \in E(G)} \mathfrak{s}_e \, .$$

This is a Hopf algebra morphism, so

$$\Psi_{\mathbf{G}} = \Psi_{\mathbf{GP}} \circ Z \, .$$

There is also a Hopf algebra embedding $Z : \mathbf{Mat} \to \mathbf{GP}$.

$$\Psi_{\mathbf{Mat}} = \Psi_{\mathbf{GP}} \circ Z \, .$$

Faces of nestohedra

For a colouring f, note that

$$\mathfrak{q}_f = {}^M\sum_{\substack{J\neq\emptyset \\ a_J>0}} (a_J \mathfrak{s}_J)_f = \text{ point } \Leftrightarrow \forall_{J:a_J>0}\ (\mathfrak{s}_J)_f = \text{ point},$$

This allows us to establish a parallel for the modular relation on graphs:

Proposition (Modular relations on nestohedra)

Consider a nestohedron $\mathfrak{q}$*,* $\{B_j | j \in T\}$ *a family of subsets on* $\{1, \cdots n\}$ *and* $\{a_j | j \in T\}$ *some positive scalars. Suppose "some magic" happens. Then,*

$$\sum_{T\subseteq J} (-1)^{\#T}\, \Psi_{\mathbf{GP}} \left[\mathfrak{q} +_M {}^M\sum_{j\in T} a_j \mathfrak{s}_{B_j} \right] = 0\,.$$

K_π^c parallel and conclusion of proof

The nestohedra that are not *extendable* are exactly

$$\mathfrak{p}^f = \sum_{J : (\mathfrak{s}_J)_f = \text{ point}} a_J \mathfrak{s}_J \, ,$$

for positive a_J.
Up to isomorphism there is only one such $\mathfrak{p}^\alpha$ for each composition α of n. Also, $\{\Psi_{\mathbf{GP}}(\mathfrak{p}^\alpha)\}_\alpha$ are linearly independent.

Theorem (RP 2017)

The modular relations and the isomorphism relations span the kernel of the restriction of $\Psi_{\mathbf{GP}}$ to the nestohedra.

Tree conjecture on graphs

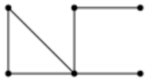


Figure: Non-isomorphic graphs with the same CSF

Conjecture (Tree conjecture -Stanley and Stembridge)

Any two non-isomorphic trees T_1, T_2 have distinct CSF.

Tree conjecture on graphs

This is a graph invariant:

$$\chi'(G) = \sum_f x_f \prod_i q_i^{\#\text{ monochromatic edges in } f \text{ of colour } i}$$

where the sum runs over all colourings.
The modular relations and isomorphism relations are in $\ker \chi'$. So

$$\ker \Psi_{\mathbf{G}} \subseteq \chi' .$$

Conjecture (Tree conjecture -Stanley and Stembridge)

Any two non-isomorphic trees T_1, T_2 have distinct χ'.

Further questions

- From nestohedra to generalised permutahedra?
- Modular relations on matroids?
- The image of the CF on graphs $\Psi_{\mathbf{G}}$ is spanned by $\{\Psi_{\mathbf{G}}(K_\lambda^c)\}_\lambda$, which forms a basis of $\operatorname{im}\Psi_{\mathbf{G}}$. Combinatorial meaning of the coefficients?

Thank you