Chromatic symmetric functions on graphs and polytopes

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Raúl Penaguião

University of Zurich

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The chromatic symmetric function on graphs

A *colouring* on a graph G is a map $f : V(G) \to \mathbb{N}$.
It is *proper* if $f(v_1) \neq f(v_2)$ when $\{v_1, v_2\} \in E(G)$.

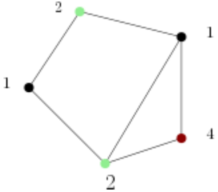


Figure: Example of a proper colouring f of a graph

Set $x_f = \prod_v x_{f(v)}$. We have $x_f = x_1^2 x_2^2 x_4$ in the figure.

The chromatic symmetric function on graphs

The *chromatic symmetric function* (CSF) of G is $\Psi_{\mathbf{G}}(G) = \sum_{f \text{ proper}} x_f$.

This is a Hopf algebra morphism between $\mathbf{G} = \mathrm{span}\{$ all graphs $\}$ and Sym.

Example:

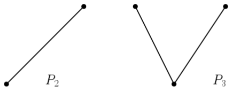


Figure: The line graph P_2 and the path P_3

Their CSF are

$$\Psi_{\mathbf{G}}(P_2) = 2 \sum_{1 \leq i < j} x_i x_j \,, \quad \Psi_{\mathbf{G}}(P_3) = 6 \left(\sum_{1 \leq i < j < k} x_i x_j x_k \right) + \left(\sum_{i \neq j} x_i^2 x_j \right) .$$

Tree conjecture on graphs

Evaluating $x_1 = \cdots = x_t = 1$ and $x_i = 0$ for $i > t$ we obtain the chromatic polynomial $\chi_G(t)$.
With the CSF, we can compute **the number of connected components**, compute the **degree sequence** for trees, etc... , but

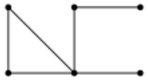


Figure: Non-isomorphic graphs with the same CSF[1]

Conjecture (Tree conjecture - Stanley and Stembridge)

Any two non-isomorphic trees T_1, T_2 have distinct CSF.
Think about the chromatic polynomial

[1] Rose Orelanna and Scott

CF on graphs - The kernel problem

Question (The kernel problem on graphs)

Compute generators of $\ker \Psi_{\mathbf{G}}$. *I.e. describe all linear relations of the form*

$$\sum_i a_i \Psi_{\mathbf{G}}(G_i) = 0\,.$$

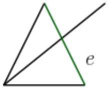
Theorem (RP-2017)

The space $\ker \Psi_{\mathbf{G}}$ *is spanned by the modular relations and isomorphism relations.*

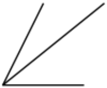
Outline

Graphs terminology

The edge deletion of a graph: $H \setminus \{e\}$.

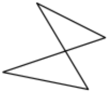


H



$H \setminus \{e\}$

The edge addition of a graph: $G + \{e\}$.



G

$G + \{e\}$

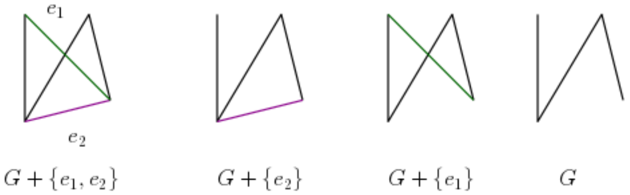
Modular relations

$$\Psi_{\mathbf{G}}(G) = \sum_{f \text{ proper on } G} x_f \,.$$

Proposition (Modular relations - Guay-Paquet, Orellana, Scott, 2013)

Let G be a graph that contains an edge e_3 and does not contain e_1, e_2 such that the edges $\{e_1, e_2, e_3\}$ form a triangle. Then,

$$\Psi_{\mathbf{G}}(G) - \Psi_{\mathbf{G}}(G + \{e_1\}) - \Psi_{\mathbf{G}}(G + \{e_2\}) + \Psi_{\mathbf{G}}(G + \{e_1, e_2\}) = 0 \,.$$



The kernel problem

For G_1, G_2 isomorphic graphs, we have $G_1 - G_2 \in \ker \Psi_{\mathbf{G}}$. These are called *isomorphism relation*.

Theorem (RP-2017)

The kernel of $\Psi_{\mathbf{G}}$ is generated by modular relations and isomorphism relations.

Let $\mathcal{M} = \langle \text{ modular relations}, \text{ isomorphism relations } \rangle$.
Goal: $\ker \Psi_{\mathbf{G}} = \mathcal{M}$.

Idea of proof - Rewriting graph combinations

Condition to be a modular relation:

$$e_3 \in G \Rightarrow G - (G + \{e_1\}) - (G + \{e_2\}) + (G + \{e_1, e_2\}) \in \mathcal{M} \,.$$

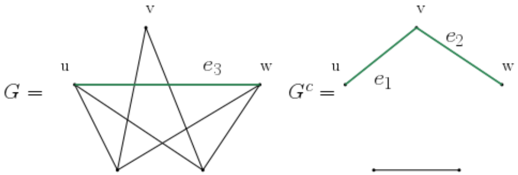
- Take $z = \sum_i G_i a_i$ in the kernel of $\Psi_{\mathbf{G}}$.
  Goal: by working on $\ker \Psi_{\mathbf{G}} / \mathcal{M}$, show that $z \in \mathcal{M}$.
- Some of the G_i can be rewritten as graphs with more edges (through modular relation). We call them *extendible*.
- The *non-extendible* graphs $\{H_1, H_2, \cdots\}$ are not a lot, and $\{\Psi_{\mathbf{G}}(H_1), \Psi_{\mathbf{G}}(H_2), \cdots\}$ is linearly independent.
- Linear algebra 'magic' $\Rightarrow$ a theorem is born.

Idea of proof - Rewriting graph combinations

$$e_3 \in G \Rightarrow G - (G + \{e_1\}) - (G + \{e_2\}) + (G + \{e_1, e_2\}) \in \mathcal{M}\,.$$

Proposition (Non-extendible graphs)

A graph is non-extendible if and only if any connected component of G^c, the complement graph of G, is a complete graph.



Idea of proof - Linear algebra magic

So, always working on $\ker \Psi_{\mathbf{G}} / \mathcal{M}$, we car rewrite:

$$z = \sum_{\lambda \in \mathcal{P}_n} K_\lambda^c a_\lambda \in \ker \Psi_{\mathbf{G}} \,,$$

Apply $\Psi_{\mathbf{G}}$ to get

$$0 = \sum_{\lambda \in \mathcal{P}_n} \Psi_{\mathbf{G}}(K_\lambda^c) a_\lambda \Rightarrow a_\lambda = 0 \,.$$

Possible to show: the set $\{\Psi_{\mathbf{G}}(K_\lambda^c)\}_{\lambda \in \mathcal{P}_n}$ is linearly independent. So $z = 0$, as desired.

Polytopes

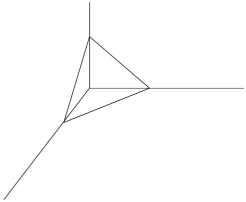
Fix a dimension n. A polytope is a bounded set of the form $\mathfrak{q} = \{x \in \mathbb{R}^n | Ax \leq b\}$.
Given a colouring $f : [n] \to \mathbb{N}$ of the **coordinates**, the face $\mathfrak{q}_f$ is

$$\mathfrak{q}_f = \arg\min_{x \in \mathfrak{q}} \sum_{i=1}^{n} x_i f(i) \, .$$

Polytopes: Examples

Simplexes and its dilations: Consider $J \subseteq [n]$ non empty.

$$\lambda \mathfrak{s}_J = \text{conv}\{\lambda e_i | i \in J\}\,.$$



The permutahedron and its generalisations

The n order permutahedron: $\mathfrak{per} = \mathrm{conv}\{(\sigma(1), \ldots, \sigma(n)) | \sigma \in S_n\}$.
Is $(n-1)$-dimensional.

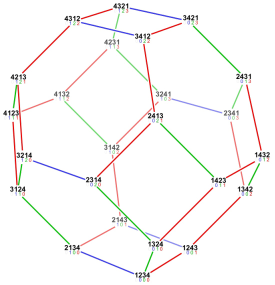


Figure: The 4-permutahedron[2]

[2] https://en.wikipedia.org/wiki/Permutohedron

The permutahedron and its generalisations

A *generalised permutahedron* is a polytope $\mathfrak{q}$ of the form

$$\mathfrak{q} = \left({}^{M}\sum_{J \neq \emptyset} a_J \mathfrak{s}_J \right) -_M \left({}^{M}\sum_{J \neq \emptyset} b_J \mathfrak{s}_J \right) ,$$

A *nestohedron* is only the positive part:

$$\mathfrak{q} = {}^{M}\sum_{J \neq \emptyset} a_J \mathfrak{s}_J \, .$$

Chromatic function and zonotopes

We define the *chromatic quasisymmetric function* (CF) as

$$\Psi_{\mathbf{GP}}(\mathfrak{q}) = \sum_{\mathfrak{q}_f = \mathrm{pt}} x_f \,.$$

Given a graph G, its zonotope is defined as

$$Z(G) = {}^{M}\sum_{e \in E(G)} \mathfrak{s}_e \,.$$

These are all Hopf algebra morphisms from the Hopf algebra $\mathbf{GP} = \mathrm{span}\{$ generalised permutahedra in $\mathbb{R}^n,\ n \geq 0\}$.
Also,

$$\Psi_{\mathbf{G}} = \Psi_{\mathbf{GP}} \circ Z \,.$$

Some relations in nestohedra

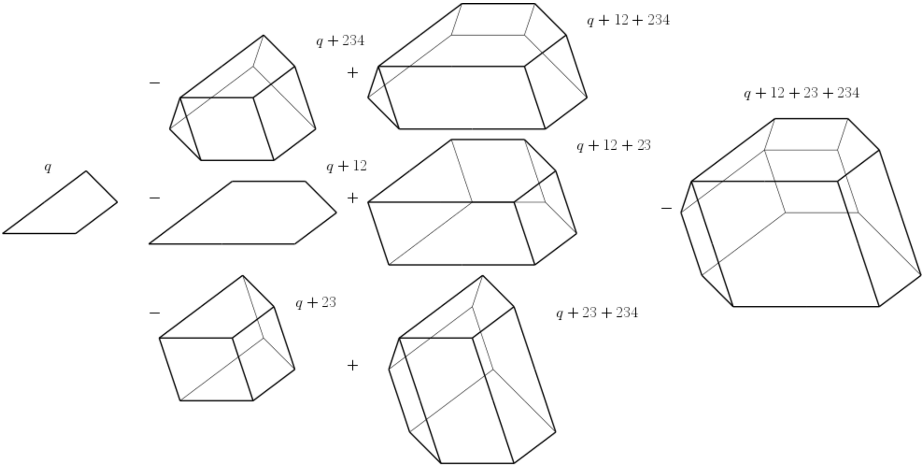
Proposition (Modular relations on nestohedra)

Consider a nestohedron $\mathfrak{q}$, $\{B_j | j \in T\}$ *a family of subsets on* $\{1, \cdots n\}$ *and* $\{a_j | j \in T\}$ *some positive scalars. Suppose "some magic" happens. Then,* $\sum_{T \subseteq J} (-1)^{\# T} \, \Psi_{\mathbf{GP}} \left[\mathfrak{q} +_M {}^M\!\sum_{j \in T} a_j \mathfrak{s}_{B_j} \right] = 0.$

Additionally, there are also the so called **simple relations** - describe that we only care about **which** coefficients are positive, not how big they are.

Some relations on nestohedra - Example

An example of a modular relation:



K_π^c parallel and conclusion of proof

Theorem (RP 2017)

The modular relations, the isomorphism relations and the simple relations span the kernel of the restriction of $\Psi_{\mathbf{GP}}$ to the nestohedra.

Tree conjecture on graphs

The following:

$$\chi'(G) = \sum_f x_f \prod_i q_i^{\#\text{ monochromatic edges in } f \text{ of colour } i}$$

is a graph invariant, where the sum runs over all colourings.
If we consider the projection of this invariant modulo the relations

$$q_i(q_i - 1)^2 = 0\,,$$

then the modular relations are in $\ker \chi'$. We obtain

$$\ker \Psi_{\mathbf{G}} = \ker \chi'\,.$$

Conjecture (Tree conjecture - χ' formulation)

Any two non-isomorphic trees T_1, T_2 have distinct χ'.

Further questions

- From nestohedra to generalised permutahedra?
- The image of the CF on graphs $\Psi_{\mathbf{G}}$ is spanned by $\{\Psi_{\mathbf{G}}(K_\lambda^c)\}_\lambda$, which forms a basis of $\operatorname{im}\Psi_{\mathbf{G}}$. Combinatorial meaning of the coefficients?

Thank you