THE FEASIBLE REGION FOR CONSECUTIVE PATTERNS OF PERMUTATIONS IS A CYCLE POLYTOPE

JACOPO BORGA AND RAUL PENAGUIAO

ABSTRACT. We study proportions of consecutive occurrences of permutations of a given size. Specifically, the limit of such proportions on large permutations forms a region, called *feasible region*. We show that this feasible region is a polytope, more precisely the cycle polytope of a specific graph called *overlap graph*. This allows us to compute the dimension, vertices and faces of the polytope, and to determine the equations that define it. Finally we prove that the limit of classical occurrences and consecutive occurrences are in some sense independent. As a consequence, the scaling limit of a sequence of permutations induces no constraints on the local limit and vice versa.

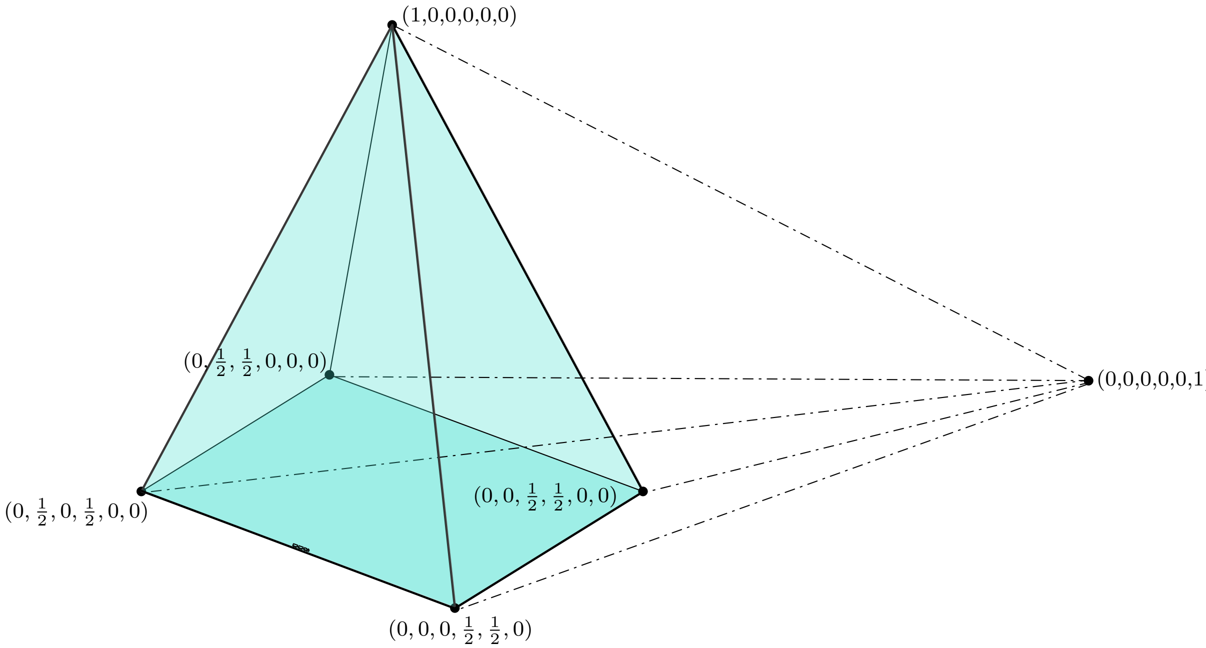


FIGURE 1. The four-dimensional polytope P_3 given by the six patterns of size three (see Eq. (1) for a precise definition). We highlight in light-blue one of the six three-dimensional facets of P_3. This facet is a pyramid with square base. The polytope itself is a four-dimensional pyramid, whose base is the highlighted facet.

2010 *Mathematics Subject Classification.* 52B11,05A05, 60C05.
Key words and phrases. Permutation patterns, cycle polytopes, overlap graphs.

1. INTRODUCTION

1.1. **Motivations.** Despite this article not containing any probabilistic result, we introduce here some motivations that come from the study of random permutations. This is a classical topic at the interface of combinatorics and discrete probability theory. There are two main approaches to the topic: the first concerns the study of statistics on permutations, and the second, more recent, looks at the typical shape of large permutations. The two approaches are not orthogonal and many results relate them, for instance Theorems 1.1 and 1.2 below.

In order to study the shape of permutations, two main notions of convergence have been defined: a global notion of convergence (called permuton convergence) and a local notion of convergence (called Benjamini–Schramm convergence, or BS-convergence, for short). The notion of permuton limit for permutations has been introduced in [19]. A permuton is a probability measure on the unit square with uniform marginals, and represents the scaling limit of a permutation seen as a permutation matrix, as the size grows to infinity. The study of permuton limits is an active and exciting research field in combinatorics, see for instance [5, 6, 7, 11, 23, 26, 29, 30]. On the other hand, the notion of BS-limit for permutations is more recent, having been introduced in [10]. Informally, in order to investigate BS-limits, we study the permutation in a neighborhood around a randomly marked point. Limiting objects for this framework are called *infinite rooted permutations* and are in bijection with total orders on the set of integer numbers. BS-limits have also been studied in some other works, see for instance [9, 11, 12].

We denote by $\mathcal{S}_n$ the set of permutations of size n, by $\mathcal{S}$ the space of all permutations, and by $\widetilde{\mathrm{occ}}(\pi,\sigma)$ (resp. $\widetilde{\text{c-occ}}(\pi,\sigma)$) the proportion of classical occurrences (resp. consecutive occurrences) of a permutation π in σ (see Section 1.8 for notation and basic definitions). The following theorems provide a relevant combinatorial characterizations of the two aforementioned notions of convergence.

Theorem 1.1 ([19])**.** *For any $n \in \mathbb{N}$, let $\sigma^n \in \mathcal{S}$ and assume that $|\sigma^n| \to \infty$. The sequence $(\sigma^n)_{n\in\mathbb{N}}$ converges to some limiting permuton P if and only if there exists a vector $(\Delta_\pi(P))_{\pi\in\mathcal{S}}$ of non-negative real numbers (that depends on P) such that, for all $\pi \in \mathcal{S}$,*

$$\widetilde{\mathrm{occ}}(\pi,\sigma^n) \to \Delta_\pi(P).$$

Theorem 1.2 ([10])**.** *For any $n \in \mathbb{N}$, let $\sigma^n \in \mathcal{S}$ and assume that $|\sigma^n| \to \infty$. The sequence $(\sigma^n)_{n\in\mathbb{N}}$ converges in the Benjamini–Schramm topology to some random infinite rooted permutation σ^∞ if and only if there exists a vector $(\Gamma_\pi(\sigma^\infty))_{\pi\in\mathcal{S}}$ of non-negative real numbers (that depends on σ^∞) such that, for all $\pi \in \mathcal{S}$,*

$$\widetilde{\text{c-occ}}(\pi,\sigma^n) \to \Gamma_\pi(\sigma^\infty).$$

A natural question, motivated by the theorems above, is the following: given a finite family of patterns $\mathcal{A} \subseteq \mathcal{S}$ and a vector $(\Delta_\pi)_{\pi\in\mathcal{A}} \in [0,1]^{\mathcal{A}}$, or $(\Gamma_\pi)_{\pi\in\mathcal{A}} \in [0,1]^{\mathcal{A}}$, does there exist a sequence of permutations $(\sigma^n)_{n\in\mathbb{N}}$ such that $|\sigma^n| \to \infty$ and

$$\widetilde{\mathrm{occ}}(\pi,\sigma^n) \to \Delta_\pi, \quad \text{for all} \quad \pi \in \mathcal{A},$$

or

$$\widetilde{\text{c-occ}}(\pi,\sigma^n) \to \Gamma_\pi, \quad \text{for all} \quad \pi \in \mathcal{A}\,?$$

We consider the classical pattern limiting sets, sometimes called the *feasible region* for (classical) patterns, defined as

$$\begin{aligned} clP_k &:= \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\operatorname{occ}}(\pi,\sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\} \\ &= \left\{ (\Delta_\pi(P))_{\pi\in\mathcal{S}_k} \middle| P \text{ is a permuton} \right\} , \end{aligned}$$

and we introduce the consecutive pattern limiting sets, called here the *feasible region* for consecutive patterns,

$$\begin{aligned} P_k &:= \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi,\sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\} \\ &= \left\{ (\Gamma_\pi(\sigma^\infty))_{\pi\in\mathcal{S}_k} \middle| \sigma^\infty \text{ is a random infinite rooted } \textit{shift-invariant} \text{ permutation} \right\} . \end{aligned} \tag{1}$$

We present the definition of *shift-invariant* permutation in Definition 3.3, and we prove the equality in Eq. (1) in Proposition 3.4.

The feasible region clP_k was previously studied in several papers (see Section 1.2). The main goal of this project is to analyze the feasible region P_k, that turns out to be connected to specific graphs called *overlap graphs* (see Section 1.4) and its corresponding cycle polytopes (see Section 1.5).

1.2. **The feasible region for classical patterns.** The feasible region clP_k was first studied in [23] for some particular families of patterns instead of the whole $\mathcal{S}_k$. More precisely, given a list of finite sets of permutations $(\mathcal{P}_1,\dots,\mathcal{P}_\ell)$, the authors considered the *feasible region* for $(\mathcal{P}_1,\dots,\mathcal{P}_\ell)$, that is, the set

$$\left\{ \vec{v} \in [0,1]^{\ell} \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \sum_{\tau\in\mathcal{P}_i} \widetilde{\operatorname{occ}}(\tau,\sigma^m) \to \vec{v}_i, \text{ for } i=1,\dots,\ell \right\} .$$

They first studied the simplest case when $\mathcal{P}_1 = \{12\}$ and $\mathcal{P}_2 = \{123, 213\}$ showing that the corresponding feasible region for $(\mathcal{P}_1,\mathcal{P}_2)$ is the region of the square $[0,1]^2$ bounded from below by the parameterized curve $(2t-t^2, 3t^2-2t^3)_{t\in[0,1]}$ and from above by the parameterized curve $(1-t^2, 1-t^3)_{t\in[0,1]}$ (see [23, Theorem 13]).

They also proved in [23, Theorem 14] that if each $\mathcal{P}_i = \{\tau_i\}$ is a singleton, and there is some value p such that, for all permutations τ_i, the final element $\tau_i(|\tau_i|)$ is equal to p , then the corresponding feasible region is convex. They remarked that one can construct examples where the feasible region is not strictly convex: e.g. in the case where $\mathcal{P}_1 = \{231, 321\}$ and $\mathcal{P}_2 = \{123, 213\}$.

They finally studied two additional examples: the feasible regions for the patterns $(\{12\},\{123\})$ (see [23, Theorem 15]) and for the patterns $(\{123\},\{321\})$ (see [23, Section 10]). In the first case, they showed that the feasible region is equal to the so-called "scalloped triangle" of Razborov [28, 27] (this region also describes the space of limit densities for edges and triangles in graphs). For the second case, they showed that the feasible region is equal to the limit of densities of triangles versus the density of anti-triangles in graphs, see [20, 21].

The set clP_k was also studied in [17], even though with a different goal. There, it was shown that clP_k contains an open ball B with dimension $|I_k|$, where I_k is the set of $\oplus$-indecomposable permutations of size at most k. Specifically, for a particular ball $B \subseteq \mathbb{R}^{I_k}$, the authors constructed permutons $P_{\vec{x}}$ such that $\Delta_\pi(P_{\vec{x}}) = \vec{x}_\pi$, for each point $\vec{x} \in B$.

This work opened the problem of finding the maximal dimension of an open ball contained in clP_k, and placed a lower bound on it. In [32] an upper bound for this maximal dimension was indirectly given as the number of so-called *Lyndon permutations* of size at most k, whose set we denote $\mathcal{L}_k$. In this article, the author showed that for any permutation π that is not a Lyndon permutation, $\widetilde{\mathrm{occ}}(\pi, \sigma)$ can be expressed as a polynomial on the functions $\{\widetilde{\mathrm{occ}}(\tau, \sigma) | \tau \in \mathcal{L}_k\}$ that does not depend on σ. It follows that clP_k sits inside an algebraic variety of dimension $|\mathcal{L}_k|$. We expect that this bound is sharp since, from our computations, this is the case for small values of k.

Conjecture 1.3. The feasible region clP_k is full-dimensional inside a manifold of dimension $|\mathcal{L}_k|$.

1.3. **First main result.** Unlike with the case of classical patterns, we are able to obtain here a full description of the feasible region P_k as the cycle polytope of a specific graph, called the *overlap graph* $\mathcal{O}v(k)$.

Definition 1.4. *The graph $\mathcal{O}v(k)$ is a directed multigraph with labeled edges, where the vertices are elements of $\mathcal{S}_{k-1}$ and for every $\pi \in \mathcal{S}_k$ there is an edge labeled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.*

The overlap graph $\mathcal{O}v(4)$ is displayed in Fig. 2.

Definition 1.5. *Let $G = (V, E)$ be a directed multigraph. For each non-empty cycle $\mathcal{C}$ in G, define $\vec{e}_{\mathcal{C}} \in \mathbb{R}^E$ so that*

$$(\vec{e}_{\mathcal{C}})_e := \frac{\text{\# of occurrences of } e \text{ in } \mathcal{C}}{|\mathcal{C}|}, \quad \text{for all} \quad e \in E.$$

We define the cycle polytope *of G to be the polytope $P(G) := \mathrm{conv}\{\vec{e}_{\mathcal{C}} | \mathcal{C}$ is a simple cycle of $G\}$.*

Our first main result is the following.

Theorem 1.6. *P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$. Its dimension is $k! - (k-1)!$ and its vertices are given by the simple cycles of $\mathcal{O}v(k)$.*

In addition, we also determine the equations that describe the polytope P_k (for a precise statement see Theorem 3.12).

In order to prove Theorem 1.6, we first prove general results for cycle polytopes of directed multigraphs (see Section 1.5) and then we transfer them to the specific case of overlap graphs.

1.4. **The overlap graph.** Overlap graphs were already studied in previous works. We give here a brief summary of the relevant literature. The overlap graph $\mathcal{O}v(k)$ is the line graph of the *de Bruijn graph for permutations* of size $k-1$. The latter was introduced in [13], where the authors studied universal cycles (sometime also called *de Bruijn cycles*) of several combinatorial structures, including permutations. In this case, a universal cycle of order n is a cyclic word of size $n!$ on an alphabet of N letters that contains all the patterns of size n as consecutive patterns. In [13] it was conjectured (and then proved in [22]) that such universal cycles always exist when the alphabet is of size $N = n+1$.

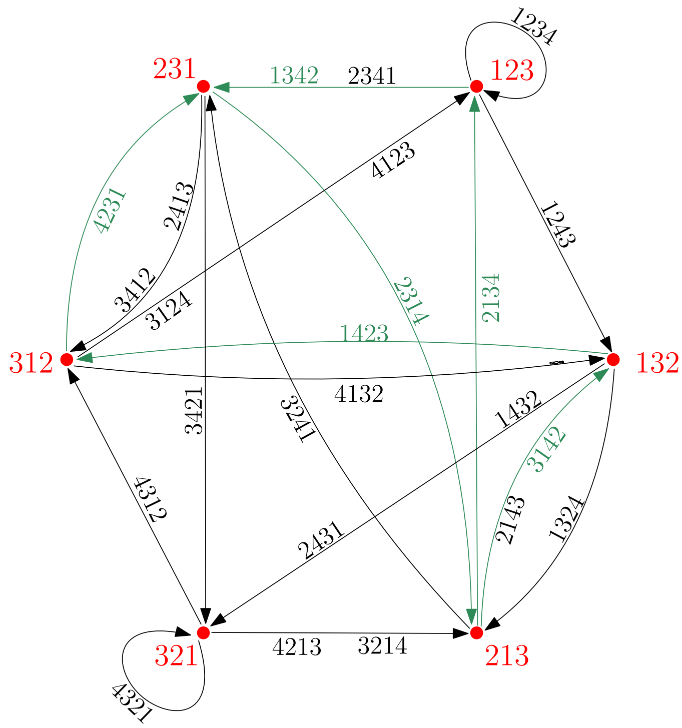


FIGURE 2. The overlap graph $\mathcal{O}v(4)$. The six vertices are painted in red and the edges are drawn as labeled arrows. Note that in order to obtain a clearer picture we did not draw multiple edges, but we use multiple labels (for example the edge $231 \to 312$ is labeled with the permutations 3412 and 2413 and should be thought of as two distinct edges labeled with 3412 and 2413 respectively). The role of the green arrows is clarified in Example 3.7.

The *de Bruijn graph for permutations* was also studied under the name of *graph of overlapping permutations* in [24] again in relation with universal cycles and universal words. Further, in [1] the authors enumerate some families of cycles of these graphs.

We mainly use overlap graphs as a tool to state and prove our results on P_k, rather than exploiting its properties. We remark that, applying the same ideas used to show the existence of Eulerian and Hamiltonian cycles in classical de Bruijn graphs (see [13]), it is easy to prove the existence of both Eulerian and Hamiltonian cycles in $\mathcal{O}v(k)$. In particular, with an Eulerian path in $\mathcal{O}v(k)$, we can construct (although not uniquely) a permutation σ of size $k! + k - 1$ such that $\text{c-occ}(\pi, \sigma) = 1$ for any $\pi \in \mathcal{S}_k$.

1.5. **Polytopes and cycle polytopes.** As said before, we obtain general results for cycle polytopes of directed multigraphs:

Theorem 1.7. *The cycle polytope of a strongly connected directed multigraph $G = (V, E)$ has dimension $|E| - |V|$.*

We also determine the equations defining the polytope (see Theorem 2.13) and we show that all its faces can be identified with some subgraphs of G (see Theorem 2.14). This gives us a description of the face poset of the polytope. Further, the computation of the dimension is generalized for any cycle polytope, even those that do not come from strongly connected graphs (see Theorem 2.12).

Some weaker versions of our results already appeared in the literature. Polytopes similar to the cycle polytopes studied here, called *unrescaled cycle polytopes* (*U-cycle polytopes* for short), were introduced in [2] in the directed version and in [14] in the undirected version[1]. Balas & Oosten [2] and Balas & Stephan [3] computed the dimension of the U-cycle polytope of the *complete graph* (that is, the complete directed graph without loops) and described the facets of the corresponding polytope. Notice that we study cycle polytopes for general directed multigraphs and we do not restrict to the case of complete graphs as in [2, 3].

The U-cycle polytopes for undirected graphs were initially considered to tackle the Simple Cycle Problem (SCP) [18], that also goes by the name of Weighted Girth Problem [8]. This problem consists in finding a minimum weighted simple cycle in an undirected graph with costs associated with each edge. The related decision problem is known to be NP-hard, as it can be reduced to the "travelling salesman" problem (TSP), that asks the following question: "Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?". The Simple Cycle Problem was also considered later in [25].

We point out that other instances of polytopes related to paths in graphs were also investigated. For instance, there is a path version of U-cycle polytopes, considered in [31]. Specifically, the (s,t)-p-path polytope of a directed graph G is the convex hull of the incidence vectors of simple directed (s,t)-paths in G of size p. There, the authors gave some characterizations of the facets of the path polytopes. More concerning this polytope can be found, for instance, in [15, 16].

Also present in the literature is the *flow polytope*, introduced by [4]. This is a polytope that is associated to a root system. For a root system of type A_n, the flow polytope can be computed from a labeled undirected graph in $[n] := \{1,\dots,n\}$: thus, if we are given a graph $G = ([n], E)$ and a flow vector $\vec{a} \in \mathbb{R}^n$, its corresponding flow polytope is

$$\mathcal{F}_G(\vec{a}) := \left\{ \vec{x} \in \mathbb{R}^E \,\middle|\, \sum_{\{j<i\}\in E} \vec{x}_{\{j<i\}} - \sum_{\{i<j\}\in E} \vec{x}_{\{i<j\}} = \vec{a}_i,\, i \in [n] \right\}.$$

Classical examples of polytopes that are flow polytopes are the *Stanley–Pitman polytope*, also called the *parking functions polytope*, and the *Chan–Robbins–Yuen polytope*, a polytope on the space of doubly stochastic square matrices. In [4], the authors obtained formulas for the volume and the number of integer points in its interior. In praticular, they recovered a formula of the volume of the Chan–Robbins–Yuen polytope, due to Zeilberger, in his very short paper [33].

[1]The cycle polytopes introduced in our paper result from a rescaling of the U-cycle polytopes: indeed, the U-cycle polytope of a directed multigraph G is defined as the convex hull of the incidence vectors of cycles of G. We *additionally rescale* the vertices so that the coordinates sum up to one. The U-cycle polytopes were considered in the literature simply under the name of *cycle polytopes*. We adapt the name of *cycle polytopes* to our family of polytopes for the sake of simplifying the terminology in our paper.

Remark 1.8. In [2, Proposition 4], the dimension of the U-cycle polytope for the complete graph on n vertices without loops is computed as $(n-1)^2$. We point out that in Theorem 1.7 we compute the dimension of the cycle polytope of the complete graph as $(n^2-n)-n=(n-1)^2-1$. This is coherent with the previous result, because a cycle polytope has an extra equation given by $\sum_e x_e = 1$ when compared with its corresponding U-cycle polytope.

1.6. **Mixing classical patterns and consecutive patterns.** We saw in Section 1.2 that the feasible region clP_k for classical pattern occurrences has been studied in several papers. In this paper we study the feasible region P_k of limiting points for consecutive pattern occurrences. A natural question is the following: what is the feasible region if we mix classical and consecutive patterns?

We answer this question showing that:

Theorem 1.9. *For any two points $\vec{v}_1 \in clP_k, \vec{v}_2 \in P_k$, there exists a sequence of permutations $(\sigma^m)_{m\in\mathbb{N}}$ such that $|\sigma^m| \to \infty$, satisfying*

$$(\widetilde{\mathrm{occ}}(\pi,\sigma^m))_{\pi\in\mathcal{S}_k} \to \vec{v}_1 \quad \text{and} \quad (\widetilde{\text{c-occ}}(\pi,\sigma^m))_{\pi\in\mathcal{S}_k} \to \vec{v}_2.$$

This result shows a sort of independence between classical patterns and consecutive patterns, in the sense that knowing the proportion of classical patterns of a certain sequence of permutations gives no constraints for the proportion of consecutive patterns of the same sequence and vice versa.

We stress that we provide an explicit construction of the sequence $(\sigma^m)_{m\in\mathbb{N}}$ in the theorem above (for a more precise and general statement, see Theorem 4.1).

We conclude this section with the following observation on local and scaling limits of permutations.

Observation 1.10. In Theorems 1.1 and 1.2 we saw that the proportion of occurrences (resp. consecutive occurrences) in a sequence of permutations $(\sigma^m)_{m\in\mathbb{N}}$ characterizes the permuton limit (resp. Benjamini–Schramm limit) of the sequence. Theorem 1.9 proves that the permuton limit of a sequence of permutations induces no constraints for the Benjamini–Schramm limit and vice versa. For instance, we can construct a sequence of permutations where the permuton limit is the decreasing diagonal and the Benjamini–Schramm limit is the classical increasing total order on the integer numbers.

We remark that a particular instance of this "independence phenomenon" for local/scaling limits of permutations was recently also observed by Bevan, who pointed out in the abstract of [9] that "the knowledge of the local structure of uniformly random permutations with a specific fixed proportion of inversions reveals nothing about their global form". Here, we prove that this is a *universal phenomenon* which is not specific to the framework studied by Bevan.

1.7. **Outline of the paper.** The paper is organized as follows:

- In Section 2 we analyze directed multigraphs and consider their *cycle polytopes*. There, we prove Theorem 1.7 and the results mentioned immediately below it.
- Our results regarding P_k come in Section 3, where we prove Theorem 1.6.
- Finally, we prove in Section 4 a more precise version of Theorem 1.9.

1.8. **Notation.** We summarize here the notation and some basic definitions used in the paper.

Permutations and patterns. We let $\mathbb{N} = \{1, 2, \dots\}$ denote the collection of strictly positive integers. For every $n \in \mathbb{N}$, we view permutations of $[n] = \{1, 2, \dots, n\}$ as words of size n, and write them using the one-line notation $\sigma = \sigma(1)\sigma(2)\dots\sigma(n)$. We denote by $\mathcal{S}_n$ the set of permutations of size n, by $\mathcal{S}_{\geq n}$ the set of permutations of size at least n, and by $\mathcal{S}$ the set of permutations of finite size.

We often view a permutation $\sigma \in \mathcal{S}_n$ as a diagram, specifically as an $n \times n$ board with n points at positions $(i, \sigma(i))$ for all $i \leq n$.

If $x_1, \dots, x_n$ is a sequence of distinct numbers, let $\mathrm{std}(x_1, \dots, x_n)$ be the unique permutation π in $\mathcal{S}_n$ whose elements are in the same relative order as $x_1, \dots, x_n$, i.e. $\pi(i) < \pi(j)$ if and only if $x_i < x_j$. Given a permutation $\sigma \in \mathcal{S}_n$ and a subset of indices $I \subseteq [n]$, let $\mathrm{pat}_I(\sigma)$ be the permutation induced by $(\sigma(i))_{i \in I}$, namely, $\mathrm{pat}_I(\sigma) := \mathrm{std}\left((\sigma(i))_{i \in I}\right)$. For example, if $\sigma = 87532461$ and $I = \{2, 4, 7\}$, then $\mathrm{pat}_{\{2,4,7\}}(87532461) = \mathrm{std}(736) = 312$.

Given two permutations, $\sigma \in \mathcal{S}_n$, $\pi \in \mathcal{S}_k$ for some positive integers $n \geq k$, we say that σ contains π as a *pattern* if there exists a *subset* $I \subseteq [n]$ such that $\mathrm{pat}_I(\sigma) = \pi$, that is, if σ has a subsequence of entries order-isomorphic to π. Denoting by $i_1, i_2, \dots, i_k$ the elements of I in increasing order, the subsequence $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)$ is called an *occurrence* of π in σ. In addition, we say that σ contains π as a *consecutive pattern* if there exists an *interval* $I \subseteq [n]$ such that $\mathrm{pat}_I(\sigma) = \pi$, that is, if σ has a subsequence of adjacent entries order-isomorphic to π. Using the same notation as above, $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)$ is then called a *consecutive occurrence* of π in σ.

We denote by $\mathrm{occ}(\pi, \sigma)$ the number of occurrences of a pattern π in σ, more precisely

$$\mathrm{occ}(\pi, \sigma) := \left| \left\{ I \subseteq [n] | \mathrm{pat}_I(\sigma) = \pi \right\} \right| .$$

We denote by $\text{c-occ}(\pi, \sigma)$ the number of consecutive occurrences of a pattern π in σ, more precisely

$$\text{c-occ}(\pi, \sigma) := \left| \left\{ I \subseteq [n] | \ I \text{ is an interval}, \mathrm{pat}_I(\sigma) = \pi \right\} \right| .$$

Moreover, we denote by $\widetilde{\mathrm{occ}}(\pi, \sigma)$ (resp. by $\widetilde{\text{c-occ}}(\pi, \sigma)$) the proportion of occurrences (resp. consecutive occurrences) of a pattern π in σ, that is,

$$\widetilde{\mathrm{occ}}(\pi, \sigma) := \frac{\mathrm{occ}(\pi, \sigma)}{\binom{n}{k}} \in [0, 1], \qquad \widetilde{\text{c-occ}}(\pi, \sigma) := \frac{\text{c-occ}(\pi, \sigma)}{n} \in [0, 1] .$$

For a fixed $k \in \mathbb{N}$ and a permutation $\sigma \in \mathcal{S}_{\geq k}$, we let $\widetilde{\mathrm{occ}}_k(\sigma), \widetilde{\text{c-occ}}_k(\sigma) \in [0, 1]^{\mathcal{S}_k}$ be the vectors

$$\widetilde{\mathrm{occ}}_k(\sigma) := \left(\widetilde{\mathrm{occ}}(\pi, \sigma)\right)_{\pi \in \mathcal{S}_k}, \qquad \widetilde{\text{c-occ}}_k(\sigma) := \left(\widetilde{\text{c-occ}}(\pi, \sigma)\right)_{\pi \in \mathcal{S}_k} .$$

Finally, we denote with $\oplus$ the direct sum of two permutations, i.e. for $\tau \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$,

$$\tau \oplus \sigma = \tau(1) \dots \tau(k)(\sigma(1) + m) \dots (\sigma(n) + m) ,$$

and we denote with $\oplus_\ell \sigma$ the direct sum of ℓ copies of σ (we remark that the operation $\oplus$ is associative).

Polytopes. Given a set $S \subseteq \mathbb{R}^n$, we define the *convex hull* (resp. *affine span*, *linear span*) of S as the set of all *convex combinations* (resp. *affine combinations*, *linear combinations*) of points in S, that is

$$\mathrm{conv}(S) = \left\{ \sum_{i=1}^{k} \alpha_i \vec{v}_i \,\middle|\, \vec{v}_i \in S, \alpha_i \in [0,1] \text{ for } i = 1, \dots k \,, \sum_{i=1}^{k} \alpha_i = 1 \right\} ,$$

$$\mathrm{Aff}(S) = \left\{ \sum_{i=1}^{k} \alpha_i \vec{v}_i \,\middle|\, \vec{v}_i \in S, \alpha_i \in \mathbb{R} \text{ for } i = 1, \dots k \,, \sum_{i=1}^{k} \alpha_i = 1 \right\} ,$$

$$\mathrm{span}(S) = \left\{ \sum_{i=1}^{k} \alpha_i \vec{v}_i \,\middle|\, \vec{v}_i \in S, \alpha_i \in \mathbb{R} \text{ for } i = 1, \dots k \right\} .$$

Definition 1.11 (Polytope)**.** *A polytope $\mathfrak{p}$ in $\mathbb{R}^n$ is a bounded subset of $\mathbb{R}^n$ described by m linear inequalities. That is, there is some $m \times n$ real matrix A and a vector $b \in \mathbb{R}^m$ such that*

$$\mathfrak{p} = \{ \vec{x} \in \mathbb{R}^n | A\vec{x} \geq b \} .$$

The dimension of a polytope $\mathfrak{p}$ in $\mathbb{R}^n$ is the dimension of $\mathrm{Aff}\, \mathfrak{p}$ as an affine space.

For any polytope, there is a unique minimal finite set of points $\mathcal{P} \subset \mathbb{R}^n$ such that $\mathfrak{p} = \mathrm{conv}\, \mathcal{P}$, see [34]. This family $\mathcal{P}$ is called the set of *vertices* of $\mathfrak{p}$.

Definition 1.12 (Faces of a polytope)**.** *Let $\mathfrak{p}$ be a polytope in $\mathbb{R}^n$. A linear form in $\mathbb{R}^n$ is a linear map $f : \mathbb{R}^n \to \mathbb{R}$. Its minimizing set on $\mathfrak{p}$ is the subset $\mathfrak{p}_f \subseteq \mathfrak{p}$ where f takes minimal values. This set always exists because $\mathfrak{p}$ is compact.*

A face of $\mathfrak{p}$ is a subset $\mathfrak{f} \subseteq \mathfrak{p}$ for which there exists a linear form f that satisfies $\mathfrak{p}_f = \mathfrak{f}$. A face is also a polytope, and any vertex of $\mathfrak{p}$ is a face of $\mathfrak{p}$. The faces of a polytope $\mathfrak{p}$ form a poset when ordered by inclusion, called the face poset.

We observe that the vertices of a polytope are exactly the singletons that are faces.

Remark 1.13 (Computing faces and vertices of a polytope). If f is a linear form in $\mathbb{R}^n$ and $\mathfrak{p} = \mathrm{conv}\, A \subseteq \mathbb{R}^n$, then

$$\mathfrak{p}_f = \mathrm{conv}\{\arg\min_{a \in A} f(a)\} . \tag{2}$$

In particular, the vertices V of $\mathrm{conv}\, A$ satisfy $V \subseteq A$. Also, when computing $\mathfrak{p}_f$, it suffices to evaluate f on A.

Directed graphs. All graphs, their subgraphs and their subtrees are considered to be directed multigraphs in this paper (and we often refer to them as directed graphs or simply as graphs). In a directed multigraph $G = (V(G), E(G))$, the set of edges $E(G)$ is a multiset, allowing for loops and parallel edges. An edge $e \in E(G)$ is an oriented pair of vertices, (v, u), often denoted by $e = v \to u$. We write $\mathrm{s}(e)$ for the starting vertex v and $\mathrm{a}(e)$ for the arrival vertex u.We often consider directed graphs G with labeled edges, and write $\mathrm{lb}(e)$ for the label of the edge $e \in E(G)$. In a graph with labeled edges we refer to edges by using their labels. Given an edge

$e = v \to u \in E(G)$, we denote by $C_G(e)$ (for "set of *continuations* of e") the set of edges $e' \in E(G)$ such that $e' = u \to w$ for some $w \in V(G)$, i.e. $C_G(e) = \{e' \in E(G) | \operatorname{s}(e') = \operatorname{a}(e)\}$.

A *walk* of size k on a directed graph G is a sequence of k edges $(e_1, \dots, e_k) \in E(G)^k$ such that for all $i \in [k-1]$, $\operatorname{a}(e_i) = \operatorname{s}(e_{i+1})$. A walk is a *cycle* if $\operatorname{s}(e_1) = \operatorname{a}(e_k)$. A walk is a *path* if all the edges are distinct, as well as its vertices, with a possible exception that $\operatorname{s}(e_1) = \operatorname{a}(e_k)$ may happen. A cycle that is a path is called a *simple cycle*. Given two walks $w = (e_1, \dots, e_k)$ and $w' = (e'_1, \dots, e'_{k'})$ such that $\operatorname{a}(e_k) = \operatorname{s}(e'_1)$, we write $w \star w'$ for the concatenation of the two walks, i.e. $w \star w' = (e_1, \dots, e_k, e'_1, \dots, e'_{k'})$. For a walk w, we denote by $|w|$ the number of edges in w.

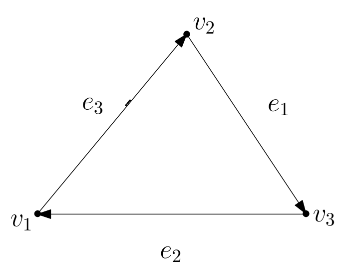
Given a walk $w = (e_1, \dots, e_k)$ and an edge e, we denote by $n_e(w)$ the number of times the edge e is traversed in w, i.e. $n_e(w) \coloneqq |\{i \leq k | e_i = e\}|$.

For a vertex v in a directed graph G, we define $\deg^i_G(v)$ to be the number of incoming edges to v, i.e. edges $e \in E(G)$ such that $\operatorname{a}(e) = v$, and $\deg^o_G(v)$ to be the number of outgoing edges in v, i.e. edges $e \in E(G)$ such that $\operatorname{s}(e) = v$. Whenever it is clear from the context, we drop the subscript G. A vertex v that satisfies $\deg^i(v) = 0$ is called a *source*.

The incidence matrix of a directed graph G is the matrix $L(G)$ with rows indexed by $V(G)$, and columns indexed by $E(G)$, such that for any edge $e = v \to u$, the corresponding column in $L(G)$ has $(L(G))_{v,e} = -1$, $(L(G)))_{u,e} = 1$ and is zero everywhere else.

For instance, we show in Fig. 3 a graph G with its incidence matrix $L(G)$.

$G =$



$$, \qquad L(G) = \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} .$$

FIGURE 3. A graph G with its incidence matrix $L(G)$.

A directed graph G is said to be *strongly connected* if for any two vertices $v_1, v_2 \in V(G)$, there is a path starting in v_1 and arriving in v_2. For instance, the graph in Fig. 3 is strongly connected.

For a graph G with a distinguished vertex r, we say that T is a *rooted spanning tree* with root r if T is tree with $T \subseteq G$ such that $V(T) = V(G)$ and any edge of T is directed away from the root.

2. THE CYCLE POLYTOPE OF A GRAPH

In this section we establish general results about the cycle polytope of a graph. Here, all graphs are considered to be directed multigraphs that may have loops and parallel edges, unless stated otherwise. We recall the definition of cycle polytope.

Definition 2.1 (Cycle polytope)**.** *Let G be a directed graph. For each no-empty cycle $\mathcal{C}$ in G, define $\vec{e}_{\mathcal{C}} \in \mathbb{R}^{E(G)}$ so that*

$$(\vec{e}_{\mathcal{C}})_e := \frac{n_e(\mathcal{C})}{|\mathcal{C}|}, \quad \textit{for all} \quad e \in E(G).$$

We define the cycle polytope *of G to be the polytope $P(G) := \operatorname{conv}\{\vec{e}_{\mathcal{C}} | \, \mathcal{C}$ is a simple cycle of $G\}$.*

2.1. **Vertices of the cycle polytope.** We start by giving a full description of the vertices of this polytope.

Proposition 2.2. *The vertices of $P(G)$ are precisely the vectors $\{\vec{e}_{\mathcal{C}} | \, \mathcal{C}$ is a simple cycle of $G\}$.*

Proof. From Remark 1.13 we only need to show that any point of the form $\vec{e}_{\mathcal{C}}$ is indeed a vertex. Consider now a simple cycle $\mathcal{C}$, and recall that vertices of a polytope are characterized by being the only singletons that are faces. Define:

$$f_{\mathcal{C}}(\vec{x}) := -\sum_{e \in \mathcal{C}} x_e, \quad \text{for all} \quad \vec{x} = (x_e)_{e \in E(G)} \in \mathbb{R}^{E(G)},$$

where we identify $\mathcal{C}$ with the set of edges in $\mathcal{C}$. We will show that $P(G)_{f_{\mathcal{C}}} = \{\vec{e}_{\mathcal{C}}\}$. That is, that $\vec{e}_{\mathcal{C}}$ is the unique minimizer of $f_{\mathcal{C}}$ in $P(G)$, concluding the proof.

It is easy to check that $f_{\mathcal{C}}(\vec{e}_{\mathcal{C}}) = -1$. From Eq. (2), we only need to establish that any simple cycle $\tilde{\mathcal{C}}$ that satisfies $f_{\mathcal{C}}(\vec{e}_{\tilde{\mathcal{C}}}) \leq -1$ is equal to $\mathcal{C}$. Take a generic simple cycle $\tilde{\mathcal{C}}$ in G such that $f_{\mathcal{C}}(\vec{e}_{\tilde{\mathcal{C}}}) \leq -1$. Then, $\sum_{e \in \mathcal{C}} (\vec{e}_{\tilde{\mathcal{C}}})_e \geq 1$. Since $\vec{e}_{\tilde{\mathcal{C}}}$ satisfies the equation $\sum_{e \in E(G)} (\vec{e}_{\tilde{\mathcal{C}}})_e = 1$ and has non-negative coordinates, we must have that $(\vec{e}_{\tilde{\mathcal{C}}})_e = 0$ for all $e \notin \mathcal{C}$. Thus $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ as sets of edges. However, because both $\tilde{\mathcal{C}}, \mathcal{C}$ are simple cycles, we conclude that $\mathcal{C} = \tilde{\mathcal{C}}$, as desired. □

2.2. **Dimension of the cycle polytope.** The goal of this section is to prove the following result.

Theorem 2.3 (Dimension of the cycle polytope)**.** *If G is a strongly connected graph, then the cycle polytope of G has dimension $|E(G)| - |V(G)|$.*

To compute the dimension of the polytope $P(G)$ we start by finding some linear relations that are satisfied in $P(G)$ and that define an affine space of dimension $|E(G)| - |V(G)|$ (see Lemma 2.5). This gives us an upper bound on the dimension of $P(G)$. For the lower bound, we first assume that the graph G has a loop lp. We find a rooted spanning tree T of G, and construct $|E(G)| - |V(G)|$ many distinct points $\vec{v}^{(e)}$ (indexed by $E(G) \setminus (E(T) \cup \{lp\})$) in a suitable translation of $P(G)$. Finally we observe that the set

$$\{\vec{v}^{(e)} | e \in E(G) \setminus (E(T) \cup \{lp\})\},$$

is linearly independent. Finally, we reduce the problem on loopless graphs G to the remaining ones.

The construction of the rooted spanning tree is done in Lemma 2.4, the construction of $\vec{v}^{(e)}$ is done with the help of Lemma 2.6, and the desired linear independence is proved in Lemma 2.10. In Theorem 2.12, we establish a generalization of this theorem, where we compute the dimension of $P(G)$ for any graph G, not only the ones that are strongly connected.

Lemma 2.4. *Let G be a directed graph that is strongly connected, and r be a vertex of G. Then, there exists a rooted spanning tree T of G with root r.*

Proof. We construct the desired tree T algorithmically. Start with a tree T with only one vertex r and no edges. Order the vertices in $V(G) \setminus \{r\}$ and successively for each v, consider the shortest path P from a vertex of the tree T to v in G. This exists because G is strongly connected and it has at most one vertex in common with T.

After going through all vertices, we obtain a spanning tree of G. Further, it is easy to see that all the edges are oriented away from r at every step of the algorithm. So this is a rooted spanning tree, as desired. □

In what follows we assume that we have a strongly connected graph G with at least one loop. We fix a particular loop lp in G, and a spanning rooted tree T with a root r, which exists by Lemma 2.4 above.

Lemma 2.5. *The points $\vec{x} \in P(G)$ satisfy the following relations:*

$$\sum_{e \in E(G)} x_e = 1 \,, \tag{3}$$

$$\sum_{\mathrm{s}(e)=v} x_e = \sum_{\mathrm{a}(e)=v} x_e \,, \forall v \in V(G) \,. \tag{4}$$

Moreover, these equations define an affine space with dimension $|E(G)| - |V(G)|$.

Proof. Because these equations are linear, to observe that any $\vec{x} \in P(G)$ satisfies Eqs. (3) and (4) we only need to show that the vertices $\{\vec{e}_{\mathcal{C}} | \, \mathcal{C}$ is a simple cycle of $G\}$ satisfy them, which is trivial.

Thus, the claim is proven once we establish the dimension of the affine space. Let A be the matrix with rows indexed by $\{\triangleleft\} \cup V(G)$, where $\triangleleft$ is a formal symbol, and columns indexed by $E(G)$ defined as

$$A_{\triangleleft,e} = 1 \qquad A_{v,e} = \begin{cases} -1, & \text{if } \mathrm{s}(e) = v, \, \mathrm{a}(e) \neq v, \\ 1, & \text{if } \mathrm{s}(e) \neq v, \, \mathrm{a}(e) = v, \\ 0, & \text{otherwise,} \end{cases}$$

for any vertex v in $V(G)$, and any edge e in $E(G)$.

Then, Eqs. (3) and (4) are equivalent to

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

We want to show that this equation defines an affine space with dimension $|E(G)| - |V(G)|$. First, we observe that this system has a non-empty set of solutions. For instance $\vec{e}_{lp}$ satisfies Eqs. (3)

and (4). Hence, it suffices to show that $\operatorname{rank}(A) = |V(G)|$. We claim that $\operatorname{rank}(A) \leq |V(G)|$. Indeed

$$\begin{bmatrix} 0 & 1 & \cdots & 1 \end{bmatrix} A = \vec{0}\,,$$

and by the rank nullity theorem on A^T, $\operatorname{rank}(A) + 1 \leq \operatorname{rank}(A^T) + \dim\ker(A^T) = |V(G)| + 1$. Then, the result is established if we find a non-singular $|V(G)| \times |V(G)|$ minor of A.

Let $V' = V(G) \setminus \{r\}$, where r is the root of the tree T in G, and consider the minor M given by the rows indexed with $\{\triangleleft\} \cup V'$ and the columns indexed with $\{lp\} \cup E(T)$. We denote by $L'(H)$ the incidence matrix of a subgraph H of G, with one row (corresponding to r) removed. We define

$$M := \begin{array}{cc} \begin{array}{cccc} lp & & E(T) & \end{array} & \\ \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & L'(T) & \\ 0 & & & \end{bmatrix} & \begin{array}{c} \triangleleft \\ \\ V' \\ \\ \end{array} \end{array}.$$

Note that because T is a spanning tree, $|E(T) \cup \{lp\}| = |V(G)| = |\{\triangleleft\} \cup V'|$, and so M is a square matrix. Observe that M is non-singular whenever $L'(T)$ is non-singular. Then, it suffices to establish that $L'(T)$ is non-singular. We proceed by induction on the size of T.

The base case is when the tree T has one vertex. Then, $L'(T)$ is the empty matrix, which is by convention non-singular. For the induction step, consider a leaf w of T. We reorder the rows and columns of $L'(T)$, so that the column corresponding to the edge e incident to w is the leftmost one, and the row corresponding to the leaf w is the uppermost one. Then we have

$$L'(T) = \begin{array}{cc} \begin{array}{cccc} e & & E(T') & \end{array} & \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & & & \\ \vdots & & L'(T') & \\ * & & & \end{bmatrix} & \begin{array}{c} w \\ \\ V(T') \\ \\ \end{array} \end{array},$$

where T' is the tree corresponding to T after deleting the vertex w, along with its incident edge e. By induction hypothesis, this is a non-singular matrix, and that completes the proof. $\square$

Lemma 2.6. *Let $e \in E(G) \setminus (E(T) \cup \{lp\})$ be an edge in G. Then there are two non-empty cycles $\mathcal{C}_1^{(e)}, \mathcal{C}_2^{(e)}$ such that $e \in \mathcal{C}_1^{(e)}$, $e \notin \mathcal{C}_2^{(e)}$ and $\{f \in E(G) | n_f(\mathcal{C}_1^{(e)}) \neq n_f(\mathcal{C}_2^{(e)})\} \subseteq E(T) \cup \{e, lp\}$.*

Recall that we denote the concatenation of walks by $\star$.

Proof. Let $v_1 = \mathrm{s}(e)$, $v_2 = \mathrm{a}(e)$ and recall that r is the root of the rooted spanning tree T. We suggest to compare what follows with Fig. 4. We can find a path P_1 (resp. P_2) from r to v_1 (resp. v_2) in T. Let D be a walk from v_2 to r, in G. Such a path exists because G is strongly connected. We further choose a minimal path D such that $e \notin D$.

Suppose that $r = v_2$. Then, the path D is the empty path (by minimality), and the cycles $\mathcal{C}_1^{(e)} = P_1 \star e$ and $\mathcal{C}_2^{(e)} = lp$ satisfy the desired properties.

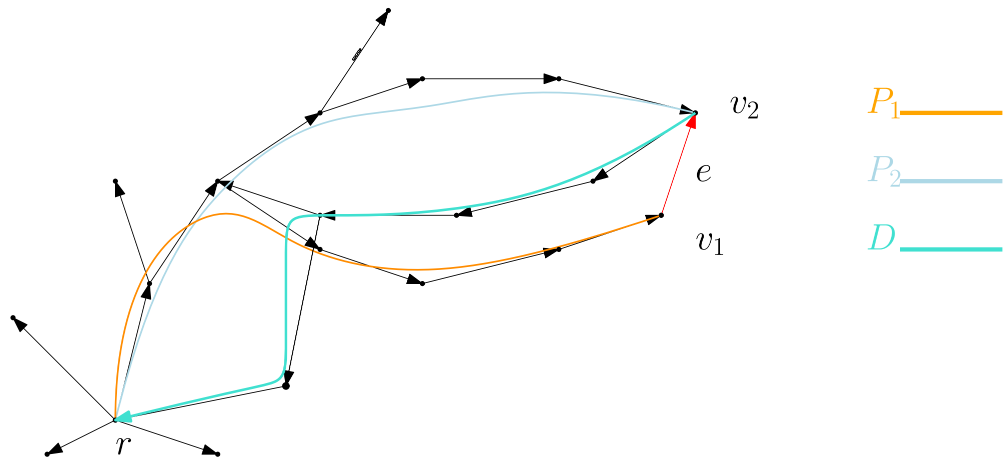


FIGURE 4. The construction of the cycles $\mathcal{C}_1^{(e)}, \mathcal{C}_2^{(e)}$.

If $r \neq v_2$, we show that the cycles $\mathcal{C}_1^{(e)} = D \star P_1 \star e$ and $\mathcal{C}_2^{(e)} = D \star P_2$ are as desired. First, observe that they satisfy $\{f \in E(G) | n_f(\mathcal{C}_1^{(e)}) \neq n_f(\mathcal{C}_2^{(e)})\} \subseteq E(T) \cup \{e, lp\}$. Indeed, any $f \notin E(T) \cup \{e, lp\}$ is neither in P_1 nor in P_2, so $f \in D$ or $f \notin \mathcal{C}_1^{(e)} \cup \mathcal{C}_2^{(e)}$. In either case we have that $n_f(\mathcal{C}_1^{(e)}) = n_f(\mathcal{C}_2^{(e)})$. In addition, $e \notin \mathcal{C}_2^{(e)}$ and $e \in \mathcal{C}_1^{(e)}$. Finally, observe that the cycles obtained are non-empty. □

Before stating the next result on the cycle polytope, we take the following detour that is useful also for later purposes.

Lemma 2.7. *Let G be a directed graph and w a walk on it. Then the multiset of edges of w can be decomposed into ℓ simple cycles (for some $\ell \geq 0$) $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ and a tail $\mathcal{T}$ that does not repeat vertices (but is possibly empty). Specifically, we have the following relation of multisets of edges of G:*

$$w = \mathcal{C}_1 \uplus \cdots \uplus \mathcal{C}_\ell \uplus \mathcal{T} .$$

When w is a cycle, then this decomposition can be further refined to include only simple cycles, that is we have the following relation of multisets of edges in G:

$$w = \mathcal{C}_1 \uplus \cdots \uplus \mathcal{C}_\ell,$$

for some $\ell \geq 0$.

Proof. This decomposition is obtained inductively on the number of edges. If w has no repeated vertices, the decomposition $w = \mathcal{T}$ satisfies the desired conditions. If w has repeated vertices, it has a simple cycle corresponding to the first repetition of a vertex. By pruning from the walk this simple cycle, we obtain a smaller walk which decomposes by the induction hypothesis. This gives us the first result.

If w is a cycle, apply to w the above decomposition for walks, and observe that the walk $w' = \mathcal{T}$ forms a smaller cycle or is the empty walk. However, $\mathcal{T}$ should not repeat vertices, so it is the empty walk, and we obtain the desired decomposition. □

Remark 2.8. The decomposition obtained above, of a walk w into cycles $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ and a tail $\mathcal{T}$, is a decomposition of the edge multiset. In particular, each of the cycles $\mathcal{C}_i$ or the tail $\mathcal{T}$ are *not* necessarily formed by consecutive sequences of edges of w. Explicit examples can be readily found.

Lemma 2.9. *For a non-empty cycle $\mathcal{C}$ in G, we have that $\vec{e}_{\mathcal{C}} \in P(G)$.*

Proof. We have seen in Lemma 2.7, that a cycle $\mathcal{C}$ has a decomposition into simple cycles as $\mathcal{C} = \mathcal{C}_1 \uplus \cdots \uplus \mathcal{C}_\ell$. It follows that, for an edge $e \in E(G)$, we have

$$\vec{e}_{\mathcal{C}} = \sum_{j=1}^{\ell} \vec{e}_{\mathcal{C}_j} \frac{|\mathcal{C}_j|}{|\mathcal{C}|} \,. \tag{5}$$

Note that $\sum_{j=1}^{\ell} |\mathcal{C}_j| = |\mathcal{C}|$. Therefore, $\vec{e}_{\mathcal{C}}$ is a convex combination of the vertices of $P(G)$, as desired. □

For $e \in E(G) \setminus (E(T) \cup \{lp\})$, consider the cycles $\mathcal{C}_1^{(e)}, \mathcal{C}_2^{(e)}$ constructed above in Lemma 2.6 and define the vector

$$\vec{v}^{(e)} := |\mathcal{C}_1^{(e)}|(\vec{e}_{\mathcal{C}_1^{(e)}} - \vec{e}_{lp}) - |\mathcal{C}_2^{(e)}|(\vec{e}_{\mathcal{C}_2^{(e)}} - \vec{e}_{lp}) \,.$$

In particular, observe that for $f \in E(G) \setminus (E(T) \cup \{lp\})$ we have

$$(\vec{v}^{(e)})_f = n_f(\mathcal{C}_1^{(e)}) - n_f(\mathcal{C}_2^{(e)}) \text{ is non-zero if and only if } e = f \,. \tag{6}$$

Lemma 2.10. *The set $\{\vec{v}^{(e)} | e \in E(G) \setminus (E(T) \cup \{lp\})\}$ is linearly independent.*

Proof. This follows immediately from Eq. (6). □

For a set $S \subseteq \mathbb{R}^{E(G)}$, recall that we defined the affine span as

$$\mathrm{Aff}(S) = \left\{ \sum_{i=1}^{k} \alpha_i \vec{v}_i \,\middle|\, \vec{v}_i \in S, \alpha_i \in \mathbb{R} \text{ for } i = 1, \dots k \,, \sum_{i=1}^{k} \alpha_i = 1 \right\} \,.$$

In particular, note that if $\vec{0} \in S$, then $\mathrm{Aff}(S) = \mathrm{span}(S)$.

Proof of Theorem 2.3. We first assume that G has a loop lp. Then, from Lemma 2.5, we know that

$$\dim P(G) = \dim \mathrm{Aff}(P(G)) \leq |E(G)| - |V(G)| \,.$$

Define the translation $P(G)' = P(G) - \vec{e}_{lp}$. Observe that $\vec{0} \in P(G)'$, hence $\mathrm{Aff}(P(G)')$ is a linear space. Furthermore, observe that for each edge $e \in E(G) \setminus (E(T) \cup \{lp\})$, each vector $\vec{v}^{(e)}$ is a linear combination of $\vec{e}_{\mathcal{C}_1^{(e)}} - \vec{e}_{lp}$ and $\vec{e}_{\mathcal{C}_2^{(e)}} - \vec{e}_{lp}$, which are both in $P(G)'$, so $\vec{v}^{(e)} \in \mathrm{Aff}(P(G)')$.

Moreover, this is a linearly independent set of vectors, from Lemma 2.10, from which we conclude that $\dim P(G)' = \dim \mathrm{Aff}(P(G)') \geq |E(G) \setminus (E(T) \cup \{lp\})| = |E(G)| - |V(G)|$. The theorem follows, for the case where G has a loop, from $\dim P(G) = \dim P(G)'$.

Now suppose that G has no loops, and consider the graph $G \cup \{lp\}$ obtained from G by adding a loop lp to one of its vertices. By the above result, the polytope $P(G \cup \{lp\})$ has dimension $|E(G)| - |V(G)| + 1$. We can write

$$P(G \cup \{lp\}) = \operatorname{conv}(P(G) \cup \vec{e}_{lp}) ,$$

where $P(G) \subseteq \mathbb{R}^{E(G)} \setminus \{\vec{0}\}$ and $\vec{e}_{lp} \in (\mathbb{R}^{E(G)})^{\perp} \setminus \{\vec{0}\}$. It follows that

$$|E(G)| - |V(G)| + 1 = \dim P(G \cup \{lp\}) = \dim P(G) + 1 ,$$

concluding the proof of the theorem. □

We now generalize Theorem 2.3 to any graph. We start with the following technical result.

Proposition 2.11. *Let $\mathfrak{a}_1 \subset \mathbb{R}^A, \mathfrak{a}_2 \subset \mathbb{R}^B$ be polytopes such that $\operatorname{Aff}(\mathfrak{a}_1), \operatorname{Aff}(\mathfrak{a}_2)$ do not contain the zero vector, and let $d(\mathfrak{a}_1), d(\mathfrak{a}_2)$ be their respective dimensions.*

Then the dimension of the polytope $\mathfrak{c} = \operatorname{conv}(\mathfrak{a}_1 \times \{\vec{0}\}, \{\vec{0}\} \times \mathfrak{a}_2) \subset \mathbb{R}^{A \sqcup B}$ is

$$d(\mathfrak{c}) = d(\mathfrak{a}_1) + d(\mathfrak{a}_2) + 1.$$

Proof. In this proof, for sake of simplicity, we will identify $\mathfrak{a}_1 \in \mathbb{R}^A$ and $\mathfrak{a}_2 \in \mathbb{R}^B$ with $\mathfrak{a}_1 \times \{\vec{0}\}$ and $\{\vec{0}\} \times \mathfrak{a}_2$, respectively. In particular, we will refer to points $\vec{x} \in \mathfrak{a}_i$ for $i = 1, 2$ as their suitable extensions $(\vec{x}, \vec{0})$ or $(\vec{0}, \vec{x})$, respectively, in $\mathbb{R}^{A \sqcup B}$ without further notice.

Suppose that $\operatorname{Aff}(\mathfrak{a}_1) = W_1 + \vec{x}_1$, $\operatorname{Aff}(\mathfrak{a}_2) = W_2 + \vec{x}_2$ and $\operatorname{Aff}(\mathfrak{c}) = W + \vec{x}_1$, where W_1, W_2, W are vector subspaces of $V := \mathbb{R}^{A \sqcup B}$ with dimension $d(\mathfrak{a}_1), d(\mathfrak{a}_2)$ and $d(\mathfrak{c})$ respectively. A choice of $\vec{x}_1, \vec{x}_2$ such $\vec{x}_i \in \mathfrak{a}_i$ for $i = 1, 2$ is always possible. Since $\vec{0} \notin \operatorname{Aff}(\mathfrak{a}_1), \vec{0} \notin \operatorname{Aff}(\mathfrak{a}_2)$, we have that $\vec{x}_1 \notin W_1, \vec{x}_2 \notin W_2$.

The dimension that we wish to compute is $d(\mathfrak{c}) = \dim(W)$. We will do this by establishing both underlying inequalities.

We start with a lower bound for $d(\mathfrak{c})$. Consider bases $\{\vec{v}_1^{(1)}, \dots, \vec{v}_{d(\mathfrak{a}_1)}^{(1)}\}$ and $\{\vec{v}_1^{(2)}, \dots, \vec{v}_{d(\mathfrak{a}_2)}^{(2)}\}$ of W_1, W_2, respectively. It is clear that $\vec{v}_i^{(1)} \in \operatorname{Aff}(\mathfrak{a}_1) - \vec{x}_1 \subseteq W$ for $i = 1, \dots, d(\mathfrak{a}_1)$, and that $\vec{x}_2 - \vec{x}_1 \in W$. In addition we have that

$$\vec{v}_i^{(2)} \in \operatorname{Aff}(\mathfrak{a}_2) - \vec{x}_2 \subseteq \operatorname{Aff}(\mathfrak{c}) - \vec{x}_2 = \operatorname{Aff}(\mathfrak{c}) + (\vec{x}_2 - \vec{x}_1) - \vec{x}_2 = W, \quad \text{for } i = 1, \dots, d(\mathfrak{a}_2).$$

This proves that $\{\vec{v}_1^{(1)}, \dots, \vec{v}_{d(\mathfrak{a}_1)}^{(1)}, \vec{v}_1^{(2)}, \dots, \vec{v}_{d(\mathfrak{a}_2)}^{(2)}, \vec{x}_2 - \vec{x}_1\} \subseteq W$. We now show that this set is linearly independent.

Because $W_1 \cap W_2 = \{\vec{0}\}$, the set $\{\vec{v}_1^{(1)}, \dots, \vec{v}_{d(\mathfrak{a}_1)}^{(1)}, \vec{v}_1^{(2)}, \dots, \vec{v}_{d(\mathfrak{a}_2)}^{(2)}\}$ forms a basis of $W_1 \oplus W_2$. Because $\vec{x}_1 \in \mathbb{R}^A \setminus W_1$ and $\vec{x}_2 \in \mathbb{R}^B \setminus W_2$, adding the vectors $\vec{x}_1, \vec{x}_2$ extends this basis. It follows that

$$\{\vec{v}_1^{(1)}, \dots, \vec{v}_{d(\mathfrak{a}_1)}^{(1)}, \vec{v}_1^{(2)}, \dots, \vec{v}_{d(\mathfrak{a}_2)}^{(2)}, \vec{x}_2 - \vec{x}_1\}$$

is linearly independent.

Observe that we found a linearly independent set with $d(\mathfrak{a}_1) + d(\mathfrak{a}_2) + 1$ many vectors in W. This gives us a lower bound for $\dim \mathfrak{c}$.

For an upper bound, observe that $\operatorname{Aff}(\mathfrak{c}) \subseteq \operatorname{span} \mathfrak{c}$, and that

$$\dim(\operatorname{span} \mathfrak{c}) \leq \dim(\operatorname{span} \mathfrak{a}_1) + \dim(\operatorname{span} \mathfrak{a}_2) = d(\mathfrak{a}_1) + d(\mathfrak{a}_2) + 2 .$$

We now prove that $0 \notin \operatorname{Aff}(\mathfrak{c})$ by contradiction. Assume otherwise that $\sum_i \alpha_i \vec{a}_i + \sum_j \beta_j \vec{b}_j = 0$, where $\vec{a}_i \in \mathfrak{a}_1$, $\vec{b}_j \in \mathfrak{a}_2$ and $\sum_i \alpha_i + \sum_j \beta_j = 1$. But $\sum_i \alpha_i \vec{a}_i \in \mathbb{R}^A$, and $-\sum_j \beta_j \vec{b}_j \in \mathbb{R}^B$, so $\sum_i \alpha_i \vec{a}_i = -\sum_j \beta_j \vec{b}_j \in \mathbb{R}^A \cap \mathbb{R}^B = \{\vec{0}\}$. Because $\sum_i \alpha_i + \sum_j \beta_j = 1$, without loss of generality we can assume that $\sum_i \alpha_i \neq 0$. Then we have $\frac{\sum_i \alpha_i \vec{a}_i}{\sum_i \alpha_i} = 0 \in \operatorname{Aff}(\mathfrak{a}_1)$, a contradiction.

Since $0 \in \operatorname{span}(\mathfrak{c})$, we conclude that $\operatorname{Aff}(\mathfrak{c}) \neq \operatorname{span} \mathfrak{c}$. It follows that

$$\dim \operatorname{Aff}(\mathfrak{c}) < d(\mathfrak{a}_1) + d(\mathfrak{a}_2) + 2 \,.$$

This concludes the proof. □

With the help of Proposition 2.11, we can generalize Theorem 2.3 to the cycle polytope of any graph: We say that a graph $G = (V, E)$ is *full* if any edge $e \in E$ is part of a cycle of G. It is easy to see that if G is not full, then $P(G) = P(H)$, where $H \subseteq G$ is the largest full subgraph of G. Equivalently, H is obtained from G by removing all edges from G that are not part of a cycle.

If G is a full graph, there are no *bridges*, that is an edge e connecting two distinct strongly connected components. Hence, we can decompose G as the disjoint union of strongly connected components and a set of isolated vertices V': $G = H_1 \sqcup \cdots \sqcup H_k \sqcup V'$. It can be seen that $P(G) = \operatorname{conv}\{P(H_i) | i = 1, \dots k\}$, where identify $P(H_i)$ with its canonical image in $\mathbb{R}^{E(G)}$. Noting that $P(H_i) \subseteq \operatorname{Aff}(P(H_i)) \subseteq \mathbb{R}^{E(H_i)}$ and that $\operatorname{Aff}(P(H_i))$ does not contain the origin for any $i = 1, \dots, k$, from Proposition 2.11 we have that

$$\dim P(G) = k - 1 + \sum_i \dim P(H_i) = k - 1 + |E| - |V \setminus V'| = |E| - |V| + |V'| + k - 1 \,.$$

Theorem 2.12. *If G is a directed multigraph and $H \subseteq G$ its largest full subgraph, then the dimension of the polytope $P(G)$ is*

$$\dim P(G) = |E(H)| - |V(G)| + |\{ \textit{ connected components of } H\}| - 1.$$

2.3. **Faces of the cycle polytope.** We now focus on the faces of a cycle polytope $P(G)$. We prove two results: in Theorem 2.13 we describe the equations that define $P(G)$, then in Theorem 2.14 we find a bijection between faces of $P(G)$ and the subgraphs of G that are full.

Theorem 2.13. *Let G be a directed graph. The polytope $P(G)$ is given by Specifically*

$$P(G) = \left\{ \vec{x} \in \mathbb{R}^{E(G)} \middle| \sum_{e \in E(G)} x_e = 1 \,, \ \sum_{\mathrm{s}(e) = v} x_e = \sum_{\mathrm{a}(e) = v} x_e \,, \forall v \in V(G) \,, \quad \vec{x} \geq \vec{0} \right\} .$$

Proof. For simplicity of notation, let $H_1 = \{\vec{x} \in \mathbb{R}^{E(G)} | \sum_e x_e = 1\}$ and

$$P_+(G) = \left\{ \vec{x} \in \mathbb{R}^{E(G)} \middle| \sum_{\mathrm{s}(e) = v} x_e = \sum_{\mathrm{a}(e) = v} x_e \,, \forall v \in V(G) \,, \quad \vec{x} \geq \vec{0} \right\} .$$

We wish to show that

$$P(G) = P_+(G) \cap H_1 \,. \tag{7}$$

The inclusion $P(G) \subseteq P_+(G) \cap H_1$ is trivial. Suppose now, for the sake of contradiction that there is a point $\vec{x} \in (P_+(G) \cap H_1) \setminus P(G)$, and pick one that minimizes the size of the edge set $\mathcal{Z}(\vec{x}) := \{e \in E(G) | x_e \neq 0\}$. First observe that $\vec{0} \notin H_1$, so $\mathcal{Z}(\vec{x}) \neq \emptyset$.

We now show that any source v of the graph $(V(G), \mathcal{Z}(\vec{x}))$ is an isolated vertex. In fact, if no edge $e \in \mathcal{Z}(\vec{x})$ satisfies $\mathrm{a}(e) = v$, we have

$$\sum_{\mathrm{s}(e)=v} x_e = \sum_{\mathrm{a}(e)=v} x_e = 0\,,$$

and because $x_e \geq 0$ for any edge e, we have that $x_e = 0$ for any edge e such that $\mathrm{s}(e) = v$. Hence, $v \in V(G)$ is an isolated vertex in $(V(G), \mathcal{Z}(\vec{x}))$.

Because $(V(G), \mathcal{Z}(\vec{x}))$ is a graph with at least one oriented edge where any source is isolated, it must have a simple cycle $\mathcal{C}$. Let $e \in \mathcal{C}$ be the edge in $\mathcal{C}$ that minimizes x_e, and consider $\vec{y} := \vec{x} - x_e |\mathcal{C}| \vec{e}_{\mathcal{C}}$.

In the case that we have $\vec{y} = \vec{0}$, then $\vec{x} = x_e |\mathcal{C}| \vec{e}_{\mathcal{C}} \in H_1$ by assumption. But $\vec{e}_{\mathcal{C}} \in H_1$, so then $\vec{x} = \vec{e}_{\mathcal{C}} \in P(G)$, which is a contradiction. In the case that $\vec{y}$ is a non-zero vector with non-negative coefficients, we have that $\sum_e y_e \neq 0$, and

$$\vec{z} := \frac{\vec{y}}{\sum_e y_e} \in P_+(G) \cap H_1\,.$$

Now suppose that $\vec{z} \in P(G)$. Then $\vec{x} = \left(\sum_e y_e\right) \vec{z} + x_e |\mathcal{C}| \vec{e}_{\mathcal{C}}$ is a convex combination of points in $P(G)$, so $\vec{x} \in P(G)$, which contradicts our assumption that $\vec{x} \in (P_+(G) \cap H_1) \setminus P(G)$. Thus $\vec{z} \notin P(G)$. But, $\mathcal{Z}(\vec{z}) = \mathcal{Z}(\vec{y}) \subseteq \mathcal{Z}(\vec{x}) \setminus \{e\}$, contradicting the minimality of $\vec{x}$. We conclude that there is no $\vec{x} \in (P_+(G) \cap H_1) \setminus P(G)$, hence $P_+(G) \cap H_1 \subseteq P(G)$, as desired. ☐

Recall that a subgraph $H = (V, E')$ of a graph G is called a *full subgraph* if any edge $e \in E'$ is part of a cycle of H.

Theorem 2.14. *The face poset of $P(G)$ is isomorphic to the poset of non-empty full subgraphs of G according to the following identification:*

$$H \mapsto P(G)_H := \{\vec{x} \in P(G) | x_e = 0 \textit{ for } e \notin E(H)\}\,.$$

Further, if we identify $P(H)$ with its image under the canonical inclusion $\mathbb{R}^{E(H)} \hookrightarrow \mathbb{R}^{E(G)}$, we have that $P(H) = P(G)_H$.

In particular, the dimension of $P(G)_H$ is $|E(H)| - |V| + |\{$ connected components of $H\}| - 1$.

Proof. From Theorem 2.13, a face of $P(G)$ is given by setting some of the inequalities of $\vec{x} \geq 0$ as equalities. So, a face is of the form $P(G)_H$ for some subgraph $H = (V(G), E(H))$, where $E(G) \setminus E(H)$ corresponds to the inequalities of $\vec{x} \geq 0$ that become equalities. It is immediate to observe that the identification $\mathbb{R}^{E(H)} \hookrightarrow \mathbb{R}^{E(G)}$ gives us that $P(H) = P(G)_H$.

We show that it suffices to take H a full subgraph: consider an edge $e_0 = v \to w$ in H that is not contained in any cycle in H. Then $(\vec{e}_{\mathcal{C}})_{e_0} = 0$ for any simple cycle $\mathcal{C}$ in H, and so $x_{e_0} = 0$ for any point $\vec{x} \in P(H) = P(G)_H$. It follows that $P(G)_H = P(G)_{H \setminus e_0}$.

Conversely, we can see that if $H_1 \neq H_2$ are two full subgraphs of G, then $P(G)_{H_1} \neq P(G)_{H_2}$, that is, H_1, H_2 correspond to two different faces of $P(G)$. Indeed, without loss of generality we

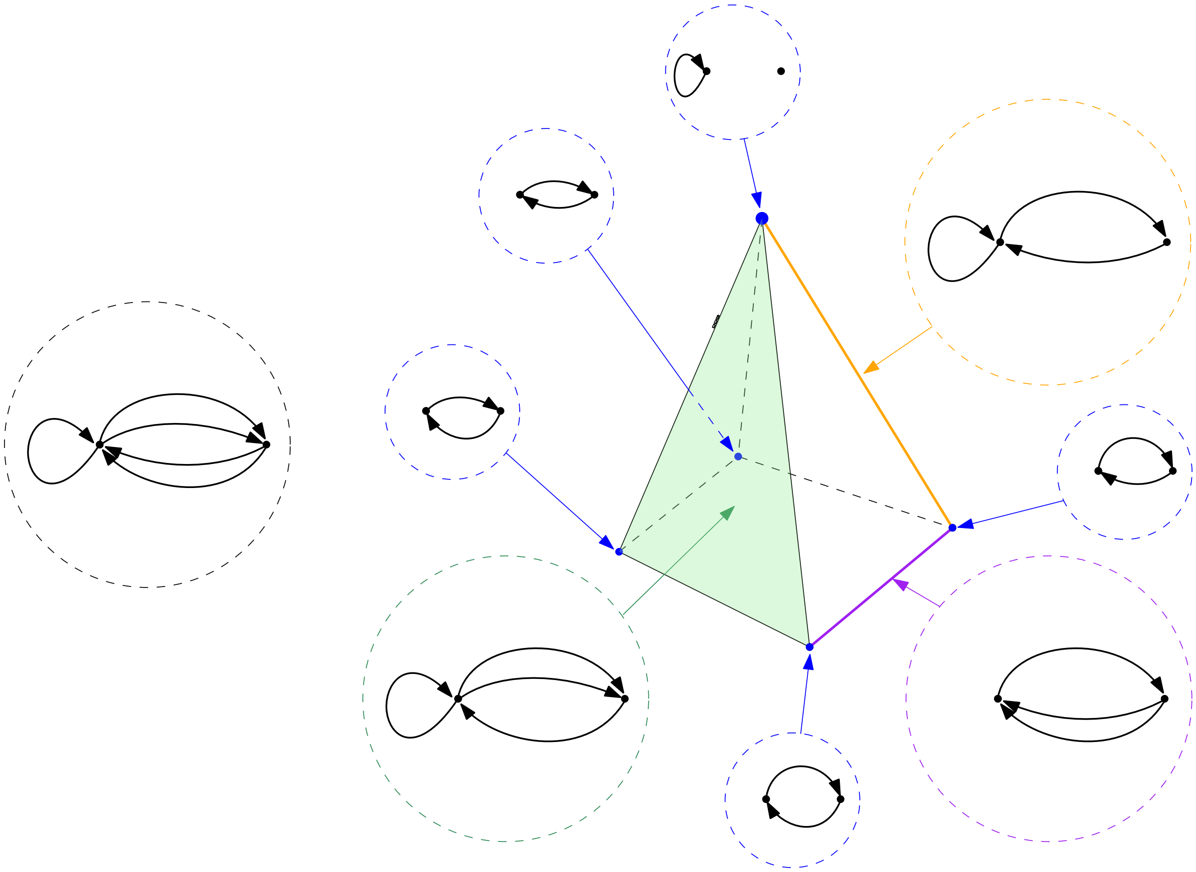


FIGURE 5. The face structure of the cycle polytope of a graph. On the left-hand side of the picture (inside the dashed black ball) we have a graph G with two vertices and five edges. On the right-hand side, we draw the associated cycle polytope $P(G)$ that is a pyramid with squared base. The blue dashed balls correspond to the simple cycles corresponding to the five vertices of the polytope. We also underline the relation between two edges of the polytope (in purple and orange respectively) and a face (in green) and the corresponding full subgraphs. Note that, for example, the graph corresponding to the green face is just the union of the three graphs corresponding to the vertices of that face.

can assume that there is an edge $e \in E(H_1) \setminus E(H_2)$. This edge is, by hypothesis, contained in a simple cycle $\mathcal{C}$, so $\vec{e}_{\mathcal{C}} \in P(G)_{H_1} \setminus P(G)_{H_2}$, so $P(G)_{H_1} \neq P(G)_{H_2}$.

It is clear that if $H_1 \subseteq H_2$ then $P(G)_{H_1} \subseteq P(G)_{H_2}$, so the identification $H \mapsto P(G)_H$ preserves the poset structure. Finally, we obtained the dimension of $P(G)_H = P(H)$ in Theorem 2.12. $\square$

Example 2.15 (Face structure of a specific cycle polytope). Consider the graph G given on the left-hand side of Fig. 5, that has two vertices and five edges. It follows that the corresponding cycle polytope has dimension three, and its face structure is partially depicted in the right-hand side Fig. 5.

In fact, from Theorem 2.14, to each face of the polytope we can associate a full subgraph of G. Some of these correspondences are highlighted in Fig. 5 and described in its caption.

Given a simple cycle $\mathcal{C}$ in a graph G, a path P is a *chord* of $\mathcal{C}$ if it is edge-disjoint from $\mathcal{C}$ and it starts and arrives at vertices of $\mathcal{C}$. In particular, given two simple cycles sharing a vertex, any one of them forms a chord of the other.

Remark 2.16 (The skeleton of the polytope $P(G)$). We want to characterize the pairs of vertices of $P(G)$ that are connected by an edge. The structure behind this is usually called the *skeleton of the polytope*. Suppose that we are given two vertices of the polytope $P(G)$, $\vec{e}_{\mathcal{C}_1}, \vec{e}_{\mathcal{C}_2}$ corresponding to the simple cycles $\mathcal{C}_1, \mathcal{C}_2$ of the graph G, according to Proposition 2.2.

With the description of the faces in Theorem 2.14, we have that a face $P(G)_H$ is an edge when it has dimension one, that is

$$|E(H)| - |V(G)| + |\{ \text{ connected components of } H\}| - 1 = 1\,.$$

This happens if and only if the undirected version of H is a forest with two edges added.

Because H is full, each connected component must contain a cycle, so H has either one or two connected components. Hence, it results either from the union of two vertex-disjoint simple cycles, or from the union of a simple cycle and one of its chords. Equivalently, $\vec{e}_{\mathcal{C}_1}, \vec{e}_{\mathcal{C}_2}$ are connected with an edge when $\mathcal{C}_1 \setminus \mathcal{C}_2$ forms a unique chord of $\mathcal{C}_2$, or when $\mathcal{C}_1, \mathcal{C}_2$ are vertex-disjoint.

For instance, in Fig. 5, there are two pairs of vertices of $P(G)$ that are not connected, and each pair corresponds to two cycles $\mathcal{C}_1, \mathcal{C}_2$ such that $\mathcal{C}_1 \setminus \mathcal{C}_2$ forms two chords of $\mathcal{C}_2$.

Remark 2.17 (Computing the volume of $P(G)$). The problem of finding the volume of a polytope is a classical one in convex geometry. We propose an algorithmic approach that uses the face description of $P(G)$ in Theorem 2.14 and the following facts:

- Let A be a polytope and v a point in space. If $v \notin \operatorname{Aff}(A)$, then
$$\operatorname{vol}(\operatorname{conv}(A \cup \{v\})) = \operatorname{vol}(A) \operatorname{dist}(v, \operatorname{Aff}(A)) \frac{1}{\dim A + 1}\,.$$
- If v is vertex of the polytope $\mathfrak{p}$ of dimension d, then we have the following decomposition of the polytope $\mathfrak{p}$:
$$\mathfrak{p} = \bigcup_{v \notin \mathfrak{q} \subsetneq \mathfrak{p}} \operatorname{conv}(\mathfrak{q} \cup \{v\})\,,$$
where the union runs over all high-dimensional faces $\mathfrak{q}$ that do not contain the vertex v. This decomposition is such that the intersection of each pair of blocks has volume zero, and each block has a non-zero $d-1$ dimensional volume.

If $\mathcal{C}$ is a simple cycle of G, the following decomposition holds:

$$P(G) = \bigcup_{\mathcal{C} \not\subseteq H \subsetneq G} \operatorname{conv}(\vec{e}_{\mathcal{C}}, P(G)_H)\,,$$

where the union runs over all maximal full proper subgraphs of G that do not contain $\mathcal{C}$.

Hence, we obtain the volume of $P(G)$ as follows:

$$\mathrm{vol}(P(G)) = \sum_{\mathcal{C}\nsubseteq H \subsetneq G} \mathrm{conv}(\vec{e}_{\mathcal{C}}, P(G)_{G\setminus e}) = \sum_{\mathcal{C}\nsubseteq H \subsetneq G} \frac{\mathrm{vol}(P(H))\,\mathrm{dist}(\vec{e}_{\mathcal{C}}, \mathrm{Aff}(P(G)_H))}{\dim P(G)+1},$$

where the sum runs over all maximal full proper subgraphs of G that do not contain $\mathcal{C}$. This gives us a recursive way of computing the volume $\mathrm{vol}(P(G))$ by computing the volume of cycle polytopes of smaller graphs. We have unfortunately not been able to find a general formula for $\mathrm{vol}\,P(G)$, and leave this as an open problem.

3. The feasible region P_k is a cycle polytope

Recall that we defined

$$P_k := \left\{\vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi,\sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\} .$$

The goal of this section is to prove that P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$ (see Theorem 3.12). We first prove that P_k is closed and convex (see Proposition 3.2), then we use a correspondence between permutations and paths in $\mathcal{O}v(k)$ (see Definition 3.5) to prove the desired result.

3.1. The feasible region P_k is convex.

We start with a preliminary result.

Lemma 3.1. *The feasible region P_k is closed.*

This is a classical consequence of the fact that P_k is a set of limit points. For completeness, we include a simple proof of the statement. Recall that $\widetilde{\text{c-occ}}_k(\sigma) := \left(\widetilde{\text{c-occ}}(\pi,\sigma)\right)_{\pi\in\mathcal{S}_k}$.

Proof. It suffices to show that, for any sequence $(\vec{v}_s)_{s\in\mathbb{N}}$ in P_k such that $\vec{v}_s \to \vec{v}$ for some $\vec{v} \in [0,1]^{\mathcal{S}_k}$, we have that $\vec{v} \in P_k$. For all $s \in \mathbb{N}$, consider a sequence of permutations $(\sigma_s^m)_{m\in\mathbb{N}}$ such that $|\sigma_s^m| \overset{m\to\infty}{\longrightarrow} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_s^m) \overset{m\to\infty}{\longrightarrow} \vec{v}_s$, and some index $m(s)$ of the sequence $(\sigma_s^m)_{m\in\mathbb{N}}$ such that for all $m \geq m(s)$,

$$|\sigma_s^m| \geq s \quad \text{and} \quad ||\widetilde{\text{c-occ}}_k(\sigma_s^m) - \vec{v}_s|| \leq \tfrac{1}{s}\,.$$

W.l.o.g. assume that $m(s)$ is increasing. For every $\ell \in \mathbb{N}$, define $\sigma^\ell := \sigma_\ell^{m(\ell)}$. It is easy to show that

$$|\sigma^\ell| \overset{\ell\to\infty}{\longrightarrow} \infty \quad \text{and} \quad \widetilde{\text{c-occ}}_k(\sigma^\ell) \overset{\ell\to\infty}{\longrightarrow} \vec{v}\,,$$

where we use that $\vec{v}_s \to \vec{v}$. Therefore $\vec{v} \in P_k$. □

We can now prove the first important result of this section.

Proposition 3.2. *The feasible region P_k is convex.*

Proof. Since P_k is closed (by Lemma 3.1) it is enough to consider rational convex combinations of points in P_k, i.e. it is enough to establish that for all $\vec{v}_1, \vec{v}_2 \in P_k$ and all $s, t \in \mathbb{N}$, we have that

$$\frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2 \in P_k.$$

Fix $\vec{v}_1, \vec{v}_2 \in P_k$ and $s, t \in \mathbb{N}$. Since $\vec{v}_1, \vec{v}_2 \in P_k$, there exist two sequences $(\sigma_1^m)_{m\in\mathbb{N}}$, $(\sigma_2^m)_{m\in\mathbb{N}}$ such that $|\sigma_i^m| \xrightarrow{m\to\infty} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_i^m) \xrightarrow{m\to\infty} \vec{v}_i$, for $i = 1, 2$.

Define $t_m \coloneqq t \cdot |\sigma_1^m|$ and $s_m \coloneqq s \cdot |\sigma_2^m|$.

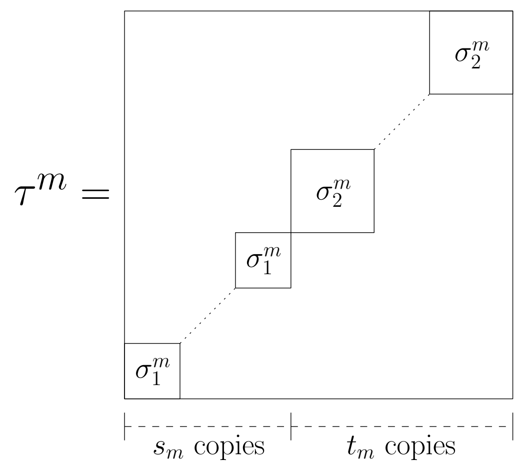


FIGURE 6. Schema for the definition of the permutation τ^m.

We set $\tau^m \coloneqq (\oplus_{s_m} \sigma_1^m) \oplus (\oplus_{t_m} \sigma_2^m)$. For a graphical interpretation of this construction we refer to Fig. 6. We note that for every $\pi \in \mathcal{S}_k$, we have

$$\text{c-occ}(\pi, \tau^m) = s_m \cdot \text{c-occ}(\pi, \sigma_1^m) + t_m \cdot \text{c-occ}(\pi, \sigma_2^m) + Er,$$

where $Er \leq (s_m + t_m - 1) \cdot |\pi|$. This error term comes from the number of intervals of size $|\pi|$ that intersect the boundary of some copies of σ_1^m or σ_2^m. Hence

$$\begin{aligned}\widetilde{\text{c-occ}}(\pi, \tau^m) &= \frac{s_m \cdot |\sigma_1^m| \cdot \widetilde{\text{c-occ}}(\pi, \sigma_1^m) + t_m \cdot |\sigma_2^m| \cdot \widetilde{\text{c-occ}}(\pi, \sigma_2^m) + Er}{s_m \cdot |\sigma_1^m| + t_m \cdot |\sigma_2^m|} \\ &= \frac{s}{s+t}\widetilde{\text{c-occ}}(\pi, \sigma_1^m) + \frac{t}{s+t}\widetilde{\text{c-occ}}(\pi, \sigma_2^m) + O\left(|\pi| \left(\tfrac{1}{|\sigma_1^m|} + \tfrac{1}{|\sigma_2^m|}\right)\right).\end{aligned}$$

As m tends to infinity, we have

$$\widetilde{\text{c-occ}}_k(\tau^m) \to \frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2,$$

since $|\sigma_i^m| \xrightarrow{m\to\infty} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_i^m) \xrightarrow{m\to\infty} \vec{v}_i$, for $i = 1, 2$. Noting also that

$$|\tau^m| \to \infty,$$

we can conclude that $\frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2 \in P_k$. This ends the proof. □

3.2. **The feasible region P_k as the limit of random permutations.** Using similar ideas to the ones used in the proof above, we can establish the equality between the sets in Eq. (1). We first recall the following.

Definition 3.3. *For a total order* $(\mathbb{Z},\preccurlyeq)$*, its* shift $(\mathbb{Z},\preccurlyeq')$ *is defined by* $i+1\preccurlyeq' j+1$ *if and only if* $i\preccurlyeq j$*. A random infinite rooted permutation, or equivalently a random total order on* $\mathbb{Z}$*, is said to be shift-invariant if it has the same distribution as its shift.*

We refer to [10, Section 2.6] for a full discussion on shift-invariant random permutations.

Proposition 3.4. *The following equality holds*

$$P_k=\left\{(\Gamma_\pi(\sigma^\infty))_{\pi\in\mathcal{S}_k}\middle|\sigma^\infty \textit{ is a random infinite rooted shift-invariant permutation}\right\}\,.$$

Proof. In [10, Proposition 2.44 and Theorem 2.45] it was proved that a random infinite rooted permutation σ^∞ is shift-invariant if and only if it is the annealed Benjamini–Schramm limit of a sequence of random permutations[2]. Furthermore, we can choose this sequence of random permutations σ^n in such a way that $|\sigma^n|=n$ a.s., for all $n\in\mathbb{N}$.

This result and Theorem 1.2 immediately imply that

$$P_k\subseteq\left\{(\Gamma_\pi(\sigma^\infty))_{\pi\in\mathcal{S}_k}\middle|\sigma^\infty \text{ is a random infinite rooted shift-invariant permutation}\right\}\,.$$

To show the other inclusion, it is enough to show that for every random infinite rooted shift-invariant permutation σ^∞, there exists a sequence of *deterministic* permutations that Benjamini–Schramm converges to σ^∞.

By the above mentioned result of [10], there exists a sequence $(\sigma^n)_{n\in\mathbb{N}}$ of *random* permutations such that $|\sigma^n|=n$ a.s., for all $n\in\mathbb{N}$, and $(\sigma^n)_{n\in\mathbb{N}}$ converges in the annealed Benjamini–Schramm sense to σ^∞. Using [10, Theorem 2.24] we know that, for every $\pi\in\mathcal{S}$,

$$\mathbb{E}[\widetilde{\text{c-occ}}(\pi,\sigma^n)]\to\Gamma_\pi(\sigma^\infty)\,. \tag{8}$$

Let, for all $n\in\mathbb{N}$ and $\rho\in\mathcal{S}_n$,

$$p^n_\rho:=\mathbb{P}(\sigma^n=\rho).$$

For every $n\in\mathbb{N}$, we can find $n!$ integers $\{q^n_\rho\}_{\rho\in\mathcal{S}_n}$ such that for every $\rho\in\mathcal{S}_n$,

$$\left|\frac{q^n_\rho}{\sum_{\theta\in\mathcal{S}_n}q^n_\theta}-p^n_\rho\right|\leq\frac{1}{n^n}. \tag{9}$$

Let us now consider the deterministic sequence of permutations of size $n\sum_{\theta\in\mathcal{S}_n}q^n_\theta$ defined as

$$\nu^n:=\bigoplus_{\rho\in\mathcal{S}_n}(\oplus_{q^n_\rho}\rho),$$

where we fixed any order on $\mathcal{S}_n$. Using the same error estimates as in the proof of Proposition 3.2, it follows that

$$\widetilde{\text{c-occ}}(\pi,\nu^n)=\frac{\sum_{\rho\in\mathcal{S}_n}q^n_\rho\cdot\text{c-occ}(\pi,\rho)+Er}{n\cdot\sum_{\theta\in\mathcal{S}_n}q^n_\theta},\quad\text{for all}\quad\pi\in\mathcal{S},$$

[2]The annealed Benjamini–Schramm convergence is an extension of the Benjamini–Schramm convergence (BS-limit) to sequences of *random* permutations. For more details see [10, Section 2.5.1].

with $Er \leq (-1 + \sum_{\theta \in \mathcal{S}_n} q^n_\theta) \cdot |\pi|$. Therefore

$$\begin{aligned}
&\left|\widetilde{\text{c-occ}}(\pi, \nu^n) - \mathbb{E}[\widetilde{\text{c-occ}}(\pi, \sigma^n)]\right| \\
&\qquad\leq \left|\sum_{\rho \in \mathcal{S}_n} \frac{q^n_\rho}{\sum_{\theta \in \mathcal{S}_n} q^n_\theta} \cdot \widetilde{\text{c-occ}}(\pi, \rho) - \sum_{\rho \in \mathcal{S}_n} p^n_\rho \cdot \widetilde{\text{c-occ}}(\pi, \rho)\right| + \left|\frac{Er}{n \cdot \sum_{\theta \in \mathcal{S}_n} q^n_\theta}\right| \\
&\qquad\qquad\leq \frac{1}{n^n} \cdot \sum_{\rho \in \mathcal{S}_n} \widetilde{\text{c-occ}}(\pi, \rho) + \frac{|\pi|}{n},
\end{aligned}$$

where in the second inequality we used the bound in Eq. (9) and the bound for Er. Since the size of π is fixed and the term $\sum_{\rho \in \mathcal{S}_n} \widetilde{\text{c-occ}}(\pi, \rho)$ is bounded by $n!$, we can conclude that $\left|\widetilde{\text{c-occ}}(\pi, \nu^n) - \mathbb{E}[\widetilde{\text{c-occ}}(\pi, \sigma^n)]\right| \to 0$. Combining this with Eq. (8) we get

$$\widetilde{\text{c-occ}}(\pi, \nu^n) \to \Gamma_\pi(\sigma^\infty), \quad \text{for all} \quad \pi \in \mathcal{S}.$$

Therefore, using Theorem 1.2 we can finally deduce that the deterministic sequence $\{\nu^n\}_{n \in \mathbb{N}}$ converges to σ^∞ in the Benjamini–Schramm topology, concluding the proof. ☐

3.3. **The overlap graph.** We now want to study the way in which consecutive patterns of permutations can overlap.

We start by introducing some more notation. For a permutation $\pi \in \mathcal{S}_k$, with $k \in \mathbb{N}_{\geq 2}$, let $\text{beg}(\pi) \in \mathcal{S}_{k-1}$ (resp. $\text{end}(\pi) \in \mathcal{S}_{k-1}$) be the patterns generated by its first $k-1$ indices (resp. last $k-1$ indices). More precisely,

$$\text{beg}(\pi) \coloneqq \text{pat}_{[1,k-1]}(\pi) \quad \text{and} \quad \text{end}(\pi) \coloneqq \text{pat}_{[2,k]}(\pi).$$

The following definition, introduced in [13], is key in the description of the feasible region P_k.

Definition 3.5 (Overlap graph). *Let $k \in \mathbb{N}_{\geq 2}$. We define the* overlap graph $\mathcal{O}v(k)$ *of size k as a directed multigraph with labeled edges, where the vertices are elements of $\mathcal{S}_{k-1}$ and for all $\pi \in \mathcal{S}_k$ we add the edge* $\text{beg}(\pi) \to \text{end}(\pi)$ *labeled by π.*

This gives us a directed graph with $k!$ many edges, and $(k-1)!$ many vertices. Informally, the continuations of an edge τ in the overlap graph $\mathcal{O}v(k)$ records the consecutive patterns of size k that can appear after the consecutive pattern τ. More precisely, for a permutation $\sigma \in \mathcal{S}_{\geq k+1}$ and an interval $I \subseteq [|\sigma|-1]$ of size k, let $\tau \coloneqq \text{pat}_I(\sigma) \in \mathcal{S}_k$, then we have that

$$\text{pat}_{I+1}(\sigma) \in C_{\mathcal{O}v(k)}(\tau), \tag{10}$$

where $I+1$ denotes the interval obtained from I shifting all the indices by $+1$, and we recall that $C_{\mathcal{O}v(k)}(\tau)$ is the set of continuations of τ.

Example 3.6. We recall that the overlap graph $\mathcal{O}v(4)$ was displayed in Fig. 2 on page 5. The six vertices (in red) correspond to the six permutations of size three and the twenty-four oriented edges correspond to the twenty-four permutations of size four.

Given a permutation $\sigma \in \mathcal{S}_m$, for some $m \geq k$, we can associate to it a walk $W_k(\sigma) = (e_1, \ldots, e_{m-k+1})$ in $\mathcal{O}v(k)$ of size $m-k+1$ defined by

$$\text{lb}(e_i) \coloneqq \text{pat}_{[i,i+k-1]}(\sigma), \quad \text{for all} \quad i \in [m-k+1]. \tag{11}$$

Note that Eq. (10) justifies that this sequence of edges is indeed a walk in the overlap graph.

Example 3.7. Take the graph $\mathcal{O}v(4)$ from Fig. 2 on page 5, and consider the permutation $\sigma = 628451793 \in \mathcal{S}_9$. The corresponding walk $W_4(\sigma)$ in $\mathcal{O}v(4)$ is

$$(3142, 1423, 4231, 2314, 2134, 1342)$$

and it is highlighted in green in Fig. 2.

Note that the map W_k is not injective (see for instance the Example 3.9 below) but the following holds.

Lemma 3.8. *Fix $k \in \mathbb{N}_{\geq 2}$ and $m \geq k$. The map W_k, from the set $\mathcal{S}_m$ of permutations of size m to the set of walks in $\mathcal{O}v(k)$ of size $m-k+1$, is surjective.*

Proof. We exhibit a greedy procedure that, given a walk $w = (e_1, \dots, e_s)$ in $\mathcal{O}v(k)$, constructs a permutation σ of size $s+k-1$ such that $W_k(\sigma) = w$. Specifically, we construct a sequence of s permutations $(\sigma_i)_{i \leq s}$, with $|\sigma_i| = i + k - 1$, in such a way that σ is equal to σ_s. For this proof, it is useful to consider permutations as diagrams.

The first permutation is defined as $\sigma_1 = \mathrm{lb}(e_1)$. To construct σ_{i+1} we add to the diagram of σ_i a final additional point on the right of the diagram between two rows, in such a way that the last k points induce the consecutive pattern $\mathrm{lb}(e_{i+1})$ (the choice for this final additional point may not be unique, but exists). Setting $\sigma \coloneqq \sigma_s$ we have by construction that $W_k(\sigma) = w$. □

We illustrate the construction above in a concrete example.

Example 3.9. Consider the walk $w = (3142, 1423, 4231, 2314, 2134, 1342)$ obtained in Example 3.7 and construct, as explained in the previous proof, a permutation σ such that $W_k(\sigma) = w$. We set $\sigma_1 = 3142$. Then, since $e_2 = 1423$, we add a point between the second and the third row of σ_1 (see Fig. 7 for the diagrams of the considered permutations), obtaining $\sigma_2 = 41523$. Note that the pattern induced by the last 4 points of σ_2 is exactly $e_2 = 1423$. We highlight that we could also add the point between the third and the fourth row of σ_1 obtaining the same induced pattern. However, in this example, we always chose to add the points in the bottommost possible place. We iterate this procedure constructing $\sigma_3 = 516342$, $\sigma_4 = 6173425$, $\sigma_5 = 71834256$, $\sigma_6 = 819452673$. Setting $\sigma \coloneqq \sigma_6 = 819452673$ we obtain that $W_4(\sigma) = w$. Note that this is not the same permutation considered in Example 3.7, indeed the map W_k is not injective.

We conclude this section with two simple results useful for the following sections.

Lemma 3.10. *If σ is a permutation and $w = W_k(\sigma) = (e_1, \dots, e_s)$ is its corresponding walk on $\mathcal{O}v(k)$, then*

$$\text{c-occ}(\pi, \sigma) = |\{i \leq s |\, \mathrm{lb}(e_i) = \pi\}|.$$

Proof. This is a trivial consequence of the definition of the map W_k. See in particular Eq. (11). □

Observation 3.11. Let π_1 and π_2 be two permutations of size $k-1 \geq 1$, and take $\tau = \pi_1 \oplus \pi_2$. Then the path $W_k(\tau)$ goes from π_1 to π_2. Consequently, $\mathcal{O}v(k)$ is strongly connected.

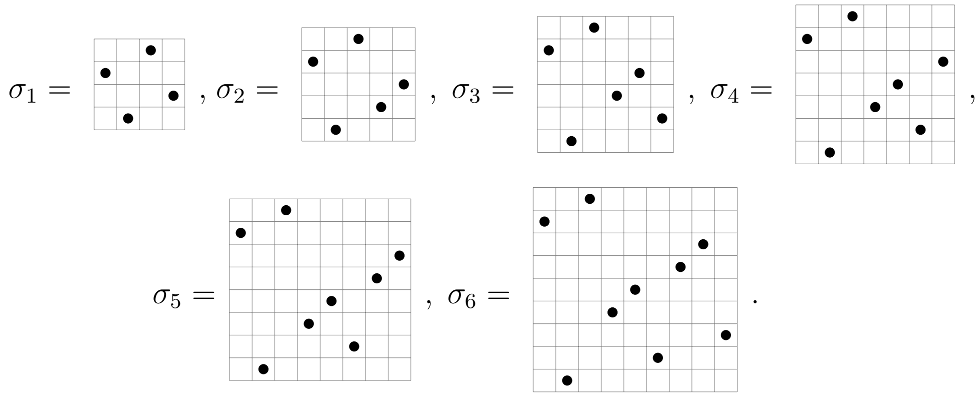


FIGURE 7. The diagrams of the six permutations considered in Example 3.9. Note that every permutation is obtained by adding a new final point to the previous one.

3.4. **A description of the feasible region P_k.** The goal of this section is to prove the following result.

Theorem 3.12. *The feasible region P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$, i.e.*

$$P_k = P(\mathcal{O}v(k)). \tag{12}$$

As a consequence, the vertices of P_k are precisely the vectors $\{\vec{e}_{\mathcal{C}} | \, \mathcal{C}$ is a simple cycle of $\mathcal{O}v(k)\}$ and the dimension of P_k is $k! - (k-1)!$. Moreover, the polytope P_k is described by the equations

$$P_k = \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \sum_{\pi \in \mathcal{S}_k} v_\pi = 1 \, , \quad \sum_{\mathrm{beg}(\pi)=\rho} v_\pi = \sum_{\mathrm{end}(\pi)=\rho} v_\pi \, , \forall \rho \in \mathcal{S}_{k-1} \, , \quad \vec{v} \geq \vec{0} \right\} .$$

Proof. The first step is to show that, for any simple cycle $\mathcal{C}$ of $\mathcal{O}v(k)$, the vector $\vec{e}_{\mathcal{C}}$ is in P_k. This, together with Proposition 3.2 implies that $P(\mathcal{O}v(k)) \subseteq P_k$.

According to Lemma 3.8, for every $m \in \mathbb{N}$, there is a permutation σ^m such that $W_k(\sigma^m)$ is the walk resulting from the concatenation of m copies of $\mathcal{C}$. We claim that $\widetilde{\text{c-occ}}_k(\sigma^m) \to \vec{e}_{\mathcal{C}}$ and $|\sigma^m| \to \infty$. The latter affirmation is trivial since, by Lemma 3.8, $|\sigma^m| = |\mathcal{C}|m + k - 1$. For the first claim, according to Lemma 3.10, we have that $\text{c-occ}(\pi, \sigma^m) = m$ for any π that is the label of an edge in the simple cycle $\mathcal{C}$, and $\text{c-occ}(\pi, \sigma^m) = 0$ otherwise. Hence

$$\widetilde{\text{c-occ}}_k(\sigma^m) = \vec{e}_{\mathcal{C}} \frac{m|\mathcal{C}|}{|\sigma^m|} \to \vec{e}_{\mathcal{C}} \, .$$

as desired.

On the other hand, suppose that $\vec{v} \in P_k$, so we have a sequence σ^m of permutations such that $|\sigma^m| \to \infty$ and $\widetilde{\text{c-occ}}_k(\sigma^m) \to \vec{v}$. We will show that $\mathrm{dist}(\widetilde{\text{c-occ}}_k(\sigma^m), P(\mathcal{O}v(k))) \to 0$. It is then immediate, since $P(\mathcal{O}v(k))$ is closed, that $\vec{v} \in P(\mathcal{O}v(k))$, proving that $P_k = P(\mathcal{O}v(k))$. We consider the walk $w^m = W_k(\sigma^m)$. Using Lemma 2.7, the edge multiset of the walk w^m can be decomposed into simple cycles and a tail (that does not repeat vertices and may be empty) as follows

$$w^m = \mathcal{C}_1^m \uplus \cdots \uplus \mathcal{C}_\ell^m \uplus \mathcal{T}^m.$$

Then from Lemma 3.10 we can compute $\widetilde{\text{c-occ}}_k(\sigma^m)$ as a convex combination of $\vec{e}_{\mathcal{C}}$ for some simple cycles $\mathcal{C}$, plus a small error term. Specifically,

$$\widetilde{\text{c-occ}}_k(\sigma^m) = \vec{e}_{\mathcal{C}_1^m}\tfrac{|\mathcal{C}_1^m|}{|\sigma^m|} + \cdots + \vec{e}_{\mathcal{C}_\ell^m}\tfrac{|\mathcal{C}_\ell^m|}{|\sigma^m|} + \vec{Er}^{\,m},$$

where $|\vec{Er}^{\,m}| \leq \frac{(k-1)!}{|\sigma^m|}$ since there are $(k-1)!$ distinct vertices in $\mathcal{O}v(k)$ and the path $\mathcal{T}^m$ does not contain repeated vertices. In particular, $|\vec{Er}^{\,m}| \to 0$ since k is constant and $|\sigma^m| \to \infty$.

Noting that

$$\widetilde{\text{c-occ}}_k(\sigma^m) = \vec{Er}^{\,m} + \frac{\sum_i |\mathcal{C}_i^m|}{|\sigma^m|}\vec{w}_m,$$

where $\vec{w}_m = \frac{1}{\sum_i |\mathcal{C}_i^m|}\left(\vec{e}_{\mathcal{C}_1^m}|\mathcal{C}_1^m| + \cdots + \vec{e}_{\mathcal{C}_\ell^m}|\mathcal{C}_\ell^m|\right) \in P(\mathcal{O}v(k))$, we can conclude that

$$\text{dist}\left(\widetilde{\text{c-occ}}_k(\sigma^m), P(\mathcal{O}v(k))\right) \leq \text{dist}\left(\vec{Er}^{\,m} + \frac{\sum_i |\mathcal{C}_i^m|}{|\sigma^m|}\vec{w}_m, \vec{w}_m\right) \to 0\,,$$

since $|\vec{Er}^{\,m}| \to 0$, $\frac{\sum_i |\mathcal{C}_i^m|}{|\sigma^m|} \to 1$ and $\vec{w}_m$ is uniformly bounded. This concludes the proof of Eq. (12).

The characterization of the vertices is a trivial consequence of Proposition 2.2. For the dimension, it is enough to note that $\mathcal{O}v(k)$ is strongly connected from Observation 3.11. So by Theorem 2.3 it has dimension $|E(\mathcal{O}v(k))| - |V(\mathcal{O}v(k))| = k! - (k-1)!$, as desired. Finally, the equations for P_k are determined using Theorem 2.13 and the definition of overlap graph. □

Remark 3.13. Since for two different simple cycles $\mathcal{C}_1, \mathcal{C}_2$ we have that $\vec{e}_{\mathcal{C}_1} \neq \vec{e}_{\mathcal{C}_2}$, enumerating the vertices corresponds to enumerating simple cycles of $\mathcal{O}v(k)$ (which seems to be a difficult problem). This problem was partially investigated in [1]. There, all the cycles of size one and two are enumerated.

4. Mixing classical patterns and consecutive patterns

In Section 1.6 we explained that a natural question is to describe the feasible region when we mix classical and consecutive patterns.

More generally, let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ be two finite sets of permutations. We consider the following sets of points

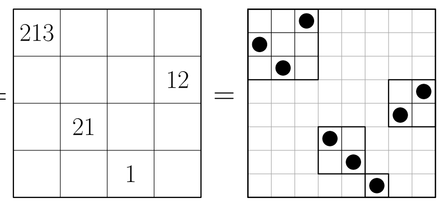
$$\begin{aligned} A &= \left\{\vec{v} \in [0,1]^{\mathcal{A}} | \exists\ (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \left(\widetilde{\text{c-occ}}(\pi, \sigma^m)\right)_{\pi\in\mathcal{A}} \to \vec{v}\right\}, \\ B &= \left\{\vec{v} \in [0,1]^{\mathcal{B}} | \exists\ (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \left(\widetilde{\text{occ}}(\pi, \sigma^m)\right)_{\pi\in\mathcal{B}} \to \vec{v}\right\}. \end{aligned} \tag{13}$$

We want to investigate the set

$$\begin{aligned} C = \Big\{\vec{v} \in [0,1]^{\mathcal{A}\sqcup\mathcal{B}} \Big| &\exists\ (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty, \\ &\left(\widetilde{\text{c-occ}}(\pi, \sigma^m)\right)_{\pi\in\mathcal{A}} \to (\vec{v})_{\mathcal{A}} \text{ and } \left(\widetilde{\text{occ}}(\pi, \sigma^m)\right)_{\pi\in\mathcal{B}} \to (\vec{v})_{\mathcal{B}}\Big\}. \end{aligned} \tag{14}$$

For the statement of the next theorem we need to recall the definition of the *substitution operation* on permutations. For $\theta, \nu^{(1)}, \dots, \nu^{(d)}$ permutations such that $d = |\theta|$, the substitution $\theta[\nu^{(1)}, ..., \nu^{(d)}]$ is defined as follows: for each i, we replace the point $(i, \theta(i))$ in the diagram of θ with the diagram of $\nu^{(i)}$. Then, rescaling the rows and columns yields the diagram of a larger permutation $\theta[\nu^{(1)}, ..., \nu^{(d)}]$. Note that $|\theta[\nu^{(1)}, ..., \nu^{(d)}]| = \sum_{i=1}^{d} |\nu^{(i)}|$ (see Fig. 8 for an example).

$4213[213, 21, 1, 12] =$



$= 76832145\,.$

FIGURE 8. Example of substitution of permutations.

Theorem 4.1. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ be finite sets of permutations, and A, B, C be defined as in Eqs.* (13) *and* (14). *It holds that*

$$A \times B = C\,. \tag{15}$$

Specifically, given two points $\vec{v}_{\mathcal{A}} \in A, \vec{v}_{\mathcal{B}} \in B$, consider two sequences $(\sigma_A^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ and $(\sigma_B^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ such that

$$\begin{aligned} &|\sigma_A^m| \to \infty \text{ and } \left(\widetilde{\text{c-occ}}(\pi, \sigma_A^m)\right)_{\pi\in\mathcal{A}} \to \vec{v}_A, \\ &|\sigma_B^m| \to \infty \text{ and } \left(\widetilde{\text{occ}}(\pi, \sigma_B^m)\right)_{\pi\in\mathcal{B}} \to \vec{v}_B, \end{aligned} \tag{16}$$

then the sequence $(\sigma_C^m)_{m\in\mathbb{N}}$ defined by

$$\sigma_C^m := \sigma_B^m[\sigma_A^m, \dots, \sigma_A^m], \quad \text{for all} \quad m \in \mathbb{N}, \tag{17}$$

satisfies

$$|\sigma_C^m| \to \infty, \quad \left(\widetilde{\text{c-occ}}(\pi, \sigma_C^m)\right)_{\pi\in\mathcal{A}} \to \vec{v}_A \quad \text{and} \quad \left(\widetilde{\text{occ}}(\pi, \sigma_C^m)\right)_{\pi\in\mathcal{B}} \to \vec{v}_B. \tag{18}$$

Proof. Let $(\sigma_C^m)_{m\in\mathbb{N}}$ be defined as in Eq. (17). The fact that the size of σ_C^m tends to infinity follows from $|\sigma_C^m| = |\sigma_A^m||\sigma_B^m| \to \infty$. For the second limit in Eq. (18), note that for every pattern $\pi \in \mathcal{A}$,

$$\widetilde{\text{c-occ}}(\pi, \sigma_C^m) = \frac{\text{c-occ}(\pi, \sigma_C^m)}{|\sigma_B^m| \cdot |\sigma_A^m|} = \frac{\text{c-occ}(\pi, \sigma_A^m) \cdot |\sigma_B^m| + Er}{|\sigma_B^m| \cdot |\sigma_A^m|} = \widetilde{\text{c-occ}}(\pi, \sigma_A^m) + \frac{Er}{|\sigma_B^m| \cdot |\sigma_A^m|},$$

where $Er \leq |\sigma_B^m| \cdot |\pi|$. This error term comes from intervals of $[|\sigma_C^m|]$ that intersect more than one copy of σ_A^m. Since $|\pi|$ is fixed and $|\sigma_A^m| \to \infty$ we can conclude the desired limit, using the assumption in Eq. (16) that $\left(\widetilde{\text{c-occ}}(\pi, \sigma_A^m)\right)_{\pi\in\mathcal{A}} \to \vec{v}_A$.

Finally, for the third limit in Eq. (18) we note that setting $n = |\sigma_C^m|$ and $k = |\pi|$,

$$\widetilde{\text{occ}}(\pi, \sigma_C^m) = \frac{\text{occ}(\pi, \sigma_C^m)}{\binom{n}{k}} = \mathbb{P}\left(\text{pat}_{\boldsymbol{I}}(\sigma_C^m) = \pi\right), \tag{19}$$

where $\boldsymbol{I}$ is a random set, uniformly chosen among the $\binom{n}{k}$ subsets of $[n]$ with k elements (we denote random quantities in **bold**). Let now E^m be the event that the random set $\boldsymbol{I}$ contains two indices $\boldsymbol{i}, \boldsymbol{j}$ of $[|\sigma_C^m|]$ that belong to the same copy of σ_A^m in σ_C^m. Denote by $(E^m)^C$ the complement of the event E^m. We have

(20)

$$\mathbb{P}\left(\mathrm{pat}_{\boldsymbol{I}}(\sigma_C^m) = \pi\right) = \mathbb{P}\left(\mathrm{pat}_{\boldsymbol{I}}(\sigma_C^m) = \pi | E^m\right) \cdot \mathbb{P}\left(E^m\right) + \mathbb{P}\left(\mathrm{pat}_{\boldsymbol{I}}(\sigma_C^m) = \pi | (E^m)^C\right) \cdot \mathbb{P}\left((E^m)^C\right).$$

We claim that

$$\mathbb{P}\left(E^m\right) \leq \binom{k}{2} \frac{1}{|\sigma_B^m|} \to 0. \tag{21}$$

Indeed, the factor $\binom{k}{2}$ counts the number of pairs i, j in a set of cardinality k and the factor $\frac{1}{|\sigma_B^m|}$ is an upper bound for the probability that given a uniform two-element set $\{\boldsymbol{i}, \boldsymbol{j}\}$ then $\boldsymbol{i}, \boldsymbol{j}$ belong to the same copy of σ_A^m in σ_C^m (recall that there are $|\sigma_B^m|$ copies of σ_A^m in σ_C^m). Note also that

$$\mathbb{P}\left(\mathrm{pat}_{\boldsymbol{I}}(\sigma_C^m) = \pi | (E^m)^C\right) = \widetilde{\mathrm{occ}}(\pi, \sigma_B^m) \to \vec{v}_B, \tag{22}$$

where the last limit comes from Eq. (16). Using Eqs. (19) to (22), we obtain that

$$(\widetilde{\mathrm{occ}}(\pi, \sigma_C^m))_{\pi \in \mathcal{B}} \to \vec{v}_B.$$

This concludes the proof of Eq. (18). The result in Eq. (15) follows from the fact that we trivially have $C \subseteq A \times B$, and for the other inclusion we use the construction above, which proves that $(\vec{v}_A, \vec{v}_B) \in C$, for every $\vec{v}_{\mathcal{A}} \in A, \vec{v}_{\mathcal{B}} \in B$. □

ACKNOWLEDGEMENTS

The authors are very grateful to Mathilde Bouvel and Valentin Féray for the various discussions during the preparation of the paper.

This work was completed with the support of the SNF grants number 200021-172536 and 200020-172515.

REFERENCES

[1] John Asplund and N. Bradley Fox. Enumerating cycles in the graph of overlapping permutations. *Discrete Mathematics*, 341(2):427–438, 2018.

[2] Egon Balas and Maarten Oosten. On the cycle polytope of a directed graph. *Networks: An International Journal*, 36(1):34–46, 2000.

[3] Egon Balas and Rüdiger Stephan. On the cycle polytope of a directed graph and its relaxations. *Networks: An International Journal*, 54(1):47–55, 2009.

[4] Welleda Baldoni and Michèle Vergne. Kostant partitions functions and flow polytopes. *Transformation Groups*, 13(3-4):447–469, 2008.

[5] Frédérique Bassino, Mathilde Bouvel, Valentin Féray, Lucas Gerin, Mickaël Maazoun, and Adeline Pierrot. Universal limits of substitution-closed permutation classes. *ArXiv preprint:1706.08333*, 2017.

[6] Frédérique Bassino, Mathilde Bouvel, Valentin Féray, Lucas Gerin, Mickaël Maazoun, and Adeline Pierrot. Scaling limits of permutation classes with a finite specification: a dichotomy. *ArXiv preprint:1903.07522*, 2019.

[7] Frédérique Bassino, Mathilde Bouvel, Valentin Féray, Lucas Gerin, and Adeline Pierrot. The Brownian limit of separable permutations. *Ann. Probab.*, 46(4):2134–2189, 2018.

[8] Petra Bauer. The circuit polytope: Facets. *Mathematics of Operations Research*, 22(1):110–145, 1997.

[9] David Bevan. Permutations with few inversions are locally uniform. *ArXiv preprint:1908.07277*, 2019.
[10] Jacopo Borga. Local convergence for permutations and local limits for uniform ρ-avoiding permutations with $|\rho| = 3$. *Probability Theory and Related Fields, to appear (https://doi.org/10.1007/s00440-019-00922-4)*, 2019.
[11] Jacopo Borga, Mathilde Bouvel, Valentin Féray, and Benedikt Stufler. A decorated tree approach to random permutations in substitution-closed classes. *ArXiv preprint:1904.07135*, 2019.
[12] Jacopo Borga and Erik Slivken. Square permutations are typically rectangular. *ArXiv preprint:1904.03080*, 2019.
[13] Fan Chung, Persi Diaconis, and Ron Graham. Universal cycles for combinatorial structures. *Discrete Math.*, 110(1-3):43–59, 1992.
[14] Collette R. Coullard and William R. Pulleyblank. On cycle cones and polyhedra. *Linear Algebra Appl.*, 114/115:613–640, 1989.
[15] Geir Dahl and Luis Gouveia. On the directed hop-constrained shortest path problem. *Oper. Res. Lett.*, 32(1):15–22, 2004.
[16] Geir Dahl and Bjørnar Realfsen. The cardinality-constrained shortest path problem in 2-graphs. *Networks*, 36(1):1–8, 2000.
[17] Roman Glebov, Carlos Hoppen, Tereza Klimošová, Yoshiharu Kohayakawa, Daniel Král', and Hong Liu. Densities in large permutations and parameter testing. *European J. Combin.*, 60:89–99, 2017.
[18] Gregory Gutin and Abraham P. Punnen. *The traveling salesman problem and its variations*, volume 12 of *Combinatorial Optimization*. Kluwer Academic Publishers, Dordrecht, 2002.
[19] Carlos Hoppen, Yoshiharu Kohayakawa, Carlos Gustavo Moreira, Balázs Ráth, and Rudini Menezes Sampaio. Limits of permutation sequences. *J. Combin. Theory Ser. B*, 103(1):93–113, 2013.
[20] Hao Huang, Nati Linial, Humberto Naves, Yuval Peled, and Benny Sudakov. On the 3-local profiles of graphs. *J. Graph Theory*, 76(3):236–248, 2014.
[21] Hao Huang, Nati Linial, Humberto Naves, Yuval Peled, and Benny Sudakov. On the densities of cliques and independent sets in graphs. *Combinatorica*, 36(5):493–512, 2016.
[22] J. Robert Johnson. Universal cycles for permutations. *Discrete Math.*, 309(17):5264–5270, 2009.
[23] Richard Kenyon, Daniel Král', Charles Radin, and Peter Winkler. Permutations with fixed pattern densities. *ArXiv preprint:1506.02340*, 2015.
[24] Sergey Kitaev, Vladimir N. Potapov, and Vincent Vajnovszki. On shortening u-cycles and u-words for permutations. *Discrete Appl. Math.*, 260:203–213, 2019.
[25] Abilio Lucena, Alexandre Salles da Cunha, and Luidi Simonetti. A new formulation and computational results for the simple cycle problem. *Electronic Notes in Discrete Mathematics*, 44:83–88, 2013.
[26] Mustazee Rahman, Balint Virag, and Mate Vizer. Geometry of permutation limits. *ArXiv preprint:1609.03891*, 2016.
[27] Alexander A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.
[28] Alexander A. Razborov. On the minimal density of triangles in graphs. *Combin. Probab. Comput.*, 17(4):603–618, 2008.
[29] Dan Romik. Permutations with short monotone subsequences. *Adv. in Appl. Math.*, 37(4):501–510, 2006.
[30] Shannon Starr. Thermodynamic limit for the Mallows model on S_n. *J. Math. Phys.*, 50(9):095208, 15, 2009.
[31] Rüdiger Stephan. Facets of the (s, t)-p-path polytope. *Discrete Appl. Math.*, 157(14):3119–3132, 2009.
[32] Yannic Vargas. Hopf algebra of permutation pattern functions. *HAL preprint: hal-01207565*, 2015.
[33] Doron Zeilberger. Proof of a conjecture of Chan, Robbins, and Yuen. *Electron. Trans. Numer. Anal*, 9(147-148):1–2, 1999.
[34] Günter M Ziegler. *Lectures on polytopes*, volume 152. Springer Science & Business Media, 2012.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH, WINTERTHURERSTR. 190, CH-8057 ZÜRICH, SWITZERLAND

Email address: `jacopo.borga@math.uzh.ch`
Email address: `raul.penaguiao@math.uzh.ch`