Probability 2

Exercise sheet nb. 11

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Due until: 3rd December at 5 p.m.

Exercise 1 (4 points). Consider the Markov chain $(X_n)_{n\geq 0}$ on $S := \{1, \dots, 7\}$ with transition matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{12} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1. Find all recurrent and transient states of the Markov chain, and find its decomposition into closed irreducible components.
2. What are the periods of each of the states?
3. Find two distinct stationary probability distributions on the Markov chain. (Hint: find one in each closed irreducible recurrent component.)

Exercise 2 (4 points). We consider the simple random walk on the vertices of a cube: at each time, the particle jumps from one vertex to one of its three neighbors, each one with probability $1/3$. Fix a vertex v_0 of the cube. The goal is to compute the mean return time to v_0, when the chain starts at v_0.

1. For $i = 0, 1, 2, 3$, let v_i be a fixed vertex at distance i from v_0. We let $T_{v_0} = \min\{n \geq 1 : X_n = v_0\}$ and

   $$m_i = \mathbb{E}_{v_i}\big[T_{v_0}\big].$$

   Show that, for j and $i \neq 0$ such that $\mathbb{P}_{v_j}[X_1 = v_i] > 0$, we have

   $$\mathbb{E}_{v_j}\big[T_{v_0} | X_1 = v_i\big] = m_i + 1.$$

   What is the value of the left-hand side for $i = 0$?

2. Show that $m_1 = 1 + \frac{2}{3}m_2$ (hint: condition on the value of X_1). Write similar equations for m_0, m_2 and m_3.

3. Solve the system.

4. Find a stationary probability distribution μ for the Markov chain. Observe that $m_0 = \mu(v_0)^{-1}$.

Exercise 3 (3 points). Fix $p \in (0,1)$ and let $q = 1-p$. Consider the Markov chain on $S = \{0, \dots, n\}$, where $n \geq 2$ with transition matrix Q defined as follows:

$$Q_{i,j} = \begin{cases} p, & \text{if } j-1 = i, \quad i \in \{0, \dots, n-1\}, \\ q, & \text{if } j = 0 \text{ and any } i, \\ 1, & \text{if } i = n,\ j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

1. Show that the Markov chain is irreducible, aperiodic and recurrent.

2. Compute the stationary measure of the Markov chain.

3. Show that $\mathbb{E}_n[T_n] = \frac{1-p^{n+1}}{p^n - p^{n+1}}$, where $T_n = \inf_{k \geq 1}\{X_k = n\}$.