Pattern Hopf algebras in combinatorial presheaves

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Slides can be found at
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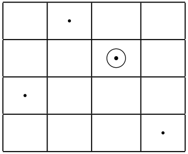
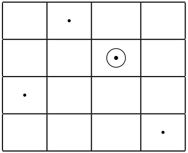
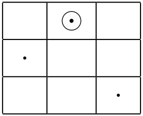
Counting occurences of a pattern

Marked permutation π^* on a set S (a pair of orders on $S \sqcup \{*\}$).

$\pi^* =$  $= 24\bar{3}1$

Set of columns of the square configuration of π^* - subset of S.
The **restriction to** I can be defined $\pi|_I$ and is a permutation in I.
We can count occurrences!

$\pi^*|_{\{1,3\}} =$  $\Big|_{\{1,3\}} =$  $= 2\bar{3}1$

Marked permutation pattern algebra

We write

$$\mathbf{p}_{2\bar{3}1}(24\bar{3}1) = 1,\ \mathbf{p}_{\bar{1}23}(\bar{1}23456) = 20,\ \mathbf{p}_{2\bar{4}13}(762341\bar{8}95) = 0\,.$$

Pattern function p_{π^*} are in the space of functions $\mathcal{F}(\mathcal{G}(\mathtt{MPer}), \mathbb{R})$
The linear span of all pattern functions - $\mathcal{A}(\mathtt{MPer})$ - is closed for pointwise multiplication.

Marked permutation pattern algebra

Adding another ingredient

$$\text{If } \tau^* = \begin{array}{|c|c|c|}\hline \tau^{lu} & & \tau^{ru} \\ \hline & \odot & \\ \hline \tau^{ld} & & \tau^{rd} \\ \hline \end{array}\text{, define:}$$

$$\tau^* \star \pi^* = \begin{array}{|c|c|c|}\hline \tau^{lu} & & \tau^{ru} \\ \hline & \pi^* & \\ \hline \tau^{ld} & & \tau^{rd} \\ \hline \end{array}\,, \qquad \text{example: } 13\bar{2}4 \star 1\bar{2}3 = 152\bar{3}46\,.$$

By magic properties of dualisation, this gives a coproduct on $\mathcal{A}(\texttt{MPer})$:

$$\Delta\, \mathbf{p}_{\pi^*} = \sum_{\pi^* = \tau_1^* \star \tau_2^*} \mathbf{p}_{\tau_1^*} \otimes \mathbf{p}_{\tau_2^*}\,,$$

We have a Hopf algebra $\mathcal{A}(\texttt{MPer})$.

Permutation pattern algebra

Proposition (Product formula)

Let $\binom{\sigma^*}{\pi^*, \tau^*}$ *count the number of covers of* σ^* *with permutations* π^**,* τ^**.*

$$\mathbf{p}_{\pi^*} \cdot \mathbf{p}_{\tau^*} = \sum_{|\sigma^*| \leq |\pi^*| + |\tau^*|} \binom{\sigma^*}{\pi^*, \tau^*} \mathbf{p}_{\sigma^*} \,,$$

where σ^* *runs over marked permutations.*

Theorem (P, 2018)

The Hopf algebra $\mathcal{A}(\mathtt{MPer})$ *is free commutative. We can explicitly describe the free generators of the algebra.*

Outline of the talk

Pattern algebra

What do we need to have a pattern Hopf algebra?

1. Assignment $S \mapsto h[S] = \{$structures over $S\}$ + notion of *relabelling*.
2. For any inclusion $V \hookrightarrow W$, a restriction map $h[W] \to h[V]$.
3. An associative *monoid* operation $*$ with unit, $* : h[I] \times h[J] \to h[I \sqcup J]$ that is compatible with restrictions.
4. A unique element of size zero.

$$\text{If } a \in h[A],\ b \in h[B],\ \mathbf{p}_a(b) := \#\{A' \subseteq A | a|_{A'} \sim b\}\,.$$

$1 + 2 =$ *combinatorial presheaf* $\to$ pattern algebra.
$1 + 2 + 3 =$ *monoid in combinatorial presheaves*.
$1 + 2 + 4 =$ *connected presheaf*.
$1 + 2 + 3 + 4 =$ Hopf algebra.

A presheaf on graphs - $\texttt{Gr}$

- For each set V we are given the set $\texttt{Gr}[V]$ of graphs with vertex set V, and for any bijection $\phi : V \to W$ we are given a relabelling of graphs $\texttt{Gr}[W] \to \texttt{Gr}[V]$.
- If $J \subseteq I$, induced subgraphs on I to $J \to$ restrictions $\texttt{Gr}[I] \to \texttt{Gr}[J]$.
- The disjoint union of graphs is an associative monoid structure. It is also **commutative**.
- The empty graph fortunately exists and is unique in $\emptyset$!

The pattern Hopf algebra

Proposition (Linear independence)

The set $\{\mathbf{p}_G \,|\, G \in \uplus_{n \geq 0} \texttt{Gr}[n]/_\sim\}$ *is linearly independent.*

This defines a basis for the patterns algebra $\mathcal{A}(\texttt{Gr})$.

Simple example - `Set`

The presheaf of sets.

For each $n \geq 0$, $\texttt{Set}[n]$ has a unique element $*_n$ of size n.

$$\mathbf{p}_{*_n}(*_m) = \binom{m}{n} \qquad \binom{*_d}{*_a, *_b} = \binom{d}{a}\binom{a}{a+b-d}.$$

So

$$\mathbf{p}_{*_a}\,\mathbf{p}_{*_b}(*_c) = \sum_{d \geq 0} \binom{d}{a}\binom{a}{a+b-d}\,\mathbf{p}_{*_d}(*_c)$$

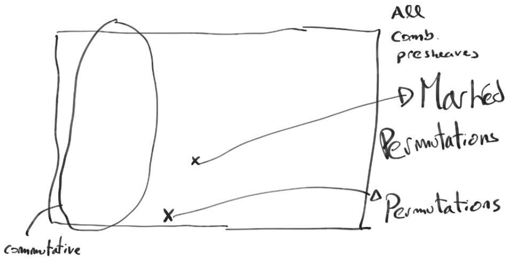
Monoidal structure - Disjoint union: $*_n \cdot *_m = *_{n+m}$.

$$\Delta\,\mathbf{p}_{*_a} = \sum_{k=0}^{a} \mathbf{p}_{*_k} \otimes \mathbf{p}_{*_{a-k}}, \qquad \mathcal{A}(\texttt{Set}) = \mathbb{R}\langle \mathbf{p}_{*_1} \rangle$$

Simple example

Theorem (P - 2019)

If $\mathtt{h}$ is a connected commutative presheaf, then $\mathcal{A}(\mathtt{h})$ is free. The free generators are the indecomposable objects with respect to the commutative product.



Connected commutative combinatorial presheafs

Proof (by example):
Graphs, with a disjoint union, form a commutative presheaf. Every graph has a unique factorization into **indecomposables** $\mathcal{I}$.
Let $L \subseteq \mathcal{I}$ be a multiset of connected graphs. Let $G = \biguplus_{l \in L} l$, the **unique factorization** of G into connected graphs.

$$\prod_{l \in L} \mathbf{p}_l = \mathbf{p}_G + \sum_{H \leq G} c_H \, \mathbf{p}_H \text{ for some total order } \leq$$

$$\Rightarrow \left\{ \prod_{l \in L} \mathbf{p}_l \,|\, L \subseteq \mathcal{I} \text{ multiset} \right\} \text{ is lin. ind.}$$

$$\Leftrightarrow \mathcal{A}(\mathtt{Gr}) \text{ is free commutative}$$

Unique factorization theorem on permutations

The $\oplus$ product on permutations provides a unique factorization theorem on permutations:

$$\pi = \tau_1 \oplus \cdots \oplus \tau_k = \begin{array}{|c|c|c|}\hline & & \tau_k \\ \hline & \iddots & \\ \hline \tau_1 & & \\ \hline\end{array}$$

- For any permutation π, there is a unique k and unique $\tau_1, \dots, \tau_k$ **indecomposable permutations** such that $\pi = \tau_1 \oplus \cdots \oplus \tau_k$.

Unique factorization theorem on permutations

There are more permutations than what the indecomposable ones can generate.
Enlarge the set $\mathcal{I}$ to $\mathcal{L}$ with the so called **Lyndon permutations**, by adding some decomposable elements.
Motivation: choose between $\pi_1 \oplus \pi_2$ and $\pi_2 \oplus \pi_1$ which one to include in the set of free generators.

Theorem (Vargas, 2014)

The Hopf algebra $\mathcal{A}(\mathtt{Per})$ is free commutative. We can explicitly describe the free generators of the algebra.

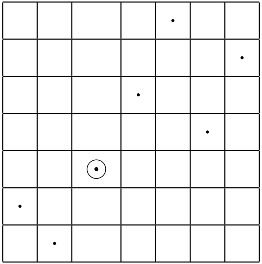
Unique factorisation theorem on marked permutations

- The factorization is **not unique.**
  If τ_1, τ_2 $\oplus$-indecomposable.

$$(\bar{1} \oplus \tau_1) \star (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) \star (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1 \, .$$

For $\tau_1 = 2413$ and $\tau_2 = 21$ we have

$$(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = 21 \oplus \bar{1} \oplus 2413 =$$



- The order of the factors **does matter** to some extent.

Unique factorization theorem on marked permutations

We can define a map from words on indecomposable marked permutations:

$$\text{Monoid morphism } \star : \mathcal{W}(\mathcal{I}) \to \mathcal{A}(\texttt{MPer})\,.$$

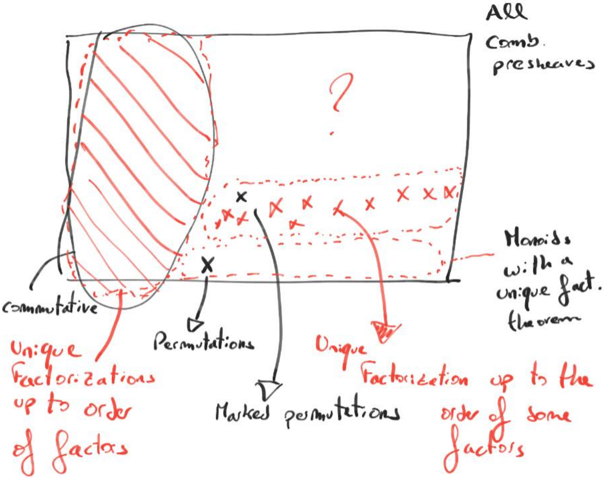
$$\oplus\text{- relations : } (\bar{1} \oplus \tau_1) \star (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) \star (\bar{1} \oplus \tau_1)\,.$$

Goal: describe $\ker \star$.

Theorem (P - 2018)

The equivalence relation $\ker \star$ is spanned by relations as the one above and their $\ominus$ equivalent.

Freeness conjecture



Further questions - *Permutons* and *feasible regions*

Permutons - Notion of patterns of a permutation π can be extended to a permuton P, that is a probability measure in $[0,1] \times [0,1]$ as follows:

$$\mathbf{p}_\pi(P) = \mathbb{P}[\text{ n indep. points with law P form pattern } \pi] \,.$$

Conjecture

Let $\mathcal{L}_q = \{\mathbf{p}_l \,|\, l$ is a Lyndon permutation with size $q\}$ be the set of free generators of $\mathcal{A}(\mathtt{Per})$. The image of the map,

$$\{\textit{Permutons}\} \to \mathbb{R}^{\#\mathcal{L}_q} : P \mapsto (\mathbf{p}_l(P))_{l \in \mathcal{L}_q} \,,$$

called feasible region, contains a full dimensional ball.

Partial results for $\mathcal{I}$ by Glebov, R., Hoppen, C. et al.

Further questions - *Algebra*

- *Character theory of pattern Hopf algebras:* simple characters can be constructed:

$$\zeta_a(\mathbf{p}_b) = \mathbf{p}_b(a)\,,$$

  and all its convolutions. Can we describe all characters?
- *Freeness conjecture:* Other examples include set compositions, polyominoes, etc. Are all pattern algebras of combinatorial presheaves free commutative?

Thank you

I am finishing my PhD studies soon...

Biblio

- Aguiar, M., & Mahajan, S. A. (2010). Monoidal functors, species and Hopf algebras (Vol. 29). *Providence, RI: American Mathematical Society*.
- Vargas, Y. (2014). Hopf algebra of permutation pattern functions. In Discrete Mathematics and Theoretical Computer Science (pp. 839-850). *Discrete Mathematics and Theoretical Computer Science*.
- Glebov, R., Hoppen, C., Klimosova, T., Kohayakawa, Y., & Liu, H. (2017). Densities in large permutations and parameter testing. European Journal of Combinatorics, 60, 89-99.