Probability 2

Exercise sheet nb. 13

Raul Penaguiao - Mailbox in **J floor**

Due until: 17th December at 5 p.m.

Exercise 1 (1 point). Consider a one dimensional Brownian motion $\{B_t\}_{t\geq 0}$, and let $p, q \in \mathbb{R}^+$. Show that $\mathrm{Cov}(B_p, B_q) = p \wedge q$.

Exercise 2 (3 points). We are given a one dimensional Brownian motion $\{B_t\}_{t\geq 0}$, and consider the process

$$X_t \coloneqq B_t - tB_1 + t(y-x) + x\,.$$

Note that, a.s., $X_0 = x$ and $X_1 = y$. The process $(X_t)_{0\leq t\leq 1}$ is called *Brownian bridge* between $x, y \in \mathbb{R}$.

1. Fix $t \in (0,1)$. What is the distribution of X_t?
2. Let $0 < t_1 < \cdots < t_n < 1$. Prove that $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian vector and find its covariance matrix. (Hint: Consider $Y_i = B_{t_{i+1}} - B_{t_i}$ and find an expression of X_{t_n} using these)

Exercise 3 (5 points). Let $\{B_u\}_{u\geq 0}$ be a one dimensional Brownian motion.

1. Let $f : [0,1] \to \mathbb{R}$ be a function, which we assume to be differentiable at some point x in $(0,1)$. Show that we can find N such that for any $n \geq N$ there exists $i = i(n)$ such that *all* the following assertions hold:
   - $|f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right)| \leq n^{-0.9}$,
   - $|f\left(\frac{i+2}{n}\right) - f\left(\frac{i+1}{n}\right)| \leq n^{-0.9}$,
   - $|f\left(\frac{i+3}{n}\right) - f\left(\frac{i+2}{n}\right)| \leq n^{-0.9}$.

   (Hint: Take an appropriate interval around x where f is approximated by a linear function.)
2. Let $i < n$ be two positive integers, show that

$$\mathbb{P}\left[|B_{(i+1)/n} - B_{i/n}| \leq n^{-0.9}\right] \leq \sqrt{\frac{2}{\pi}}n^{-0.4}\,.$$

3. Assume that $i \leq n-3$. Find an upper bound for the probability of the event

$$E_i^{(n)} = \{|B_{(i+1)/n} - B_{i/n}| \leq n^{-0.9}\} \cap \{|B_{(i+2)/n} - B_{(i+1)/n}| \leq n^{-0.9}\} \\ \cap \{|B_{(i+3)/n} - B_{(i+2)/n}| \leq n^{-0.9}\}.$$

   Conclude that $\lim_{n\to\infty} \mathbb{P}\big[\cup_{i=0}^{n-3} E_i\big] = 0$.

4. Show that, a.s., the function $t \mapsto B_t$ is nowhere differentiable in $(0,1)$.

Note: This is a result due to Paley, Wiener and Zygmund from 1933.