What is... a combinatorial presheaf

Zurich Graduate Coloquium 2019, Zurich Switzerland

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26th November, 2019

Slides can be found at

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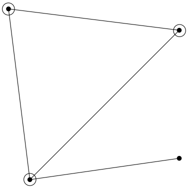
Patterns on graphs

A graph G on the vertex set V is a pair (V, E).
If $I \subseteq V$, we can define the **restriction** $G|_I$ by considering only the edges between points in I.
We can count how many ways we can restrict a graph G to obtain a graph isomorphic to a pattern H: $\mathbf{p}_H(G)$.

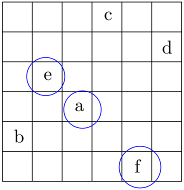
$\mathbf{p}_{\triangle}(G)$ counts triangles in G .



Patterns on permutations

A permutation π on the set X is a pair of orders $(\leq_P, \leq_V)$.
If $I \subseteq X$, we can define the **restriction** $\pi|_I$ by considering only the orders in I.
We can count how many ways we can restrict a permutation G to obtain a permutation isomorphic to a pattern τ: $\mathbf{p}_\tau(\pi)$.

$\mathbf{p}_{321}(G)$ counts decreasing seqs. of size 3 in the permutation π .

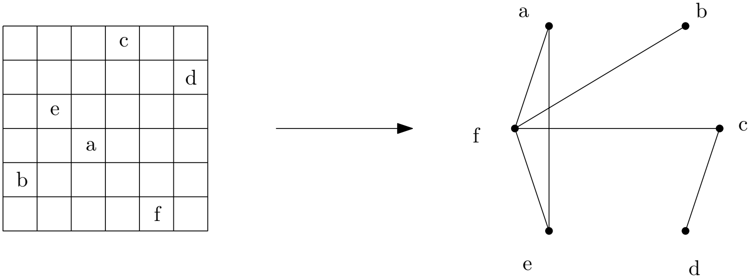


$$= (b <_P e <_P a <_P c <_P f <_P d, f <_V b <_V a <_V e <_V d <_V c)$$

$$= 243615$$

The inversion graphs

Given a permutation π in X, we can consider its **inversion graph**.



This mapping **is stable wrt patterns.** That is, if π is a patterns in τ, then $\mathbf{Inv}(\pi)$ is a pattern in $\mathbf{Inv}(\tau)$ (corresponding to the same set of indices).

Phenomenal phenomena on pattern algebras

For any graphs G, H_1, H_2 there exists a finite family of graphs $J_1, \dots, J_k$ and coefficients $\binom{J_i}{H_1, H_2}$ (independent of G) such that

$$\mathbf{p}_{H_1}(G)\,\mathbf{p}_{H_2}(G) = \sum_{i=1}^{k} \binom{J_i}{H_1, H_2} \mathbf{p}_{J_i}(G)\,.$$

Example: if H_1 is the path with one edge, and H_2 is the path with two edges, then

$\mathbf{p}_{H_1}(G)\,\mathbf{p}_{H_2}(G) = 4\,\mathbf{p}$  $(G) + 6\,\mathbf{p}$  $(G) + 8\,\mathbf{p}$  $(G) + 4\,\mathbf{p}$  (G)

Phenomenal phenomena on combinatorial patterns

For any permutations σ, π_1, π_2 there exists a finite family of graphs $\tau_1, \ldots, \tau_k$ and coefficients $\binom{\tau_i}{\pi_1, \pi_2}$ (independent of σ) such that

$$\mathbf{p}_{\pi_1}(\sigma)\,\mathbf{p}_{\pi_2}(\sigma) = \sum_{i=1}^{k} \binom{\sigma_i}{\pi_1, \pi_2} \mathbf{p}_{\tau_i}(\sigma)\,.$$

Example: if $\pi_1 = 1$, $\pi_2 = 21$ then

$$\begin{aligned}\mathbf{p}_{\pi_1}(\sigma)\,\mathbf{p}_{\pi_2}(\sigma) = {} & 2\,\mathbf{p}_{21}(\sigma) + 3\,\mathbf{p}_{321}(\sigma) \\ & + \mathbf{p}_{213}(\sigma) + 2\,\mathbf{p}_{231}(\sigma) + \mathbf{p}_{132}(\sigma) + 2\,\mathbf{p}_{312}(\sigma)\,.\end{aligned}$$

Outline of the talk

1. Introduction
2. Algebraic concepts
   - Hopf algebra
   - Species and category theory
   - Combinatorial presheaves
3. Free pattern Hopf algebras
   - Marked permutations
4. The freeness conjecture

Hopf algebras - Algebras

Let k be a field and A a vector space over $\mathbb{K}$.

- Associative map $\cdot : A \otimes A \to A$
- Unit map $1 : \mathbb{K} \to A$

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\cdot \otimes id"] \arrow[d, "id \otimes \cdot"] & A \otimes A \arrow[d, "\cdot"] \\
A \otimes A \arrow[r, "\cdot"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes A \arrow[r, "1 \otimes id"] \arrow[dr, leftrightarrow, "\cong"'] & A \otimes A \arrow[d, "\cdot"] & A \otimes k \arrow[l, "id \otimes 1"'] \arrow[dl, leftrightarrow, "\cong"] \\
& A &
\end{tikzcd}
$$

Ex: The polynomial algebra $k[x]$.

Hopf algebras - Coalgebras and Bialgebras

- Cossociative map $\Delta : A \to A \otimes A$
- Counit map $\epsilon : A \to k$

$$\begin{tikzcd}
A \otimes A \otimes A & & A \otimes A \arrow[ll, "\Delta \otimes id"'] \\
A \otimes A \arrow[u, "id \otimes \Delta"] & & A \arrow[ll, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes A & A \otimes A \arrow[l, "\epsilon \otimes id"'] \arrow[r, "id \otimes \epsilon"] & A \otimes k \\
& A \arrow[ul, leftrightarrow, "\cong"] \arrow[u, "\Delta"'] \arrow[ur, leftrightarrow, "\cong"'] &
\end{tikzcd}$$

A **bialgebra** is both an algebra and a coalgebra, where Δ, ϵ are multiplicative maps.
Ex: The polynomial algebra $k[x]$ with the coproduct $\Delta(x) \coloneqq 1 \otimes x + x \otimes 1$ and counit $\epsilon(p) = p(0)$.

Hopf algebras - Antipodes

H is a Hopf algebra if $(H, \cdot, 1, \Delta, \epsilon)$ is a bialgebra and has an antipode S such that

$$\begin{tikzcd}
 & A \otimes A \arrow[r, "id \otimes S"] & A \otimes A \arrow[dr, "\cdot"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\epsilon"] & & k \arrow[r, "1"] & A \\
 & A \otimes A \arrow[r, "S \otimes id"'] & A \otimes A \arrow[ur, "\cdot"'] &
\end{tikzcd}$$

Example: $k[x]$ is a Hopf algebra with $S(x^n) = (-x)^n$.

Hopf algebras - Examples

- $k\{G|$ graphs $\}$. Product: disjoint union $\uplus$.

$$\Delta G = \sum_{I \subseteq V} G|_I \otimes G|_{I^c} \, .$$

- $k\{\pi|$ permutations $\}$. Product: sum of all the shuffles of two permutations.

$$\Delta \pi = \sum_{\pi = \tau_1 \oplus \tau_2} \tau_1 \otimes \tau_2 \, ,$$

where $\oplus$ is the **diagonal product of permutations**.

Category theory - Functors between categories

Categories are a triple $(\mathcal{O}, \mathcal{M}, \circ)$ of objects, morphisms and compositions.

- $\underline{Set}$ is the category of all sets and all functions.
- $\underline{fSet}$ is the category of all finite sets and all functions between them.
- $\underline{fSet}^{\hookrightarrow}$ is the category of all finite sets and all injective functions.
- $\underline{fSet}^{\times}$ is the category of all finite sets and all bijections.

Functors $F : \mathcal{C} \to \mathcal{D}$ map objects and morphisms. It is a **covariant** functor if $F(f) \circ F(g) = F(f \circ g)$, and **contravariant** if $F(f) \circ F(g) = F(g \circ f)$.

Functors - Examples

Example: T a topological space. $\mathcal{C}(T, \mathbb{R})$ the space of all continuous real functions. If $f : T_1 \to T_2$ is a continuous map, this defines a function

$$\mathcal{C}(f, \mathbb{R}) : \mathcal{C}(T_2, \mathbb{R}) \to \mathcal{C}(T_1, \mathbb{R}),$$

so the functor $\mathcal{C}(\cdot, \mathbb{R})$ is contravariant.

Natural transformations

For $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ functors, $\mu : F_1 \Rightarrow F_2$ is a natural transformation if for $A \in \mathcal{O}(\mathcal{C})$ object it assigns a morphism $\mu(A) : F_1(A) \to F_2(A)$ and for any $f \in \mathcal{M}(\mathcal{C})$ morphism we have

$$\begin{tikzcd} A \arrow[d, "f"'] & F_1(A) \arrow[r, "\mu_A"] \arrow[d, "F_1(f)"] & F_2(A) \arrow[d, "F_2(f)"] \\ B & F_1(B) \arrow[r, "\mu_B"'] & F_2(B) \end{tikzcd}$$

Natural Transformations - Example

Category of groups $\underline{Gr}$, the identity functor id and the op functor that sends a group $G = (G, *)$ to the opposite group G^{op}, with group operation defined as $a *^{op} b = b * a$.

" **Any group is naturally isomorphic to its opposite group** "

This means that there is a natural transformation μ, where each μ_G is an isomorphism, between op and id. This natural transformation is $\mu_G(a) := a^{-1}$.

Species and monoids in category theory

A **combinatorial species** is a contravariant functor $a : \underline{fSet}^{\times} \to \underline{fSet}$.
Examples:

$$\mathtt{Gr}[I] = \{ \text{graphs with vertex set } I \} \,,$$

$$\mathtt{Per}[I] = \{ \text{permutations on the set } I \}.$$

Product of species and monoids in species
Given a, b species, its product,

$$a \odot b[I] = \biguplus_{I = A \uplus B} a[A] \times b[B] \,.$$

A product structure on a species a is, thus, a natural transformation $a \odot a \Rightarrow a$. Examples:

$$\uplus : \mathtt{Gr} \odot \mathtt{Gr} \to \mathtt{Gr}, \qquad \oplus : \mathtt{Per} \odot \mathtt{Per} \to \mathtt{Per} \,.$$

Species and monoids in category theory

A **combinatorial presheaf** is a contravariant functor $a : \underline{fSet}^{\hookrightarrow} \to \underline{fSet}$.
Examples:

$$\mathtt{Gr}[I] = \{ \text{ graphs with vertex set } I\} \, ,$$

$$\mathtt{Per}[I] = \{ \text{ permutations on the set } I\}.$$

If $A \subseteq B$, the inclusion map $i : A \to B$ corresponds to a map $\mathtt{Per}[i] : \mathtt{Per}[B] \to \mathtt{Per}[A$. Thus, we can define for $b \in a[B]$,

$$\mathbf{p}_a(b) \coloneqq \{I \subseteq B \Big| b|_I \cong a\} \, .$$

Notation: $\mathcal{G}(a) = \frac{\uplus_I a[I]}{\cong}$

Algebras on combinatorial presheaves

Theorem

Fix a combinatorial presheaf h. For any objects $a, b_1, b_2 \in \mathcal{G}(h)$ there exists a family $c_1, \ldots, c_k$ and coefficients $\binom{c_i}{b_1, b_2}$ such that

$$\mathbf{p}_{b_1}(a)\,\mathbf{p}_{b_2}(a) = \sum_{i=1}^{k} \binom{c_i}{b_1, b_2} \mathbf{p}_{c_i}(a)\,,$$

*In particular, $\mathcal{A}(h) := k\{\mathbf{p}_a\}_{a \in \mathcal{G}(h)}$ is an algebra. This is the **pattern algebra** of a combinatorial presheaf.*

Algebras on combinatorial presheaves

Sketch of proof of theorem

Fix $x \in h[I]$, and note that $\mathbf{p}_a(x)\,\mathbf{p}_b(x)$ counts the following

$$\begin{aligned}
\mathbf{p}_a(x)\,\mathbf{p}_b(x) &= \#\{A \subseteq I \text{ s.t. } x|_A \sim a\} \times \{B \subseteq I \text{ s.t. } x|_B \sim b\} \\
&= \#\{(A, B) \text{ s.t. } A, B \subseteq I,\, x|_A \sim a,\, x|_B \sim b\} \\
&= \sum_{C \subseteq I} \#\{(A, B) \text{ s.t. } A \cup B = C,\, x|_A \sim a,\, x|_B \sim b,\} \\
&= \sum_{C \subseteq I} \binom{x|_C}{a, b} = \sum_{c \in \mathcal{G}(h)} \binom{c}{a, b} \mathbf{p}_c(x)\,.
\end{aligned}$$

Hopf algebras on combinatorial presheaves

If $h = (h, *, 1)$ is an associative presheaf, then we can define a coproduct Δ on $\mathcal{A}(h)$

$$\Delta \mathbf{p}_a = \sum_{a = a_1 * a_2} \mathbf{p}_{a_2} \otimes \mathbf{p}_{a_2} \ .$$

Theorem

Fix an associative presheaf h. Then $\mathcal{A}(h) := k\{\mathbf{p}_a\}_{a \in \mathcal{G}(h)}$ is a Hopf algebra.

Simple example - The presheaf of sets $\mathtt{Set}$

For each $n \geq 0$, $\mathtt{Set}[n]$ is defined to have a unique element $*_n$ of size n.

$$\mathbf{p}_{*_n}(*_m) = \binom{m}{n} \qquad \binom{*_d}{*_a, *_b} = \binom{d}{a}\binom{a}{a+b-d}.$$

So

$$\mathbf{p}_{*_a}\,\mathbf{p}_{*_b}(*_c) = \sum_{d \geq 0} \binom{d}{a}\binom{a}{a+b-d}\,\mathbf{p}_{*_d}(*_c)$$

Monoidal structure - Disjoint union: $*_n \cdot *_m = *_{n+m}$.

$$\Delta\,\mathbf{p}_{*_a} = \sum_{k=0}^{a} \mathbf{p}_{*_k} \otimes \mathbf{p}_{*_{a-k}}, \quad \mathcal{A}(\mathtt{Set}) = k[\mathbf{p}_{*_1}]$$

Graphs and permutations

Graphs and permutations - The inversion graph

Inversion graph, that can be seen as a natural transformation $\mathbf{Inv} : \mathtt{Per} \Rightarrow \mathtt{Gr}$.
For any set I, $\mathbf{Inv}_I$ is a map from permutations on the set I to graphs with vertex set I.
This is a natural transformation that preserves the products: sends $\pi \oplus \tau$ to $\mathbf{Inv}(\pi) \uplus \mathbf{Inv}(\tau)$.

$$\mathbf{Inv} : \mathcal{A}(\mathtt{Gr}) \to \mathcal{A}(\mathtt{Per})\,,$$

$$\mathbf{Inv}(\mathbf{p}_G) = \sum_{\mathbf{Inv}(\pi)=G} \mathbf{p}_\pi \ .$$

Pattern functions on marked permutations

Marked permutation π^* on a set S (a pair of orders on $S \sqcup \{*\}$).

$$\pi^* = \begin{array}{|c|c|c|c|}\hline & \cdot & & \\ \hline & & \odot & \\ \hline \cdot & & & \\ \hline & & & \cdot \\ \hline\end{array} = 24\bar{3}1$$

The **restriction to** I is $\pi|_I$, a marked permutation in I.

$$\pi^*|_{\{1,3\}} = \left.\begin{array}{|c|c|c|c|}\hline & \cdot & & \\ \hline & & \odot & \\ \hline \cdot & & & \\ \hline & & & \cdot \\ \hline\end{array}\right|_{\{1,3\}} = \begin{array}{|c|c|c|}\hline & \odot & \\ \hline \cdot & & \\ \hline & & \cdot \\ \hline\end{array} = 2\bar{3}1$$

We can count occurrences! We have a combinatorial presheaf.

Marked permutation pattern algebra

We write

$$\mathbf{p}_{2\bar{3}1}(24\bar{3}1) = 1,\ \mathbf{p}_{\bar{1}23}(\bar{1}23456) = 20,\ \mathbf{p}_{2\bar{4}13}(762341\bar{8}95) = 0\,.$$

Pattern function p_{π^*} are in the space of functions $\mathcal{F}(\mathcal{G}(\mathtt{MPer}), \mathbb{R})$
The linear span of all pattern functions - $\mathcal{A}(\mathtt{MPer})$ - is closed for pointwise multiplication.

Unique factorization theorem on graphs

Any graph can be uniquely decomposed into the disjoint union of connected graphs

$$G = \biguplus_i G_i \,.$$

So the product of the pattern functions $\mathbf{p}_{G_i}$ decomposes as

$$\prod_i \mathbf{p}_{G_i} = \mathbf{p}_G + \text{terms that have fewer connected components} \,.$$

Thus

$$\left\{ \prod_i \mathbf{p}_{G_i} \middle| G_i \text{ connected graphs} \right\} ,$$

is linearly independent.

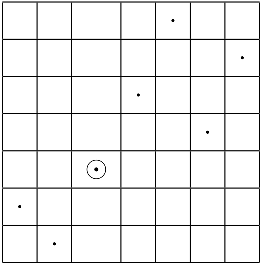
Unique factorization theorem on permutations

The $\oplus$ product on permutations provides a unique factorization theorem on permutations:

$$\pi = \tau_1 \oplus \cdots \oplus \tau_k = \begin{array}{|c|c|c|}\hline & & \tau_k \\ \hline & \iddots & \\ \hline \tau_1 & & \\ \hline \end{array}$$

- For any permutation π, there is a unique k and unique $\tau_1, \ldots, \tau_k$ **indecomposable permutations** such that $\pi = \tau_1 \oplus \cdots \oplus \tau_k$.

Unique factorization theorem on marked permutations

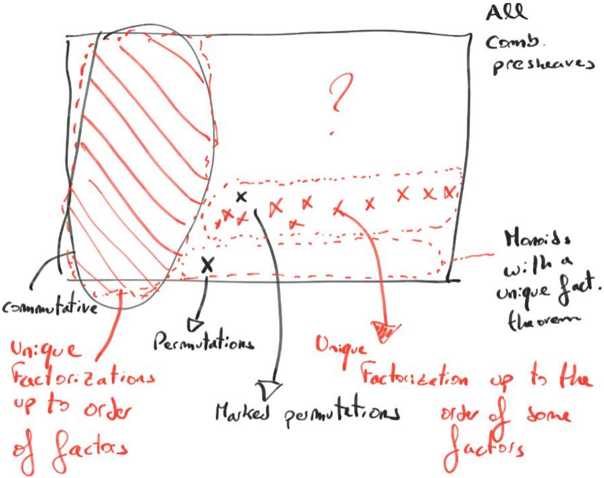
$\sigma =$ 

- The factorization is **not unique** as $\sigma = 21\bar{3} \star \bar{1}3524 = \bar{1}3524 \star 21\bar{3}$. For any permutations τ_1, τ_2,

$$(\bar{1} \oplus \tau_1) \star (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) \star (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1 \, .$$

- The order of the factors **does matter** to some extent.

Freeness conjecture - current state



The end