Research Plan - Algebraic combinatorics in patterns and polytopes

Applications of convex geometry in pattern functions and Hopf monoids

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1 Summary

In algebraic combinatorics we wish to study combinatorial objects like graphs and posets, through the properties of algebraic structures. At the heart of this effort lie Hopf algebras and Hopf monoids, objects in the interface of category theory and combinatorics. Intuitively, these algebraic operations reflect how combinatorial objects can be merged and split. Fundamental examples are the symmetric functions Hopf algebra and the shuffle Hopf algebra. In this research project we will focus on Hopf monoids and Hopf algebras that describe two topics with historical roots in combinatorics: geometric combinatorics and the study of substructures.

Specifically, this project will be split into three parts: algebras of **pattern functions**, that count substructures and form Hopf algebras; the **limit of pattern proportions** in big permutations; and the investigation of the Hopf monoid of **type B generalized permutahedra and type B Hopf monoids**.

On pattern functions we investigate the freeness of the so called **pattern Hopf algebras**. These are algebras constructed from a combinatorial presheaf a, *i.e.* a collection of combinatorial objects together with some notion of restriction. For some pattern Hopf algebras, it was shown that they are free commutative. Previous works include the proof of Vargas in the permutation case in [Var14], and the marked permutations and marked graph case in [Pen19b]. We wish to settle a specific conjecture regarding the freeness of these algebras.

On the **limit of proportion of patterns** project, we study the interplay of the proportion of different consecutive patterns in a big permutation. It has been observed that the proportion of consecutive patterns in a big permutation is deeply related to the *cycle polytope* of a particular graph. This raises many algebraic combinatorics questions: can we relate the geometry of the polytope to some intrinsic properties of permutations? Can we find other interesting graphs that have meaningful cycle polytopes? Our goal is to find how the cycle polytopes arise in the study of patterns of permutations or other patterns in general. Further, similar questions arise in patterns on graphs, where the same methods used for the consecutive patterns in permutations may have some applications.

On the project of **type B generalized permutahedra and type B Hopf monoids**, we are motivated by generalized permutahedra, combinatorial objects that are well behaved enough to have a **combinatorial Hopf monoid structure**, while being rich enough to embed several combinatorial objects of interest. The construction of the family of generalized permutahedra depends on the action of the symmetric group, so

natural extensions of this notion have been proposed, where an action of other Coxeter groups is considered. With this, the family of **type B generalized permutahedra** was constructed. Our proposed project intends to find a **type B combinatorial Hopf monoid** structure, a set of algebraic axioms that would mimic the well known *Hopf monoids* while admitting an algebraic structure describing the generalized permutaheda of type B.

2 Research Plan

2.1 Current state of research in the field

We first establish some basic language used throughout this research plan. A Hopf algebra is an algebra H over a field k endowed with a product μ and an additional coproduct $\Delta : H \to H \otimes H$ that is compatible with the multiplication in H, among other maps s, ι, ϵ that satisfy some additional properties, usually depicted as commutative diagrams, see an example in Fig. 1. The main topics and definitions related to Hopf algebras, as well as its applications to combinatorics, can be found in the survey work [GR14].

In this vein, there is a widespread interest in the algebraic combinatorics community in endowing combinatorial objects with compatible products and coproducts.

$$\begin{tikzcd}
& H \otimes H \arrow[rr] & & H \otimes H \arrow[dr] & \\
H \arrow[ur] \arrow[rr] \arrow[dr] & & k \arrow[rr] & & H \\
& H \otimes H \arrow[rr] & & H \otimes H \arrow[ur] &
\end{tikzcd}$$

Figure 1: One of the compatibility axioms on Hopf algebras

In [AM10], an algebraic structure with the same flavor is described, called a **Hopf monoid**. This is a category theory parallel of Hopf algebras detailed in [AA17], of which we stress the particular case of Hopf species. In fact, **combinatorial objects** are better described as set species, so this language was introduced in algebraic combinatorics. A **set species** a is a functor that maps finite sets X to sets $a[X]$ for each set X, together with bijective relabeling functions between $a[X]$ and $a[Y]$, whenever $\#X = \#Y$. The elements of $a[X]$ are called combinatorial object.

To a set species, we may add a Hopf monoid structure, which are maps that describe how to split and merge objects, in a similar way as Hopf algebras. A simple example is the Hopf species of graphs, that is defined as $\mathtt{G}[A] := \{\text{graphs supported in the vertex set } A\}$, where the product of two graphs is their disjoint union, and the coproduct is a tensor of its restrictions.

The main feature relating Hopf algebras and Hopf species in combinatorics is that there are functors, called **Fock functors**, that map a given Hopf species to a Hopf algebra. In fact, many of the Hopf algebras in vouge in combinatorics can be lifted in this way to Hopf species. This motivates us to work on the more combinatorially flavoured Hopf monoids instead of the more classical algebraic objects of Hopf algebras, while still enjoying most of the machinery built in the Hopf algebra world.

Pattern Hopf algebras in combinatorial objects

The study of substructures is widespread in combinatorics. In graphs, the notion of minors and subgraphs are the main examples. Permutation patterns is a significant topic on itself, see for instance the book [Bón16].

If π, σ are permutations, we can define the number of occurrences of π in σ as the number of sets I such that $\sigma|_I = \pi$, where we consider restrictions in the columns of the diagram of σ. This gives us functions

$$\mathrm{occ}_\pi(\sigma) = \#\{\text{occurrences of } \pi \text{ in } \sigma\}\,.$$

Theorem 1 (Vargas' pattern algebra, [Var14])**.** *The vector space $\mathcal{A}(\texttt{Per})$ spanned by the permutation pattern functions is closed under pointwise multiplication and has a unit. So, it forms an algebra, called the pattern algebra. In fact, we have the product rule*

$$\mathrm{occ}_{\pi_1} \cdot \mathrm{occ}_{\pi_2} = \sum_\tau \binom{\tau}{\pi_1, \pi_2} \mathrm{occ}_\tau\,, \tag{1}$$

where the coefficients $\binom{\tau}{\pi_1, \pi_2}$ represent the number of quasi-shuffles of π_1, π_2 that result in τ.

We can further endow this algebra with a Hopf algebra structure by adding a suitable coproduct. We are interested in studying the algebraic properties of $\mathcal{A}(\texttt{Per})$. A very descriptive theorem was obtained in [Var14]:

Theorem 2 (Freeness of the pattern Hopf algebra, [Var14])**.** *The permutation pattern Hopf algebra $\mathcal{A}(\texttt{Per})$ is free commutative. Its free generators are the so called Lyndon permutations.*

Limiting proportion of patterns

This project is motivated by a growing area of discrete probability theory that studies the limit of big combinatorial objects, and some of its statistics. A whole area of combinatorics studies patterns in permutations- Specifically, the notion of a **permuton** arose with the motivation of finding limiting objects that generalize permutations, and for which we can still talk about pattern functions. A permuton is a probability measure in the square $[0,1] \times [0,1]$ with uniform marginals, and it mimics the behavior of a permutation of big size, see an introduction of the topic in [HKM$^+$13]. Furthermore, these objects come equipped with a topology and a notion of convergence. Let $\widetilde{\mathrm{occ}}_\sigma(\pi)$ be the **proportion** of patterns of σ in π. This can be extended to a function on permutons. The following feature is says that the extension of $\widetilde{\mathrm{occ}}$ to permutons is a continuous extension:

Theorem 3. *If $\{\pi_n\}_n \to P$ is a sequence of permutations converging to the permuton P, with $|\pi_n| \to \infty$, and σ is another permutation, then*

$$\widetilde{\mathrm{occ}}_\sigma(\pi_n) \to \widetilde{\mathrm{occ}}_\sigma(P)\,.$$

Consider a family of patterns $\mathcal{F}$ and the following *feasible region*:

$$\Phi_{\mathcal{F}} := \{(\widetilde{\mathrm{occ}}_\pi(P))_{\pi\in\mathcal{F}} | P \text{ a permuton}\} = \{\lim(\widetilde{\mathrm{occ}}_\pi(\sigma^n))_{\pi\in\mathcal{F}} | \sigma^n \text{ permutations}, |\sigma_n| \to \infty\} \subseteq \mathbb{R}^{\#\mathcal{F}}\,, \tag{2}$$

that is, the image of permutons through the function $P \mapsto (\widetilde{\mathrm{occ}}_\pi(P))_{\pi\in\mathcal{F}}$, also called the **pattern signature** of a permuton.

In Fig. 2 one can see $\Phi_{\mathcal{F}}$ for the family of permutations $\mathcal{F} = \{12, 123\}$, as well as some permutons that are in the preimage of the respective points. Specifically, they find the permuton that maximizes the entropy, among all in the preimage.

With only two indecomposable permutations, we already obtain a complicated set, so in the literature we find several geometric properties with some isolated explicit descriptions.

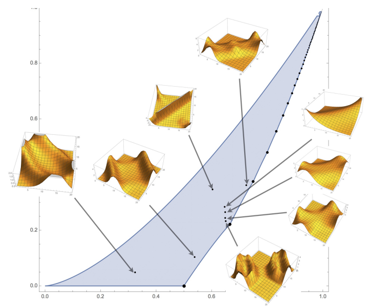

Figure 2: Image of permutons through permutations 123 and 12, from [KKRW15].

In [GHK$^+$17], the problem of finding the dimension of the feasible region was approached.

Theorem 4 (From [GHK$^+$17])**.** *Consider the family $\mathcal{I}_q$ of indecomposable permutations of size $\leq q$. Then, the image $\Phi_{\mathcal{I}_q}$ of permutions has a non-empty interior. It follows that there are no meaningful equation relating the pattern functions of indecomposable permutations.*

We denote by $\widetilde{\text{c-occ}}_\pi(\tau)$ the **proportion** of consecutive occurrences of the permutation π in τ. More recently, another limiting object for permutations was also constructed, called **infinite rooted permutation**, in [Bor18]. This limiting object has the following counterpart to Theorem 3, that is related to the $\widetilde{\text{c-occ}}_\pi$ functions:

Theorem 5. *If $\pi_n \to \mathcal{O}$ is a sequence of permutations converging to the infinite rooted permutation $\mathcal{O}$, and σ is another permutation, then*

$$\widetilde{\text{c-occ}}_\sigma(\pi_n) \to \widetilde{\text{c-occ}}_\sigma(\mathcal{O})\,.$$

This motivates us to consider the counterpart of $\Phi_{\mathcal{S}}$ for proportion of consecutive patterns, $\Phi^c_{\mathcal{S}}$.

Type B generalized permutahedra and type B Hopf monoids

The third part of the project follows a different path, related to Hopf monoids. On [ABS06], a universal property of the Hopf algebra of quasisymmetric functions *QSym* is established: for each **combinatorial Hopf algebra** $(\mathbf{H}, \eta)$ it is shown that there is a unique combinatorial Hopf algebra morphism $\mathbf{\Psi_H} : \mathbf{H} \to$ `QSym`.

This map $\mathbf{\Psi}_h$, called the **chromatic morphism**, stems from chromatic problems in graphs. Incidentally, Stanley's chromatic symmetric function, introduced in [Sta95], is a particular case of a map from the Hopf algebra of graphs to the Hopf algebra of symmetric functions, and it can be seen to be precisely $\mathbf{\Psi_G}$, where $\mathbf{G}$ is the graph Hopf monoid. On several other combinatorial Hopf monoids, the map $\mathbf{\Psi}_h$ was also in one way or another studied, see for example [BJR09] for matroids.

Generalized permutahedra (of type A) form a family of polytopes that are general enough to encode other combinatorial objects, while being restrictive enough to allow for a Hopf monoid structure that was introduced in [AA17].

The Hopf monoid of *word quasisymmetric functions* is related to the face structure of the permutahedron of type A, in that it has bases indexed by the faces of the permutahedron of type A.

The extension to generalized permutahedra of type B was natural and found some interesting applications. In [ACEP19], several combinatorial objects were embedded in the family of generalized permutahedra of several types. Specifically, there it is constructed a Coxeter-graphic polytope, Coxeter matroids and Coxeter generalized associahedra. This shows that this typed versions of generalized permutahedra are versatile enough to encompass most of the interesting objects that we already want to study.

2.2 Current state of own research

Pattern Hopf algebras in combinatorial objects

The notion of **combinatorial presheaf** combines the study of substructures in combinatorics with the Hopf monoid world.

Definition 6 (Combinatorial presheaf, from [Pen19b]). A **combinatorial presheaf** is a coutervariant functor a from finite sets with injective maps to sets. Thus, a is given by the following information: for each X finite set, $a[X]$ is a set, called the set of a-objects in X; for each pair of sets X, Y with $\#X = \#Y$, we are given a bijection $a[X] \to a[Y]$; and restriction maps $a[X] \to a[Y]$ whenever $Y \subseteq X$.

It turns out that from any combinatorial presheaf a we can construct an algebra $\mathcal{A}(a)$, generalizing the work in [Var14] in the permutation combinatorial presheaf. The algebra $\mathcal{A}(a)$ is the linear span of the following functions $p_t : \biguplus_X a[X] \to \mathbb{Q}$ that are defined, for each t combinatorial object, as:

$$p_t : b \mapsto p_t(b) = \#\{\text{substructures of } t \text{ inside } b\} .$$

We can further enrich the combinatorial presheaf a with a **monoid structure**, *i.e.* a map that describes how to merge objects, that is compatible with the presheaf structure. These objects are called **monoids in combinatorial presheaves**. It turns out that this follows very closely the motivation of finding Hopf algebras supported on combinatorial objects alluded in the introduction, in light of the following result:

Theorem 7. *The algebra $\mathcal{A}(a)$ endowed with a coproduct dual to the monoid structure is a Hopf algebra.*

The freeness of the pattern algebra in permutations, Theorem 1, was established by finding a **unique factorisation theorem** for permutations, and was extended to marked permutations in [Pen19b]. There the major issue is that there was no unique factorization theorem, but a close factorization theorem gives us the following:

Theorem 8. *The algebra $\mathcal{A}(\texttt{MPer})$ is free commutative, with the Lyndon marked permutations as its free generators.*

The first general proposition regarding the freeness of patterns algebras is the following result, established in [Pen19b].

Theorem 9. *If a is a commutative presheaf, then the algebra $\mathcal{A}(a)$ is free commutative.*

Limiting proportion of patterns

A *cycle polytope* of a directed multigraph G is the polytope $P(G)$, defined as the convex hull of specific vectors:

$$P(G) := \mathrm{conv}\left\{ \frac{1}{|\mathcal{C}|} \sum_{h \text{ edge in } \mathcal{C}} \vec{e}_h \,|\, \mathcal{C} \text{ a simple cycle in } G \right\} \subseteq \mathbb{R}^{E(G)} .$$

When describing the feasible region $\Phi^c_{\mathcal{S}_n}$ of the limiting proportion of consecutive patterns in permutations, we obtain in [BP19] that it is a polytope associated to a particular graph, called the overlap graph, or $\mathcal{O}v(n)$, see Fig. 3.

Theorem 10 ([BP19])**.** $\Phi^c_{\mathcal{S}_n} = P(\mathcal{O}v(n))$.

Some general results were obtained for these cycle polytopes.

Theorem 11 (Dimension theorem, from [BP19])**.** *The dimension of the cycle polytope $P(G)$ of a strongly connected directed multigraph G is precisely $|E(G)| - |V(G)|$.*

A *full graph* is a directed multigraph where each edge is part of a cycle. This allows us to describe the face structure of the cycle polytope of a graph G, see Fig. 4, for a particular example.

Theorem 12 (Face structure theorem, from [BP19])**.** *The face poset of the cycle polytope $P(G)$ of a directed multigraph G is isomorphic to the poset of full subgraphs of G.*

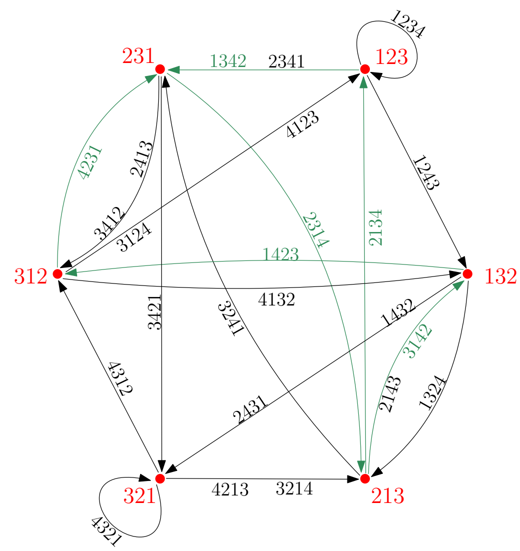

Figure 3: $\mathcal{O}v(4)$, the overlap graph of size four, from [BP19].

Type B generalized permutahedra and type B Hopf monoids

The relation between the Hopf algebra of word quasisymmetric functions $\mathtt{WQSym}$ and the face structure of the permutahedron of type A has been already alluded in [Pen18]. Specifically, the faces of the permutahedra and bases of the word quasisymmetric functions are both indexed by set compositions.

The following theorem is a general construction of Hopf monoid morphisms to any Hopf monoid.

Theorem 13 ([Pen18])**.** *Let* $\mathbf{h}$ *be a combinatorial Hopf monoid. Then there is a unique combinatorial Hopf monoid morphism* $\mathbf{\Psi_h} : \mathbf{h} \to \mathtt{WQSym}$.

For example, when $\mathbf{h}$ is the Hopf algebra on graphs, the map $\mathbf{\Psi_G}$ is the chromatic symmetric function on graphs, introduced in [Sta95]. This is the universal property of *QSym*, proved in [ABS06], on the Hopf monoid category.

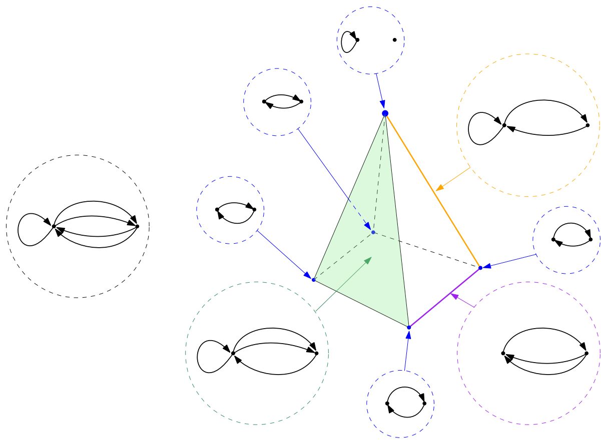

Figure 4: The cycle polytope of a directed graph, and its face strucutre, from [BP19].

We have explored the map $\Psi^B_{\texttt{BGPer}}$ as it is done for its type A counterpart $\Psi_{\texttt{BGPer}}$ in [Pen18]. Particularly, in [Pen18] we find generators of the kernel of $\Psi_{\texttt{BGPer}}$, when we restrict to the space of **hypergraphic polytopes**. We introduce type B hypergraphic polytopes and extend the results on hypergraphic polytopes to their type B counterpart, as described in [Pen19a].

With the geometry of $\mathbf{B}Per_n$ in mind, we construct a type B equivalent of the word quasisymmetric functions, `BWQSym`, in [Pen19a] that generalizes the Hopf monoids of word quasisymmetric functions `WQSym` in a way that is consistent with other work on the topic, specifically the one of Chow on his PhD thesis in [Cho01], where a concept of quasisymmetric functions of type B is explored.

We describe a natural candidate for $\Psi^B_{\texttt{BGPer}}$ in [Pen19a], where this map satisfies the following commutative diagram:

$$\begin{tikzcd} \texttt{GPer} \arrow[r, hook] \arrow[d, "\mathbf{\Psi_{GPer}}"] & \texttt{BGPer} \arrow[d, "\mathbf{\Psi^B_{BGPer}}"] \\ \texttt{WQSym} & \texttt{BWQSym} \arrow[l, two heads] \end{tikzcd} . \tag{3}$$

2.3 Detailed research plan

Pattern Hopf algebras

The first general goal of this project is to explore pattern Hopf algebras, find algebraic and coalgebraic generators, enumerate the dimension of primitive elements, describe their coradical filtration, and study their character group.

Since the concept of monoids in combinatorial presheaves is a very general one, we can easily find other examples of pattern algebras. In particular, we hope to find connections between the Hopf algebras indexed by set partitions and set compositions, like **WQSym** and **WSym**, and the relevant pattern algebras. It is interesting to ask the freeness question for other pattern Hopf algebras.

Problem 1. *Is the pattern Hopf algebra on set compositions and set partitions free? Describe a family of algebraic generators in these cases. How about other combinatorial presheaves?*

It is a long term goal of the project to establish a generic result, for instance to describe an infinite family of pattern Hopf algebras that are free, or a broad sufficient conditions for freeness. One first result in this direction is Theorem 9, already presented in [Pen19b], where it is shown that all the so called *commutative combinatorial presheaves* have a free pattern Hopf algebra.

We also studied two other pattern algebras that do not fit the general result in Theorem 9. The freeness in Theorem 8 obtained in [Pen19b] and in Theorem 2 obtained in [Var14] use methods that have the same structure. Specifically, we start by finding a **unique factorization theorem** of our objects under a given product. For instance, in the permutation case studied by Vargas, the $\oplus$ decomposition is used and in the marked permutation case, a unique factorisation for the inflation product, from [AA05], is used. Factorization theorems of this form are enough to establish freeness, through a beautiful application of the Lyndon word theory in [Var14]. Finally, some tools developed in the marked permutation case in [Pen19b] allow us some flexibility on the **factorization theorem**, where uniqueness is no longer a requirement.

Problem 2. *Construct a combinatorial presheaf that has a non-free pattern algebra.*

Consider a presheaf a and the set of a-objects that are irreducible, $\mathcal{I}$. For instance, in the permutation pattern Hopf algebra from Vargas, the $\oplus$- indecomposable permutations are used to construct the free basis, *Lyndon permutations*. In the graph case, the indecomposable marked graphs are precisely the ones that cannot be partitioned into two disjoint vertex sets without any edges in between. These graphs are precisely the ones that form a free basis.

If we consider the map $*$, from words in the alphabet $\mathcal{I}$ to a-objects, given by the monoidal structure of the presheaf, we obtain a surjective map. We say that a family of relations of words in $\mathcal{I}$ are a *unique factorization theorem* if these relations generate the equivalence classes of $\ker *$. One specific way of reading the result from [Var14] is that if $\ker *$ is the equivalence class with only singletons, the finest equivalence class in the set of words of $\mathcal{I}$, then the pattern algebra is free. If $\ker *$ is the equivalence class that results from setting for any letters $a, b \in \mathcal{I}$ the relation

$$ab \sim ba \, ,$$

then according to [Pen19b], the pattern algebra is free. We have a specific conjecture in mind for this project.

Conjecture 3. *If the equivalence class* $\ker *$ *is finer than* comu*, then the pattern algebra is free.*

Limiting proportion of patterns

Following the work that has been done in classical permutations, specifically Theorem 4, we wish to improve some results regarding the dimension of the set $\Phi_{\mathcal{S}}$. In [Var14] it was established that the *Lyndon permutations* form an algebraic basis of the pattern function Hopf algebra $\mathcal{A}(\texttt{Pat})$, so we know that there are no algebraic relations between pattern functions of Lyndon permutations. It is natural, then, to wonder if no analytic relations exist, and one of the goals of this project is to establish that:

Problem 4. *Establish, for some big set $\mathcal{J}$ such that $\mathcal{I}_q \subseteq \mathcal{J} \subseteq \mathcal{L}_q$, that $\Phi_{\mathcal{J}}$ has a non-empty interior. Does the set $\Phi_{\mathcal{L}_q}$ have a non-empty interior?*

The corresponding problem in graphs has been already fully solved in [ELS79], dealing with *graphons* instead of permutons. There, it was shown that the family of pattern functions of connected graphs map the *graphons* to a set with non-empty interior. However, nothing in this direction is known for the marked graph case or the marked permutation case, where the free algebraic elements are presented in [Pen19b].

Problem 5. *Apply the same study from [BP19] on consecutive patterns to study the proportion of patterns in marked graphs. Do we still get a polytope? Can we describe it?*

In Theorem 10, we show the relation of the polytope $P(\mathcal{O}v(n))$ with consecutive patterns of permutations. We plan to endeavor in a coming work in a deeper description of this polytope. Specifically:

Problem 6. *Find a description of the volume for the cycle polytope.*

In the discussion following (2), we observed that in previous works on the feasible region, there is some interest in finding the typical object in the preimage of a point from the feasible region. On consecutive patterns, this is still unnanswered:

Problem 7. *Describe the typical shift invariant random order that is in the preimage of each point of $P(\mathcal{O}v(n))$. That is, find the typical limit object of sequences of permutations that approaches a particular* ***consecutive pattern signature*** *in $P(\mathcal{O}v(n))$.*

Type B generalized permutahedra and type B Hopf monoids

With this, our first task is to find a **Hopf-like structure**, distinct from the Hopf monoids on species discussed above, where `BWQSym` has a universal property as it is the case with `WQSym` on type A as described in Theorem 13. This would be some category motivated but distinct from the category of linear species, where the usual Hopf monoid structures are studied.

Problem 8. *Can we describe the category of type B combinatorial Hopf monoids that has* `BWQSym` *as its terminal object, and that admits a type B combinatorial Hopf monoid of BGPer?*

Problem 9 (Type B kernel problem)**.** *What new elements can we find in the kernel of the map $\Psi^B_{\mathtt{BGPer}}$? That is, can we compute the kernel of $\Psi_{\mathtt{BGPer}}$ without taking the restriction to the space of type B hypergraphic polytopes?*

A consequence of the **kernel problem** in type B is its applications in the kernel problem in type A, originally discussed in [Pen18]. Specifically, information on the kernel of the map $\boldsymbol{\Psi}^B_{\mathbf{BGPer}}$ gives us information about the kernel of the map $\boldsymbol{\Psi}_{\mathbf{GPer}}$, according to (3), which is the original unanswered question in [Pen18].

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