The kernel of chromatic quasisymmetric functions on graphs and hypergraphic zonotopes

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Abstract

The chromatic symmetric function on graphs is a celebrated graph invariant. Analogous chromatic maps can be defined on other objects, as presented by Aguiar, Bergeron and Sottile. The problem of identifying the kernel of some of these maps was addressed by Féray, for the Gessel quasisymmetric function on posets.

On graphs, we show that the modular relations and isomorphism relations span the kernel of the chromatic symmetric function. This helps us to construct a new invariant on graphs, which may be helpful in tackling the tree conjecture. We also address the kernel problem in the Hopf algebra of generalized permutahedra, introduced by Aguiar and Ardila. We present a solution to the kernel problem on the Hopf algebra spanned by hypergraphic zonotopes, which is a subfamily of generalized permutahedra that contains a number of polytope families.

Finally, we consider the non-commutative analogues of these quasisymmetric invariants, and establish that the word quasisymmetric functions, also called non-commutative quasisymmetric functions, is the terminal object in the category of combinatorial Hopf monoids, recovering a result from Aguiar and Mahajan using a character language. As a corollary, we show that there is no combinatorial Hopf monoid morphism between the combinatorial Hopf monoid of posets and that of hypergraphic zonotopes.

1 Introduction

Chromatic function on graphs

For a graph G with vertex set $V(G)$, a coloring f of the graph G is a function $f : V(G) \to \mathbb{N}$. A coloring is *proper* in G if no edge is monochromatic.

We denote by $\mathbf{G}$ the graph Hopf algebra, which is a vector space freely generated by the graphs whose vertex sets are of the form $[n]$ for some $n \geq 0$. This can be endowed with a Hopf algebra structure, as described by Schmitt in [Sch94], and also presented below in Section 2.2.

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Stanley defines in [Sta95] the *chromatic symmetric function* of G in commuting variables $\{x_i\}_{i\geq 1}$ as

$$\Psi_{\mathbf{G}}(G) = \sum\nolimits_f x_f \,, \tag{1}$$

where we write $x_f = \prod_{v\in V(G)} x_{f(v)}$, and the sum runs over proper colorings of G. Note that $\Psi_{\mathbf{G}}(G)$ is in the ring *Sym* of symmetric functions. The ring *Sym* is a Hopf subalgebra of *QSym*, the ring of quasisymmetric functions introduced by Gessel in [Ges84]. A long standing conjecture in this subject, commonly referred to as the *tree conjecture*, is that if two trees T_1, T_2 are not isomorphic, then $\Psi_{\mathbf{G}}(T_1) \neq \Psi_{\mathbf{G}}(T_2)$.

When $V(G) = [n]$, the natural ordering on the vertices allows us to consider a non-commutative analogue of $\Psi_{\mathbf{G}}$, as done by Gebhard and Sagan in [GS01]. They define the chromatic symmetric function on non-commutative variables $\{\mathbf{a}_i\}_{i\geq 1}$ as

$$\mathbf{\Upsilon}_{\mathbf{G}}(G) = \sum\nolimits_f \mathbf{a}_f \,,$$

where we write $\mathbf{a}_f = \mathbf{a}_{f(1)} \dots \mathbf{a}_{f(n)}$, and we sum over the proper colorings f of G.

Note that $\mathbf{\Upsilon}_{\mathbf{G}}(G)$ lies homogeneous and symmetric in the variables $\{\mathbf{a}_i\}_{i\geq 1}$. Such power series are called *word symmetric functions*. The ring of word symmetric functions, **WSym** for short, was introduced in [RS06], and is sometimes called the ring of symmetric functions in non-commutative variables, or **NCSym**, for instance in [BZ09]. Here we adopt the former name to avoid confusion with the ring of non-commutative symmetric functions.

In this paper we describe generators for $\ker \Psi_{\mathbf{G}}$ and $\ker \mathbf{\Upsilon}_{\mathbf{G}}$. A similar problem was already considered for posets. In [Fér15], Féray studies $\Psi_{\mathbf{Pos}}$, the Gessel quasisymmetric function defined on the poset Hopf algebra, and describes a set of generators of its kernel.

Some elements of the kernel of $\Psi_{\mathbf{G}}$ have already been constructed in [GP13] by Guay-Paquet and independently in [OS14] by Orellana and Scott. These relations, called *modular relations*, extend naturally to the non-commutative case. We introduce them now.

Given a graph G and an edge set E that is disjoint from $E(G)$, let $G\cup E$ denote the graph G with the edges in E added. If we have edges $e_3 \in G$ and $e_1, e_2 \notin G$ such that $\{e_1, e_2, e_3\}$ forms a triangle, then we have

$$\mathbf{\Upsilon}_{\mathbf{G}}(G) - \mathbf{\Upsilon}_{\mathbf{G}}(G\cup\{e_1\}) - \mathbf{\Upsilon}_{\mathbf{G}}(G\cup\{e_2\}) + \mathbf{\Upsilon}_{\mathbf{G}}(G\cup\{e_1,e_2\}) = 0\,. \tag{2}$$

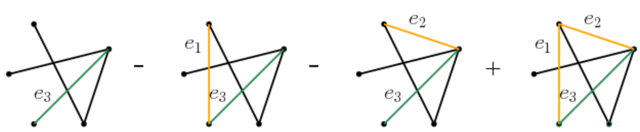

Figure 1: Example of a modular relation.

We call the formal sum $G - G\cup\{e_1\} - G\cup\{e_2\} + G\cup\{e_1,e_2\}$ in $\mathbf{G}$ a *modular relation on graphs*. An example is given in Fig. 1. Our first result is that these modular relations span the kernel of the chromatic symmetric function in non-commuting variables.

Theorem 1 (Kernel and image of $\mathbf{\Upsilon}_{\mathbf{G}} : \mathbf{G} \to \mathbf{WSym}$)**.** *The modular relations span* $\ker \mathbf{\Upsilon}_{\mathbf{G}}$. *The image of* $\mathbf{\Upsilon}_{\mathbf{G}}$ *is* ***WSym***.

Two graphs G_1, G_2 are said to be isomorphic if there is a bijection between the vertices that preserves edges. For the commutative version of the chromatic symmetric function, if two isomorphic graphs G_1, G_2 are given, it holds that $\Psi_{\mathbf{G}}(G_1)$ and $\Psi_{\mathbf{G}}(G_2)$ are the same. The formal sum in $\mathbf{G}$ given by $G_1 - G_2$ is called an *isomorphism relation on graphs.*

Theorem 2 (Kernel and image of $\Psi_{\mathbf{G}} : \mathbf{G} \to Sym$)**.** *The modular relations and the isomorphism relations generate the kernel of the commutative chromatic symmetric function $\Psi_{\mathbf{G}}$. The image of $\Psi_{\mathbf{G}}$ is Sym.*

It was already noticed that $\Psi_{\mathbf{G}}$ is surjective. For instance, in [CvW16], several bases of Sym_n are constructed, which are the chromatic symmetric function of graphs, namely are of the form $\{\Psi_{\mathbf{G}}(G_\lambda) | \lambda \vdash n\}$ for suitable graphs G_λ on n vertices. Here we present a new family of graphs with such property.

At the end of Section 3 we introduce a new graph invariant $\tilde{\Psi}$, called the *augmented chromatic invariant*. We observe that modular relations on graphs are in the kernel of the augmented chromatic invariant. It follows from Theorem 2 that $\ker \Psi_G = \ker \tilde{\Psi}$. This reduces the tree conjecture in $\Psi_{\mathbf{G}}$ to a similar conjecture on this new invariant $\tilde{\Psi}$, which contains seemingly more information.

Generalized Permutahedra

The remaining goal of this paper is to look at other kernel problems of chromatic flavour. In particular, we establish similar results to Theorems 1 and 2 in the combinatorial Hopf algebra of hypergraphic polytopes, which is a Hopf subalgebra of generalized permutahedra.

Generalized permutahedra is a family of polytopes that include permutahedra, associahedra and graph zonotopes. This family has been studied, for instance, in [PRW08], and we introduce it now.

The Minkowski sum of two polytopes $\mathfrak{a}, \mathfrak{b}$ is set as $\mathfrak{a} + \mathfrak{b} = \{a + b | \, a \in \mathfrak{a}, \, b \in \mathfrak{b}\}$. The Minkowski difference $\mathfrak{a} - \mathfrak{b}$ is only sometimes defined: it is the unique polytope $\mathfrak{c}$ that satisfies $\mathfrak{b} + \mathfrak{c} = \mathfrak{a}$, if it exists. We denote as $\sum_i \mathfrak{a}_i$ the Minkowski sum of several polytopes.

If we let $\{e_i | i \in I\}$ be the canonical basis of $\mathbb{R}^I$, a *simplex* is a polytope of the form $\mathfrak{s}_J = conv\{e_j | j \in J\}$ for non-empty $J \subseteq I$. A generalized permutahedron in $\mathbb{R}^I$ is a polytope given by real numbers $\{a_J\}_{\emptyset \neq J \subseteq I}$ as follows: Let $A_+ = \{J | a_J > 0\}$ and $A_- = \{J | a_J < 0\}$, then, the corresponding generalized permutahedron is

$$\mathfrak{q} = \left(\sum_{J \in A_+} a_J \mathfrak{s}_J \right) - \left(\sum_{J \in A_-} |a_J| \mathfrak{s}_J \right) , \tag{3}$$

if the Minkowski difference exists. We identify a generalized permutahedron $\mathfrak{q}$ with the list $\{a_J\}_{\emptyset \neq J \subseteq I}$. Note that not every list of real numbers will give us a generalized permutahedron, since the Minkowski difference is not always defined.

In [Pos09], generalized permutahedra are introduced in a different manner. A polytope is said to be a generalized permutahedron if it can be described as

$$\mathfrak{q} = \{x \in \mathbb{R}^n | \sum_{i \in I} x_i \leq z_I \text{ for } I \subsetneq [n] \text{ non-empty} \; ; \sum_{i \in [n]} x_i = z_{[n]}\} ,$$

for reals $\{z_J\}_{\emptyset \neq J \subseteq I}$.

A third definition of generalized permutahedra is present in [AA17]. Here, a generalized permutahedron is a polytope whose normal fan coarsens the one of the permutahedron. These three definitions are equivalent, and a discussion regarding this can be seen in Section 2.

A *hypergraphic polytope* is a generalized permutahedron where the coefficients a_J in (3) are non-negative. For a hypergraphic polytope $\mathfrak{q}$, we denote by $\mathcal{F}(\mathfrak{q}) \subseteq 2^I \setminus \{\emptyset\}$ the family of sets $J \subseteq I$ such that $a_J > 0$. A *fundamental hypergraphic polytope* on $\mathbb{R}^I$ is a hypergraphic polytope $\sum_{\emptyset \neq J \subseteq I} a_J \mathfrak{s}_J$ such that $a_J \in \{0, 1\}$. Finally, for a set $A \subseteq 2^I \setminus \{\emptyset\}$, we write $\mathcal{F}^{-1}(A)$ for the hypergraphic polytope $\mathfrak{q} = \sum_{J \in A} \mathfrak{s}_J$. Note that a fundamental hypergraphic polytope is of the form $\mathcal{F}^{-1}(A_+)$ for some family $A_+ \subseteq 2^{[n]} \setminus \{\emptyset\}$. We remark that **HGP** and its substructures has been studied, for instance, in [AA17, Part 4].

One can easily note that the hypergraphic polytope $\mathfrak{q}$ and $\mathcal{F}^{-1}(\mathcal{F}(\mathfrak{q}))$ are, in general, distinct, so some care will come with this notation. However, the face structure is the same, and we give an explicit combinatorial equivalence in Proposition 29. This observation is not true for generalized permutahedra in general, and this is a relevant observation that, in particular, does not allow us to state Theorem 3 in the generalized permutahedra setting. If $\mathfrak{q}$ is an hypergraphic polytope such that $\mathcal{F}(\mathfrak{q})$ is a building set, then $\mathfrak{q}$ is called a *nestohedron*, see [Pil17] and [AA17]. Hypergraphic polytopes and its subfamilies are studied in [AA17], where they are also called y-positive generalized permutahedra.

In [AA17], Aguiar and Ardila define **GP**, a Hopf algebra structure on the linear space generated by generalized permutahedra in $\mathbb{R}^n$ for $n \geq 0$. The Hopf subalgebra **HGP** is the linear subspace generated by hypergraphic polytopes. When we address sums of polytopes, we will let context clarify if we are taking a Minkowski sum or a dilation on polytopes, or an algebraic operation in **GP**.

In [Gru17], Grujić introduced a quasisymmetric map in generalized permutahedra $\Psi_{\mathbf{GP}} : \mathbf{GP} \to QSym$ that was extended to a weighted version in [GPS17]. For a polytope $\mathfrak{q} \subseteq \mathbb{R}^I$, Grujić defines a function $f : I \to \mathbb{N}$ to be $\mathfrak{q}$-generic if the face of $\mathfrak{q}$ that minimises $\sum_{i \in I} f(i) x_i$, denoted $\mathfrak{q}_f$, is a point. Equivalently, f is $\mathfrak{q}$-generic if it lies in the interior of the normal cone of some vertex of $\mathfrak{q}$. Then, Grujić defines for a set $\{x_i\}_{i \geq 1}$ of commutative variables, the quasisymmetric function:

$$\Psi_{\mathbf{GP}}(\mathfrak{q}) = \sum_{f \text{ is } \mathfrak{q}\text{-generic}} x_f \,. \tag{4}$$

This quasisymmetric function is called the *chromatic quasisymmetric function on generalized permutahedra*, or simply *commutative chromatic quasisymmetric function.*

We discuss now a non-commutative version of $\Psi_{\mathbf{GP}}$, where we will establish an analogue of Theorem 1 for hypergraphic polytopes. For that, consider the Hopf algebra of word quasisymmetric functions **WQSym**, a version of $QSym$ in non-commutative variables introduced in [NT06] that is also called non-commutative quasisymmetric functions, or **NCQSym**, for instance in [BZ09]. For a generalized permutahedron $\mathfrak{q}$ and non-commutative variables $\{\mathbf{a}_i\}_{i \geq 1}$, let $\mathbf{a}_f = \mathbf{a}_{f(1)} \cdots \mathbf{a}_{f(n)}$ and define

$$\mathbf{\Upsilon}_{\mathbf{GP}}(\mathfrak{q}) = \sum_{f \text{ is } \mathfrak{q}\text{-generic}} \mathbf{a}_f \,.$$

We will see from Proposition 15 that $\mathbf{\Upsilon_{GP}}(\mathfrak{q})$ is a word quasisymmetric function. Moreover, a straightforward computation shows that $\mathbf{\Upsilon_{GP}}$ defines a Hopf algebra morphism between $\mathbf{GP}$ and $\mathbf{WQSym}$. Let us call $\Psi_{\mathbf{HGP}}$ and $\mathbf{\Upsilon_{HGP}}$ the restrictions of $\Psi_{\mathbf{GP}}$ and $\mathbf{\Upsilon_{GP}}$ to $\mathbf{HGP}$, respectively.

Our next theorems describe the kernel of the maps $\Psi_{\mathbf{HGP}}$ and $\mathbf{\Upsilon_{HGP}}$, using two types of relations:

- the *simple relations*, which are presented in Proposition 29, and convey that $\mathbf{\Upsilon_{GP}}(\mathfrak{q})$ only depends on which coefficients a_I are positive;
- the *modular relations*, which are exhibited in Theorem 30. These generalize the ones for graphs, that is the image of modular relations on graphs by the graph zonotope embedding Z are modular relations on hypergraphic polytopes.

Theorem 3 (Kernel and image of of $\mathbf{\Upsilon_{HGP}} : \mathbf{HGP} \to \mathbf{WQSym}$)**.** *The space* $\ker \mathbf{\Upsilon_{HGP}}$ *is generated by the simple relations and the modular relations on hypergraphic polytopes. The image of* $\mathbf{\Upsilon_{HGP}}$ *is* $\mathbf{SC}$*, a proper subspace of* ***WQSym*** *introduced in Definition 24 below.*

Let us denote by $\mathbf{WQSym}_n$ the linear space of homogeneous word quasisymmetric functions of degree n, and let $\mathbf{SC}_n = \mathbf{SC} \cap \mathbf{WQSym}_n$. A monomial basis for $\mathbf{SC}$ is presented in Definition 24. An asymptotic for the dimension of $\mathbf{SC}_n$ is computed in Proposition 34, where in particular it is shown that it is exponentially smaller than the dimension of $\mathbf{WQSym}_n$.

Two generalized permutahedra $\mathfrak{q}_1, \mathfrak{q}_2$ are isomorphic if one can be obtained from the other by permuting the coordinates. If $\mathfrak{q}_1, \mathfrak{q}_2$ are isomorphic, the commutative chromatic quasisymmetric functions $\Psi_{\mathbf{GP}}(\mathfrak{q}_1)$ and $\Psi_{\mathbf{GP}}(\mathfrak{q}_2)$ are the same. We say that $\mathfrak{q}_1 - \mathfrak{q}_2$ is an *isomorphism relation on hypergraphic polytopes.*

Theorem 4 (Kernel and image of $\Psi_{\mathbf{HGP}} : \mathbf{HGP} \to QSym$)**.** *The linear space* $\ker \Psi_{\mathbf{HGP}}$ *is generated by the simple relations, the modular relations and the isomorphism relations. The image of* $\Psi_{\mathbf{HGP}}$ *is* $QSym$*.*

In [AA17], Aguiar and Ardila define the graph zonotope, a Hopf algebra morphism $Z : \mathbf{G} \to \mathbf{GP}$ that is injective. Remarkably, we have that $\Psi_{\mathbf{G}} \circ Z = \Psi_{\mathbf{GP}}$. They also define other polytopal embeddings from other combinatorial Hopf algebras $\mathbf{h}$, like matroids, to $\mathbf{GP}$. One associates a universal morphism $\Psi_{\mathbf{h}}$ to these algebras that also satisfy $\Psi_{\mathbf{h}} \circ Z = \Psi_{\mathbf{GP}}$. These universal morphisms are discussed below.

In particular $Z(\ker \Psi_{\mathbf{h}}) = \ker \Psi_{\mathbf{GP}} \cap Z(\mathbf{h})$. This relation between $\ker \Psi_{\mathbf{h}}$ and $\ker \Psi_{\mathbf{GP}}$ is the main motivation to describe $\ker \Psi_{\mathbf{GP}}$, and indicates that $\ker \Psi_{\mathbf{GP}}$ is the kernel problem that deserves more attention. In this paper we leave the description of $\ker \Psi_{\mathbf{GP}}$ as an open problem.

Most of the combinatorial objects embedded in $\mathbf{GP}$ are also embedded in $\mathbf{HGP}$, such as graphs and matroids, so a description of $\ker \Psi_{\mathbf{HGP}}$ is already interesting.

We remark that a description of the generators of $\ker \Psi_{\mathbf{GP}}$ or $\ker \Psi_{\mathbf{HGP}}$ does not entail a description of the generators of $\ker \Psi_{\mathbf{h}}$, for the corresponding Hopf subalgebras. In particular, for that reason the kernel problem on matroids and simplicial complexes is still open, despite these Hopf algebras being realised as Hopf subalgebras of $\mathbf{HGP}$.

Universal morphisms

For a Hopf algebra $\mathbf{h}$, a *character* η of $\mathbf{h}$ is a linear map $\eta : \mathbf{h} \to \mathbb{K}$ that preserves the multiplicative structure and the unit of $\mathbf{h}$. We define a *combinatorial Hopf algebra* as a pair $(\mathbf{h}, \eta)$ where $\mathbf{h}$ is a Hopf algebra and $\eta : \mathbf{h} \to \mathbb{K}$ a character of $\mathbf{h}$. For instance, consider the ring of quasisymmetric functions $QSym$ introduced in [Ges84] with its monomial basis $\{M_\alpha\}$, indexed by compositions. Then, $QSym$ has a combinatorial Hopf algebra structure $(QSym, \eta_0)$, by setting $\eta_0(M_\alpha) = 1$ whenever α has one or zero parts.

In [ABS06], Aguiar, Bergeron, and Sottile showed that any combinatorial Hopf algebra $(\mathbf{h}, \eta)$ has a unique combinatorial Hopf algebra morphism $\Psi_{\mathbf{h}} : \mathbf{h} \to QSym$, i.e. a Hopf algebra morphism that satisfies $\eta_0 \circ \Psi_{\mathbf{h}} = \eta$. In other words, $(QSym, \eta_0)$ is a terminal object in the combinatorial Hopf algebras. We will refer to these maps as universal maps to $QSym$.

The commutative invariants previously shown on graphs $\Psi_{\mathbf{G}}$, on posets $\Psi_{\mathbf{Pos}}$ and on generalized permutahedra $\Psi_{\mathbf{GP}}$ can be obtained as universal maps to $QSym$. If we take a character $\eta(G) = \mathbb{1}[G \text{ has no edges}]$ on the graphs Hopf algebra, the unique combinatorial Hopf algebra morphism $\mathbf{G} \to QSym$ is exactly the map $\Psi_{\mathbf{G}}$. With the Hopf algebra structure imposed on $\mathbf{GP}$ in [AA17], if we consider the character $\eta(\mathfrak{q}) = \mathbb{1}[\mathfrak{q} \text{ is a point}]$, then $\Psi_{\mathbf{GP}}$ is the universal map from $\mathbf{GP}$ to $QSym$. On posets, the Hopf algebra structure considered is the one presented in [GR14] and the character that is considered is $\eta(P) = \mathbb{1}[P \text{ is an antichain}]$.

To see the maps $\mathbf{\Upsilon}_{\mathbf{G}} : \mathbf{G} \to \mathbf{WSym}$ and $\mathbf{\Upsilon}_{\mathbf{GP}} : \mathbf{GP} \to \mathbf{WQSym}$ as universal maps, we need a parallel of the universal property of $QSym$ in the non-commutative world. The fitting property is better described as a property of the category of Hopf monoids in vector species. The Hopf monoid $\overline{\mathbf{WQSym}}$ is presented in [AM10] as the Hopf monoid of faces. It is seen that there is a unique Hopf monoid morphism $\mathbf{\Upsilon}_{\overline{\mathbf{h}}}$ between a combinatorial Hopf monoid $\overline{\mathbf{h}}$ and $\overline{\mathbf{WQSym}}$. In the last chapter we establish another proof of this fact, and expand on that showing that instead of a connected Hopf monoid we can instead take any combinatorial Hopf monoid, as long as it coincides with the counit in the lower degree terms.

The relationship between Hopf algebras and Hopf monoids is very well captured with the so called Fock functors, mapping Hopf monoids to Hopf algbras, and Hopf monoid morphisms to Hopf algebra morphisms. In particular, the full Fock functor $\mathcal{K}$ satisfies $\mathcal{K}(\overline{\mathbf{WQSym}}) = \mathbf{WQSym}$. Then, the universal property of $\overline{\mathbf{WQSym}}$ gives us a Hopf algebra morphism $\mathcal{K}(\mathbf{\Upsilon}_{\overline{h}})$ from $\mathcal{K}(\overline{h})$ to $\mathbf{WQSym}$. The maps $\mathbf{\Upsilon}_{\mathbf{G}}, \mathbf{\Upsilon}_{\mathbf{GP}}$ arise precisely in this way, when applying $\mathcal{K}$ to the unique combinatorial Hopf monoid morphism from the Hopf monoid on graphs $\overline{\mathbf{G}}$ and of generalized permutahedra $\overline{\mathbf{GP}}$ to $\overline{\mathbf{WQSym}}$. In particular, we observe that $\mathcal{K}(\overline{\mathbf{G}}) = \mathbf{G}$ and $\mathcal{K}(\overline{\mathbf{GP}}) = \mathbf{GP}$. If we start with the poset Hopf monoid $\overline{\mathbf{Pos}}$ we obtain the non-commutative analogue $\mathbf{\Upsilon}_{\mathbf{Pos}}$ of the Gessel invariant, presented in [Fér15]. In particular, $\mathcal{K}(\overline{\mathbf{Pos}}) = \mathbf{Pos}$. We will refer to these Hopf algebra morphisms as universal maps to $\mathbf{WQSym}$.

Finally, our previous results have an interesting consequence. We show that, because $\mathbf{\Upsilon}_{\mathbf{HGP}}$ is not surjective, there is no combinatorial Hopf monoid morphism from the Hopf monoid on posets to the Hopf monoid on hypergraphic polytopes. However, in [AA17] a Hopf monoid morphism from posets to extended generalized permutahedra is constructed. With this result we obtain that this map cannot be restricted extended generalized permutahedra to generalized permutahedra.

There are natural maps $\mathbf{WSym} \to Sym$ and $\mathbf{WQSym} \to QSym$ by allowing the variables to commute. We denote these maps by *comu*.

Note: for sake of clarity, we have been using boldface for non-commutative Hopf algebras, their elements, and the associated combinatorial objects, like word symmetric functions. We try and maintain that notational convention throughout the paper.

This paper is organized as follows: In Section 2 we address the preliminaries, where the reader can find the linear algebra tools that we use, the main Hopf algebras are presented, and where we prove equivalence of the several definitions of a generalized permutahedra. In Section 3 we prove Theorem 1 and Theorem 2, and we study the augmented chromatic invariant. In Section 4 we prove Theorem 3 and Theorem 4, and we present asymptotics for the dimension of the graded Hopf algebra **SC**. In Section 5 we present the universal property of $\overline{\mathbf{WQSym}}$. In Appendix A we find relations between the coefficients of the augmented chromatic symmetric function and the coefficients of the original chromatic symmetric function on graphs.

2 Preliminaries

For an equivalence relation $\sim$ on a set A, we write $[x]_\sim$ for the equivalence class of x in $\sim$, and we write $[x]$ when $\sim$ is clear from context. We write both $\mathcal{E}(\sim)$ and $A/\sim$ for the set of equivalence classes of $\sim$.

2.1 Linear algebra preliminaries

The following linear algebra lemmas will be useful to compute generators of the kernels and the images of Ψ and $\mathbf{\Upsilon}$. These lemmas describe a sufficient condition for a set $\mathcal{B}$ to span the kernel of a linear map $\phi : V \to W$.

Lemma 5. *Let V be a finite dimensional vector space with a basis $\{a_i | i \in [m]\}$, $\phi : V \to W$ be a linear map, and $\mathcal{B} = \{b_j | j \in J\} \subseteq \ker \phi$ be a family of relations.*

Assume that there exists $I \subseteq [m]$ such that:

- *the family $\{\phi(a_i)\}_{i\in I}$ is linearly independent in W,*
- *for $i \in [m] \setminus I$ we have $a_i = b + \sum_{k=i+1}^{m} \lambda_{k,i} a_k$ for some $b \in \mathcal{B}$ and some scalars $\lambda_{k.i}$;*

Then $\mathcal{B}$ spans $\ker \phi$. Additionally, we have that $\{\phi(a_i)\}_{i\in I}$ is a basis of the image of ϕ.

The following lemma will help us dealing with the composition $\Psi = \mathrm{comu} \circ \mathbf{\Upsilon}$: we give a sufficient condition for a natural enlargement of the set $\mathcal{B}$ to generate $\ker \Psi$, given that it already generates $\ker \mathbf{\Upsilon}$.

Lemma 6. *We will use the same notation as in Lemma 5. Additionally, consider $\phi_1 : W \to W'$ linear map and write $\phi' = \phi_1 \circ \phi$. Take an equivalence relation $\sim$ in $\{a_i\}_{i\in[m]}$ that satisfies $\phi'(a_i) = \phi'(a_j)$ whenever $a_i \sim a_j$. Let $\mathcal{C} = \{a_i - a_j | \, a_i \sim a_j\}$ and write $\phi'([a_i]) = \phi'(a_i)$ with no ambiguity.*

Assume the hypothesis in Lemma 5 and, additionally, suppose that the family $\{\phi'([a_i])\}_{[a_i]\in\mathcal{E}(\sim)}$ is linearly independent.

Then $\ker \phi'$ *is generated by* $\mathcal{B} \cup \mathcal{C}$*. Furthermore,* $\{\phi'([a_i])\}_{[a_i]\in\mathcal{E}(\sim)}$ *is a basis of* $\operatorname{im} \phi'$*.*

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook] & V \arrow[r, "\phi'"] \arrow[dr, "\phi'"'] & W \arrow[d, "\phi_1"] \\ & & W' \end{tikzcd} \tag{5}$$

Proof of Lemma 5. Suppose, for sake of contradiction, that there is some element $c \in \ker \phi \setminus \langle \mathcal{B} \rangle$. In particular $c \neq 0$. Write

$$c = \sum_{k=1}^{m} \tau_k a_k \, . \tag{6}$$

Note that if $\tau_i = 0$ for every $i \notin I$, then

$$0 = \phi(c) = \phi\left(\sum_{k\in I} \tau_k a_i\right) = \sum_{k\in I} \tau_k \phi(a_k),$$

which, by linear independence of $(\phi(a_k))_{k\in \bar{I}}$, implies that $\tau_k = 0$ for every $k \in I$, contradicting $c \neq 0$. Therefore, we have $\lambda_i \neq 0$ for some $i \notin I$.

Consider the smallest index $i_c \in [m] \setminus I$ such that τ_i is non-zero. Choose $c \in \ker \phi \setminus \langle \mathcal{B} \rangle$ that maximizes i_c.

Thus, we can write

$$c = \sum_{j\in I} \tau_j a_j + \sum_{\substack{j\in[m]\setminus I \\ j\geq i_c}} \tau_j a_j \, . \tag{7}$$

By hypotheses, because $i_c \notin I$, there is some $b' \in \mathcal{B}$ such that:

$$a_{i_c} = b' + \sum_{j=i_c+1}^{m} \lambda_{j,i_c} a_j \, .$$

So applying this to (7) gives us:

$$\begin{aligned} c - \tau_{i_c} b' &= \sum_{j\in I} \tau_j a_j + \sum_{\substack{j\in[m]\setminus I \\ j\geq i_c}} \tau_j a_j - \tau_{i_c} a_{i_c} + \sum_{k=i_c+1}^{m} \tau_{i_c} \lambda_{k,i_c} a_k \\ &= \sum_{j\in I} \tau_j a_j + \sum_{\substack{j\in[m]\setminus I \\ j> i_c}} \tau_j a_j + \sum_{j=i_c+1}^{m} \tau_{i_c} \lambda_{j,i_c} a_j \, . \end{aligned} \tag{8}$$

Note that $c - \lambda_{i_c} b_j \in \ker \phi \setminus \langle \mathcal{B} \rangle$ which contradicts the maximality of i_c.

From this we conclude that there are no elements c in $\ker \phi \setminus \langle \mathcal{B} \rangle$. $\square$

Proof of Lemma 6. Define $I' = \{\max A \cap I | \, A \in \mathcal{E}(\sim)\}$. Note that for every $j \in I \setminus I'$ there is $c \in \mathcal{C}$ such that $a_j = c + \sum_{k=j+1}^{m} \lambda_{k,j} a_k$. Indeed it is enough to choose $i \sim j$ with $i > j$ and $i \in I$, to write $a_j = \underbrace{a_j - a_i}_{\in \mathcal{C}} + a_i$.

So the set $I' \subseteq [m]$ satisfies both that:

- We have that $(\phi'(a_i))_{i\in I'} = (\phi'([a_i]))_{i\in I'}$ is linearly independent in W';
- For $i \in [m] \setminus I'$ we can write $a_i = b + \sum_{k=j+1}^{m} \lambda_{k,i} a_k$ for some $b \in \mathcal{B} \cup \mathcal{C}$ and some scalars $\lambda_{k,i}$.

Now applying Lemma 5 to I' instead of I, to ϕ' instead of ϕ and to $\mathcal{B}\cup\mathcal{C}$ instead of $\mathcal{B}$ tells us that $\mathcal{B} \cup \mathcal{C}$ generates $\ker \phi'$, and that $\{\phi'([a_i])|i \in I'\} = \{\phi'(a_i)|i \in I'\}$ spans the image of ϕ', as desired. □

2.2 Hopf algebras and associated combinatorial objects

In the following, all the Hopf algebras $\mathbf{H}$ have a grading, denoted by $\mathbf{H} = \oplus_{n\geq 0}\mathbf{H}_n$.

An *integer composition*, or simply a composition, of n, is a list $\alpha = (\alpha_1, \cdots, \alpha_k)$ of positive integers which sum is n. We write $\alpha \models n$. We denote by $l(\alpha)$ for the length of the list and we denote by $\mathcal{C}_n$ the set of compositions of size n.

An *integer partition*, or simply a partition, of n, is a non-increasing list of positive integers $\lambda = (\lambda_1, \cdots, \lambda_k)$ which sum is n. We write $\lambda \vdash n$. We denote by $l(\lambda)$ the length of the list and we denote by $\mathcal{P}_n$ the set of partitions of size n. By disregarding the order of the parts on a composition α we obtain a partition $\lambda(\alpha)$.

A *set partition* $\boldsymbol{\pi} = \{\boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_k\}$ of a set I is a collection of non-empty disjoint subsets of I, called *blocks*, that cover I. We write $\boldsymbol{\pi} \vdash I$. We denote the number of parts of the set partition by $l(\boldsymbol{\pi})$, and call it its length. We write $\mathbf{P}_I$ for the family of set partitions of I, or simply $\mathbf{P}_n$ if $I = [n]$. By counting the elements on each block we obtain an integer partition denoted by $\lambda(\boldsymbol{\pi}) \vdash \#I$. We identify a set partition $\boldsymbol{\pi} \in \mathbf{P}_I$ with an equivalence relation $\sim_{\boldsymbol{\pi}}$ on I, where $x \sim_{\boldsymbol{\pi}} y$ if $x, y \in I$ are on the same block of $\boldsymbol{\pi}$.

A *set composition* $\vec{\boldsymbol{\pi}} = S_1|\cdots|S_k$ of I is a list of non-empty disjoint subsets of I that cover I. We write $\vec{\boldsymbol{\pi}} \models I$. We denote by $l(\vec{\boldsymbol{\pi}})$ the size of the set composition. We call $\mathbf{C}_I$ to the family of set compositions of I, or simply $\mathbf{C}_n$ if $I = [n]$. By disregarding the order of a set composition $\vec{\boldsymbol{\pi}}$, we obtain a set partition $\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}) \vdash I$. By counting the elements on each block we obtain a composition denoted by $\alpha(\vec{\boldsymbol{\pi}}) \models \#I$. A set composition is naturally identified with a total preorder $R_{\vec{\boldsymbol{\pi}}}$ on I, where $xR_{\vec{\boldsymbol{\pi}}}y$ if $x \in S_i, y \in S_j$ for $i \leq j$.

Permutations act on set compositions and set partitions as follows: for a set composition $\vec{\boldsymbol{\pi}} = (S_1, \cdots, S_k)$, a set partition $\pi = \{\pi^{(1)}, \cdots, \pi^{(k)}\}$ on I and a permutation $\phi : I \to I$, we define the automorphism $\phi(\vec{\boldsymbol{\pi}}) = (\phi(S_1), \cdots, \phi(S_k))$ and $\phi(\pi) = \{\phi(\pi^{(1)}), \cdots, \phi(\pi^{(k)})\}$ as expected.

A *coloring* of the set I is a function $f : I \to \mathbb{N}$. The set composition type $\vec{\boldsymbol{\pi}}(f)$ of a coloring $f : I \to \mathbb{N}$ is the set composition obtained after deleting the empty sets of $f^{-1}(1)|f^{-1}(2)|\cdots$.

We recall that in partitions and in set partitions, we define the classical *coarsening orders* $\leq$ with the same notation, where we say that $\pi \leq \tau$ (resp. $\boldsymbol{\pi} \leq \boldsymbol{\tau}$) if τ is obtained from π by adding some parts of the original parts together (resp. if $\boldsymbol{\tau}$ is obtained from $\boldsymbol{\pi}$ by merging some blocks).

These objects relate to the Hopf algebras $QSym, Sym, \mathbf{WSym}$ and $\mathbf{WQSym}$. The homogeneous components $QSym_n$ (resp. Sym_n, $\mathbf{WSym}_n$ and $\mathbf{WQSym}_n$) of these Hopf algebras have monomial bases indexed by these objects, which we will denote by $\{M_\alpha\}_{\alpha\in\mathcal{C}_n}$ (resp. $\{m_\lambda\}_{\lambda\in\mathcal{P}_n}$, $\{\mathbf{m}_{\boldsymbol{\pi}}\}_{\boldsymbol{\pi}\in\mathbf{P}_n}$ and $\{\mathbf{M}_{\vec{\boldsymbol{\pi}}}\}_{\vec{\boldsymbol{\pi}}\in\mathbf{C}_n}$).

2.3 Hopf algebras on graphs and posets

Of interest will also be the Hopf algebra on graphs $\mathbf{G}$ and on posets $\mathbf{Pos}$, which are graded and connected, and whose homogeneous components $\mathbf{G}_n$, resp. $\mathbf{Pos}_n$, are the linear span of the graphs with vertex set $[n]$, resp. partial orders in the set $[n]$.

In these graded vector spaces, we define the products and coproducts in the basis elements. For that, when A, B are sets of integers with the same cardinality, we let $rl_{A,B}$ be the canonical relabelling of combinatorial objects on A to combinatorial objects on B.

Recall that the disjoint union of graphs $(V_1, E_1), (V_2, E_2)$, where $V_1 \cap V_2 = \emptyset$, is $(V_1 \sqcup V_2, E_1 \sqcup E_2)$, and the restriction of a graph $G|_I$ is $(I, E(G) \cap \binom{I}{2})$. Denote $[m] = \{1, \dots, m\}$ as usual, and $[m, n] = \{m, m+1, \dots, n\}$ for $n \geq m$. Given two graphs G_1, G_2 with vertex labelled in $[n], [m]$ respectively, the product is the relabelled disjoint union

$$G_1 \cdot G_2 = G_1 \sqcup rl_{[m],[m+1,m+n]}(G_2)\,.$$

For the coproduct, let G be a graph labelled in $[n]$, then

$$\Delta G = \sum_{[n]=I \sqcup J} rl_{I,[\#I]}(G|_I) \otimes rl_{J,[\#J]}(G|_J)\,.$$

For posets, consider two posets $P_1 = (S_1, R_{P_1}), P_2 = (S_2, R_{P_2})$, where R represents the set of pairs (x, y) such that $x \leq y$ in the respective poset. The disjoint union of posets is written $P_1 \sqcup P_2$ and defined as $(S_1 \sqcup S_2, R_{P_1} \sqcup R_{P_2})$, and the restriction of a poset $P = (S, R_P)$ is written $P|_I$ and defined as $(I, R_P \cap (I \times I))$. Recall that S is an ideal of P if whenever $x \leq y$ and $x \in S$, then $y \in S$. Define the product

$$P \cdot Q = P \sqcup rl_{[n],[n+1,n+m]}(Q)\,.$$

And the coproduct

$$\Delta P = \sum_{S \text{ ideal of } P} rl_{S,[\#S]}(P|_S) \otimes rl_{S^c,[n-\#S]}(P|_{S^c})\,.$$

These operations indeed define a Hopf algebra structure on $\mathbf{G}$ and $\mathbf{Pos}$.

Recall from the introduction that, for graphs, Gebhard and Sagan defined in [GS01] the non-commutative chromatic morphism on graphs, and obtained an expression related with proper set partitions of graphs:

Lemma 7 ([GS01, Proposition 3.2])**.** *For a graph G we say that a set partition $\boldsymbol{\tau}$ of $V(G)$ is proper if no block of $\boldsymbol{\tau}$ contains an edge. Then have that*

$$\boldsymbol{\Upsilon}_{\mathbf{G}}(G) = \sum_{\boldsymbol{\tau}} \mathbf{m}_{\boldsymbol{\tau}}\,,$$

where the sum runs over all proper set partitions of $V(G)$.

2.4 Faces and a Hopf algebra structure of generalized permutahedra

In the following we identify I with $[n]$. For a set composition $\vec{\boldsymbol{\pi}} = S_1 | \dots | S_k$ on $[n]$ and a non-empty set $J \subseteq [n]$, recall that $R_{\vec{\boldsymbol{\pi}}}$ is a total preorder on $[n]$. We define the set $J_{\vec{\pi}} = \{\text{minima of J in } R_{\vec{\pi}}\} = J \cap S_i$ where i is the smallest possible index with $J \cap S_i \neq \emptyset$.

In the cartesian space $\mathbb{R}^{[n]}$, we denote $\mathfrak{s}_J = \operatorname{conv}\{e_v | \, v \in J\}$ for the simplex on $J \subseteq [n]$. Recall that a *generalized permutahedron* is a Minkowski sum and difference of the form

$$\mathfrak{q} = \left(\sum_{J \in A_+} a_J \mathfrak{s}_J \right) - \left(\sum_{J \in A_-} |a_J| \mathfrak{s}_J \right) , \tag{9}$$

for reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset \neq J \subseteq [n]}$ that can be either positive, negative or zero, and $A_+ = \{J | a_J > 0\}$ and $A_- = \{J | a_J < 0\}$.

Recall as well that a *hypergraphic polytope* is a generalized permutahedron of the form

$$\mathfrak{q} = \sum_{J \neq \emptyset} a_J \mathfrak{s}_J ,$$

for non-negative reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset \neq J \subseteq [n]}$.

In the following, we will identify colorings of $[n]$, that is functions $f : [n] \to \mathbb{R}$, and linear functions $f : \mathbb{R}^n \to \mathbb{R}$ by

$$x \mapsto \sum_{i=1}^{n} f(i) x_i .$$

For a polytope $\mathfrak{q}$ and a coloring f on $[n]$, we denote by $\mathfrak{q}_f$ the subset of $\mathfrak{q}$ on which f is minimized, that is

$$\mathfrak{q}_f := \arg\min_{x \in \mathfrak{q}} \sum_{i \in I} f(i) x_i .$$

A face of $\mathfrak{q}$ is the solution to a linear optimisation problem on $\mathfrak{q}$ for some coloring f.

Example 8. Consider the hypergraphic polytope $\mathfrak{q} = \mathfrak{s}_{\{1,2,3\}} + \mathfrak{s}_{\{1,2\}}$ in $\mathbb{R}^3$. If we take the coloring of $\{1, 2, 3\}$ given by $f(1) = f(2) = 1$ and $f(3) = 3$, then $\mathfrak{q}_f = 2\mathfrak{s}_{\{1,2\}}$. If we consider the coloring $g(1) = g(3) = 2$ and $g(2) = 1$, then $\mathfrak{q}_g = 2\mathfrak{s}_{\{2\}}$ is a point, so g is $\mathfrak{q}$- generic.

In particular, note that if $J_1 \subseteq J_2$, then $\mathfrak{s}_{J_1}$ is a face of $\mathfrak{s}_{J_2}$, and in fact we have that $\mathfrak{s}_{J_1} = (\mathfrak{s}_{J_2})_f$ whenever f is a coloring that is constant and minimal exactly in J_1. In fact, for a coloring $f : [n] \to \mathbb{N}$ the face corresponding to f of a simplex is another simplex, specifically

$$(\mathfrak{s}_J)_f = \mathfrak{s}_{J_{\vec{\pi}(f)}} . \tag{10}$$

The following fact describes faces of the Minkowski sums and differences of polytopes:

Lemma 9. *Let f be a coloring and $\mathfrak{a}, \mathfrak{b}$ two polytopes. Then $(\mathfrak{a} + \mathfrak{b})_f = \mathfrak{a}_f + \mathfrak{b}_f$ and, if the difference $\mathfrak{a} - \mathfrak{b}$ is well defined, $(\mathfrak{a} - \mathfrak{b})_f = \mathfrak{a}_f - \mathfrak{b}_f$*

Proof. Suppose that $m_{\mathfrak{a}}, m_{\mathfrak{b}}$ are the minima of f in the polytopes $\mathfrak{a}, \mathfrak{b}$. Let $x \in \mathfrak{a} + \mathfrak{b}$. So $x = a + b$ for some $a \in \mathfrak{a}, b \in \mathfrak{b}$.

Then $f(x) = f(a) + f(b) \geq m_{\mathfrak{a}} + m_{\mathfrak{b}}$. We have equality precisely when $a \in \mathfrak{a}_f, b \in \mathfrak{b}_f$, that is when $x \in \mathfrak{a}_f + \mathfrak{b}_f$.

Now $(\mathfrak{a} - \mathfrak{b})_f = \mathfrak{a}_f - \mathfrak{b}_f$ follows because $(\mathfrak{a} - \mathfrak{b})_f + \mathfrak{b}_f = \mathfrak{a}_f$ by the above. □

Definition 10 (Normal fan of a polytope). A cone is a subset of an $\mathbb{R}$-vector space that is closed for addition and multiplication by positive scalars. For a polytope $\mathfrak{q}$ and $F \subseteq \mathfrak{q}$ one of its faces, its corresponding normal cone is the set of colorings f whose minimum in $\mathfrak{q}$ is attained in the face F, explicitly

$$\mathcal{N}_{\mathfrak{q}}(F) := \{f : [n] \to \mathbb{R} | \ \mathfrak{q}_f = F\} .$$

This is a cone in the dual space of $\mathbb{R}^n$. Moreover, the normal cones of all the faces of $\mathfrak{q}$ partition $(\mathbb{R}^n)^*$ into cones $\mathcal{N}_{\mathfrak{q}} = \{\mathcal{N}_{\mathfrak{q}}(F) | \, F \text{ is a face of } \mathfrak{q}\}$. This is the *normal fan* of $\mathfrak{q}$.

Example 11 (The normal cone of the n-permutahedron - The braid fan). The faces of the permutahedron are indexed by $\mathbf{C}_n$, and in particular the corresponding normal cone of the face $F_{\vec{\pi}}$ corresponding to $\vec{\pi} \in \mathbf{C}_n$ is precisely $\mathcal{N}(F_{\vec{\pi}}) = \{f : [n] \to \mathbb{R} | \vec{\pi}(f) = \vec{\pi}\}$.

In the introduction we referred two other definitions of generalized permutahedra that are present in the literature. we now justify their equivalence:

Lemma 12 (Definition 1 of generalized permutahedra, see [AA17])**.** *A polytope is a generalized permutahedron in the sense of* (9) *if and only if its normal fan coarsens the one from the permutahedron. Specifically, for any two colorings f_1, f_2 with the same set composition type $\vec{\pi}(f_1) = \vec{\pi}(f_2) = \vec{\pi}$, the corresponding face is the same $\mathfrak{q}_{f_1} = \mathfrak{q}_{f_2} = \mathfrak{q}_{\vec{\pi}}$.*

Lemma 13 (Definition 2 of generalized permutahedra, see [Pos09])**.** *Define the polytope $\mathcal{P}_n^z(\{z_I\}_{I\subseteq[n]})$ in the plane $\sum_i x_i = z_{[n]}$ given by the inequalities*

$$\sum_{i\in I} x_i \geq z_I \, ,$$

for some real numbers $\{z_I\}_{I\subseteq[n]}$. Then a polytope is a generalized permutahedron in the sense of (9) *if and only if it can be expressed as $\mathcal{P}_n^z(\{z_I\}_{I\subseteq[n]})$ for real numbers $\{z_I\}_{I\subseteq[n]}$. Moreover, such real numbers can be chosen so that they satisfy*

$$z_I + z_J \geq z_{I\cup J} + z_{I\cap J} \, ,$$

for all sets $I, J \subseteq [n]$.

In [AA17, Theorem 12.3], it is shown that a polytope is of the form $\mathcal{P}_n^z(\{z_I\}_{I\subseteq[n]})$ for real numbers $\{z_I\}_{I\subseteq[n]}$ such that $z_I + z_J \geq z_{I\cup J} + z_{I\cap J}$ for all sets $I, J \subseteq [n]$ if and only if its normal fan coarsens the one from the permutahedron.

In [ABD10, Proposition 2.4], Ardila, Benedetti and Doker show that any polytope of the form $\mathcal{P}_n^z(\{z_I\}_{I\subseteq[n]})$ can be expressed as Eq. (3).

In the following we establish that the normal fan of a polytope of the form Eq. (3) coarsens the one of the n-permutahedron, establishing a proof of both **??** and Lemma 13 that the three definitions of generalized permutahedra presented are equivalent.

Proposition 14. *Let $\mathfrak{q}$ be a polytope of the form*

$$\mathfrak{q} = \left(\sum_{J\in A_+} a_J \mathfrak{s}_J \right) - \left(\sum_{J\in A_-} |a_J| \mathfrak{s}_J \right) \, ,$$

for reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset\neq J\subseteq[n]}$ that can be either positive, negative or zero, and $A_+ = \{J | a_J > 0\}$ and $A_- = \{J | a_J < 0\}$. then its normal fan coarsens the one of the permutahedron.

Proof. As a consequence of Example 11 and as discussed in the end of Lemma 9, we just need to establish that $\mathfrak{q}_f$ depends only on the set composition type of f. The face of a generalized permutahedron $\mathfrak{q}$ that minimises f is

$$\mathfrak{q}_f = \left(\sum_{J \in A_+} a_J \mathfrak{s}_{J_{\vec{\pi}(f)}} \right) - \left(\sum_{J \in A_-} |a_J| \mathfrak{s}_{J_{\vec{\pi}(f)}} \right) , \tag{11}$$

and indeed only depends on the set composition type of the coloring f.

□

We denote by $\mathfrak{q}_{\vec{\pi}}$ the face on $\mathfrak{q}$ that is the solution to any linear optimisation problem on $\mathfrak{q}$ for a coloring f that satisfies $\vec{\pi}(f) = \vec{\pi}$, so

$$\mathfrak{q}_{\vec{\pi}} = \left(\sum_{J \in A_+} a_J \mathfrak{s}_{J_{\vec{\pi}}} \right) - \left(\sum_{J \in A_-} |a_J| \mathfrak{s}_{J_{\vec{\pi}}} \right) .$$

We have immediately the following:

Proposition 15. *If* $\mathfrak{q}$ *is a generalized permutahedron, then*

$$\Upsilon_{\mathbf{GP}}(\mathfrak{q}) = \sum_{f\ \mathfrak{q}\text{-generic}} \mathbf{a}_f = \sum_{\mathfrak{q}_{\vec{\pi}} = \mathrm{pt}} \mathbf{M}_{\vec{\pi}} \in \mathbf{WQSym}_n . \tag{12}$$

We now turn away from the face structure of a generalized permutahedron and debate its Hopf monoid structure. Here we recover the structure described in [AA17] As usual, consider a generalized permutahedron $\mathfrak{q}$ given by

$$\mathfrak{q} = \left(\sum_{J \in A_+} a_J \mathfrak{s}_J \right) - \left(\sum_{J \in A_-} |a_J| \mathfrak{s}_J \right) , \tag{13}$$

for reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset \neq J \subseteq [n]}$ that can be either positive, negative or zero, and $A_+ = \{J | a_J > 0\}$ and $A_- = \{J | a_J < 0\}$. If $\vec{\pi} = A|B$ is a set composition, then $\mathfrak{q}_{\vec{\pi}}$ can be written as a cartesian product, so that

$$\mathfrak{q}_{\vec{\pi}} =: \mathfrak{q}|_A + \mathfrak{q}\backslash_A ,$$

where $\mathfrak{q}|_A$ is a generalized permutahedron in $\mathbb{R}^A \times \{0\}^B$ and $\mathfrak{q}\backslash_A$ is a generalized permutahedron on $\mathbb{R}^B \times \{0\}^A$. These will be identified with generalized permutahedra in $\mathbb{R}^A$ and $\mathbb{R}^B$ without further notice. Note that $B = A^c$ so the dependence of $\mathfrak{q}|_A$ and $\mathfrak{q}\backslash_A$ on B is implicit.

We can obtain explicit expressions for $\mathfrak{q}|_A$ and $\mathfrak{q}\backslash_A$:

$$\mathfrak{q}|_A = \left(\sum_{\substack{J \in A_+ \\ J \not\subseteq B}} a_J \mathfrak{s}_{J\cap A} \right) - \left(\sum_{\substack{J \in A_- \\ J \not\subseteq B}} |a_J| \mathfrak{s}_{J\cap A} \right) , \mathfrak{q}\backslash_A = \left(\sum_{\substack{J \in A_+ \\ J \subseteq B}} a_J \mathfrak{s}_J \right) - \left(\sum_{\substack{J \in A_- \\ J \subseteq B}} |a_J| \mathfrak{s}_J \right) .$$

We have now all the material to endow the space of generalized permutahedra with a Hopf algebra structure according to [AA17]: let $\mathbf{GP} = \oplus_{n \geq 0} \mathbf{GP}_n$, where $\mathbf{GP}_n$ is the free linear space on generalized permutahedra in $\mathbb{R}^n$.

The **GP** linear space has the following product, when $\mathfrak{q}_1, \mathfrak{q}_2$ are generalized permutahedra in $\mathbb{R}^n, \mathbb{R}^m$ respectively:

$$\mathfrak{q}_1 \cdot \mathfrak{q}_2 = \mathfrak{q}_1 \times rl_{[m],[m+1,m+n]}(\mathfrak{q}_2)\,.$$

The **GP** linear space has the following coproduct, when $\mathfrak{q}$ is a generalized permutahedron in $\mathbb{R}^n$:

$$\Delta\mathfrak{q} = \sum_{A\subseteq[n]} rl_{A,[\#A]}(\mathfrak{q}|_A) \otimes rl_{A^c,[n-\#A]}(\mathfrak{q}\backslash_A)\,.$$

Remark 16. Note that the span of the fundamental hypergraphic polytopes does not form a Hopf algebra, as it is not stable for the coproduct in generalized permutahedra.

3 The chromatic symmetric function on graphs

3.1 Main theorems on graphs

In this section we will prove Theorem 1 and Theorem 2, which will be applications of Lemma 5 and Lemma 6. We discuss as well an application of Theorem 2 on the tree conjecture, by constructing a new graph invariant $\tilde{\Psi}(G)$ that satisfies the modular relations.

For a set partition $\boldsymbol{\pi}$, we define the graph $K_{\boldsymbol{\pi}}$ where $\{i,j\} \in E(K_{\boldsymbol{\pi}})$ if $i \sim_{\boldsymbol{\pi}} j$. This graph is the disjoint union of the complete graphs on the blocks of $\boldsymbol{\pi}$. We denote the complement of $K_{\boldsymbol{\pi}}$ as $K^c_{\boldsymbol{\pi}}$. Note that a set partition $\boldsymbol{\tau}$ is proper in $K^c_{\boldsymbol{\pi}}$ if and only if $\boldsymbol{\tau} \leq \boldsymbol{\pi}$ in the coarsening order on set partitions. Hence, as a consequence of Lemma 7,

$$\boldsymbol{\Upsilon}_{\mathbf{G}}(K^c_{\boldsymbol{\pi}}) = \sum_{\boldsymbol{\tau}\leq\boldsymbol{\pi}} \mathbf{m}_{\boldsymbol{\tau}}. \tag{14}$$

We now show that the kernel of $\boldsymbol{\Upsilon}_{\mathbf{G}}$ is spanned by the modular relations.

Proof of Theorem 1. Recall that $\mathbf{G}_n$ is spanned by graphs with vertex set $[n]$. We choose an order $\tilde{\geq}$ in this family of graphs in a way that the number of edges is non-decreasing.

From (14), we know that the transition matrix of $\{\boldsymbol{\Upsilon}_{\mathbf{G}}(K^c_{\boldsymbol{\pi}})|\boldsymbol{\pi} \in \mathbf{P}_n\}$ over the monomial basis of **WSym** is upper triangular, hence forms a basis set of **WSym**. In particular, $\operatorname{im} \boldsymbol{\Upsilon}_{\mathbf{G}} = \mathbf{WSym}$.

In order to apply Lemma 5 to the set of modular relations on graphs, it suffices to show the following: if a graph G is not of the form $K^c_{\boldsymbol{\pi}}$, then we can find a formal sum $G - G\cup\{e_1\} - G\cup\{e_2\} + G\cup\{e_1, e_2\}$ that is a modular relation. Indeed, G is the graph with least edges in that expression, so it is the smallest in the order $\tilde{\geq}$. If the above holds, Lemma 5 implies that the modular relations generate the space $\ker \boldsymbol{\Upsilon}_{\mathbf{G}}$.

To find the desired modular relation, it is enough to find a triangle $\{e_1, e_2, e_3\}$ such that $e_1, e_2 \notin E(G)$ and $e_3 \in E(G)$. Consider $\boldsymbol{\tau}$, the set partition given by the connected components of G^c. By hypothesis, $G \neq K^c_{\boldsymbol{\tau}}$, so there are vertices u, w in the same block of $\boldsymbol{\tau}$ that are not neighbours in G^c. Without loss of generality we can take such u, w that are at distance 2 in G^c, so they have a common neighbour v in G^c (see example in Fig. 2).

The edges $e_1 = \{v, u\}$, $e_2 = \{v, w\}$ and $e_3 = \{u, w\}$ form the desired triangle, concluding the proof. □

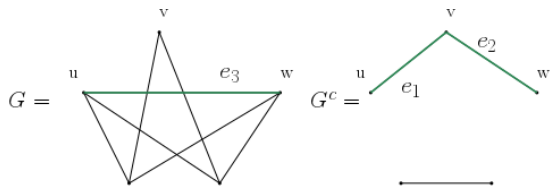

Figure 2: Choice of edges in proof of Theorem 1

Proof of Theorem 2. It is clear that $\Psi_{\mathbf{G}}$ is surjective, since $\boldsymbol{\Upsilon}_{\mathbf{G}}$ is surjective. Now, our goal is to apply Lemma 6 to the map $\Psi_{\mathbf{G}} = \mathrm{comu} \circ \boldsymbol{\Upsilon}_{\mathbf{G}}$ and the equivalence relation corresponding to graph isomorphism. First, note that if $\lambda(\boldsymbol{\pi}) = \lambda(\boldsymbol{\tau})$ then $K^c_{\boldsymbol{\pi}}$ and $K^c_{\boldsymbol{\tau}}$ are isomorphic graphs. Define without ambiguity $r_{\lambda(\boldsymbol{\pi})} = \Psi_{\mathbf{G}}(K^c_{\boldsymbol{\pi}})$.

From the proof of Theorem 1, the hypotheses of Lemma 5 are satisfied. Therefore, to apply Lemma 6 it is enough to establish that the family $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ is linearly independent. Indeed, it would follow that $\ker \Psi_{\mathbf{G}}$ is generated by the modular relations and the isomorphism relations, and $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ is a basis of $\mathrm{im}\, \Psi_{\mathbf{G}}$ concluding the proof.

The linear independence of $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ follows from the fact that its transition matrix to the monomial basis is upper triangular under the coarsening order in integer partitions. Indeed, from (14), if we let $\boldsymbol{\tau}$ run over set partitions and σ run over integer partitions, we have

$$r_{\lambda(\boldsymbol{\pi})} = \Psi_{\mathbf{G}}(K^c_{\boldsymbol{\pi}}) = \sum_{\boldsymbol{\tau} \leq \boldsymbol{\pi}} m_{\lambda(\boldsymbol{\tau})} = \sum_{\sigma \leq \lambda(\boldsymbol{\pi})} a_{\boldsymbol{\pi},\sigma}\; m_\sigma \,,$$

where $a_{\boldsymbol{\pi},\sigma} = \#\{\boldsymbol{\tau} \vdash [n] | \lambda(\boldsymbol{\tau}) = \sigma,\, \boldsymbol{\tau} \leq \boldsymbol{\pi}\}$. Note that $a_{\boldsymbol{\pi},\lambda(\boldsymbol{\pi})} = 1$, so $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ is linearly independent. $\square$

Remark 17. We have obtained in the proof of Theorem 2 that $(r_\lambda)_{\lambda \vdash n}$ is a basis for Sym_n. This basis is different from other "chromatic bases" proposed in [CvW16]. The proof gives us a recursive way to compute the coefficients $\zeta_\lambda(G)$ on the span $\Psi_{\mathbf{G}}(G) = \sum_\lambda \zeta_\lambda(G) r_\lambda$. It is then natural to ask if combinatorial properties can be obtained for these coefficients, which are isomorphism invariants.

Similarly in the non-commutative case, we obtain that $\mathbf{WSym}_n$ is spanned by $(\boldsymbol{\Upsilon}_{\mathbf{G}}(K^c_{\boldsymbol{\pi}}))_{\boldsymbol{\pi} \vdash [n]}$, and so other coefficients arise. We can again ask for combinatorial properties of these coefficients.

3.2 The augmented chromatic invariant

Consider the ring of power series $\mathbb{K}[[x_1, x_2, \dots; q_1, q_2, \dots]]$ on two countably infinite collections of commuting variables, and let R be such ring modulo the relations $q_i(q_i - 1)^2 = 0$.

Consider the graph invariant $\tilde{\Psi}(G) = \sum_f x_f \prod_i q_i^{c_G(f,i)}$ in R, where the sum runs over **all** colorings f, and $c_G(f,i)$ stands for the number of monochromatic edges of color i in the coloring f (i.e. edges $\{v_1, v_2\}$ such that $f(v_1) = f(v_2) = i$).

For instance, if $G = K_2$, then $\tilde{\Psi}(G) = 2 \sum_{1 \leq i < j} x_i x_j + \sum_{1 \leq i} x_i^2 q_i$. If we consider $G = K_3$ then we have

$$\tilde{\Psi}(G) = 6 \sum_{1 \leq i < j < k} x_i x_j x_k + 3 \sum_{i \neq j} x_i x_j^2 q_j + \sum_{1 \leq i} x_i^3 q_i^3 \,.$$

Note that we can simplify further with $q_i^3 = 2q_i^2 - q_i$.

A main property of this graph invariant is that it can be specialised to the chromatic symmetric function, by evaluating each variable q_i to zero. Another property of this graph invariant is that its kernel is $\ker \Psi_{\mathbf{G}}$, as it will be shown latter, using Theorem 2:

Proposition 18. *We have that* $\ker \tilde{\Psi} = \ker \Psi_{\mathbf{G}}$. *In particular, for graphs* G_1, G_2 *we have* $\Psi_{\mathbf{G}}(G_1) = \Psi_{\mathbf{G}}(G_2)$ *if and only if* $\tilde{\Psi}(G_1) = \tilde{\Psi}(G_2)$.

Take, for instance, the celebrated tree conjecture introduced in [Sta95]:

Conjecture 19 (Tree conjecture on chromatic symmetric functions)**.** *If two trees* T_1, T_2 *are not isomorphic, then* $\Psi_{\mathbf{G}}(T_1) \neq \Psi_{\mathbf{G}}(T_2)$.

Consequently, from Proposition 18, the tree conjecture is equivalent to the following conjecture:

Conjecture 20. *If two trees* T_1, T_2 *are not isomorphic, then* $\tilde{\Psi}(T_1) \neq \tilde{\Psi}(T_2)$.

One strategy that has been employed to show that a family of non-isomorphic trees is distinguished by their chromatic symmetric function is to reconstruct trees from the coefficients of their chromatic symmetric function over several bases, see for instance [OS14], [SST15], [MMW08] and [APZ14]. The graph invariant $\tilde{\Psi}$ provides more coefficients to reconstruct a tree, because Ψ results from $\tilde{\Psi}$ after the specialization $q_i = 0$. So employing the same strategy to prove Conjecture 20 *a priori* easier than approaching Conjecture 19.

The kernel method can also give us some light on other graph invariants: they may look stronger than Ψ, but are in fact only as strong as Ψ if they satisfy the modular relations.

Proof of Proposition 18. Note that we have $\tilde{\Psi}(G)|_{q_i=0,\ i=1,2,\dots} = \Psi_{\mathbf{G}}(G)$. This readily yields $\ker \tilde{\Psi} \subseteq \ker \Psi_{\mathbf{G}}$. To show that $\ker \tilde{\Psi} \supseteq \ker \Psi_{\mathbf{G}}$, we need only to show that the modular relations and the isomorphism relations belong to $\ker \tilde{\Psi}$. for the isomorphism relations, this is trivial.

Let $l = G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ be a modular relation on graphs, i.e. $\{e_1, e_2, e_3\}$ are edges that form a triangle between vertices $\{v_1, v_2, v_3\}$, with $e_3 \in G, e_1, e_2 \notin G$. Say that $e_1 = \{v_2, v_3\}$, $e_2 = \{v_3, v_1\}$ and $e_3 = \{v_1, v_2\}$. The proposition is proved if we show that $\tilde{\Psi}(l) = 0$.

For a coloring f on a graph H and a monochromatic edge e in H, define $c(e)$ the color of the vertices of e. Abbreviate $\mathbb{1}[e \text{ is monochromatic}] = m(e)$. With this, we use the abuse of notation $q_{c(e)}^{m(e)}$ even when e is not monochromatic. In that case, $q_{c(e)}^{m(e)} = q_{c(e)}^0 = 1$. Then

$$\prod_i q_i^{c_H(f,i)} = \prod_{e \text{ monochromatic}} q_{c(e)} = \prod_{e \in E(H)} q_{c(e)}^{m(e)} . \tag{15}$$

Set

$$\begin{aligned} s_f := &\prod_i q_i^{c_G(f,i)} - \prod_i q_i^{c_{G\cup\{e_1\}}(f,i)} - \prod_i q_i^{c_{G\cup\{e_2\}}(f,i)} + \prod_i q_i^{c_{G\cup\{e_1,e_2\}}(f,i)} \\ = &\prod_{e\in E(G)} q_{c(e)}^{m(e)} \left(1 - q_{c(e_1)}^{m(e_1)} - q_{c(e_2)}^{m(e_2)} + q_{c(e_1)}^{m(e_1)} q_{c(e_2)}^{m(e_2)}\right) , \end{aligned} \tag{16}$$

and observe that $\tilde{\Psi}(l) = \sum_f x_f s_f$. We now show that s_f is always zero. Fix a coloring f, then exactly one of the following happens:

- **None of the edges** e_1, e_2 **is monochromatic**.

  Then, the contribution of monochromatic edges to $c_H(f,i)$ in (15) comes only from edges in G, and then grouping the common factors yields:

$$s_f = \prod_{e \in E(G)} q_{c(e)}^{m(e)} (1 - 1 - 1 + 1) = 0$$

- **The edge** e_1 **is monochromatic of color** a, **and** $f(v_1) = b \neq a$.

  Then, again grouping the common factors in (15) gives:

$$s_f = \prod_{e \in E(G)} q_{c(e)}^{m(e)} (1 - q_a - 1 + q_a) = 0$$

- **The edge** e_2 **is monochromatic of color** a, **and** $f(v_2) = b \neq a$.

  Then,

$$s_f = \prod_{e \in E(G)} q_{c(e)}^{m(e)} (1 - 1 - q_a + q_a) = 0$$

- **The triangle** $\{v_1, v_2, v_3\}$ **is monochromatic of color** a.

  Then, in the ring R we immediately have

$$s_f = \prod_{e \in E(G)} q_{c(e)}^{m(e)} \left(q_a - q_a^2 - q_a^2 + q_a^3\right) = 0\,.$$

So $\tilde{\Psi}(l) = \sum_f x_f s_f = 0$.

In conclusion, any modualr relation and any isomorphism relation is in $\ker \tilde{\Psi}$. From Theorem 2 we have that $\ker \Psi_{\mathbf{G}} \subseteq \ker \tilde{\Psi}$, so we conclude the proof. $\square$

It is clear that Proposition 18 was established in an indirect way, by studying the kernel of the maps $\tilde{\Psi}$ and Ψ, instead of relating the coefficients of both invariants when expressed in their natural bases.

In Appendix A we relate the coefficients of both invariants in Corollary 54. Our original goal of establishing Proposition 18 without using Theorem 2 directly is not accomplished, which lends more strength to this indirect kernel method.

4 The chromatic quasisymetric function on hypergraphic polytopes

4.1 Poset structures on compositions

In this chapter we consider generalized permutahedra and hypergraphic polytopes in $\mathbb{R}^n$, and colorings on $[n]$. Recall that with a set composition $\vec{\pi} = S_1|\dots|S_k \in \mathbf{C}_n$ we have the associated total preorder $R_{\vec{\pi}}$, and for a non-empty set $A \subseteq [n]$, we define the set $A_{\vec{\pi}} = A \cap S_i$ where i is the smallest index possible so that $A \cap S_i \neq \emptyset$. We refer to $A_{\vec{\pi}}$ as the minima of $R_{\vec{\pi}}$ in A.

Finally, recall as well that, for a hypergraphic polytope $\mathfrak{q}$, we denote by $\mathcal{F}(\mathfrak{q}) \subseteq 2^I \setminus \{\emptyset\}$ the family of sets $J \subseteq I$ such that the coefficients in (3) satisfy $a_J > 0$, and for a set $A \subseteq 2^I \setminus \{\emptyset\}$, we write $\mathcal{F}^{-1}(A)$ for the hypergraphic polytopes $\mathfrak{q} = \sum_{J \in A} \mathfrak{s}_J$. We write $a = \mathrm{pt}$ whenever a is a point polytope.

The face $\mathfrak{q}_f$ only depends on the set composition $\vec{\boldsymbol{\pi}}(f)$. For a generalized permutahedron $\mathfrak{q}$ and a coloring $f : [n] \to \mathbb{N}$ of set composition type $\vec{\boldsymbol{\pi}}$, we write $\mathfrak{q}_{\vec{\boldsymbol{\pi}}}$ for the face $\mathfrak{q}_f$, without ambiguity.

Definition 21 (Fundamental hypergraphic polytopes and a preorder in set compositions). For $\vec{\boldsymbol{\pi}} \in \mathbf{C}_n$, we define the *basic hypergraphic polytope* as the fundamental hypergraph polytope $\mathfrak{p}^{\vec{\boldsymbol{\pi}}} = \mathcal{F}^{-1}\{A | \# A_{\vec{\boldsymbol{\pi}}} = 1\}$.

On set compositions of $[n]$, we write that $\vec{\boldsymbol{\pi}}_1 \preceq \vec{\boldsymbol{\pi}}_2$ whenever for any non-empty $A \subseteq [n]$ we have $\# A_{\vec{\boldsymbol{\pi}}_1} = 1 \Rightarrow \# A_{\vec{\boldsymbol{\pi}}_2} = 1$. Equivalently, $\vec{\boldsymbol{\pi}}_1 \preceq \vec{\boldsymbol{\pi}}_2$ if $\mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}_1}) \subseteq \mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}_2})$. With this, $\preceq$ is a preorder, called *singleton commuting preorder* or *SC preorder*. This terminology comes from Proposition 23 below.

Additionally, we define the equivalence relation $\sim$ in $\mathbf{C}_n$ as $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$ whenever $\# A_{\vec{\boldsymbol{\pi}}} = 1 \Leftrightarrow \# A_{\vec{\boldsymbol{\tau}}} = 1$ for all non-empty sets $A \subseteq [n]$. Note that $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$ is equivalent to $\mathfrak{p}^{\vec{\boldsymbol{\pi}}} = \mathfrak{p}^{\vec{\boldsymbol{\tau}}}$. We write $[\vec{\boldsymbol{\pi}}]$ for the equivalence class of $\vec{\boldsymbol{\pi}}$ under $\sim$, and denote $\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]} = \mathfrak{p}^{\vec{\boldsymbol{\pi}}}$ without ambiguity.

Example 22. We see here the preorder $\preceq$ for $n = 3$, and the corresponding order in the equivalence classes of $\sim$. We first study the set compositions $\vec{\boldsymbol{\pi}}$ such that $\lambda(\vec{\boldsymbol{\pi}}) = (1, 1, 1)$. Since these are in bijection with permutations on $\{1, 2, 3\}$, we call these the *permutations in* $\mathbf{C}_3$. For a permutation $\vec{\boldsymbol{\pi}}$ in $\mathbf{C}_3$, we have $\# A_{\vec{\boldsymbol{\pi}}} = 1$ for all A, so the permutations are maximal elements in the singleton commuting preorder, and are equivalent in $\sim$.

We also observe that if $\# A_{23|1} = 1$, then $\{2, 3\} \not\subseteq A$ and so we have that $\# A_{1|23} = 1$. It follows that $1|23 \succeq 23|1$. The remaining structure of the preorder in $\mathbf{C}_3$ can be seen in Fig. 3, where we collapse equivalence classes into vertices and draw the corresponding poset in its Hasse diagram.

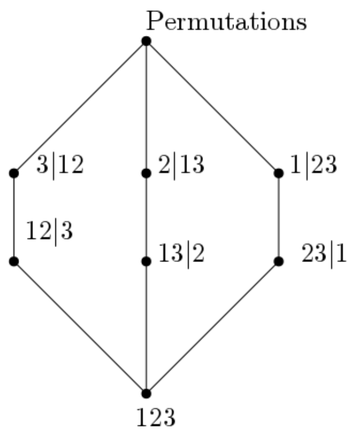

Figure 3: The SC order in $\mathbf{C}_3/_\sim$.

For $n = 4$, things are more interesting as we have non-trivial equivalence classes. For instance, we have that $12|3|4 \sim 12|4|3$.

Proposition 23. *Let $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\tau}} \in \mathbf{C}_n$. Then we have that $\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}} \Rightarrow \boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}) \geq \boldsymbol{\lambda}(\vec{\boldsymbol{\tau}})$.*

Additionally, $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$ if and only if all the following happens:

1. *We have* $\boldsymbol{\pi} = \boldsymbol{\tau}$ *and;*
2. *Each* $a, b \in [n]$ *that satisfies both* $a\, R_{\vec{\boldsymbol{\pi}}}\, b$ *and* $b\, R_{\vec{\boldsymbol{\tau}}}\, a$ *are either singletons or in the same block in* $\boldsymbol{\pi}$*.*

In particular, $\alpha(\vec{\boldsymbol{\pi}}) = \alpha(\vec{\boldsymbol{\tau}})$*, whenever* $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$*.*

Property 2. will be called the *SC property*.

The equivalence classes of $\sim$ have a clear combinatorial description via Proposition 23. In particular, we see that $\vec{\boldsymbol{\pi}}_1 \sim \vec{\boldsymbol{\pi}}_2$ if all blocks are the same and in the same order, with possible exceptions on the singletons that can transpose between each other. For instance, we have $12|3|4|5|67 \sim 12|3|5|4|67$ but $12|3|4|5|67 \not\sim 3|12|4|5|67$.

We are also told in Proposition 23 that the map $\boldsymbol{\lambda} : \mathbf{C}_n \to \mathbf{P}_n$ flips the SC preorder with respect to the coarsening order $\leq$ in $\mathbf{P}_n$.

Proof of Proposition 23. Write $\boldsymbol{\pi}, \boldsymbol{\tau}$ for the underlying set partitions $\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}), \boldsymbol{\lambda}(\vec{\boldsymbol{\tau}})$, respectively. Suppose that $\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}}$ and take i, j elements of $[n]$ such that $i \sim_{\boldsymbol{\tau}} j$. Then $\{i, j\} \notin \mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\tau}}}) \supseteq \mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}})$. This implies that $\#\{i, j\}_{\vec{\boldsymbol{\pi}}} \neq 1$, hence $\{i, j\}_{\vec{\boldsymbol{\pi}}} = \{i, j\}$, so we have $i \sim_{\boldsymbol{\pi}} j$. Since i, j are generic, we have that $\boldsymbol{\pi} \geq \boldsymbol{\tau}$. This concludes the first part.

For the second part, we first show the direct implication. Suppose that $\mathfrak{p}^{\vec{\boldsymbol{\pi}}} = \mathfrak{p}^{\vec{\boldsymbol{\tau}}}$, so it follows from the first part of the proposition that $\boldsymbol{\pi} = \boldsymbol{\tau}$. Our goal is to show the SC property.

Take a, b that are in distinct blocks in $\boldsymbol{\pi}$, such that both $a\, P_{\vec{\boldsymbol{\pi}}}\, b$ and $b\, P_{\vec{\boldsymbol{\tau}}}\, a$. For sake of contradiction let $c \neq a$ be such that $c \sim_{\boldsymbol{\pi}} a$. Then $\{a, b, c\}_{\vec{\boldsymbol{\pi}}} = \{a, c\} \neq pt$, but since $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$ by hypothesis, we have that $\{a, b, c\}_{\vec{\boldsymbol{\tau}}} = \{b\} \neq pt$, which is a contradiction. This contradicts the assumption that $a \sim_{\boldsymbol{\pi}} c$, so we conclude that a is a singleton in $\boldsymbol{\pi}$. Similarly we obtain that b is also a singleton in $\boldsymbol{\tau} = \boldsymbol{\pi}$. This shows the SC property.

For the reverse implication, suppose that $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\tau}}$ are such that $\boldsymbol{\pi} = \boldsymbol{\tau}$ that satisfy the SC property. Our goal is to show that $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$, so it is enough to show that $\#A_{\vec{\boldsymbol{\pi}}} = 1 \Leftrightarrow \#A_{\vec{\boldsymbol{\tau}}} = 1$ for any non empty $A \subseteq [n]$. Take some nonempty set $A \subseteq [n]$ such that $A_{\vec{\boldsymbol{\pi}}} = \{a\}$ is a singleton, but $\#A_{\vec{\boldsymbol{\tau}}} \neq 1$, for sake of contradiction. Finally, take an element $b \in A_{\vec{\boldsymbol{\tau}}}$.

Observe that, since $\boldsymbol{\pi} = \boldsymbol{\tau}$, either $A_{\vec{\boldsymbol{\pi}}} = A_{\vec{\boldsymbol{\tau}}}$ or $A_{\vec{\boldsymbol{\pi}}} \cap A_{\vec{\boldsymbol{\tau}}} = \emptyset$. Since, $A_{\vec{\boldsymbol{\pi}}} \neq A_{\vec{\boldsymbol{\tau}}}$, they are disjoint and $a \neq b$. Note that we have $a\, P_{\vec{\boldsymbol{\pi}}}\, b$, $b\, P_{\vec{\boldsymbol{\tau}}}\, a$. So, by the SC property we conclude that b is a singleton in $\boldsymbol{\tau}$, contradicting that $\#A_{\vec{\boldsymbol{\tau}}} \neq 1$.

That $\#A_{\vec{\boldsymbol{\tau}}} = 1 \Rightarrow \#A_{\vec{\boldsymbol{\pi}}} = 1$ follows similarly, concluding the proof.

From the SC property, it imediately follows that $\alpha(\vec{\boldsymbol{\pi}}) = \alpha(\vec{\boldsymbol{\tau}})$ □

The following definition focus on the algebraic counterpart of the equivalence relation $\sim$.

Definition 24. Consider the quasisymmetric functions $\mathbf{N}_{[\vec{\boldsymbol{\pi}}]} = \sum_{\vec{\boldsymbol{\tau}} \sim \vec{\boldsymbol{\pi}}} \mathbf{M}_{\vec{\boldsymbol{\tau}}}$, which are linearly independent. The *singleton commuting space*, or **SC** for short, is the graded vector subspace of **WQSym** spanned by $\biguplus_n \{\mathbf{N}_{[\vec{\boldsymbol{\pi}}]} : [\vec{\boldsymbol{\pi}}] \in \mathbf{C}_n / \sim\}$.

In Lemma 27, we show that **SC** is the image of the Hopf algebra morphism $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$. As a consequence of Theorem 3, **SC** is a Hopf algebra.

We turn to some properties of the faces of hypergraphic polytopes in the next lemma:

Lemma 25 (Vertices of a hypergraphic polytope)**.** *Let* $\mathfrak{q}$ *be a hypergraphic polytope and let* f *be a coloring.*

Then, we have that $\mathfrak{q}_f = \mathrm{pt}$ *if and only if* $\#A_{\vec{\boldsymbol{\pi}}(f)} = 1$ *for each* $A \in \mathcal{F}(\mathfrak{q})$*. In particular, if* $\vec{\boldsymbol{\pi}}(f_1) \sim \vec{\boldsymbol{\pi}}(f_2)$ *then* $\mathfrak{q}_{f_1} = \mathrm{pt} \Leftrightarrow \mathfrak{q}_{f_2} = \mathrm{pt}$.

Proof. Write $\mathfrak{q} = \sum_J a_J \mathfrak{s}_J$ for some coefficients $a_J \geq 0$. Computing on both sides the face that minimizes f, we obtain that $\mathfrak{q}_f = \mathrm{pt}$ if and only if

$$\sum_J a_J (\mathfrak{s}_A)_f = \mathrm{pt}\,,$$

or equivalently, if $(\mathfrak{s}_A)_f = \mathrm{pt}$ for each $A \in \mathcal{F}(\mathfrak{q})$.

We observed in Eq. (10) that $(\mathfrak{s}_A)_f = \mathfrak{s}_{A_{\vec{\boldsymbol{\pi}}(f)}}$. Hence, we conclude that $\mathfrak{q}_f = \mathrm{pt}$ if and only if $\#A_{\vec{\boldsymbol{\pi}}(f)} = 1$ for each $A \in \mathcal{F}(\mathfrak{q})$, as desired.

To show the last part of the lemma, just observe that $\#A_{\vec{\boldsymbol{\pi}}} = 1$ only depends on the equivalence class of $\vec{\boldsymbol{\pi}}$ by definition of $\sim$. $\square$

The following corollary is immediate from (12) and Lemma 25.

Corollary 26. *The image of* $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$ *is contained in the* ***SC*** *space, i.e. for any hypergraphic polytopes* $\mathfrak{q}$ *we have that*

$$\boldsymbol{\Upsilon}_{\mathbf{HGP}}(\mathfrak{q}) \in \mathbf{SC}\,.$$

Another consequence of Lemma 25 is that we have $\mathfrak{p}^{\vec{\boldsymbol{\pi}}}_{\vec{\boldsymbol{\tau}}} = \mathrm{pt}$ precisely when $\#A_{\vec{\boldsymbol{\pi}}} = 1 \Rightarrow \#A_{\vec{\boldsymbol{\tau}}} = 1$, i.e. when $\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}}$. It follow from (12) that:

$$\boldsymbol{\Upsilon}_{\mathbf{GP}}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}}) = \sum_{\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}}} \mathbf{M}_{\vec{\boldsymbol{\tau}}}\,. \tag{17}$$

As presented, (17) seems to shows that the transition matrix of $(\boldsymbol{\Upsilon}_{\mathbf{GP}}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}}))_{\vec{\boldsymbol{\pi}} \in \mathbf{C}_n}$ over the monomial basis is upper triangular. Since $\preceq$ is not an order, that is not necessarily the case, but we obtain a related result with this reasoning:

Lemma 27. *The family* $(\boldsymbol{\Upsilon}(\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]}))_{[\vec{\boldsymbol{\pi}}] \in \mathbf{C}_n/\sim}$ *forms a basis of* ***SC****. In particular, we have* $\mathrm{im}\, \boldsymbol{\Upsilon}_{\mathbf{HGP}} = \mathbf{SC}$.

Proof of Lemma 27. From (17) we have the following triangularity relation:

$$\boldsymbol{\Psi}(\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]}) = \sum_{\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}} M_{\vec{\boldsymbol{\tau}}} + \sum_{\vec{\boldsymbol{\pi}} \prec \vec{\boldsymbol{\tau}}} M_{\vec{\boldsymbol{\tau}}} = N_{[\vec{\boldsymbol{\pi}}]} + \sum_{[\vec{\boldsymbol{\pi}}] \prec [\vec{\boldsymbol{\tau}}]} N_{[\vec{\boldsymbol{\tau}}]}\,, \tag{18}$$

where the preorder $\preceq$ projects naturally into an order in $\mathbf{C}_n/\sim$.

Thus, $\{\boldsymbol{\Upsilon}_{\mathbf{HGP}}(\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]})\}_{[\vec{\boldsymbol{\pi}}] \in \mathbf{C}_n/\sim}$ is another basis of $\mathbf{SC}$. From Corollary 26, we conclude that $\mathrm{im}\, \boldsymbol{\Upsilon}_{\mathbf{HGP}} = \mathbf{SC}$. $\square$

Since $\mathbf{SC}$ is the image of a bialgebra morphism, and it is a connected graded bialgebra, then it is a Hopf algebra.

In the commutative case, we wish to carry the triangularity of the monomial transition matrix in (18) into a new smaller basis in $QSym$.

For that, we project the order $\preceq$ into an order $\leq'$ in $\mathcal{C}_n$ as follows: we say that $\alpha \leq' \beta$ if we can find set compositions $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\tau}}$ that satisfy $\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}}$ and also $\alpha(\vec{\boldsymbol{\pi}}) = \alpha$, $\alpha(\vec{\boldsymbol{\tau}}) = \beta$.

Lemma 28. *The relation $\leq'$ on $\mathcal{C}_n$ satisfies $\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}} \Rightarrow \alpha(\vec{\boldsymbol{\pi}}) \leq' \alpha(\vec{\boldsymbol{\tau}})$, and is an order.*

We recover the action of permutations on set compositions. If $\vec{\boldsymbol{\pi}} = S_1|\dots|S_k \in \mathbf{C}_n$ and $\phi \in S_n$, then $\phi(\vec{\boldsymbol{\pi}}) \in \mathbf{C}_n$ is the set composition $\phi(S_1)|\dots|\phi(S_k)$, where $\phi(A) = \{\phi(a)|a \in A\}$.

Proof. We only need to check that $\leq'$ as defined is indeed an order, as it is straightforward that $\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}} \Rightarrow \alpha(\vec{\boldsymbol{\pi}}) \leq' \alpha(\vec{\boldsymbol{\tau}})$.

Reflexivity of $\leq'$ trivially follows from the definition of $\preceq$. To show antisymmetry of $\leq'$, it is enough to establish that if $\vec{\boldsymbol{\pi}}_1 \preceq \vec{\boldsymbol{\tau}}_2$, $\vec{\boldsymbol{\tau}}_1 \preceq \vec{\boldsymbol{\pi}}_2$ are set compositions such that $\alpha := \alpha(\vec{\boldsymbol{\pi}}_1) = \alpha(\vec{\boldsymbol{\pi}}_2)$ and $\beta := \alpha(\vec{\boldsymbol{\tau}}_1) = \alpha(\vec{\boldsymbol{\tau}}_2)$, then $\alpha = \beta$.

Indeed, if $\alpha(\vec{\boldsymbol{\pi}}_1) = \alpha(\vec{\boldsymbol{\pi}}_2)$ then there is a permutation ϕ in $[n]$ that satisfies $\phi(\vec{\boldsymbol{\pi}}_1) = \vec{\boldsymbol{\pi}}_2$. Then, ϕ lifts to a bijection between $\mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}_1})$ and $\mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}_2})$; in particular, they have the same cardinality.

Similarly, $\mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\tau}}_1})$ and $\mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\tau}}_2})$ have the same cardinality. But since $\vec{\boldsymbol{\pi}}_1 \preceq \vec{\boldsymbol{\tau}}_2$, $\vec{\boldsymbol{\tau}}_1 \preceq \vec{\boldsymbol{\pi}}_2$, i.e. $\mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}_1}) \subseteq \mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\tau}}_2})$ and $\mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\tau}}_1}) \subseteq \mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}_2})$, it follows $\mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}_1}) = \mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\tau}}_1})$, and so $\vec{\boldsymbol{\pi}}_1 \sim \vec{\boldsymbol{\tau}}_1$. From Proposition 23 we have $\alpha = \alpha(\vec{\boldsymbol{\pi}}_1) = \alpha(\vec{\boldsymbol{\tau}}_1) = \beta$, and antisymmetry follows.

To show transitivity, take compositions such that $\alpha \leq' \beta$ and $\beta \leq' \sigma$, i.e. there are set compositions $\vec{\boldsymbol{\pi}} \preceq \vec{\boldsymbol{\tau}}_1$ and $\vec{\boldsymbol{\tau}}_2 \preceq \vec{\boldsymbol{\gamma}}$ such that $\alpha(\vec{\boldsymbol{\pi}}) = \alpha$, $\alpha(\vec{\boldsymbol{\tau}}_1) = \alpha(\vec{\boldsymbol{\tau}}_2) = \beta$ and $\alpha(\vec{\boldsymbol{\gamma}}) = \sigma$. Take a permutation ϕ in $[n]$ such that $\phi(\vec{\boldsymbol{\tau}}_1) = \vec{\boldsymbol{\tau}}_2$ and call $\vec{\boldsymbol{\delta}} = \phi(\vec{\boldsymbol{\pi}})$, note that $\alpha(\vec{\boldsymbol{\delta}}) = \alpha(\vec{\boldsymbol{\pi}}) = \alpha$.

We claim that $\vec{\boldsymbol{\delta}} \preceq \vec{\boldsymbol{\tau}}_2$. It follows that $\vec{\boldsymbol{\delta}} \preceq \vec{\boldsymbol{\gamma}}$ and $\alpha \leq' \sigma$, so the transitivity of $\leq'$ also follows. Take $A \subseteq [n]$ nonempty such that $\#A_{\vec{\boldsymbol{\delta}}} = 1$. Then $\#\phi^{-1}(A)_{\vec{\boldsymbol{\pi}}} = 1$ and from $\vec{\boldsymbol{\pi}} \leq \vec{\boldsymbol{\tau}}_1$ it follows that $\#\phi^{-1}(A)_{\vec{\boldsymbol{\tau}}_1} = 1$. From $\vec{\boldsymbol{\tau}}_2 = \phi(\vec{\boldsymbol{\tau}}_1)$ we have that $\#A_{\vec{\boldsymbol{\tau}}_2} = 1$. Since A is generic such that $\#A_{\vec{\boldsymbol{\delta}}} = 1$, we conclude that $\vec{\boldsymbol{\delta}} \preceq \vec{\boldsymbol{\tau}}_2$, as envisaged. $\square$

4.2 The kernel and image problem on hypergraphic polytopes

Recall that a fundamental hypergraphic polytope on $\mathbb{R}^I$ is a hypergraphic polytope $\sum_{J \subseteq 2^I \setminus \emptyset} a_J \mathfrak{s}_J$ such that $a_J \in \{0, 1\}$.

In the following proposition, we reduce the problem of describing the kernel of $\mathbf{\Upsilon_{HGP}}$ to the space spanned by the fundamental hypergraphic polytopes. We have:

Proposition 29 (Simple relations for $\mathbf{\Upsilon_{HGP}}$)**.** *If $\mathfrak{q}_1, \mathfrak{q}_2$ are two hypergraphic polytopes such that $\mathcal{F}(\mathfrak{q}_1) = \mathcal{F}(\mathfrak{q}_2)$, then*

$$\mathbf{\Upsilon_{HGP}}(\mathfrak{q}_1) = \mathbf{\Upsilon_{HGP}}(\mathfrak{q}_2) \,.$$

It remains to discuss the kernel of the map $\mathbf{\Upsilon_{HGP}}$ when restricted to the span of fundamental hypergraphic polytopes. For non-empty sets $A \subseteq [n]$, define $\operatorname{Orth} A = \{\vec{\boldsymbol{\pi}} \in \mathbf{C}_n | \#A_{\vec{\boldsymbol{\pi}}} = 1\}$. We will now exhibit some non-trivial relations involving hypergraphic polytopes.

Theorem 30 (Modular relations for $\mathbf{\Upsilon_{HGP}}$)**.** *Let $\{A_k|k \in K\}$ and $\{B_j|j \in J\}$ be two disjoint families of non-empty subsets of $[n]$. Take the following families of set compositions: $\mathcal{K} = \cup_{k \in K}(\operatorname{Orth} A_k)^c$, and $\mathcal{J} = \cup_{j \in J} \operatorname{Orth} B_j$. Consider the fundamental hypergraphic polytope $\mathfrak{q} = \mathcal{F}^{-1}\{A_k|k \in K\}$.*

Suppose that $\mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$. Then,

$$\sum\nolimits_{S \subseteq J} (-1)^{\#S}\, \boldsymbol{\Upsilon}_{\mathbf{HGP}} \left[\mathfrak{q} + \mathcal{F}^{-1}\{B_j | j \in S\}\right] = 0\,.$$

The sum $\sum_{S\subseteq J}(-1)^{\#S}\left[\mathfrak{q} + \mathcal{F}^{-1}\{B_j|j \in S\}\right]$ is called a *modular relation on hypergraphic polytopes*. An example can be observed in Fig. 4 for $n = 4$, where we take the families $\{A_k|k \in K\} = \{\{1, 4\}, \{1, 2, 4\}\}$, $\{B_k|j \in J\} = \{\{1, 2\}, \{2, 3\}, \{2, 3, 4\}\}$.

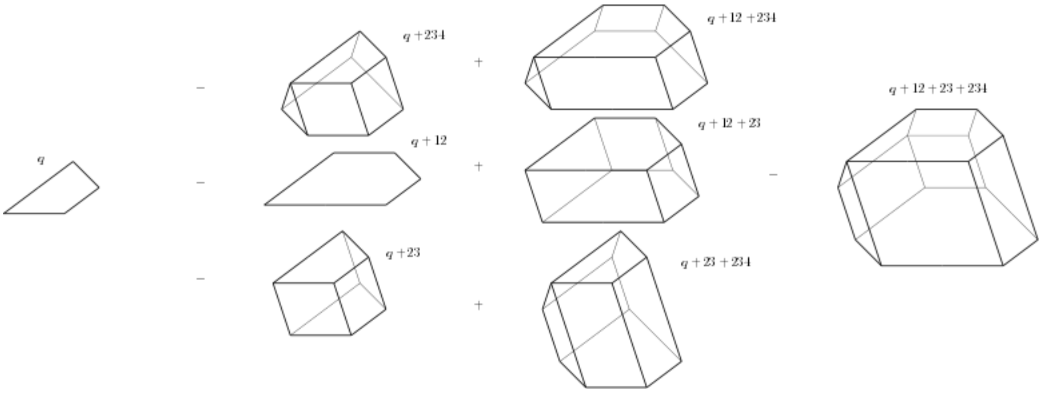

Figure 4: A modular relation on **HGP**, where $\mathfrak{q} = \mathfrak{s}_{\{\{1,4\}} + \mathfrak{s}_{\{1,2,4\}}$.

Proof of Theorem 30. Define $\eta_f(\mathfrak{q}) = \mathbb{1}[f \text{ is } \mathfrak{q} - \text{generic }]$. Recall the expansion of $\boldsymbol{\Upsilon}_{\mathbf{GP}}$ for general hypergraphic polytopes $\mathfrak{q}$ in (12), given by

$$\boldsymbol{\Psi}_{\mathbf{HGP}}(\mathfrak{q}) = \sum_f x_f \eta_f(\mathfrak{q}) = \sum_{f \text{ is } \mathfrak{q}\text{-generic}} x_f\,,$$

and write, for short, the modular relation for hypergraphic polytopes as

$$\mathrm{MRN} = \sum_{S \subseteq J} (-1)^{\#S} \left[\mathfrak{q} + \sum_{j \in S} \mathfrak{s}_{B_j}\right]\,,$$

where the first sum is considered to be the algebraic sum in the Hopf algebra of generalized permutahedra, and the inner sum is taken to be the Minkowski sum. Hence:

$$\begin{aligned} \boldsymbol{\Upsilon}_{\mathbf{HGP}}(\mathrm{MRN}) &= \sum_{S \subseteq J} (-1)^{\#S} \boldsymbol{\Upsilon}_{\mathbf{HGP}} \left[\mathfrak{q} + \sum_{j \in S} \mathfrak{s}_{B_j}\right] \\ &= \sum_{S \subseteq J} (-1)^{\#S} \sum_f x_f \eta_f \left(\mathfrak{q} + \sum_{j \in S} \mathfrak{s}_{B_j}\right) \\ &= \sum_f x_f \left[\sum_{S \subseteq J} (-1)^{\#S} \eta_f \left(\mathfrak{q} + \sum_{j \in S} \mathfrak{s}_{B_j}\right)\right]\,. \end{aligned} \tag{19}$$

We note from Lemmas 9 and 25 that for hypergraphic polytopes $\mathfrak{q}, \mathfrak{p}$, if a coloring f is not $\mathfrak{q}$-generic, then f is not $(\mathfrak{q} + \mathfrak{p})$-generic. Hence, if $\eta_f(\mathfrak{q}) = 0$ then it follows that $\eta_f\left(\mathfrak{q} + \sum_{j \in S} \mathfrak{s}_{B_j}\right) = 0$, so we can restrict the sum to $\mathfrak{q}$-generic colorings.

Further, define $B(f) = \{j \in J | f \text{ is } \mathfrak{s}_{B_j} - \text{ generic } \}$. Again according to Lemmas 9 and 25, we have that $\eta_f \left(\mathfrak{q} + \sum_{j \in S} \mathfrak{s}_{B_j} \right) = 1$ exactly when $S \subseteq B(f)$ and f is $\mathfrak{q}$-generic, so the equation (19) becomes

$$\begin{aligned} \boldsymbol{\Upsilon}_{\mathbf{HGP}}(MRN) &= \sum_{f \ \mathfrak{q}\text{- generic}} x_f \left[\sum_{S \subseteq J} (-1)^{\#S} \eta_f \left(\mathfrak{q} + \sum_{j \in S} \mathfrak{s}_{B_j} \right) \right] \\ &= \sum_{f \ \mathfrak{q}\text{- generic}} x_f \left[\sum_{S \subseteq B(f)} (-1)^{\#S} \right] = \sum_{\substack{f \ \mathfrak{q}\text{- generic} \\ B(f) = \emptyset}} x_f \, . \end{aligned} \tag{20}$$

It suffices to show that no coloring f is both $\mathfrak{q}$-generic and satisfies $B(f) = \emptyset$. Suppose otherwise, and take such f. If f is $\mathfrak{q}$-generic, we have $\vec{\boldsymbol{\pi}}(f) \notin \mathcal{K}$. Hence, if $B(f) = \emptyset$ we have that $\vec{\boldsymbol{\pi}}(f) \notin \operatorname{Orth} B_j$ for any j, hence $\vec{\boldsymbol{\pi}}(f) \notin \mathcal{J}$. Given that $\vec{\boldsymbol{\pi}}(f) \in \mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$, we have a contradiction.

Hence, no coloring f satisfies both that f is $\mathfrak{q}$-generic and $B(f) = \emptyset$. So Eq. (20) becomes $\boldsymbol{\Psi}_{\mathbf{HGP}}(\mathrm{MRN}) = 0$, which concludes the proof. $\square$

Remark 31. It can be noted that, if $l = G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ is a modular relation on graphs, then $Z(l)$ is the modular relation on hypergraphic polytopes corresponding to $\mathfrak{q} = Z(G)$ (i.e. $\{A_k | k \in K\} = E(G)$), $B_1 = e_1$ and $B_2 = e_2$. In this case, the condition $\mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$ follows from the fact that no proper coloring of G is monochromatic in both e_1 and e_2, which is imposed by $e_3 \in G$.

Recall that we set $\mathfrak{p}^{\vec{\boldsymbol{\pi}}} = \mathcal{F}^{-1}\{A \subseteq [n] | \#A_{\vec{\boldsymbol{\pi}}} = 1\}$, which only depends on the equivalence class $[\vec{\boldsymbol{\pi}}]$ under $\sim$, and we denote simply $\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]}$. These are called basic hypergraphic polytopes and are a special case of fundamental hypergraphic polytopes.

We now prove Theorem 3. We follow here roughly the same idea as in the graph case: We apply Lemma 5 to the family of hypergraphic polytopes $(\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]})_{[\vec{\boldsymbol{\pi}}] \in \mathbf{C}_n/\sim}$, whose image by $\boldsymbol{\Upsilon}_{\mathbf{GP}}$ is linearly independent and is rich enough to span the image.

Proof of Theorem 3. First recall that $\mathbf{HGP}_n$ is a linear space generated by the hypergraphic polytopes in $\mathbb{R}^n$. According to Proposition 29, to compute the kernel of $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$, it suffices to study the span of the fundamental hypergraphic polytopes. We choose a total order $\tilde{\geq}$ on fundamental hypergraphic polytopes $\mathfrak{q}$ so that $\#\mathcal{F}(\mathfrak{q})$ is non decreasing.

We will apply Lemma 5 with Theorem 30 to this subspace of $\mathbf{HGP}_n$.

Lemma 27 guarantees that $(\boldsymbol{\Upsilon}_{\mathbf{GP}}(\mathfrak{p}^{[\vec{\boldsymbol{\pi}}]}))_{[\vec{\boldsymbol{\pi}}] \in \mathbf{C}_n/\sim}$ is linearly independent. Therefore, it suffices to show that for any fundamental hypergraphic polytope $\mathfrak{q}$ that is not a basic hypergraphic polytope, we can write some modular relation b as $b = \mathfrak{q} + \sum_i \lambda_i \mathfrak{q}_i$, where $\#\mathcal{F}(\mathfrak{q}) < \#\mathcal{F}(\mathfrak{q}_i) \, \forall i$. Indeed, it would follow from Lemma 5 that the simple relations and the modular relations on hypergraphic polytopes span $\ker \boldsymbol{\Upsilon}_{\mathbf{HGP}}$.

To obtain the desired modular relation, apply Theorem 30 to $\{A \subseteq 2^{[n]} | A \in \mathcal{F}(\mathfrak{q})\}$ and $\{B \subseteq 2^{[n]} | B \notin \mathcal{F}(\mathfrak{q}), B \neq \emptyset\}$. Let us write $\mathcal{K} = \cup_{A \in \mathcal{F}(\mathfrak{q})} (\operatorname{Orth} A)^c$ and $\mathcal{J} = \cup_{B \in 2^{[n]} \setminus \mathcal{F}(\mathfrak{q}) \cup \emptyset} \operatorname{Orth} B$. We will first show that we have $\mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$.

Take, for sake of contradiction, some $\vec{\pi} \notin \mathcal{K} \cup \mathcal{J}$. Note that $\vec{\pi} \notin \mathcal{K}$ is equivalent to $\# A_{\vec{\pi}} = 1$ for every $A \in \mathcal{F}(\mathfrak{q})$. Note as well that $\vec{\pi} \notin \mathcal{J}$ is equivalent to $B_{\vec{\pi}} \neq pt$ for every $B \notin \mathcal{F}(\mathfrak{q})$. Therefore, if $\vec{\pi} \notin \mathcal{K} \cup \mathcal{J}$, then $\mathfrak{q} = \mathfrak{p}^{\vec{\pi}}$, contradicting the assumption that $\mathfrak{q}$ is not a basic hypergraphic polytope. We obtain that $\mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$. Finally, note that

$$\mathfrak{q} + \sum\nolimits_{\substack{T \subseteq \mathcal{F}(\mathfrak{q})^c \\ T \neq \emptyset}} (-1)^{\# T} \left[\mathfrak{q} + \mathcal{F}^{-1}(T) \right] ,$$

is a modular relation of the desired form, showing that the hypotheses of Lemma 5 are satisfied. The fact that $\mathbf{SC}$ is the image of $\mathbf{\Upsilon_{HGP}}$ has been already established in Lemma 27 $\square$

For the commutative case we will apply Lemma 6. Note that we already have a generator set of $\ker \mathbf{\Upsilon_{HGP}}$, so similarly to the proof of Theorem 2, we just need to establish some linear independence.

Recall that two hypergraphic polytopes $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are isomorphic if there is a permutation matrix P such that $x \in \mathfrak{q}_2 \Leftrightarrow Px \in \mathfrak{q}_1$. Since we are in the commutative case now, if $\vec{\pi}_1$ and $\vec{\pi}_2$ have the same composition type, then $\mathfrak{p}^{\vec{\pi}_1}$ and $\mathfrak{p}^{\vec{\pi}_2}$ are isomorphic, and we have $\Psi_{\mathbf{HGP}}(\mathfrak{p}^{\vec{\pi}_1}) = \Psi_{\mathbf{HGP}}(\mathfrak{p}^{\vec{\pi}_2})$. Set $R_{\alpha(\vec{\pi})} := \Psi_{\mathbf{HGP}}(\mathfrak{p}^{\vec{\pi}})$ without ambiguity.

Proof of Theorem 4. We will apply Lemma 6 to the map $\Psi_{\mathbf{HGP}} = \text{comu} \circ \mathbf{\Upsilon_{HGP}}$ on the equivalence relation corresponding to the isomorphism of hypergraphic polytopes.

From the proof of Theorem 3, to apply Lemma 6 it is enough to establish that the family $(R_\alpha)_{\alpha \in \mathcal{C}_n}$ is linearly independent. It would follow that $\ker \Psi_{\mathbf{HGP}}$ is generated by the modular relations and the isomorphism relations, and $(R_\alpha)_{\alpha \in \mathcal{C}_n}$ is a basis of $\operatorname{im} \Psi_{\mathbf{G}}$, concluding the proof.

To show the linear independence of $(R_\alpha)_{\alpha \in \mathcal{C}_n}$, we write R_α on the monomial basis of $QSym$, and use the order $\leq'$ studied in Lemma 28.

As a consequence of (17), if we write $A_{\vec{\pi},\beta} = \#\{\vec{\tau} \in \mathbf{C}_n | \vec{\pi} \preceq \vec{\tau},\, \alpha(\vec{\tau}) = \beta\}$, we have:

$$R_{\alpha(\vec{\pi})} = \Psi_{\mathbf{GP}}(\mathfrak{p}^{\vec{\pi}}) = \sum_{\vec{\pi} \preceq \vec{\tau}} M_{\alpha(\vec{\tau})} = A_{\vec{\pi},\alpha(\vec{\pi})} M_{\alpha(\vec{\pi})} + \sum_{\alpha(\vec{\pi}) <' \beta} A_{\vec{\pi},\beta} M_\beta \, , \tag{21}$$

It is clear that $A_{\vec{\pi},\alpha(\vec{\pi})} > 0$, so independence follows, which completes the proof. $\square$

Remark 32. We have obtained in the proof of Theorem 4 that $(R_\alpha)_{\alpha \models n}$ is a basis for $QSym_n$. The proof gives us a recursive way to compute the coefficients $\zeta_\alpha(\mathfrak{q})$ on the span $\Psi_{\mathbf{HGP}}(\mathfrak{q}) = \sum_{\alpha \models n} \zeta_\alpha(\mathfrak{q}) R_\alpha$. It is then natural to ask if combinatorial properties can be obtained for these coefficients, which are isomorphic invariants.

Similarly, in the non-commutative case, we can write the chromatic quasisymmetric function of a hypergraphic polytope as

$$\mathbf{\Upsilon_{HGP}}(\mathfrak{q}) = \sum_{[\vec{\pi}] \in \mathbf{C}_n / \sim} \zeta_{[\vec{\pi}]}(\mathfrak{q}) \mathbf{\Upsilon_{HGP}}(\mathfrak{p}^{[\vec{\pi}]}) \, ,$$

and ask for the combinatorial meaning of the coefficients $(\zeta_{[\vec{\pi}]}(\mathfrak{q}))_{[\vec{\pi}] \in \mathbf{C}_n / \sim}$. These questions are not answered in this paper.

4.3 The dimension of SC space

Let $sc_n := \dim \mathbf{SC}_n$. Recall that, from Definition 24, the elements of the Hopf algebra $\mathbf{SC}$ are of the form $\sum_{\vec{\pi} \models [n]} \mathbf{M}_{\vec{\pi}} a_{\vec{\pi}}$, where $a_{\vec{\pi}_1} = a_{\vec{\pi}_2}$ whenever $\vec{\pi}_1 \sim \vec{\pi}_2$ in the SC equivalence relation. Hence sc_n counts the equivalence classes of $\sim$.

The goal of this section is to compute the asymptotics of sc_n, by using the combinatorial interpretation provided in Proposition 23.

Proposition 33. *Let $F(x) = \sum_{n \geq 0} sc_n \frac{x^n}{n!}$ be the exponential power series of the dimension of $\mathbf{SC}_n$. Then*

$$F(x) = \frac{e^x}{1 + (1+x)e^x - e^{2x}} .$$

Proposition 34. *The dimension of $\mathbf{SC}_n$, sc_n, has an asymptotic growth of*

$$sc_n = n! \gamma^{-n} \left(\tau + o(\delta^{-n}) \right) ,$$

where $\delta < 1$ is some real number, $\gamma \approxeq 0.814097 \approxeq 1.1745 \log(2)$ is the unique positive root of the equation

$$e^{2x} = 1 + (1+x)e^x ,$$

and $\tau = \mathrm{Res}_\gamma(F) \approxeq 0.588175$.

In particular, $\dim \mathbf{SC}_n$ is exponentially smaller than $\dim \mathbf{WQSym}_n = \# \mathbf{C}_n$, which is asymptotically

$$n! \log(2)^{-n} \left(\frac{1}{2\log(2)} + o(1) \right) ,$$

according to [Bar80]. Before we prove these statements, we introduce some relevant notation.

A bared set composition of $[n]$ is a set composition of $[n]$ where we put some bars over some of the blocks. For instance, $13|\overline{45}|2$ and $12|4|\overline{35}$ are bared set compositions of $\{1, 2, 3, 4, 5\}$.

A bared set composition is *integral*, if

- No two bared blocks occur consecutively;
- Every block of size one is bared;

An integral bared set composition is also called *IBSC*. What follows are the IBSCs of small size:

n	IBSC	Equivalence classes of $\sim$
0	$\emptyset$	$\{()\}$
1	$\overline{1}$	$\{1\}$
2	$\overline{12}, 12$	$\{1\|2, 2\|1\}, \{12\}$
3	$\overline{123}, \overline{1}\|23, \overline{2}\|13, \overline{3}\|12,$ $12\|\overline{3}, 13\|\overline{2}, 23\|\overline{1}, 123$	$[1\|2\|3]_\sim, \{1\|23\}, \{2\|13\}, \{3\|12\},$ $\{12\|3\}, \{13\|2\}, \{23\|1\}, \{123\}$

Table 1: Small IBSC and equivalence classes of $\sim$

n	0	1	2	3	4	5	6	7	8	9	10	11
sc_n	1	1	2	8	40	242	1784	15374	151008	1669010	20503768	277049126
$\boldsymbol{\pi}_n$	1	1	3	13	75	541	4683	47293	545835	7087261	102247563	1622632573

Table 2: First elements of the sequence sc_n and $\boldsymbol{\pi}_n = \dim \boldsymbol{WQSym}_n$.

According to Proposition 23, there is a natural map from equivalence classes of $\sim$ and integral bared set compositions, where we squeeze all consecutive singletons into one bared block. This map is a bijection, as is inverted by splitting all bared blocks into singletons. So, for instance, $\overline{13}|24 \leftrightarrow \{1|3|24, 3|1|24\}$ and $13|24 \leftrightarrow \{13|24\}$, see Table 1.

Proof of Proposition 33. We will use the framework developed in [FS09] of labelled combinatorial classes. In the following, calligraphic style letters denote combinatorial classes, and the associate upper case letters denote their exponential generating functions. Let $\mathcal{B}$ and $\mathcal{U}$ be the collections $\{\overline{1}, \overline{12}, \cdots\}$ and $\{12, 123, 1234, \cdots\}$, with exponential generating functions $B(x) = e^x - 1$ and $U(x) = e^x - 1 - x$, respectively. Additionally, let $\mathcal{O} = \{\emptyset\}$ with $O(x) = 1$.

Let $\mathcal{F}$ be the class of IBSCs. Let us denote by $\mathcal{F}^o$ the class of IBSCs that start with an unbarred set, and let $\overline{\mathcal{F}}$ be the class of IBSCs that start with a barred set. Note the decomposition $\mathcal{F} = \overline{\mathcal{F}} \sqcup \mathcal{F}^o \sqcup \mathcal{O}$, which implies that $F = \overline{F} + F^o + O$.

We can recursively describe $\overline{\mathcal{F}}$ and $\mathcal{F}^o$ as $\overline{\mathcal{F}} = \mathcal{B} \times (\mathcal{F}^o \sqcup \mathcal{O})$ and $\mathcal{F}^o = \mathcal{U} \times \mathcal{F}$.

According to the dictionary rules in [FS09], this implies that

$$\begin{aligned} \overline{F}(x) &= (e^x - 1)(F^o(x) + 1)\,, \\ F^o(x) &= (e^x - 1 - x)(\overline{F}(x) + F^o(x) + 1)\,, \end{aligned}$$

The unique solution of the system has $F^o(x) = \frac{e^{2x} - (1+x)e^x}{1 - 1e^{2x} + (x+1)e^x}$ which implies that

$$F(x) = \frac{e^x}{1 + (1+x)e^x - e^{2x}}$$

as desired. ☐

With this we can easily compute the dimension of $\mathbf{SC}_n$ for small n, and a comparison with $\dim \boldsymbol{WQSym}_n$, as done in Table 2.

Proof of Proposition 34. We will find the dominant singularity of $F(x)$ and show that it is the unique positive real root of a real function, so that we can find this singularity through numerical methods.

Let $l(x) := e^{2x} - (1+x)e^x - 1$, then $F(x) = -\frac{e^x}{l(x)}$ is the quotient of two entire functions with non'vanishing numerator, so $F(x)$ is meromorphic and its the poles are the zeroes of $l(x)$. Note that $F(x)$ is an exponential generating funtion of a combinatorial class around zero, so it has positive coefficients. By Pringsheim's Theorem as in [FS09], one of the dominant singularities of $F(x)$ is a positive real number, call it γ.

We see now that any other singularity $z \neq \gamma$ of F has to satisfy $|z| > |\gamma|$, thus showing that γ is the unique dominant singularity and allowing us to compute a simple asymptotic formula. Suppose, that z is a singularity of F distinct from γ, such that $|z| = |\gamma|$. So, we have that $l(z) = 0$ and that $z \notin \mathbb{R}^+$. The equation $l(z) = 0$ can easily be rewritten as

$$1 = l(z) + 1 = \sum_{n \geq 1} z^n \frac{2^n - 1 - n}{n!} .$$

Note that $2^n \geq n + 1$ for $n \geq 1$. Now we apply the strict triangular inequality on the right hand side to obtain

$$1 < \sum_{n \geq 1} |z|^n \frac{2^n - n - 1}{n!} = \sum_{n \geq 1} \gamma^n \frac{2^n - n - 1}{n!} = l(\gamma) + 1 = 1 ,$$

where we note that the inequality on the left is strict becuase for $z \notin \mathbb{R}_+$ there are some summands that are not collinear. We obtain a contradiction, as desired.

We additionally see that γ is the unique positive real root, so we can easily approximate it by some numerical method, for instance the bisection method: the function l in the positive real line satisfies $\lim_{x \to +\infty} l(x) = +\infty$ and $l(0) = -1$, so it has at least one zero. Note that such zero is unique, as $l'(x) > 0$ for x positive. Also, since $l'(\gamma) > 0$, the zero γ is simple.

Since the function $F(x)$ is meromorphic on $\mathbb{C}$, and γ is the dominant singularity, we conclude that

$$\frac{sc_n}{n!} = \gamma^{-n} \left(\mathrm{Res}_\gamma(F) + o(\delta^{-n}) \right) ,$$

for any δ such that $1 > \delta > |\gamma/\gamma_2|$, where γ_2 is a second smallest singularity of F, if it exists, and arbitrarily large otherwise.

We can easily approximate $\gamma \approxeq 0.814097$. We can also estimate the residue of $F(x)$ at γ as $\tau = \mathrm{Res}_\gamma(F) = \frac{e^\gamma}{l'(\gamma)} \approxeq 0.588175$. This proves the desired asymptotic formula. $\square$

5 Hopf species and the non-commutative universal property

In [ABS06], a character in a Hopf algebra is defined as a multiplicative linear map that preserves unit, and a combinatorial Hopf algebra (or CHA, for short) is a Hopf algebra endowed with a character. For instance, a character η_0 in $QSym$ is $\eta_0(M_\alpha) = \mathbb{1}[l(\alpha) = 1]$. In fact, the CHA of quasisymmetric functions $(QSym, \eta_0)$ is a terminal object in the category of CHA's, i.e. for each CHA (h, η) there is a unique combinatorial Hopf algebra morphism $\Psi_h : h \to QSym$.

Our goal here is to draw a parallel for Hopf monoids in vector species. We see that the Hopf species $\overline{\mathbf{WQSym}}$ plays the role of $QSym$. Specifically, we build a unique Hopf monoid morphism from any combinatorial Hopf monoid $\overline{h}$ to $\overline{\mathbf{WQSym}}$, in line with what was done in [ABS06] and [Whi16].

In the last section we will investigate the consequence of this universal property on the Hopf structure of hypergraphic polytopes and posets. We will use Theorem 3 and Proposition 34 to obtain that no combinatorial Hopf monoid map from $\overline{\mathbf{HGP}}$ to $\overline{\mathbf{Pos}}$ exists.

Remark 35. The category of combinatorial Hopf monoids was introduced in two non equivalent ways, by [AA17] in vector species, and by [Whi16] on pointed set species. Here we consider the notion of [AA17]. We also note that a stronger notion of character (or rather of stable objects) is employed for the comonoidal combinatorial Hopf monoid.

Indeed, in [Whi16], White shows that a comonoidal combinatorial Hopf monoid in coloring problems is a terminal object on the category of CCHM. Nevertheless, it is already advanced there, that if we consider the weaker notion of combinatorial Hopf monoid, the terminal object in such category, as here proposed, is indexed by set compositions. No counterpart of $\overline{\mathbf{WQSym}}$ on pointed set species was found.

5.1 Hopf monoids in vector species

We recall the basic notions on Hopf monoids in vector species here, introduced in [AM10, Chapter 8]. We write $Set^{\times}$ for the category of finite sets with bijections as only morphisms, and write $Vec_{\mathbb{K}}$ for the category of vector spaces over $\mathbb{K}$ with linear maps as morphisms. A *vector species* is a functor $\overline{a} : Set^{\times} \to Vec_{\mathbb{K}}$, and this forms a category $Sp_{\mathbb{K}}$, where functors are natural transformations.

We can multiply species $\overline{a}, \overline{b}$ to obtain a new vector species via the Cauchy product as follows:

$$(\overline{a} \cdot \overline{b})[I] = \bigoplus_{I = S \sqcup T} \overline{a}[S] \otimes \overline{b}[T]\,.$$

Two basic vector species are of interest. The first one $\overline{I}$ acts as the identity for the Cauchy product, and is defined as $\overline{I}[\emptyset] = \mathbb{K}$ and as $\overline{I}[A] = 0$ if $A \neq \emptyset$. Morphisms are mapped to the identity by the functor $\overline{I}$. We also define the exponential vector species $\overline{E}$ as $\overline{E}[A] = \mathbb{K}$ for any set A, and maps morphisms to the identity.

A vector species $\overline{a}$ is called a bimonoid if there are natural transformations $\mu : \overline{a} \cdot \overline{a} \Rightarrow \overline{a}$, $\iota : \overline{I} \Rightarrow \overline{a}$, $\Delta : \overline{a} \Rightarrow \overline{a} \cdot \overline{a}$ and $\epsilon : \overline{a} \Rightarrow \overline{I}$ that satisfy some properties which we will recover here only informally. We address the reader to [AM10, Section 8.2 - 8.3] for a detailed introduction of bimonoids in species.

- The natural transformation μ is associative.
- The natural transformation ι acts as unit on both sides.
- The natural transformation Δ is coassociative.
- The natural transformation ϵ acts as a counit on both sides.
- The natural transformations satisfy some coherence relations which are analogues of Hopf algebra axioms. In particular it satisfies Diagram (22) below, which enforces that the multiplicative and comultiplicative structures agree.

$$\begin{tikzcd} \overline{a}[I] \otimes \overline{a}[J] \arrow[r, "\mu_{I,J}"] \arrow[d, "\Delta_{R,T} \otimes \Delta_{U,V}"] & \overline{a}[S] \arrow[d, "\Delta_{M,N}"] \\ \overline{a}[R] \otimes \overline{a}[T] \otimes \overline{a}[U] \otimes \overline{a}[V] \arrow[r, "(\mu_{R,U} \otimes \mu_{T,V}) \circ \mathrm{twist}"] & \overline{a}[M] \otimes \overline{a}[N] \end{tikzcd} \tag{22}$$

We consider the canonical isomorphism $\beta : V \otimes W \to W \otimes V$, and also refer to any composition of tensors of maps β and identities as *twists*. Whenever needed, we will consider a suitable twist function without defining it explicitly, by letting the source and the target of the maps clarify its precise definition. For instance, above in Diagram (22).

A bimonoid is determined by the maps $\mu_{A,B} : \overline{a}[A] \otimes \overline{a}[B] \to \overline{a}[I]$ and $\Delta_{A,B} : \overline{a}[I] \to \overline{a}[A] \otimes \overline{a}[B]$, where $A \sqcup B = I$, and the maps $\epsilon_{\emptyset}, \iota_{\emptyset}$. All maps of the form ϵ_I, ι_I are the zero map for $I \neq \emptyset$. We say that a bimonoid $\overline{a}$ is *connected* if the dimension of $\overline{a}[\emptyset]$ is one.

A bimonoid is called a *Hopf monoid* if there is a natural transformation, called the *antipode*, $s : \overline{h} \Rightarrow \overline{h}$ that satisfies

$$\mu \circ (id_{\overline{h}} \cdot s) \circ \Delta = \iota \circ \epsilon = \mu \circ (s \cdot id_{\overline{h}}) \circ \Delta\,.$$

We will denote vector species as a lowercase letter with a bar, and when we talk about Hopf monoids, we reserve the use of the lower case '$\overline{h}$'. We will use $\mu_{A,B}$ and $\cdot_{A,B}$ for the monoidal product.

Proposition 36 (Proposition 8.10 in [AM10])**.** *If $\overline{h}$ is a connected bimonoid, then there is an antipode on $\overline{h}$ that makes it a Hopf monoid.*

Example 37 (Hopf monoids).

- The exponential vector species can be endowed with a trivial product and coproduct. This is a connected bimonoid, hence it is a Hopf monoid.
- The vector spaces $\overline{\mathbf{G}}[I]$ ($\overline{\mathbf{Pos}}[I]$, $\overline{\mathbf{GP}}[I]$, $\overline{\mathbf{HGP}}[I]$, resp.) with basis given by all graphs on the vertex set I (posets on the vertex set I, generalized permutahedra in R^I, hypergraphic polytopes on R^I, resp.) defines a Hopf monoid with the operations introduced in Section 2.

5.2 Combinatorial Hopf monoids and Fock functors

The notion of characters in Hopf monoids was already brought to light in [AA17], where it is used to settle, for instance, a conjecture of Humpert and Martin [HM12] on graphs.

Definition 38. Let $\overline{h}$ be a Hopf monoid. A *Hopf monoid character* $\eta : \overline{h} \Rightarrow \overline{E}$, or simply a *character*, is a monoid morphism such that $\eta_\emptyset = \epsilon_\emptyset$ and the following diagram commutes:

$$\begin{array}{ccc} \overline{h}[I] \otimes \overline{h}[J] & \xrightarrow{\mu_{I,J}} & \overline{h}[A] \\ \Big\downarrow{\scriptstyle \eta_I \otimes \eta_J} & & \Big\downarrow{\scriptstyle \eta_A} \\ \mathbb{K} \otimes \mathbb{K} & \xrightarrow{\cong} & \mathbb{K} \end{array} \tag{23}$$

A *combinatorial Hopf monoid* is a pair $(\overline{h}, \eta)$ where $\overline{h}$ is a Hopf monoid, and η a character on $\overline{h}$.

The condition that η and ϵ coincide in the empty set is commonly observed in the framework of Hopf monoids in combinatorics. In particular, this condition is always verified for connected Hopf monoids.

Example 39 (Combinatorial Hopf monoids). From the examples on Hopf monoids above, we define character by describing its action on the basis elements.

- The Hopf monoid on graphs $\overline{\mathbf{G}}$ has a character $\eta(G) = \mathbb{1}[G \text{ has no edges}]$.
- The Hopf monoid on posets $\overline{\mathbf{Pos}}$ has a character $\eta(P) = \mathbb{1}[P \text{ is antichain}]$.
- The Hopf monoid on generalized permutahedra $\overline{\mathbf{GP}}$ has a character given by $\eta(\mathfrak{q}) = \mathbb{1}[\mathfrak{q} \text{ is a point }]$.

A combinatorial Hopf algebra morphism $\alpha : (\overline{h}_1, \eta_1) \Rightarrow (\overline{h}_2, \eta_2)$ is a natural transformation $\alpha : \overline{h}_1 \Rightarrow \overline{h}_2$ that preserves the combinatorial Hopf monoid structure. In particular, the following diagram commutes:

$$\begin{tikzcd} \overline{h}_1 \arrow[rr, bend left, "\alpha"] \arrow[dr, "\eta_1"'] & & \overline{h}_2 \arrow[dl, "\eta_2"] \\ & \overline{E} & \end{tikzcd} \tag{24}$$

We introduce the Fock functors, that transform Hopf monoids in vector species into Hopf algebras, and more generally vector species into graded vector spaces. The topic is carefully developed in [AM10, Sections 3.1 and 15.1].

Definition 40 (Fock functors). Call $gVec_{\mathbb{K}}$ the category of graded vector spaces over $\mathbb{K}$. We will focus on the following Fock functors $\mathcal{K}, \overline{\mathcal{K}} : Sp \to gVec_{\mathbb{K}}$, called full Fock functor and bosonic Fock functor, respectively, defined as:

$$\mathcal{K}(q) := \bigoplus_{n\geq 0} q[\{1, \dots, n\}] \text{ and } \overline{\mathcal{K}}(q) := \bigoplus_{n\geq 0} q[\{1, \dots, n\}]_{S_n} ,$$

where V_{S_n} stands for the vector space of coinvariants on V over the action of S_n, i.e. the quotient of V under all relations of the form $x - \sigma(x)$, for $\sigma \in S_n$.

If $(\overline{h}, \eta)$ is a combinatorial Hopf monoid in species with structure morphisms $\mu, \iota, \Delta, \epsilon$, then there is a Hopf algebra structure on $\mathcal{K}(\overline{h})$ and $\overline{\mathcal{K}}(\overline{h})$.

Example 41 (Fock functors of some Hopf monoids).

- The Hopf algebra $\mathcal{K}(\overline{I})$ is the linear Hopf algebra $\mathbb{K}$.
- The Hopf algebras $\mathcal{K}(\overline{\mathbf{G}})$, $\mathcal{K}(\overline{\mathbf{Pos}})$ and $\mathcal{K}(\overline{\mathbf{GP}})$ are the Hopf algebras of graphs $\mathbf{G}$, of posets $\mathbf{Pos}$ and of generalized permutahedra $\mathbf{GP}$ introduced earlier.
- The Hopf algebras $\mathcal{K}(\overline{\mathbf{HGP}})$ is the Hopf algebra $\mathbf{HGP}$, which is a Hopf subalgebra of $\mathbf{GP}$.

5.3 The word quasisymmetric function combinatorial Hopf monoid

Recall that a coloring of a set I is a map $f : I \to \mathbb{N}$, and we let $\mathfrak{C}_I$ be the set of colorings of I. Recall as well that a set composition $\vec{\pi} = S_1 | \cdots | S_l$ can be identified with a total preorder $R_{\vec{\pi}}$, where we say $a \, R_{\vec{\pi}} \, b$ if $a \in S_i$ and $b \in S_j$ satisfy $i \leq j$. For a set composition $\vec{\pi}$ of A and a non-empty subset $I \subseteq A$, we define $\vec{\pi}|_I$ as the set composition of I obtained by restricting the preorder $R_{\vec{\pi}}$ to I.

If I, J are disjoint sets, and $f \in \mathfrak{C}_I$ and $g \in \mathfrak{C}_J$, then we set $f * g \in \mathfrak{C}_{I \sqcup J}$ as the unique coloring in $I \sqcup J$ that satisfies both $f * g|_I = f$ and $f * g|_J = g$.

For a set composition $\vec{\pi} \in \mathbf{C}_I$, let

$$\mathbb{M}_{\vec{\pi}} = \sum_{\substack{f \in \mathfrak{C}_I \\ \vec{\pi}(f) = \vec{\pi}}} [f]$$

be a formal infinite sum of colorings, and define $\overline{\mathbf{WQSym}}[I]$ as the $\mathbb{K}$-span of $\{\mathbb{M}_{\vec{\pi}}\}_{\vec{\pi} \in \mathbf{C}_I}$. This gives us a $\mathbb{K}$-linear space with basis enumerated by $\mathbf{C}_I$.

We define the monoidal product operation with

$$\mathbb{M}_{\vec{\pi}} \cdot_{A,B} \mathbb{M}_{\vec{\tau}} = \sum_{\substack{f \in \mathfrak{C}_A \\ \vec{\pi}(f)=\vec{\pi}}} \sum_{\substack{g \in \mathfrak{C}_B \\ \vec{\pi}(g)=\vec{\tau}}} [f * g] = \sum_{\substack{\vec{\lambda}|_A=\vec{\pi} \\ \vec{\lambda}|_B=\vec{\tau}}} \mathbb{M}_{\vec{\lambda}} . \tag{25}$$

We write $I <_{\vec{\pi}} J$ whenever there is no $i \in I$ and $j \in J$ such that $j \, R_{\vec{\pi}} \, i$. The coproduct $\Delta_{I,J} M_{\vec{\pi}}$ is defined as

$$\mathbb{M}_{\vec{\pi}|_I} \otimes \mathbb{M}_{\vec{\pi}|_J} , \tag{26}$$

whenever $I <_{\vec{\pi}} J$, and is zero otherwise.

The unit $\iota_\emptyset(1) = \mathbb{M}_{\vec{\emptyset}}$ is the basis element indexed by the empty composition, and the counit acts on the basis as $\epsilon(\mathbb{M}_{\vec{\pi}}) = \mathbb{1}[\vec{\pi} = \vec{\emptyset}]$. This is the dual Hopf monoid of faces $\Sigma^* = \Sigma_1^*$ in [AM10].

Proposition 42 ([AM10, Definition 12.19])**.** *With these operations, the vector species* $\overline{\mathbf{WQSym}}$ *becomes a Hopf monoid.*

Proposition 43 ([AM10, Section 17.3.1])**.** *We have the folowing Hopf algebra morphisms:* $\mathcal{K}(\overline{\mathbf{WQSym}}) \cong \mathbf{WQSym}$, *and* $\overline{\mathcal{K}}(\overline{\mathbf{WQSym}}) \cong QSym$.

We identify $\mathcal{K}(\overline{\mathbf{WQSym}})$ and $\mathbf{WQSym}$ by identifying a coloring $f : [n] \to \mathbb{N}$ with the noncommutative monomial $\prod_{i=1}^n a_{f(i)} =: a_f$, and extend this to identify $\mathbb{M}_{\vec{\pi}}$ with $\mathbf{M}_{\vec{\pi}}$.

Proposition 44 (Combinatorial Hopf monoid on $\overline{\mathbf{WQSym}}$)**.** *Take the character* $\eta : \overline{\mathbf{WQSym}} \Rightarrow \overline{E}$ *defined in the basis elements as*

$$\eta_0[I](\mathbb{M}_{\vec{\pi}}) = \mathbb{1}[l(\vec{\pi}) \leq 1] . \tag{27}$$

This makes $(\overline{\mathbf{WQSym}}, \eta_0)$ *into a combinatorial Hopf monoid.*

Proof. That η_0 is a natural transformation is trivial, and also $\eta_{0,\emptyset}(\mathbb{M}_{\vec{emptyset}}) = 1$, so it preserves unit and satisfies $\eta_{0,\emptyset} = \epsilon_\emptyset$.

To show that η_0 is multiplicative, we just need to check that the diagram (23) commutes for the basis elements, i.e. if $A = I \sqcup J$, then

$$\eta_{0,I}(\mathbb{M}_{\vec{\pi}})\eta_{0,J}(\mathbb{M}_{\vec{\tau}}) = \eta_{0,A}(\mathbb{M}_{\vec{\pi}}\mathbb{M}_{\vec{\tau}}) = \sum_{\substack{\vec{\gamma} \in \mathbf{C}_A \\ \vec{\gamma}|_I=\vec{\pi} \\ \vec{\gamma}|_J=\vec{\tau}}} \eta_{0,A}(\mathbb{M}_{\vec{\gamma}}) . \tag{28}$$

Now note that if $\vec{\gamma}$ is a set composition of A such that $\vec{\gamma}|_I = \vec{\pi}$, then trivially we have that $l(\vec{\pi}) \leq l(\vec{\gamma})$, so from (27), $\eta_{0,I}(\mathbb{M}_{\vec{\pi}}) = 0 \Rightarrow \eta_{0,A}(\mathbb{M}_{\vec{\gamma}}) = 0$. Similarly, if $\vec{\gamma}|_J = \vec{\tau}$, we have $\eta_{0,J}(\mathbb{M}_{\vec{\tau}}) = 0 \Rightarrow \eta_{0,A}(\mathbb{M}_{\vec{\gamma}}) = 0$.

So it is enough to consider the case where $\eta_{0,I}(\mathbb{M}_{\vec{\pi}}) = \eta_{0,J}(\mathbb{M}_{\vec{\tau}}) = 1$, i.e. $l(\vec{\pi}), l(\vec{\tau}) \leq 1$. Now, if γ is the set partition with one block, we naturally have $\gamma|_I = \vec{\pi}$ and $\gamma|_J = \vec{\tau}$, so there is a unique $\vec{\gamma}$ on the right hand side of (28) that satisfies $l(\vec{\gamma}) \leq 1$, and this concludes the proof. $\square$

5.4 Universality of word quasisymmetric function combinatorial Hopf monoid

We now state and prove the universal property of $\overline{\mathbf{WQSym}}$.

Theorem 45 (Terminal object in combinatorial Hopf monoids)**.** *Given a Hopf monoid $\overline{h}$ with a character $\eta : \overline{h} \Rightarrow \overline{E}$, there is a unique combinatorial Hopf monoid morphism $\boldsymbol{\Upsilon}_{\overline{h}} : \overline{h} \Rightarrow \overline{\mathbf{WQSym}}$.*

For connected Hopf monoids, this proposition is a corollary of [AM10, Theorem 11.22]. We extend slightly this result to combinatorial Hopf monoids here.

A notion of multi-character will be useful in the proof of the universality theorem.

Definition 46 (Multi-character and other notations). For a set composition on a non-empty set I, $\vec{\boldsymbol{\pi}} = S_1|\cdots|S_k$, denote for short

$$\overline{h}[\vec{\boldsymbol{\pi}}] = \bigotimes_{i=1}^{k} \overline{h}[S_i]\,,$$

and similarly define for a natural transformation $\zeta : \overline{h} \Rightarrow \overline{b}$ the linear transformation $\zeta[\vec{\boldsymbol{\pi}}] : \overline{h}[\vec{\boldsymbol{\pi}}] \to \overline{b}[\vec{\boldsymbol{\pi}}]$ as $\zeta[\vec{\boldsymbol{\pi}}] = \bigotimes_{i=1}^{k} \zeta[S_i]$. If ζ is a character, we further identify $\zeta[\vec{\boldsymbol{\pi}}] : \overline{h}[\vec{\boldsymbol{\pi}}] \to \mathbb{K}^{\otimes k} \cong \mathbb{K}$.

For a set composition $\vec{\boldsymbol{\pi}}$ on I of length k, let us define $\Delta_{\vec{\boldsymbol{\pi}}}$ as a map

$$\Delta_{\vec{\boldsymbol{\pi}}} : \overline{h}[I] \to \overline{h}[\vec{\boldsymbol{\pi}}]\,,$$

inductively as follows:

- If the length of $\vec{\boldsymbol{\pi}}$ is 1, then $\Delta_{\vec{\boldsymbol{\pi}}} = id_{\overline{h}[I]}$.
- If $\vec{\boldsymbol{\pi}} = S_1|\cdots|S_k$ for $k > 1$, let $\vec{\boldsymbol{\tau}} = S_1|\cdots|S_{k-1}$ and define

$$\Delta_{\vec{\boldsymbol{\pi}}} = (\Delta_{\vec{\boldsymbol{\tau}}} \otimes id_{S_k}) \circ \Delta_{I\setminus S_k, S_k}\,. \tag{29}$$

Note that this definition of $\Delta_{\vec{\boldsymbol{\pi}}}$ is independent of the chosen order in the inductive definition in (29), i.e. for any non empty set I, a decomposition $I = A \sqcup B$ with $A \neq \emptyset \neq B$, and set composition $\vec{\boldsymbol{\pi}} \in \mathbf{C}_I$ such that $A <_{\vec{\boldsymbol{\pi}}} B$, we have

$$\Delta_{\vec{\boldsymbol{\pi}}} = (\Delta_{\vec{\boldsymbol{\pi}}|_A} \otimes \Delta_{\vec{\boldsymbol{\pi}}|_B}) \circ \Delta_{A,B}\,, \tag{30}$$

by coassociativity of Δ. We can define $f_{\vec{\boldsymbol{\pi}},\eta} : \overline{h}[I] \to \mathbb{K}$ as the composition

$$\overline{h}[I] \xrightarrow{\Delta_{\vec{\boldsymbol{\pi}}}} \overline{h}[\vec{\boldsymbol{\pi}}] \xrightarrow{\eta[\vec{\boldsymbol{\pi}}]} \mathbb{K}^{\otimes k} \cong \mathbb{K}\,,$$

Finally, if $A = I \sqcup J$ and $\vec{\boldsymbol{\pi}} \in \mathbf{C}_I, \vec{\boldsymbol{\tau}} \in \mathbf{C}_J$, then we write both $\vec{\boldsymbol{\pi}}|\vec{\boldsymbol{\tau}}$ and $(\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\tau}})$ for the unique set composition $\vec{\boldsymbol{\gamma}} \in \mathbf{C}_A$ such that $\vec{\boldsymbol{\gamma}}|_I = \vec{\boldsymbol{\pi}}$, $\vec{\boldsymbol{\gamma}}|_J = \vec{\boldsymbol{\tau}}$, and $I <_{\vec{\boldsymbol{\gamma}}} J$.

Example 47. In the graph Hopf monoid $\overline{\mathbf{G}}$, recall that we take the character defined in the basis elements as $\eta(G) = \mathbb{1}[G \text{ has no edges}]$.

Take the labelled cycle C_5 on $\{1,2,3,4,5\}$ given in Fig. 5, denote K_J the complete graph on the labels J and 0_J the empty graph on the labels J. Consider the set compositions $\vec{\boldsymbol{\pi}}_1 = 13|2|45$ and $\vec{\boldsymbol{\pi}}_2 = 24|13|5$. Note that:

$$\Delta_{\vec{\boldsymbol{\pi}}_1}(C_5) = 0_{\{1,3\}} \otimes 0_{\{2\}} \otimes K_{\{4,5\}}\,,$$

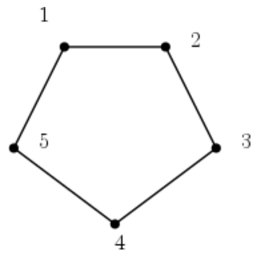

Figure 5: Cycle on the set $\{1, 2, 3, 4, 5\}$

$$\Delta_{\vec{\boldsymbol{\pi}}_2}(C_5) = 0_{\{2,4\}} \otimes 0_{\{1,3\}} \otimes 0_{\{5\}} \,,$$

in particular, $f_{\vec{\boldsymbol{\pi}}_1,\eta}(C_5) = 0$ and $f_{\vec{\boldsymbol{\pi}}_2,\eta}(C_5) = 1$. Generally,

$$f_{\vec{\boldsymbol{\pi}},\eta}(G) = \mathbb{1}[\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}) \text{ is a stable set partition on } G] \,. \tag{31}$$

With this, we can generalise diagram (22).

Proposition 48. *Consider a Hopf monoid $(\overline{h}, \mu, \iota, \Delta, \epsilon)$. Let $\vec{\boldsymbol{\gamma}} = C_1|\dots|C_l$ be a set composition on S that decomposes on non empty sets $S = I \sqcup J$. Call $A_i := C_i \cap I$ and $B_i := C_i \cap J$, and let $\vec{\boldsymbol{\pi}} := (\vec{\boldsymbol{\gamma}}|_I, \vec{\boldsymbol{\gamma}}|_J) = A_1|\dots|A_l|B_1|\dots|B_l$, possibly erasing empty blocks.*

Define $\mu_{(\vec{\boldsymbol{\gamma}},I,J)} : \overline{h}[\vec{\boldsymbol{\pi}}] \to \overline{h}[\vec{\boldsymbol{\gamma}}]$ as the tensor product of all maps

$$\overline{h}[A_i] \otimes \overline{h}[B_i] \xrightarrow{\mu_{A_i,B_i}} \overline{h}[C_i] \,,$$

composed with the necessary twist so that it maps $\overline{h}[\vec{\boldsymbol{\pi}}] \to \overline{h}[\vec{\boldsymbol{\gamma}}]$.

Then the following diagram commutes:

$$\begin{tikzcd} \overline{h}[I] \otimes \overline{h}[J] \arrow[r, "\mu_{I,J}"] \arrow[d, "(\Delta_{\vec{\boldsymbol{\gamma}}|_I} \otimes \Delta_{\vec{\boldsymbol{\gamma}}|_J})"] & \overline{h}[S] \arrow[d, "\Delta_{\vec{\boldsymbol{\gamma}}}"] \\ \overline{h}[\vec{\boldsymbol{\pi}}] \arrow[r, "\mu_{(\vec{\boldsymbol{\gamma}},I,J)}"] & \overline{h}[\vec{\boldsymbol{\gamma}}] \end{tikzcd} \tag{32}$$

Note that diagram (22) corresponds to diagram (32) for $\vec{\boldsymbol{\gamma}} = M|N$. In general, Diagram (32) is obtained by gluing diagrams of the form of Diagram (22), as seen in the proof of Proposition 48, which we do now.

Proof of Proposition 48. We will use induction on the length of $\vec{\boldsymbol{\gamma}}$, $l := l(\vec{\boldsymbol{\gamma}}) \geq 2$. If $l(\vec{\boldsymbol{\gamma}}) = 2$ we recover diagram (22).

Consider now the case $l(\vec{\boldsymbol{\gamma}}) \geq 3$. From (22) applied to $I = A_1 \sqcup A_2, J = B_1 \sqcup B_2$ we have the following commuting diagram:

$$\begin{tikzcd} \overline{h}[A_1 \sqcup A_2|B_1 \sqcup B_2] \arrow[r, "\mu_{A_1 \sqcup A_2, B_1 \sqcup B_2}"] \arrow[d, "\Delta_{A_1,A_2} \otimes \Delta_{B_1,B_2}"] & \overline{h}[C_1 \sqcup C_2] \arrow[d, "\Delta_{C_1,C_2}"] \\ \overline{h}[A_1|A_2|B_1|B_2] \arrow[r, "(\mu_{A_1,B_1} \otimes \mu_{A_2,B_2}) \circ \text{twist}"] & \overline{h}[C_1|C_2] \end{tikzcd} \tag{33}$$

Call $I' = I \setminus (C_1 \sqcup C_2)$ and $J' = J \setminus (C_1 \sqcup C_2)$, let $\vec{\boldsymbol{\gamma}}' = C_3|\dots|C_l$ and take $\vec{\boldsymbol{\gamma}}^o = C_1 \sqcup C_2|C_3|\dots|C_l = C_1 \sqcup C_2|\vec{\boldsymbol{\gamma}}'$. By tensoring diagram (33) with the following diagram

$$\begin{array}{ccc}
\overline{h}[A_3|B_3|A_4|\dots|B_l] & \xrightarrow{\mu_{(\vec{\gamma}',I',J')}} & \overline{h}[C_3|\dots|C_l] \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\
\overline{h}[A_3|B_3|A_4|\dots|B_l] & \xrightarrow{\mu_{(\vec{\gamma}',I',J')}} & \overline{h}[C_3|\dots|C_l]
\end{array} \tag{34}$$

we have:

$$\begin{array}{ccc}
\overline{h}[A_1 \sqcup A_2|B_1 \sqcup B_2|A_3|B_3|\dots] & \xrightarrow{\mu_{A_1 \sqcup A_2, B_1 \sqcup B_2} \otimes \mu_{(\vec{\gamma}',I',J')}} & \overline{h}[\vec{\gamma}^o] \\
\downarrow{\scriptstyle \Delta_{A_1,A_2} \otimes \Delta_{B_1,B_2} \otimes id} & & \downarrow{\scriptstyle \Delta_{C_1,C_2} \otimes id} \\
\overline{h}[A_1|A_2|B_1|B_2|A_3|B_3|A_4|\dots] & \xrightarrow{(\mu_{A_1,B_1} \otimes \mu_{A_2,B_2} \otimes \mu_{(\vec{\gamma}',I',J')}) \circ \text{twist}} & \overline{h}[\vec{\gamma}]
\end{array} \tag{35}$$

Note that

$$\left(\mu_{A_1,B_1} \otimes \mu_{A_2,B_2} \otimes \mu_{(\vec{\gamma}',I',J')}\right) \circ \text{twist} = \mu_{(\vec{\gamma},I,J)} \, ,$$

$$\left(\mu_{A_1 \sqcup A_2, B_1 \sqcup B_2} \otimes \mu_{(\vec{\gamma}',I',J')}\right) \circ \text{twist} = \mu_{(\vec{\gamma}^o,I,J)} \, .$$

So, consider (32) for the set composition $\vec{\gamma}^o = C_1 \sqcup C_2 | C_3 | \dots | C_l$, which commutes by induction hypothesis, and apply the necessary twists so as to glue with diagram (35). Then we obtain the commuting diagram:

$$\begin{array}{ccc}
\overline{h}[I] \otimes \overline{h}[J] & \xrightarrow{\mu_{I,J}} & \overline{h}[S] \\
\downarrow{\scriptstyle \text{twist} \circ (\Delta_{\vec{\gamma}^o|_I} \otimes \Delta_{\vec{\gamma}^o|_J})} & & \downarrow{\scriptstyle \Delta_{\vec{\gamma}^o}} \\
\overline{h}[A_1 \sqcup A_2|B_1 \sqcup B_2|A_3|B_3|\dots|B_l] & \xrightarrow{\mu_{(\vec{\gamma}^o,I,J)}} & \overline{h}[\vec{\gamma}'] \\
\downarrow{\scriptstyle \text{twist} \circ (\Delta_{A_1,A_2} \otimes \Delta_{B_1,B_2} \otimes id)} & & \downarrow{\scriptstyle \Delta_{C_1,C_2} \otimes id} \\
\overline{h}[A_1|B_1|A_2|\dots|B_l] & \xrightarrow{\mu_{(\vec{\gamma},I,J)} \circ \text{twist}} & \overline{h}[\vec{\gamma}]
\end{array} \tag{36}$$

We note that absorbing the twist with the downward arrow and erasing the middle line gives us the desired diagram. □

Proposition 49. *Consider a combinatorial Hopf monoid $(\overline{h}, \eta)$. Let $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\tau}}$ be set compositions of the disjoint sets I and J, respectively, and take $\vec{\boldsymbol{\lambda}}$ set composition of $I \sqcup J$. Suppose that $I, J \neq \emptyset$. Take $a \in \overline{h}[I]$, $b \in \overline{h}[J]$, $c \in \overline{h}[I \sqcup J]$. Then we have that*

$$f_{\vec{\boldsymbol{\lambda}},\eta}(a \cdot_{I,J} b) = f_{\vec{\boldsymbol{\lambda}}|_I,\eta}(a) f_{\vec{\boldsymbol{\lambda}}|_J,\eta}(b) \, , \tag{37}$$

and that

$$f_{\vec{\boldsymbol{\pi}},\eta} \otimes f_{\vec{\boldsymbol{\tau}},\eta}(\Delta_{I,J} a) = f_{(\vec{\boldsymbol{\pi}},\vec{\boldsymbol{\tau}}),\eta}(a) \, . \tag{38}$$

Proof. Note that (37) reduces to

$$f_{\vec{\boldsymbol{\lambda}},\eta} \circ \mu_{I,J} = f_{\vec{\boldsymbol{\lambda}}|_I,\eta} \otimes f_{\vec{\boldsymbol{\lambda}}|_J,\eta}. \tag{39}$$

Now Proposition 48 tells us that

$$\Delta_{\vec{\boldsymbol{\lambda}}} \circ \mu_{I,J} = \mu_{(\vec{\boldsymbol{\lambda}},I,J)} \circ (\Delta_{\vec{\boldsymbol{\lambda}}|_I} \otimes \Delta_{\vec{\boldsymbol{\lambda}}|_J}) \, . \tag{40}$$

Suppose that $\vec{\boldsymbol{\lambda}} = C_1 | \dots | C_l$. Then by tensoring diagrams of the form (23) for each decomposition $(I \cap C_i) \sqcup (J \cap C_i) = C_i$, we obtain

$$\eta[\vec{\boldsymbol{\lambda}}] \circ \mu_{(\vec{\boldsymbol{\lambda}},I,J)} = \eta[(\vec{\boldsymbol{\lambda}}|_I, \vec{\boldsymbol{\lambda}}|_J)] \, . \tag{41}$$

From (40) and (41) we get that

$$\eta[\vec{\boldsymbol{\lambda}}] \circ \Delta_{\vec{\boldsymbol{\lambda}}} \circ \mu_{I,J} = \eta[\vec{\boldsymbol{\lambda}}] \circ \mu_{(\vec{\boldsymbol{\lambda}},I,J)} \circ (\Delta_{\vec{\boldsymbol{\lambda}}|_I} \otimes \Delta_{\vec{\boldsymbol{\lambda}}|_J}) = \eta[(\vec{\boldsymbol{\lambda}}|_I, \vec{\boldsymbol{\lambda}}|_J)] \circ (\Delta_{\vec{\boldsymbol{\lambda}}|_I} \otimes \Delta_{\vec{\boldsymbol{\lambda}}|_J}) \,.$$

So

$$f_{\vec{\boldsymbol{\lambda}},\eta} \circ \mu_{I,J} = (\eta[\vec{\boldsymbol{\lambda}}|_I] \otimes \eta[\vec{\boldsymbol{\lambda}}|_J]) \circ (\Delta_{\vec{\boldsymbol{\lambda}}|_I} \otimes \Delta_{\vec{\boldsymbol{\lambda}}|_J}) = f_{\vec{\boldsymbol{\lambda}}|_I,\eta} \otimes f_{\vec{\boldsymbol{\lambda}}|_J,\eta} \,.$$

This concludes the proof of (39). Remains to show (38), which is equivalent to

$$(f_{\vec{\boldsymbol{\pi}},\eta} \otimes f_{\vec{\boldsymbol{\tau}},\eta}) \circ \Delta_{I,J} = f_{(\vec{\boldsymbol{\pi}},\vec{\boldsymbol{\tau}}),\eta} \,,$$

which follows from (30) via

$$(f_{\vec{\boldsymbol{\pi}},\eta} \otimes f_{\vec{\boldsymbol{\tau}},\eta}) \circ \Delta_{I,J} = (\eta[\vec{\boldsymbol{\pi}}] \otimes \eta[\vec{\boldsymbol{\tau}}]) \circ (\Delta_{\vec{\boldsymbol{\pi}}} \otimes \Delta_{\vec{\boldsymbol{\tau}}}) \circ \Delta_{I,J} = \eta[(\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\tau}})] \circ \Delta_{(\vec{\boldsymbol{\pi}},\vec{\boldsymbol{\tau}})} = f_{(\vec{\boldsymbol{\pi}},\vec{\boldsymbol{\tau}}),\eta} \,,$$

whenever both I and J are non empty. □

Proof of Theorem 45. Let $a \in \overline{h}[I]$. We define

$$\boldsymbol{\Upsilon}_{\overline{h}}(a) = \sum_{\vec{\boldsymbol{\pi}} \in \mathbf{C}_I} \mathbb{M}_{\vec{\boldsymbol{\pi}}} f_{\vec{\boldsymbol{\pi}},\eta}(a) \,,$$

where we convention that, for the unique set composition $\vec{\emptyset}$ in $\emptyset$, we have $f_{\vec{\emptyset},\eta} = \epsilon_{\emptyset}$.

The fact that $\boldsymbol{\Upsilon}_{\overline{h}}$ preserves the characters follows because $f_{\vec{\boldsymbol{\pi}},\eta} = \eta$ whenever $\vec{\boldsymbol{\pi}}$ has length one or zero. We show now the remaining axioms of a combinatorial Hopf monoid morphism:

- $\boldsymbol{\Upsilon}_{\overline{h}}(a \cdot_{I,J} b) = \boldsymbol{\Upsilon}_{\overline{h}}(a) \cdot \boldsymbol{\Upsilon}_{\overline{h}}(b)$.
- $\boldsymbol{\Upsilon}_{\overline{h}}(\Delta_{I,J} a) = \Delta_{I,J} \circ \boldsymbol{\Upsilon}_{\overline{h}}(a)$.
- $\boldsymbol{\Upsilon}_{\overline{h}} \circ \iota_{\overline{h}} = \iota_{\overline{\mathbf{WQSym}}}$.
- $\epsilon_{\overline{\mathbf{WQSym}}} \circ \boldsymbol{\Upsilon}_{\overline{h}} = \epsilon_{\overline{h}}$.

Taking the coefficients on the monomial basis for the first two items, this reduces to Proposition 49. The case where I or J are empty can be dealt with directly. The other two items are immediate. This concludes that $\boldsymbol{\Upsilon}_{\overline{h}}$ is a combinatorial Hopf monoid morphism.

We now justify uniqueness. Suppose that $\phi : \overline{h} \Rightarrow \overline{\mathbf{WQSym}}$ is a combinatorial Hopf monoid morphism.

Let I be non empty. For each $a \in \overline{h}[I]$, write $\phi[I](a) = \sum_{\vec{\boldsymbol{\pi}}} \mathbb{M}_{\vec{\boldsymbol{\pi}}} \phi_{\vec{\boldsymbol{\pi}}}(a)$ and apply $\Delta_{\vec{\boldsymbol{\pi}}}$ on both sides, recalling the fact that ϕ is a comonoid morphism:

$$\phi[\vec{\boldsymbol{\pi}}] \Delta_{\vec{\boldsymbol{\pi}}}(a) = \Delta_{\vec{\boldsymbol{\pi}}} \phi[I](a) = \sum_{\vec{\boldsymbol{\tau}}} \phi_{\vec{\boldsymbol{\tau}}}(a) \Delta_{\vec{\boldsymbol{\pi}}} \mathbb{M}_{\vec{\boldsymbol{\tau}}} \,, \tag{42}$$

However, since $\eta_0 \phi = \eta$, we have that $\eta_0[\vec{\boldsymbol{\pi}}] \phi[\vec{\boldsymbol{\pi}}] = \eta[\vec{\boldsymbol{\pi}}]$, so

$$\eta_0[\vec{\boldsymbol{\pi}}] \phi[\vec{\boldsymbol{\pi}}] \Delta_{\vec{\boldsymbol{\pi}}}(a) = \eta[\vec{\boldsymbol{\pi}}] \Delta_{\vec{\boldsymbol{\pi}}}(a) = f_{\vec{\boldsymbol{\pi}},\eta}(a) \,, \tag{43}$$

whereas

$$\eta_0[\vec{\boldsymbol{\pi}}] \sum_{\vec{\boldsymbol{\tau}}} \phi_{\vec{\boldsymbol{\tau}}}(a) \Delta_{\vec{\boldsymbol{\pi}}} \mathbb{M}_{\vec{\boldsymbol{\tau}}} = \sum_{\vec{\boldsymbol{\tau}}} \phi_{\vec{\boldsymbol{\tau}}}(a) \eta_0[\vec{\boldsymbol{\pi}}] \Delta_{\vec{\boldsymbol{\pi}}} \mathbb{M}_{\vec{\boldsymbol{\tau}}} = \phi_{\vec{\boldsymbol{\pi}}}(a) \,, \tag{44}$$

because it can be directly computed from Eq. (26) that $\eta_0[\vec{\boldsymbol{\pi}}]\Delta_{\vec{\boldsymbol{\pi}}}\mathbb{M}_{\vec{\boldsymbol{\tau}}} = 0$ whenever $\vec{\boldsymbol{\pi}} \neq \vec{\boldsymbol{\tau}}$, and $\eta_0[\vec{\boldsymbol{\pi}}]\Delta_{\vec{\boldsymbol{\pi}}}\mathbb{M}_{\vec{\boldsymbol{\pi}}} = 1$.

Let us denote $\eta_{0,\vec{\boldsymbol{\pi}}}$ the multi-character of η_0 of type $\vec{\boldsymbol{\pi}}$. Applying $\eta_0[\vec{\boldsymbol{\pi}}]$ on (42), together with (43), and (44) we have that

$$\phi_{\vec{\boldsymbol{\pi}}}(a) = f_{\vec{\boldsymbol{\pi}},\eta}(a)\,,$$

which concludes the uniqueness for I non empty.

Consider now the case where I is the empty set. Then $\eta_{0,\emptyset}$ is a bijection, and we have both that $\eta_{0,\emptyset} \circ \Upsilon_{\overline{h},\emptyset} = \eta_\emptyset = \epsilon_\emptyset$ and $\eta_{0,\emptyset} \circ \phi_\emptyset = \eta_\emptyset = \epsilon_\emptyset$. It follows that $\Upsilon_{\overline{h},\emptyset} = \eta_{0,\emptyset}^{-1} \circ \epsilon_\emptyset = \phi_\emptyset$. This concludes the uniqueness. □

We now observe the relation between the Hopf algebras previously studied and the Fock functors of the Hopf monoids here described. First note that from (31) and from Lemma 7 we have that $\mathcal{K}(\boldsymbol{\Upsilon}_{\overline{\mathbf{G}}}) = \boldsymbol{\Upsilon}_{\mathbf{G}}$ is the chromatic symmetric function in non-commutative variables, and that $\overline{\mathcal{K}}(\boldsymbol{\Upsilon}_{\overline{\mathbf{G}}}) = \Psi_{\mathbf{G}}$ is the chromatic symmetric function.

Similarly, we obtain $\mathcal{K}(\boldsymbol{\Upsilon}_{\overline{\mathbf{Pos}}}) = \boldsymbol{\Upsilon}_{\mathbf{Pos}}$, $\overline{\mathcal{K}}(\boldsymbol{\Upsilon}_{\overline{\mathbf{Pos}}}) = \Psi_{\mathbf{Pos}}$, $\mathcal{K}(\boldsymbol{\Upsilon}_{\overline{\mathbf{GP}}}) = \boldsymbol{\Upsilon}_{\mathbf{GP}}$ and $\overline{\mathcal{K}}(\boldsymbol{\Upsilon}_{\overline{\mathbf{GP}}}) = \Psi_{\mathbf{GP}}$.

5.5 Hypergraphic polytopes and posets

In the following, we will see that the universal map that we constructed earlier to $\overline{\mathbf{WQSym}}$ behaves well with respect to combinatorial Hopf monoid morphisms.

Lemma 50. *If $\phi : \overline{h}_1 \Rightarrow \overline{h}_2$ is a combinatorial Hopf monoid morphism between two Hopf monoids, then the following diagram commutes*

$$\begin{tikzcd} \overline{h}_1 \arrow[rr, "\phi"] \arrow[dr, "\boldsymbol{\Upsilon}_{\overline{h}_1}"] & & \overline{h}_2 \arrow[dl, "\boldsymbol{\Upsilon}_{\overline{h}_2}"] \\ & \overline{\mathbf{WQSym}} & \end{tikzcd} \tag{45}$$

We simply observe that $\boldsymbol{\Upsilon}_{\overline{h}_2} \circ \phi$ is a combinatorial Hopf monoid morphism to the combinatorial Hopf species $\overline{\mathbf{WQSym}}$, so the universal property tells us that $\boldsymbol{\Upsilon}_{\overline{h}_2} \circ \phi = \boldsymbol{\Upsilon}_{\overline{h}_1}$, as desired.

Corollary 51. *There are no combinatorial Hopf morphisms between the Hopf monoid of $\overline{\mathbf{Pos}}$ and the Hopf monoid of $\overline{\mathbf{HGP}}$, that commute with the given characters.*

Proof. For sake of contradiction, take such map $\phi : \overline{\mathbf{Pos}} \Rightarrow \overline{\mathbf{HGP}}$ that satisfies $\mathcal{K}(\boldsymbol{\Upsilon}_{\overline{\mathbf{HGP}}}) \circ \mathcal{K}(\phi) = \mathcal{K}(\boldsymbol{\Upsilon}_{\overline{\mathbf{Pos}}})$, according to Lemma 50, so

$$\boldsymbol{\Upsilon}_{\mathbf{Pos}} = \boldsymbol{\Upsilon}_{\mathbf{HGP}} \circ \mathcal{K}(\phi)\,.$$

However, $\boldsymbol{\Upsilon}_{\mathbf{Pos}}$ is surjective, whereas we have seen in Proposition 34 that $\boldsymbol{\Upsilon}_{\mathbf{HGP}}$ is not surjective. So there is no such map $\mathcal{K}(\phi)$. This is a contradiction with the assumption that a combinatorial Hopf monoid morphism ϕ exists. □

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A Computing the augmented chromatic symmetric function on graphs

Recall that the ring R is the quotient of the power series in $\mathbb{K}[[x_1, \dots ; q_1, \dots]]$ through the relations $q_i^2(q_i - 1) = 0$. We define a map $\tilde{\Psi} : \mathbf{G} \to R$, and we observed that $\ker \tilde{\Psi} = \ker \Psi$ in Proposition 18.

Here, we consider some specialisations of $\tilde{\Psi}$ and obtain back a linear combination of chromatic symmetric function of smaller graphs, in Theorem 52. The main goal is to explore how to obtain Proposition 18 without using Theorem 1, but in a more direct way. We should clarify that we do not have a complete proof, and this illustrates the strength of the kernel approach. Let us first set up some necessary notation.

For an element $f \in R$, denote by $f\big|_{q_i=a}$ the specialisation (or evaluation) of the variable q_i to a in f, whenever defined (for $a = 0$ or $a = 1$). Additionally, denote by $f\big|_{q_i=1'}$ the specialisation of the variable q_i to 1 in $\frac{\partial}{\partial q_i} f$. This is naturally an abuse of notation that allows us to compose several specialisations in a more compact way. We also use this notation for the x_i variables. In particular, by $f\big|_{x_i=0''}$ we mean $\frac{\partial^2}{\partial x_i^2} f\big|_{x_i=0}$.

We note that we can specialise infinitely many variables to zero. We note that, in general, taking infinitely many specialisation of the form $q_i = 1$ is not allowed, as the reader can readily check. Taking specialisations of q_i to $a \notin \{0, 1, 1'\}$ is not well defined in the quotient ring.

For an edge $e \in E(G)$, $G \setminus \mathcal{N}(e)$ is the graph resulting after both endpoints of e are deleted from G, along with all incident edges.

We say that a tuple of edges $m = (e_1, \dots, e_j)$ is an ordered matching if no two edges share a vertex, and call $\mathcal{M}_k(G)$ the set of ordered matchings of size k on a graph G. We write $G \setminus \mathcal{N}(m)$ for the graph resulting after removing all vertices in the matching m from G.

Finally, for a symmetric function f over the variables $x_1, x_2, \dots$, let $f \uparrow_k$ be the symmetric function over the variables $x_{k+1}, x_{k+2}, \dots$ with each index in f shifted up by k.

We will obtain now a formula for some specialisations of $\tilde{\Psi}(G)$. Said specialisations depend only on some $\Psi_{\mathbf{G}}(H_i)$, where each H_i is a subgraph of G.

Theorem 52. *Let $k \geq 0$. We have the following relation between the graph invariant $\tilde{\Psi}$ and the original chromatic symmetric function $\Psi_{\mathbf{G}}$:*

$$\frac{1}{2^k} \tilde{\Psi}(G)\Big|_{\substack{q_i=1' \ i=1,\dots,k \\ q_i=0 \ i>k \\ x_i=0'' \ i=1,\dots,k}} = \sum_{m \in \mathcal{M}_k(G)} \Psi_{\mathbf{G}}(G \setminus \mathcal{N}(m)) \uparrow_k . \tag{46}$$

Proof. Recall that $c_G(f,i)$ counts the monochromatic edges with a color i.

$$\tilde{\Psi}(G)\Big|_{q_i=1'\ i=1,\dots,k} = \sum_{f:V(G)\to\mathbb{N}} x_f \left(\prod_{i>k} q_i^{c_G(f,i)}\right)\prod_{i=1}^{k} c_G(f,i)\,.$$

Say that a coloring is k-proper if all monochromatic edges have color $j \le k$. Note that for a fixed coloring f, $\prod_{i=1}^{k} c_G(i,f)$ counts ordered matchings $(e_1,\dots,e_k)$ in G with $f(e_i)=\{i\}$. Then, it is clear that

$$\begin{aligned}\tilde{\Psi}(G)\Big|_{\substack{q_i=1'\ i=1,\dots,k\\ q_i=0\ i>k}} &= \sum_{f \text{ is k - proper}} \left[x_f \prod_{i=1}^{k} c_G(i,f)\right] = \sum_{f \text{ is k - proper}} \left[x_f \sum_{\substack{m\in\mathcal{M}_k(G)\\ m=(e_1,\dots,e_k)\\ f(e_i)=\{i\}}} 1\right]\\ &= \sum_{\substack{m\in\mathcal{M}_k(G)\\ m=(e_1,\dots,e_k)}} \left[\sum_{\substack{f \text{ is k - proper}\\ f(e_i)=\{i\}}} x_f\right]\\ &= \sum_{\substack{m\in\mathcal{M}_k(G)\\ m=(e_1,\dots,e_k)}} \sum_{\substack{g \text{ coloring of } G\setminus\mathcal{N}(m)\\ g \text{ k - proper}}} x_g(x_1\cdots x_k)^2\,.\end{aligned} \tag{47}$$

So after the specialization $x_i = 0''$ for $i=1,\dots,k$, all colorings of $G\setminus\mathcal{N}(m)$ that use a color $j\le k$ vanish, so

$$\tilde{\Psi}(G)\Big|_{\substack{q_i=1'\ i=1,\dots,k\\ q_i=0\ i>k\\ x_i=0''\ i=1,\dots,k}} = \sum_{\substack{m\in\mathcal{M}_k(G)\\ m=(e_1,\dots,e_k)}} \sum_{\substack{g \text{ proper}\\ \operatorname{im} g\subseteq\mathbb{Z}_{>k}}} 2^k x_g = 2^k \sum_{m\in\mathcal{M}_k(G)} \Psi_{\mathbf{G}}(G\setminus\mathcal{N}(m))\uparrow_k\,,$$

as desired. □

In fact, the right hand side of (46) can be determined by $\Psi_{\mathbf{G}}$.

Proposition 53. *If G_1, G_2 are two graphs such that $\Psi_{\mathbf{G}}(G_1)=\Psi_{\mathbf{G}}(G_2)$, and k a positive number, then*

$$\sum_{m\in\mathcal{M}_k(G_1)} \Psi_{\mathbf{G}}(G_1\setminus\mathcal{N}(m)) = \sum_{m\in\mathcal{M}_k(G_2)} \Psi_{\mathbf{G}}(G_2\setminus\mathcal{N}(m))\,.$$

Proof. We will use the power-sum basis $\{p_\lambda\}_{\lambda\vdash n}$ of Sym_n introduced, for instance, in [Sta86].

It suffices to show that, for a generic graph H the coefficients in the expression $\sum_{m\in\mathcal{M}_k(H)} \Psi_{\mathbf{G}}(H\setminus\mathcal{N}(m))$ in the power-sum basis can be expressed by the coefficients of $\Psi_{\mathbf{G}}(H)$ in the power-sum basis. Once this is established, the proposition follows.

For a graph G and a set of edges $S\subseteq E(G)$, write $\tau(S)$ for the integer partition recording the sizes of the connected components of the graph $(V(G),S)$. In [Sta95] the following expression for the coefficients in the power-sum basis is shown:

$$\Psi_{\mathbf{G}}(G) = \sum_{\lambda\vdash n} p_\lambda \sum_{\substack{S\subseteq E(G)\\ \tau(S)=\lambda}} (-1)^{\#S}\,.$$

Suppose that $\Psi_{\mathbf{G}}(H) = \sum_{\lambda \vdash n} c_\lambda p_\lambda$. For a number partition λ write $m_2(\lambda)$ for the number of parts of size two, and define $\lambda \cup (2^k)$ as the partition with k extra parts of size two. Then we have that

$$\begin{aligned}\sum_{m \in \mathcal{M}_k(H)} \Psi_{\mathbf{G}}(H \setminus \mathcal{N}(m)) &= \sum_{m \in \mathcal{M}_k(H)} \sum_{\lambda \vdash n} p_\lambda \sum_{\substack{S \subseteq E(H \setminus \mathcal{N}(m)) \\ \tau(S) = \lambda}} (-1)^{\#S} \\ &= \sum_{\lambda \vdash n} p_\lambda \sum_{m \in \mathcal{M}_k(H)} \sum_{\substack{S \subseteq E(H \setminus \mathcal{N}(m)) \\ \tau(S) = \lambda}} (-1)^{\#S} .\end{aligned} \tag{48}$$

Reorder the sum with respect to $R = S \cup m$, and note that for each $R \subseteq E(H)$ such that $\tau(R) = \lambda \cup (2^k)$, there are exactly $\binom{m_2(\lambda)+k}{k} k!$ pairs (S, m) of $m \in \mathcal{M}_k(H)$ and $S \subseteq E(H)$ such that $S \cup m = R$, $\tau(S) = \lambda$ and $S \subseteq E(H \setminus \mathcal{N}(m))$. Hence:

$$\begin{aligned}\sum_{m \in \mathcal{M}_k(H)} \Psi_{\mathbf{G}}(H \setminus \mathcal{N}(m)) &= \sum_{\lambda \vdash n} p_\lambda \sum_{\substack{R \subseteq E(H) \\ \tau(R) = \lambda \cup 2^k}} \binom{m_2(\lambda) + k}{k} k! (-1)^{\#R - k} \\ &= \sum_{\lambda \vdash n} p_\lambda (-1)^k c_{\lambda \cup (2^k)} \binom{m_2(\lambda) + k}{k} k! \ .\end{aligned} \tag{49}$$

Therefore, the sum $\sum_{m \in \mathcal{M}_k(H)} \Psi_{\mathbf{G}}(H \setminus \mathcal{N}(m))$ is determined by $\Psi_{\mathbf{G}}(H)$. $\square$

It follows from Theorem 52 and Proposition 53 that:

Corollary 54. *If G_1, G_2 are graphs such that $\Psi_{\mathbf{G}}(G_1) = \Psi_{\mathbf{G}}(G_2)$, then for every integer $k \geq 0$ we have:*

$$\tilde{\Psi}(G_1)\Big|_{\substack{q_i = 1' \ i = 1, \dots, k \\ q_i = 0 \ i > k \\ x_i = 0'' \ i = 1, \dots, k}} = \tilde{\Psi}(G_2)\Big|_{\substack{q_i = 1' \ i = 1, \dots, k \\ q_i = 0 \ i > k \\ x_i = 0'' \ i = 1, \dots, k}} .$$

The following fact is immediate from the definition of R:

Proposition 55. *Suppose that $f_1, f_2 \in R$ are such that for every pair of finite disjoint sets $I, J \subseteq \mathbb{N}$ we have*

$$f_1\Big|_{\substack{q_i = 1' \ i \in I \\ q_i = 0 \ i \in P \\ q_i = 1 \ i \in J}} = f_2\Big|_{\substack{q_i = 1' \ i \in I \\ q_i = 0 \ i \in P \\ q_i = 1 \ i \in J}}$$

where $P = \mathbb{N} \setminus (I \uplus J)$. Then $f_1 = f_2$ in R.

In conclusion, to get an alternative proof of Proposition 18 we need to establish a generalisation of Corollary 54 that introduces specialisations of the type $q_i = 1$, in order to apply Proposition 55. Such a generalisation has not been found by the author.

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