Probability 2

Exercise sheet nb. 8

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Due until: 12th November at 5 p.m.

Exercise 1 (5 points). Let $(X_n)_{n\geq 1}$ be i.i.d random variables such that $X_1 \sim \mathrm{Binom}(10, 0.3)$.

For the following random processes, prove or refute that we have a Markov chain, and in the affirmative case give the state space and the transition matrix:

1. The largest value so far, $M_n = \max_{k=1,\dots,n}\{X_k\}$.
2. The number of 0's so far, $N_n = \#\{k | X_k = 0\}$.
3. The difference between the last two results (with the convention that $X_0 = 0$), that is $P_n = X_n - X_{n-1}$.
4. The time since we last saw a 0 or since we start, whatever is the shortest, that is $Q_n = n - \max_k\{X_k = 0 \text{ or } k = 0\}$.
5. The length of the current run of equal values, that is $S_n = \max_k\{Y_i = Y_n \text{ for all } i = n-1, \dots, n-k+1\}$.

Exercise 2 (2 points). Let $\{X_n\}_{n\geq 1}$ be independent random variables with distribution $X_n \sim \mathrm{Unif}(\{1, \dots, n\})$, and define $S_n = \sum_{k=1}^{n} X_n$, with $S_0 = 0$.

1. Show that $\{S_n\}_{n\geq 0}$ is a Markov process and compute the transition matrices Q_i.
2. Is $\{X_n\}_{n\geq 1}$ a Markov process?
3. Are $\{S_n\}_{n\geq 0}$ or $\{X_n\}_{n\geq 1}$ Markov chains?

Exercise 3 (2 points). Consider a Markov chain $\{X_n\}_{n\geq 0}$ on the state space A and $E \subseteq A$, and let $T = \min_k\{X_k \in E\}$. Show that $\{X_{n\wedge T}\}_{n\geq 0}$ is also a Markov chain. Describe its transition matrix with respect to the transition matrix Q of $\{X_n\}_{n\geq 0}$.

Exercise 4 (2 points). Let $\{X_{2n}\}_{n\geq 0}$ be i.i.d random variables such that

$$\mathbb{P}[X_0 = 1] = \mathbb{P}[X_0 = -1] = \frac{1}{2}\,.$$

For $n \geq 0$, set $X_{2n+1} \coloneqq X_{2n}X_{2n+2}$.

1. Show that $\{X_n\}_{n\geq 0}$ are pairwise independent (that is, for every $i, j \geq 0$ we have that X_i and X_j are independent).

2. For any $n \geq 0$, $\varepsilon_0, \varepsilon_1$ in $\{-1, +1\}$, compute $\mathbb{P}[X_{n+1} = \varepsilon_1 | X_n = \varepsilon_0]$. Is the random process $\{X_n\}_{n\geq 0}$ a Markov chain?