Probability 2

Exercise sheet nb. 9

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Due until: 19th November at 5 p.m.

Exercise 1 (2 points). Consider a Markov chain $\{X_n\}_{n\geq 0}$ with state space $S = \{1, 2, 3, 4, 5\}$, with the following transition matrix:

$$Q = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Describe the transient and the recurrent states of the Markov chain. What are the closed irreducible sets? Assume that $X_0 = 2$. Compute the distribution of $T = \min_n\{X_n = 5\} \in \mathbb{Z} \cup \{+\infty\}$. That is, compute $F^{(n)}_{2,5}$ for all $n \geq 0$.

Exercise 2 (3 points). Let (X_n) be a Markov chain. We define the last exit probabilities as follows: for states $x, y \in S$ and $n \geq 0$, let

$$L^{(n)}_{x,y} := \mathbb{P}_x\big(X_n = y, X_k \neq x \text{ for } 1 \leq k < n\big)$$

be the probability that, starting from x, the chain X visits y at time n without revisiting x in the midtime.

1. Set $L_{x,y}(t) = \sum_{n\geq 1} t^n\, L^{(n)}_{x,y}$. Show that, for $x \neq y$ and $t \in [0, 1)$, we have

$$Q_{x,y}(t) = Q_{x,x}(t)L_{x,y}(t).$$

2. Assume that x and y are distinct states such that $Q_{x,x}(t) = Q_{y,y}(t)$. Prove that, for all $n \geq 1$, we have

$$L^{(n)}_{x,y} = F^{(n)}_{x,y}.$$

Exercise 3 (4 points). We consider the simple random walk X_n on $\mathbb{Z}$, i.e. $X_0 = 0$ and $X_n = Y_1 + \ldots Y_n$ where the Y_i are i.i.d. random variables with distribution

$$\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \tfrac{1}{2}.$$

1. Show that if the chain starts at 0, the probability that the chain visits 0 at time $2n$ is
$$Q_{0,0}^{2n} = \frac{1}{2^{2n}}\binom{2n}{n}.$$

2. Using Stirling equivalence $n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$, compute an asymptotic equivalent of $Q_{0,0}^{2n}$. Decide whether 0 is recurrent or transient.

3. Now, consider $d \geq 2$, and define the Markov chain $Z_n^{(d)} = (X_n^{(1)}, \ldots, X_n^{(d)})$, where $X_n^{(1)}, \ldots, X_n^{(d)}$ are d independent copies of the Markov chain X. Is the state $\mathbf{0}_d := (0, \ldots, 0) \in \mathbb{Z}^d$ recurrent or transient for the Markov chain $Z^{(d)}$?

   (Hint: compare $Q_{\mathbf{0}_d,\mathbf{0}_d}^{2n}$ to $Q_{0,0}^{2n}$.)