Probability 2

Exercise sheet nb. 7

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Due until: 5th November at 5 p.m.

Exercise 1 (2 points). Let $(X_n)_{n\geq 0}$ be a nonnegative supermartingale, and let X_∞ be its almost sure limit (whose existence was proved in the lecture). Show that, for all $n \geq 0$,

$$\mathbb{E}[X_\infty|\mathcal{F}_n] \leq X_n \text{ a.s.}$$

Exercise 2 (5 points). We consider an urn that contains initially one black and one white ball. At each step, we take a ball at random in the urn and replace it by two balls of the same color. So at time 0,

- with probability 1/2, we draw a white ball and in this case the urn will be left with 2 white balls and 1 black one at the end of the first step.
- with probability 1/2, we draw a black ball and in this case the urn will be left with 1 white ball and 2 black ones at the end of the second step.

If at some point the urn contains 3 white ball and 7 black ones. Then

- with probability 3/10 we draw a white ball and in this case the urn will be left with 4 white balls and 7 black ones.
- with probability 7/10 we draw a black ball and in this case the urn will be left with 3 white ball and 8 black ones.

We denote Y_n and $X_n = Y_n/(n+2)$ the number and proportion of white balls in the urn at time n, respectively. Set $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

1. Compute $\mathbb{E}(Y_{n+1}|\mathcal{F}_n)$.
2. Show that X_n is a martingale and that there exists a random variable U such that X_n tends almost surely and in L^1 to U.
3. Set, for $r \geq 1$, and for all $n \geq 1$,

$$Z_{n;r} = \frac{Y_n(Y_n+1)\dots(Y_n+r-1)}{(n+2)(n+3)\dots(n+r+1)}.$$

   Show that, for each fixed $r \geq 1$, the process $(Z_{n;r})_{n\geq 0}$ is a martingale that converges almost surely and in L^1 to U^r (as $n \to \infty$).

4. Compute $\mathbb{E}(Z_{r;n})$ and $\mathbb{E}(U^r)$. Let V be a uniform random variable in $[0,1]$. Show that for all $r \geq 1$,
$$\mathbb{E}(U^r) = \mathbb{E}(V^r).$$

One can show that 4. implies U and V have the same distribution, so that we have proved that the proportion of white balls in the urn is asymptotically uniformly distributed in $[0,1]$.

Exercise 3 (4 points). Let $(\Theta_i)_{i\geq 1}$ be i.i.d random variables that take values in $\mathbb{Z}_{\geq 0}$, define $S_n = \sum_{k=1}^n \Theta_k$, and consider the backward filtration

$$\mathcal{F}_{-n} = \sigma(S_n, S_{n+1}, \dots)\,.$$

Fix N, b integers such that $0 < b < N$. Define the negative r.v.

$$T = -\max\left(\{k \in \{1,\dots,N\} | S_k \geq k\} \cup \{1\}\right)\,.$$

Let A be the event $\{S_k \geq k \text{ for some } 1 \leq k \leq N\}$. The goal of this exercise is to show the **ballot theorem**: if $\mathbb{P}[S_N = b] \neq 0$, then $\mathbb{P}[A|S_N = b] = \frac{b}{N}$. Let $X_n := \frac{S_{-n}}{-n}$ for $n \leq -1$.

1. Recall from the lecture that $\{X_n\}_{n\leq -1}$ is a backward martingale with respect to the backward filtration $\{\mathcal{F}_n\}_{n\leq -1}$. Show that T is a stopping time with respect to the backward filtration. That is, $\{T = -n\} \in \mathcal{F}_{-n}$ for all $n \geq 1$.

2. Show that we have that $\mathbb{E}\left[\frac{S_{-T}}{-T}|\mathcal{F}_{-N}\right] = \frac{S_N}{N}$. (Hint: start by proving that $\{X_{T\wedge n}\}_{n\leq -1}$ is a backward martingale)

3. Show that $\frac{S_{-T}}{-T}\mathbb{1}[S_N = b] = \mathbb{1}[A]\mathbb{1}[S_N = b]$, compute $\mathbb{E}[\frac{S_N}{N}\mathbb{1}[S_N = b]]$, and conclude that $\mathbb{P}[A|S_N = b] = \frac{b}{N}$.