Probability 2 - Revision exercise sheet

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Exercise 1 (3 points). Let $(X_n)_{n\geq 0}$ be a sequence of random variables and $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ the associated filtration of the probability space. We assume that (X_n) is a Gaussian martingale, i.e. a martingale such that, for each $k \geq 1$, the vector $(X_0, \dots, X_k)$ is a Gaussian vector.

1. Show that X_n has *centered independent increments*, i.e. for all n, $X_{n+1} - X_n$ has mean 0 and is independent from $(X_1 - X_0, X_2 - X_1, \dots, X_n - X_{n-1})$.

   (*Hint:* Recall that two components of a Gaussian vector are independent if and only if their covariance is zero.)

2. Let θ be a real number and denote $\sigma_n^2 = \mathrm{Var}(X_n)$ for $n \geq 0$. Show that

$$M_n := \exp\left(\theta\, X_n - \theta^2\, \sigma_n^2/2\right)$$

   is a martingale as well (with respect to $\mathcal{F}_n$). (Reminder: the Laplace transform of a Gaussian r.v. $Z \sim \mathcal{N}(0, \sigma^2)$ is $\mathbb{E}[e^{\theta Z}] = \exp(\frac{\theta^2 \sigma^2}{2})$.)

Note: in general, if X_n has centered independent increments, then X_n is a martingale. Question 1 shows that the converse holds for *Gaussian* martingales (but it's not true in general!).