Probability 2

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Due until: 8th October at 5 p.m.

Exercises marked with * should be easier after attending the lecture on Thursday.

Exercise 1 (2 points). Let X be a r.v. in $[0,1]$ with distribution $2xdx$ and Y be a uniform r.v. in $[0,X]$. Compute $\mathbb{E}[Y|X]$ and $\mathbb{E}[X|Y]$.

(Hint: to compute the density $p(x,y)$ of the pair (X,Y), you can multiply the density of X and the density of Y knowing X; this is Proposition 4.2 from the lecture used in the other direction.)

Exercise 2 (3 points). Consider an unknown quantity X. We would like to estimate the value of X, but each measurement comes with a noise Z_i, i.e. we only get the value of $Y_i := X + Z_i$. After n measurements, a natural estimator for X is the conditional expectation $\hat{X}_n := \mathbb{E}[X|Y_1,\dots,Y_n]$. We will discuss its correctness when n tends to infinity in a Gaussian model.

We assume that the quantity X to measure and all noises $(Z_i)_{i\geq 1}$ follow independent centered normal distributions of variance σ^2 and $(c_i^2)_{i\geq 1}$, respectively.

1. Show that the vector $(X, Y_1, \dots, Y_n)$ is a centered Gaussian vector, whose covariance matrix is to be determined.
2. Compute $\hat{X}_n$.

   Hint: recall that $\hat{X}_n = \sum_{i=1}^n \lambda_i Y_i$ for some real numbers λ_i (why?); the λ_i can be determined by using the orthogonality between $X - \hat{X}_n$ and Y_i, for $i = 1, \dots, n$.
3. Compute $\mathbb{E}\big[(X - \hat{X}_n)^2\big]$. Conclude that $\hat{X}_n$ tends to X in L^2 if and only if $\sum_{i\geq 1} c_i^{-2} = \infty$.

Exercise 3 (1 point*). Let $\{X_n\}_{n\geq 0}$ be a random process with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$.

Show that $T = \min_n\{X_n > 0\}$ is a stopping time.

Exercise 4 (2 points). Let $p \in (0,1)$ and $\{Y_n\}_{n\geq 0}$ be a sequence of independent random variables with

$$\mathbb{P}[Y_n = 1] = p,\ \mathbb{P}[Y_n = -1] = 1 - p.$$

Let $\{X_n\}_{n\geq 0}$ be $X_n = \sum_{k=0}^{n} Y_k$. Consider the filtration $\mathcal{F}_n = \sigma(Y_0, \dots, Y_n)$.

1. Show that $\{X_n\}_{n\geq 0}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$ if $p = 0.5$.
2. Define $Z_n = \left(\frac{1-p}{p}\right)^{X_n}$. Show that $\{Z_n\}_{n\geq 0}$ is a martingale.

Exercise 5 (2 points*). Let T be a stopping time for a sequence of σ-algebras $\{\mathcal{F}_n\}_{n\geq 0}$. We define the *stopping time σ-algebra* of T as

$$\mathcal{F}_T = \{A \in \mathcal{F} | A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n \geq 0\}.$$

1. Show that T is $\mathcal{F}_T$-measurable.
2. Show that, if S, T are stopping time with $S(\omega) \leq T(\omega)$ for all $\omega \in \Omega$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.