Patterns on marked permutations and other objects
Pattern Hopf algebras

Raul Penaguiao
University of Zurich, Raul Penaguiao[1]
University of Zurich, Switzerland

Abstract

In this article we study pattern Hopf algebras in combinatorial structures.

We adapt the algebraic framework of species to the study of substructures, where we consider the functions that count the number of patterns of objects and endow the linear span of said functions with a product and a coproduct. In this way, your favourite combinatorial objects generate a Hopf algebra. For example, the Hopf algebra on permutations studied by Vargas in [Var14] and the Hopf algebra on word quasisymmetric functions are particular cases of this construction.

Further, we also study a particular case of such a Hopf algebra structure, defined on marked permutations, i.e. permutations with a marked element. These objects have an inherent multiplication structure called inflation product, an operation that is motivated from some work on permutation patterns. In this paper we show that this Hopf algebra is a free algebra.

Keywords: marked permutations, presheafs, species, Hopf algebras, free algebras.

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[1]raul.penaguiao@math.uzh.ch

1. Introduction and motivation

The notion of substructures is important in mathematics, and particularly in combinatorics. In graph theory, minors and induced subgraphs are the main examples of studied substructures. Other kinds of substructures also got some attention: set partitions, trees, paths and, to a bigger extent, permutations, where the study of patterns leads us to the concept of permutation class.

A priori unrelated, Hopf algebras are a natural tool in algebraic combinatorics to study graphs, set compositions and permutations. For instance, the celebrated Hopf algebra on permutations named after Malvenuto and

Reutenauer sheds some light on the structure of shuffles in permutations. Other examples of Hopf algebras in combinatorics are the word quasisymmetric functions with a basis indexed by set compositions, and the permutation pattern Hopf algebra introduced by Vargas in [Var14].

With that in mind, we build upon the notion of species, as presented in [AM10] by Aguiar and Mahajan, in order to connect these two areas of algebraic combinatorics. By enriching a species with restriction maps we obtain a presheaf, and we will show an automatic construction of a pattern algebra from any given presheaf.

Main examples of combinatorial presheaves are patterns on words, graphs and permutations. Lesser known examples that we introduce here are posets, which result from the presheaf of oriented graphs, simplicial complexes, which contain the graphs and the matroids, generalised permutahedra, and some marked counterparts like marked graphs.

The algebras obtained are always commutative. In analyzing Hopf algebras, it is of particular interest to show that such algebras are free commutative (henceforth, we say simply *free*), and to construct free generators of the algebra structure. This allows us, for instance, to describe the characters of the Hopf algebra. This freeness problem is addressed for the shuffle algebra $\Bbbk\langle A\rangle$ over a generic alphabet, see [Reu03]. In other combinatorial algebras this was also tackled in [BZ09] and [CFL58]. In this paper, we show that any commutative combinatorial presheaf gives rise to a free pattern algebra. Further, we also establish the freeness and enumerate the primitive elements of the pattern Hopf algebra on marked permutations. This follows the methods used to establish the freeness of the shuffle algebra in [CFL58], see also [GR14, Chapter 6].

1.1. Pattern Hopf algebra on monoids in presheafs

A species is a map from finite sets I to families $h[I]$, together with relabeling maps. Species occur very naturally in combinatorics as a way of describing the combinatorial structures of type h on the set I, for instance graph structures on a vertex set.

A *combinatorial presheaf* is a species enriched with restriction maps $\mathbf{res}_J : h[I] \to h[J]$ whenever $J \subseteq I$. In this way we see combinatorial presheafs as contravariant functors from the category of finite sets with injective maps as morphisms. The notion and the name of presheaves has been around in category theory and geometry for some time, where it generally refers to contravariant functors from the open sets of a topology with inclusions as morphisms. Main examples of combinatorial presheafs are graphs and set compositions, or permutations, see Example 1. In general, any combinatorial object that admits a notion of restriction admits a presheaf structure.

Example 1 (The presheaf on permutations)**.** For us, a permutation π on I is a pair of total orders $(\leq_P, \leq_V)$ on the set I, as discussed in [ABF18]. The natural notion of restriction and relabeling recovers the usual concept of pattern commonly present in the literature. We introduce `Per` as the resulting presheaf on permutations.

It will be useful to represent permutations in I as square diagrams labeled by I. This is done in the following way: we place the elements of I in a n $|I|\times|I|$ grid so that the elements occur horizontally according to the $\leq_P$ order, and vertically according to the $\leq_V$ order. For instance, the permutation $\pi = \{1 <_P 3 <_P 2, 2 <_V 1 <_V 3\}$ in $\{1,2,3\}$ can be represented as

$$\begin{array}{|c|c|c|}\hline & & 3 \\ \hline 1 & & \\ \hline & 2 & \\ \hline \end{array}\ . \tag{1}$$

Up to isomorphism, we can represent a permutation as a diagram with one dot in each column and row.

In presheaves, the notions of *isomorphic object* is defined naturally, through the relabeling maps. The collection of objects of a presheaf h up to isomorphism is written $\mathcal{G}(h)$. For instance, when we consider the presheaf `Per` of permutations, $\mathcal{G}(\texttt{Per})$ stands for the set of permutations up to isomorphism, so that there are $n!$ many of size n, although there are $(n!)^2$ many permutations in a set I of size n. The notion of a *pattern* is also naturally defined in presheaves, and we recover the ones that are already known in the literature from permutations and graphs. By counting the number of patterns of b that are isomorphic to an object a, we define the *pattern function* $\mathbf{p}_a(b)$.

If h is a combinatorial presheaf, then the linear span of the pattern functions is a linear subspace $\mathcal{A}(h) \subseteq \mathcal{F}(\mathcal{G}(h), \mathbb{R})$ of the space of real functions in $\mathcal{G}(h)$.

Theorem 2. The vector space $\mathcal{A}(h)$ is closed under pointwise multiplication and has a unit. So, it forms an algebra, called the *pattern algebra.* In fact, we have the product rule

$$\mathbf{p}_a\,\mathbf{p}_b = \sum_c \binom{c}{a,b} \mathbf{p}_c\,, \tag{2}$$

where we define the coefficients $\binom{c}{a,b}$ below in (8) as the number of quasi-shuffles of a, b that result in c.

Quasi-shuffles have been studied in several contexts. For details on quasi-shuffles of combinatorial objects, the interested reader can see [AM10] and [].

We assume now that our combinatorial objects have an additional product structure. In our setting, this corresponds to a monoid in the category of presheaves, i.e. a presheaf together with a compatible associative multiplication map and an identity. Basic examples are the disjoint union of graphs and the direct sum of permutations. The main example that we study in this paper is the inflation of marked permutations, defined below.

With a product $\cdot$ in our combinatorial objects, we can define the following coproduct in the pattern algebra

$$\Delta\, \mathbf{p}_a = \sum_{\substack{a=b\cdot c \\ b,c \text{ up to iso}}} \mathbf{p}_b \otimes \mathbf{p}_c \,. \tag{3}$$

The main property and motivation for this specific coproduct in $\mathcal{A}(h)$ is that the natural inclusion of function algebras $\mathcal{F}(A, B)^{\otimes 2} \hookrightarrow \mathcal{F}(A \times A, B)$ allows us to say that

$$\Delta\, \mathbf{p}_a(b, c) = \mathbf{p}_a(b \cdot c) \,. \tag{4}$$

This is shown in Theorem 8, and is central in establishing that Δ is indeed a coproduct, and that it is compatible with the multiplication above.

Theorem 3. If (h, μ, ι) is a monoid in the category of combinatorial presheaves such that $\#h[\emptyset] = 1$, then the pattern algebra together with this coproduct forms a Hopf algebra.

It is common to use category theory tools to construct Hopf algebras in a mechanical way, as it happens with Fock functors, see for instance [AM10, Chapter 15], as it gives us more algebraic tools to understand combinatorial objects.

Some known Hopf algebras can be constructed as the pattern algebra of some combinatorial presheaf. An example is $WQSym$, the Hopf algebra of *word quasisymmetric functions*, also known as quasisymmetrc functions in non commutative variables, introduced in []. This Hopf algebra has a basis indexed by set compositions, and corresponds to the pattern Hopf algebra of set compositions. The pattern Hopf algebra corresponding to permutations was introduced by Vargas in [Var14]. Some other Hopf algebras here constructed, like the one of marked permutations below, are new.

Remark 4. In [AM10], the structure of a Hopf monoid is introduced. In fact, a cocomutative Hopf monoid in sets is precisely a monoid in presheaves. Further, the coalgebra structure of the pattern Hopf algebras that we construct here is a subcoalgebra of the dual algebra of the so called *bosonic Fock functor* of these comonoids in linearized set species. In general, the algebra structure is different.

1.2. Free pattern Hopf algebras

We focus on the freeness of some pattern algebras. The first case that we want to explore is the one of commutative presheaves. A monoid in combinatorial presheaves is called *commutative* if its monoid operation is commutative, that is for any $a \in h[I], b \in h[J]$ we have that $a \cdot b$ and $b \cdot a$ are isomorphic.

As it turns out, this is the simplest case for our freeness problem.

Theorem 5 (Commutative combinatorial presheaves are free)**.** The pattern Hopf algebra of a commutative monoid is free.

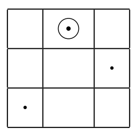
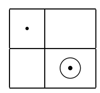
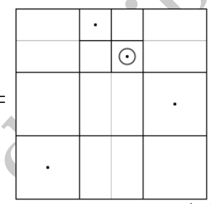
Example 6 (Permutations and their pattern Hopf algebra)**.** To the presheaf $\texttt{Per}$ it corresponds a pattern algebra $\mathcal{A}(\texttt{Per})$ as discussed above. In fact, we can further consider $\texttt{Per}$ with a monoid structure via the direct sum of permutations $\oplus$, defined as follows: If $\pi \in \texttt{Per}[I], \tau \in \texttt{Per}[I]$ are two permutations, the permutation $\pi \oplus \tau \in \texttt{Per}[I \sqcup J]$ results from adding to the order relation of π, τ that $i \leq j$ for any $i \in I, j \in J$. As mentioned above, the pattern Hopf algebra on permutations is the one discussed by Vargas in [Var14], where it is shown that it is free. We note that this is not a commutative presheaf: in general, $\pi \oplus \tau$ is a different permutation than $\tau \oplus \pi$.

In this paper we will focus on the following Hopf algebra.

Example 7 (Marked permutations and their pattern Hopf algebra)**.** A marked permutation π^* on I is a pair of orders $(\leq_P, \leq_V)$ on the set $I \uplus \{*\}$. Intuitively, this gives us a rearrangement of the elements of $I \uplus \{*\}$, where one element is special and marked. Again, a natural notion of restrictions and relabelings arises, parallel to the one in permutations, giving us a presheaf $\texttt{MPer}$.

Motivated by the inflation procedure on permutations described in [AAK03], we further consider an inflation product $\star$ in marked permutations: Given two marked permutations $\tau^* \in \texttt{MPer}[I]$ and $\pi^* \in \texttt{MPer}[J]$, the inflation product $\tau^* \star \pi^* \in \texttt{MPer}[I \sqcup J]$ is a marked permutation resulting from replacing in the diagram of τ^* the marked element with the diagram of π^*.

Hence, the inflation of two marked permutations looks like this:

$\tau^* = 1\bar{3}2 =$ , $\pi^* = 2\bar{1} =$ , $\tau^* \cdot \pi^* = 14\bar{3}2 =$  (5)

This is a presheaf monoid on the species of marked permutations, hence it gives us a Hopf algebra on marked permutations $\mathcal{A}(\texttt{MPat})$, according to Theorem 8. Again, this is not a commutative presheaf. However, we still have the following:

Theorem 8 (Freeness of the marked permutation pattern Hopf algebra)**.** The Hopf algebra $\mathcal{A}(\texttt{MPat})$ is free.

To establish the freeness of $\mathcal{A}(\texttt{MPat})$ we present a unique factorization theorem on marked permutations with the inflation product in Theorem 51. This is an analogue of [AAK03, Theorem 1]. With it, we find generators of the algebra on marked permutations, and use tools from word combinatorics, specifically the Lyndon factorization of words in [CFL58], to show that these generators are free generators.

Finally, we enumerate the primitive elements of the pattern Hopf algebra $\mathcal{A}(\texttt{MPer})$, which correspond to the *simple marked permutations*, the marked permutations that are irreducible with respect to the inflation product.

1.3. Strategy for establishing the freeness of a pattern algebra

We now discuss the general strategy that we employ when establishing the freeness of a pattern Hopf algebra. In particular, we clarify what is the relation between unique factorisation theorems and freeness of the algebra of interest. Let $\mathcal{S} \subseteq \mathcal{G}(h)$ be a collection of objects in a presheaf h. Then the set $\{\mathbf{p}_s \,|s \in \mathcal{S}\}$ is a free generator of $\mathcal{A}(h)$ if the set

$$\left\{ \prod_{s \in S} \mathbf{p}_s \,|S \text{ multiset of elements of } \mathcal{S} \right\},$$

is linearly independent. This is usually established by connecting this set with the set $\{p_a | a \in \mathcal{G}(h)\}$, which is know to be linearly independent by Remark 21. This connection is done with two ingredients and a property:

- An order $\preceq$ in $\mathcal{G}(h)$.
- A bijection α between $\{\prod_{s \in S} \mathbf{p}_s \,| S$ multiset of elements of $\mathcal{S}\}$ and $\mathcal{G}(h)$, which is usually phrased in terms of a *unique factorization theorem* with a bijection between m. See for example Theorem 52.
- These ingredients satisfy, for any S multiset of $\mathcal{S}$,

$$\prod_{s \in S} \mathbf{p}_s = \sum_{t \preceq \alpha(S)} c_{t,S}\, \mathbf{p}_t \,, \tag{6}$$

  with coefficients $c_{t,s}$ such that $c_{\alpha(S),S} \neq 0$.

For the case of marked permutations, as it is seen in the proof of Theorem 52, these are enough to establish the desired linear independence. In the commutative presheaf case, we see in Theorem 5 that the unique factorization theorem comes for free, see Theorem 34.

2. Preliminaries

2.1. Species and monoidal functors

Let $\mathtt{Set}_{\hookrightarrow}$ be the category whose objects are finite sets and morphisms are all the injective maps between finite sets. Let also $\mathtt{Set}_{\times}$ be the category whose objects are finite sets and morphisms are all the bijective maps between finite sets. Write $\mathtt{Set}$ for the usual category on finite sets.

Definition 9 (Monoidal category of presheafs)**.** A *combinatorial presheaf*, or a *presheaf* for simplicity, is a contravariant functor $h : \mathtt{Set}_{\hookrightarrow}^{op} \to \mathtt{Set}$. If $\#h[\emptyset] = 1$, we say that h is a *connected presheaf*.

In particular, we are given a collection $h[I]$, for each finite set I, of elements that we call *objects* that are thought of as the objects of type h based on the set I. Additionally, for each bijection $\sigma : I \to J$ we are also given $h[\sigma] : h[J] \to h[I]$, the relabelling maps. If $\#I = \#J$, and $a \in h[I], b \in h[J]$, we write $a \sim b$ if there is a bijection $\sigma : I \to J$ such that $h[\sigma](b) = a$. Note that this defines an equivalence relation on the objects of type h. We define a *coinvariant* as an equivalence class of $\sim$, and write $h[I]_\sim$ for the coinvariants in $h[I]$. If $a \in h[I]$, we define the *size* of a as $|a| := \#I$.

The above information describes a set species. A presheaf further extends such notion by setting, given $J \subseteq I$ and $a \in h[I]$, a restriction $a|_J := h[J \hookrightarrow I](a)$. This is how we see patterns and substructures in the objects of type h. The restriction maps, together with the species structure, define the presheaf structure.

Example 10. The species $\mathcal{E}$ with only one object $*$, which has size zero, has a unique presheaf structure. The presheaf of graphs

$$\mathtt{G}[I] = \{\text{graphs with vertex set } I\} ,$$

results from the usual species structure by adding the natural graph restrictions.

Definition 11 (Patterns in presheafs)**.** Let $a \in h[I]$. If $b \sim a|_J$ for some $J \subseteq I$, we call b a *pattern* of a and write $b \leq a$. In this case we call J an *occurrence* of b in a.

In what follows we work on the category of combinatorial presheafs, which is a monoidal category with the usual Cauchy product $\odot$ as defined in [AM10]. In particular, a monoid in this category is a combinatorial presheaf h together with natural transformations $\eta : h \odot h \Rightarrow h$ and $\iota : \mathcal{E} \Rightarrow h$ that satisfy the associativity and unit conditions. We use, for $a \in h[I]$ and $b \in h[J]$, the notation $\eta_{I,J}(a, b) = a \cdot b$. We also denote $1 := \iota[\emptyset](*)$.

Take disjoint sets I, J, K. The naturality of η, the associativity and unit conditions correspond to, respectively,

- For $a \in h[I], b \in h[J]$ and $A \subseteq I, B \subseteq J$ then $(a \cdot b)|_{A \uplus B} = a|_A \cdot b|_B$.
- For $a \in h[I], b \in h[J], c \in h[K]$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- For $a \in h[I]$, we have $a \cdot 1 = 1 \cdot a = a$.

2.2. Preliminaries on permutations and marked permutations

To fit the framework of presheafs, we use a rather unusual definition of permutations introduced in [ABF18].

Definition 12 (Permutations)**.** A permutation on a set I is a pair $(\leq_P, \leq_V)$ of total orders in I. We say that π is a permutation on I, or that $\mathbb{X}(\pi) = I$. If we add the natural concept of restrictions, this defines the presheaf `Per` of permutations. In particular, two permutations π and τ, in I_1, I_2 respectively, are said to be isomorphic if there is a bijection $f : I_1 \to I_2$ that maps both orders of π to both orders of τ, respectively.

This relates to the usual notion of a permutation as follows: if we order the elements of $I = \{a_1 \leq_P \cdots \leq_P a_k\} = \{b_1 \leq_V \cdots \leq_V b_k\}$, then this defines a bijection $a_i \mapsto b_i$ in I, that we identify with the original permutation. Conversely, for any bijection f on I, there are several pairs of orders $(\leq_P, \leq_V)$ that correspond to the bijection f, all of which are isomorphic.

We can write a permutation in its two-line notation, as $\genfrac{}{}{0pt}{}{a_1,\dots,a_k}{b_1,\dots,b_k}$ where $a_1 \leq_V a_2 \leq_V \dots$ and $b_1 \leq_P b_2 \leq_P \cdots \leq_P b_k$. If we identify $b_1, \dots, b_k$ with $1, \dots, k$, respectively, we can disregard the bottom line. This also disregards the indexing set I, and in fact any two isomorphic permutations have the same representation.

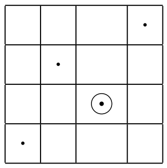
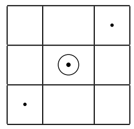
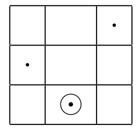
Definition 13 (Marked permutations)**.** A marked permutation on a set I is a pair $\pi^* = (\leq_P, \leq_V)$ of total orders in $I \uplus \{*\}$. We say that π^* is a marked permutation on I, or that $\mathbb{X}(\pi^*) = I$. Again, with the intrinsic notion of restrictions, we have a natural presheaf structure `MPer`. In particular, two marked permutations π^* and τ^*, in I_1, I_2 respectively, are said to be isomorphic is there is a bijection $f : I_1 \to I_2$ that maps both orders of π^* to both orders of τ^*. Note that this corresponds to the notion of permutation isomorphism, where we require the relabelling to map the marked element to the marked element.

We can also write marked permutations in a one line notation, where we add a marker over the position of $*$. The resulting notation disregards the indexing set I, and so any two isomorphic marked permutations have the same representation. Note that for each permutation of size n it corresponds n different non-isomorphic marked permutations, one for each possible marked position.

Example 14. If we consider $(1 <_P 2 <_P * <_P 4,\ 1 <_V * <_V 4 <_V 2)$, a marked permutation on $\{1, 2, 4\}$, its one line representation is $14\bar{2}3$.

The marked permutation $\tau^* = (73 <_P x <_P * <_P 47,\ 73 <_V * <_V x <_V 47)$ is based on the set $I = \{x, 47, 73\}$ and has a one line representation $13\bar{2}4$.

It is also possible to denote a marked permutation as a diagram, where we mark the block corresponding to $*$. So, for instance, the marked permutation τ^* corresponds to the diagram below

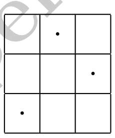
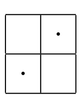
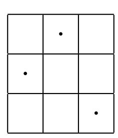
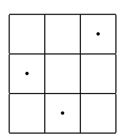
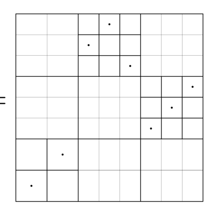
$\tau^* =$  , $\pi^* =$  , $\sigma^* =$  . (7)

We write $\texttt{MPer}[I]$ for the marked permutations in the set I. Then $\texttt{MPer}[0] = \{\bar{1}\}$ and $1\bar{2}3 \in \texttt{MPer}[2]$. In general, $\texttt{MPer}[n]_\sim$ has $(n+1)(n+1)!$ elements.

Remark 15 (Patterns in marked permutations)**.** Let π^* and τ^* be marked permutations. Recall that, if $\pi^* \sim \tau^*|_J$ for some $J \subseteq I$, we say that π^* is a pattern of τ^*, and J is an occurrence of π^* in τ^*. This corresponds to the usual notion of patterns in permutations, with the added restriction that the marked element has to be in the occurrence of the pattern.

Example 16. Consider $\tau^* = 13\bar{2}4$ marked permutation on $\{1 <_P 2 <_P 3\}$, $\pi^* = 1\bar{2}3$ and $\sigma^* = 2\bar{1}3$ marked permutations in $\{1 <_P 2\}$, as in (7).Then, we have that $\pi^*, \sigma^* \leq \tau^*$, because $J = \{1, 3\}$ is an occurrence of π^* in τ^*, and $J' = \{2, 3\}$ is an occurrence of σ^* in τ^*.

Definition 17 (Inflation of permutation and marked permtuations)**.** Let $\pi = (\leq_P, \leq_V)$ be a permutation of size n, and take $\sigma_1, \dots, \sigma_n$ permutations in the disjoint sets $I_1, \dots, I_n$, possibly empty. Then we define the inflation permutation $\pi[\sigma_1, \dots, \sigma_n]$ as a permutation in $I_1 \uplus \cdots \uplus I_n$ by replacing the i-th element on the permutation π, according to $\leq_P$, with the orders given by σ_i, obtaining two new orders. So, if $\pi = 132$, $\sigma_1 = 12$, $\sigma_2 = 231$ and $\sigma_3 = 123$ then $\pi[\sigma_1, \sigma_2, \sigma_3] = 12786345$.

$\pi =$  , $\sigma_1 =$  , $\sigma_2 =$  , $\sigma_3 =$  , $\pi[\sigma_1, \sigma_2, \sigma_3] =$  ,

If π^* and τ^* are marked permutations, we can define the inflation product $\pi^* \cdot \tau^*$ by the permutation π inflated by τ^* in the position of the marked element of π^*. If we take $\tau^* = 1\bar{3}2$ and $\pi^* = 2\bar{1}$, then $\tau^* \cdot \pi^* = 14\bar{3}2$.

Finally, if a, b are two permutations, we recall that the $\oplus$ and $\ominus$ operations act as $a \oplus b := 12[a, b]$ and $a \ominus b := 21[a, b]$. If exactly one of a or b is a marked permutation, we obtain in $a \oplus b$ and $a \ominus b$ marked permutations defined as expected.

Note that the inflation of permutations and of marked permutations is stable under the equivalence relation $\sim$.

Remark 18 (Inflations and quasi-shuffle)**.** We remark that the inflation of marked permutations is a quasi shuffle of the given marked permutations.

This is in fact a general claim for any monoidal combinatorial presheaf.

3. Substructure algebras

Here we establish a general framework that encompasses the algebra on marked permutations, the one on permutations studied in [Var14], as well as a pattern algebra on graphs and on marked graphs discussed in [Pen].

3.1. Natural family and pattern algebras

Definition 19 (The family of natural objects)**.** Given a combinatorial presheaf h, the *natural family* is the categorical colimit of the functor h restricted to the diagram $\mathtt{Set}^{\times}$. Equivalently, and for sake of fixing notation of this paper, it can be described as the family

$$\mathcal{G}(h) = \biguplus_{n \geq 0} h[n]_{\sim} \,.$$

Definition 20 (Pattern functions)**.** Let h be a presheaf, and consider $a \in h[I]$. We define the *pattern function* $\mathbf{p}_a : \biguplus_J h[J] \to \mathbb{Q}$ as follows: if $b \in h[J]$, then

$$\mathbf{p}_a(b) = \#\{J' \subseteq J \text{ s.t. } b|_{J'} \sim a\} \,.$$

Observe that this function depends only on the equivalence classes of a and b, so we may further convention that $\mathbf{p}_a \in \mathcal{F}(\mathcal{G}(h), \mathbb{Q})$ for $a \in \mathcal{G}(h)$. Finally, write

$$\mathcal{A}(h) := \operatorname{span}\{\mathbf{p}_a \mid a \in \mathcal{G}(h)\} \subseteq \mathcal{F}(\mathcal{G}(h), \mathbb{Q}) \,,$$

for the linear space spanned by all pattern functions.

Remark 21. If two objects a, b are such that $|a| \geq |b|$ and $a \not\sim b$, then $\mathbf{p}_a(b) = 0$. We have $\mathbf{p}_b(b) = 1$. Hence, the set $\{p_a | a \in \mathcal{G}(h)\}$ is a basis of $\mathcal{A}(h)$.

For a, b generic objects and $c \in h[C]$, define the *quasi-shuffle* number as:

$$\binom{c}{a, b} = \#\{(I, J) \, s.t. \; I \cup J = C \, , \; c|_I \sim a, \, c|_J \sim b\} \, , \tag{8}$$

which is invariant under the equivalence classes of $\sim$. This corresponds to the number of occurrences of the equivalence class of c in the multiset of quasi-shuffles of a, b, as defined in [Pen].

We now observe that $\mathcal{A}(h)$ is a subalgebra of $\mathcal{F}(\mathcal{G}(h), \mathbb{Q})$ with the pointwise multiplication structure.

Theorem 22. Let h be a presheaf. Then the pattern functions satisfy the following identity:

$$\mathbf{p}_a \, \mathbf{p}_b = \sum_{c \in \mathcal{G}(h)} \binom{c}{a, b} \mathbf{p}_c \; . \tag{9}$$

In particular, the pattern functions of h span a subalgebra of the function algebra $\mathcal{F}(\mathcal{G}(h), \mathbb{Q})$, where the unit is $\sum_{c \in h[\emptyset]} \mathbf{p}_c$. We say that $\mathcal{A}(h)$ is the *pattern algebra* of h.

Proof. Fix $x \in h[I]$, and note that $\mathbf{p}_a(x) \, \mathbf{p}_b(x)$ counts the following

$$\begin{aligned} \mathbf{p}_a(x) \, \mathbf{p}_b(x) &= \#\{A \subseteq I \text{ s.t. } x|_A \sim a\} \times \{B \subseteq I \text{ s.t. } x|_B \sim b\} \\ &= \#\{(A, B) \text{ s.t. } A, B \subseteq I, \, x|_A \sim a, \, x|_B \sim b\} \\ &= \sum_{C \subseteq I} \#\{(A, B) \text{ s.t. } A \cup B = C, \, x|_A \sim a, \, x|_B \sim b, \} \\ &= \sum_{C \subseteq I} \binom{x|_C}{a, b} = \sum_{c \in \mathcal{G}(h)} \binom{c}{a, b} \mathbf{p}_c(x) \, . \end{aligned} \tag{10}$$

Hence, the space $\mathcal{A}(h)$ is closed for the product of functions. Further, it is easy to observe that $\sum_{a \in h[\emptyset]} \mathbf{p}_a$ is the unit, so this is an algebra, concluding the proof. $\square$

3.2. Coproduct on presheafs

In this section we consider a presheaf monoid (h, η, ι). Namely, our combinatorial presheaf h is endowed with an associative product $\cdot$ and a unit $1 \in h[\emptyset]$. Given $a \in h[I]$, we do not necessarily have that $a|_\emptyset = 1$ or that $(a \cdot b)|_I = a$. This is so, however, if h is a *connected presheaf*, i.e. a presheaf that has $\# h[\emptyset] = 1$.

Remark 23. If a is an object, we will denote its equivalence class under $\sim$ by $\bar{a}$ for the realm of this remark. If (h, η, ι) is a presheaf monoid, then $\mathcal{G}(h)$ inherits an associative product as follows:

Let $\bar{a} \in h[n_1]_\sim, \bar{b} \in h[n_2]_\sim$ and define $C = \{n_1+1, \dots, n_1+n_2\}$. Consider $\bar{b'} \in h[C]_\sim$ such that $b = h[st](b')$, where st is the order preserving map between $[n_2]$ and C. Then we define the product in $\mathcal{G}(h)$ as $\bar{a} \cdot \bar{b} := \overline{a \cdot b'} \in h[n_1+n_2]_\sim$.

Definition 24. Let $a \in h[I]_\sim$. Then, define

$$\Delta \, \mathbf{p}_a := \sum_{\substack{a \sim b \cdot c \\ b,c \in \mathcal{G}(h)}} \mathbf{p}_b \otimes \mathbf{p}_c \, .$$

Note that the right hand side is a finite sum, so this is well defined. Define further the map $\epsilon : \mathcal{A}(h) \to \mathbb{Q}$ that sends $\mathbf{p}_a$ to $\mathbb{1}[a = 1]$.

We recover here and prove Theorem 8.

Theorem 25 (The pattern Hopf algebra)**.** Let (h, η, ι) be an associative presheaf. Then, the map Δ, together with multiplication of functions, defines a bialgebra structure in $\mathcal{A}(h)$.

Further, $\mathcal{A}(h)$ is a Hopf algebra if and only if $h[\emptyset]$ is a group with the operation $\eta_{\emptyset,\emptyset}$.

Proof of Theorem 8. Usually, when establishing that a structure is a bialgebra, the hardest part is to establish that the product and the coproduct are compatible. In this case, that follows after we observe that, for $a, x, y \in \mathcal{G}(h)$,

$$\Delta \, \mathbf{p}_a(x \otimes y) = \mathbf{p}_a(x \cdot y) \, , \tag{11}$$

using that $\mathcal{F}(\mathcal{G}(h), \mathbb{Q})^{\otimes 2} \subseteq \mathcal{F}(\mathcal{G}(h)^2, \mathbb{Q})$.

We establish (11) here, and outline the remaining of the proof later. Take $x \in h[n_1]_\sim$ and $y \in h[n_2]_\sim$, and write $B = [n_1]$, $C = \{n_1+1, \dots, n_1+n_2\}$. Let st be the order preserving map between C and $[n_2]$. Then

$$\begin{aligned}
\mathbf{p}_a(x \cdot y) &= \#\{J \subseteq [n_1+n_2] \text{ s.t. } (x\cdot y)|_J \sim a\} \\
&= \#\{J \subseteq [n_1+n_2] \text{ s.t. } x|_{J\cap B} \cdot y|_{st(J\cap C)} \sim a\} \\
&= \sum_{b\cdot c = a} \#\{J \subseteq [n_1+n_2] \text{ s.t. } x|_{J\cap B} \sim b,\ y|_{st(J\cap C)} \sim c\} \\
&= \sum_{b\cdot c = a} \#\{J \subseteq B \text{ s.t. } x|_J \sim b\}\ \#\{J \subseteq C \text{ s.t. } y|_{st(J)} \sim c\} \\
&= \sum_{b\cdot c = a} \mathbf{p}_b(x)\, \mathbf{p}_c(y) = \Delta\, \mathbf{p}_a(x \otimes y)\,.
\end{aligned} \tag{12}$$

Since both function take the same values at $\mathcal{G}(h)^2$, we conclude that (11) holds. It is now trivial to conclude that $\mathcal{A}(h)$ is a bialgebra.

Further, since $\mathcal{A}(h)$ is a graded bialgebra, it is a Hopf algebra if and only if the zero degree component $\mathcal{A}(h)_0$ is a Hopf algebra. Note the following isomorphism of bialgebras $\mathcal{A}(h)_0 \cong \mathbb{Q}h[\emptyset]^*$, and is well known that $\mathbb{Q}h[\emptyset]$ is a Hopf algebra if and only if $h[\emptyset]$ is a group, with antipode $s(g) = g^{-1}$. TODO □

3.3. Examples of monoids in presheaves

There are several interesting presheaves, most of which have a monoid structure that interests us.

Example 26 (Patterns in graphs)**.**

Example 27 (Patterns in posets)**.**

Example 28 (Patterns in simplicial complexes)**.**

Example 29 (Patterns in set partitions)**.**

Example 30 (Patterns in generalized permutahedra)**.**

4. Comutative combinatorial presheaves

In this section we will explore other pattern algebras. In particular, we will show that, if the algebra has a product with a unique factorization theorem, then it is free. The main examples are marked graphs, simplicial complexes and posets.

We also debate other

4.1. Marked graphs

Definition 31. Let G^* be a marked graph. We say that G^* is **markedly connected** if G^o, the graph resulting from removing the marked vertex and its incident edges from G^*, is a connected graph.

We define the joint product $\vee$ on marked graphs as follows:

Theorem 32 (Unique factorization in marked graphs)**.** If G^* is a marked graph, there is a unique factorization on markedly connected graphs $G_1^*, \ldots, G_k^*$ such that

$$G^* = \bigvee_{i=1}^{k} G_k^* \,.$$

Proof. First, observe that G^o decomposes uniquely into connected graphs $G^o = \biguplus_{i=1}^{k} H_i$, so define G_i^* marked graphs as follows TODO □

Corollary 33 (Freeness of the pattern algebra on graphs)**.** The patterns algebra on marked graphs is free, and its free generators are the markedly connected graphs.

The proof of this corolary is the result of Theorem 32 and the general result Theorem 35 that we introduce now. First, we describe what we understand by a commutative monoid structure in combinatorial presheaves, and what are irreducibles in that context. Then, we present a result that is very general and can be applied in several pattern algebras.

4.2. Pattern algebras of commutative monoids in presheaves

Let h be a connected presheaf. A *commutative monoid* over h is a monoid structure $(h, \times, \iota)$ such that $\times : h \cdot h \Rightarrow h$ is preserved through the braiding, that is $\times_{A,B} = \times_{B,A} \circ \text{ twist }$.

An object $o \in \mathcal{G}(h)$ is called *indecomposable* if there are no two objects $a \in h[A], b \in h[B]$ such that $\times_{A,B}(a, b) \sim o$. The family of indecomposable objects with respect to the product $\times$ is denoted $\mathcal{H}_\times \subseteq \mathcal{G}(h)$.

For us a factorization of an object $o \in h[X]$ is a partition of X, for instance $A_1, \ldots A_k$, such that $o = \times_{A_1,\ldots A_k}(o|_{A_1}, \ldots, o|_{A_k})$. Not all partitions yield a factorization, and the irreducible elements are precisely the ones that have no other factorization other than the trivial partition.

Theorem 34. Let $a \in \mathcal{G}(h)$ be an object. Then, it has a unique factorization into irreducibles $l_1, \dots, l_k \in \mathcal{H}$, that is

$$a = \times_{A_1,\dots,A_{k(a)}}(l_1, \dots, l_{k(a)}) \in h[X]\,,$$

where $k(a)$ is the well defined number of irreducible factors and $\sqcup_i A_i = X$.

Proof. Incidentally, it is trivial to establish that some factorization exists. Suppose now that a has two distinct factorizations

$$a = \times_{A_1,\dots,A_k}(l_1, \dots, l_k) = \times_{B_1,\dots,B_s}(r_1, \dots, r_s)\,, \tag{13}$$

where $\pi = \{A_1, \dots, A_k\}$ and $\tau = \{B_1, \dots, B_s\}$ are set partitions of X, and note that

$$l_j = a|_{A_j} = \times_{B_1\cap A_j,\dots,B_s\cap A_j}(r_1|_{B_1\cap A_j}, \dots, r_s|_{B_s\cap A_j})\,.$$

By irreducibility, for each j there is exactly one i such that $B_i \cap A_j \neq \emptyset$, so π is coarser than τ. By a symmetrical argument, we obtain that π is finner than τ, so we conclude that $\pi = \tau$, and the factorizations in (13) are the same. In particular, $k(a)$ is a well defined integer, concluding the proof. $\square$

Theorem 35 (Freeness of pattern algebras with commutative products)**.** Let h be a connected presheaf, and $(h, \iota, \times)$ a commutative monoid in the category of combinatorial presheaves. Let $\mathcal{H} = \mathcal{H}_\times \subseteq \mathcal{G}(h)$ be the family of irreducible elements with respect to $\times$.

Then $\{\mathbf{p}_l \,|\, l \in \mathcal{H}\}$ is a set of free generators of $\mathcal{A}(h)$.

Proof. The proof will follow the lines of Theorem 53: we build an order $\leq_{fac}$ in $\mathcal{G}(h)$ that stems from the unique factorization theorem in Theorem 34, and an equation of the type (15) arises.

Define the following partial order $\leq_p$ in $\mathcal{G}(h)$: we say that $a \leq_p b$:

- if $|a| < |b|$, or;
- if $|a| = |b|$ and $k(a) \leq k(b)$.

Now consider a total order $\leq_{fac}$ in $\mathcal{G}(h)$ that extends $\leq_p$. This will be the order that we will use to establish freeness.

We will now show that $\{\mathbf{p}_a \,|\, a \in \mathcal{H}\}$ is a free generator set, that is, the family

$$\left\{ \prod_{l\in\mathcal{L}} \mathbf{p}_l \,|\, \mathcal{L} = \{l_1, \dots, l_k\} \text{ multiset of elements in } \mathcal{H} \right\},$$

is a basis for $\mathcal{A}$. Then it sufices to show that, if the unique factorization of a into indecomposables is $a = \times_{A_1,\dots,A_{k(a)}}(l_1, \dots, l_{k(a)})$, then

$$\prod_{i=1}^{k} \mathbf{p}_{l_i} = \sum_{b \leq_{fac} a} c_b \, \mathbf{p}_b \,, \tag{14}$$

where each $c_b \geq 0$ and $c_a \geq 1$. Indeed, observe that a is a quasi-shuffle of $l_1, \dots, l_{k(a)}$ by considering the patterns $A_1, \dots, A_{k(a)}$; so we indeed have $c_a \geq 1$.

Now consider the maximal b that is a quasi-shuffle of $l_1, \dots, l_{k(a)}$, we establish (14) if we show that such b is a. Let $b = \times_{C_1,\dots,C_{k(b)}}(s_1, \dots, s_{k(b)})$ be the unique factorization into irreducibles, suppose that $b \in h[Y]$ and consider $B_1, \dots, B_{k(a)}$ sets such that

$$b|_{B_i} = a|_{A_i} = l_i \ \ \forall_{i=1,\dots,k(a)} \,,$$

and in particular such that $l_i = \times_{C_1 \cap B_i, \dots, C_{k(b)} \cap B_i}(s_1|_{C_1 \cap B_i}, \dots, s_{k(b)}|_{C_{k(b)} \cap B_i})$.

By irreducibility of l_i, we have that, for each i, there is exactly one j such that $C_j \cap B_i \neq \emptyset$. From $\sqcup_j C_j = \bigcup_i B_i = Y$, we get that each B_i is contained in some C_j. So we can define a map $f : [k(a)] \to [k(b)]$ such that $B_i \subseteq C_{f(i)}$. Also, we have that $C_j = \bigcup_{i \in f^{-1}(j)} B_i$.

First observe that $|a| = \sum_i \# A_i = \sum_i \# B_i$ and $|b| = \sum_j \# C_j$ therefore $|b| \leq \sum_j \sum_{i \in f^{-1}(j)} \# B_i = |a|$, so by maximality we have equality and so, the family $\{B_1, \dots, B_{k(a)}\}$ is disjoint.Further, f is a surjection, so we imediatelly have that $k(a) \geq k(b)$ and by maximality we reach an equality, and f is a bijective map and we conclude that $C_{f(i)} = B_i$ for each $i = 1, \dots, k(a)$.

We then conclude that

$$s_i = b|_{B_i} = b|_{C_{f(i)}} = l_{f(i)} \,,$$

and so, by commutativity, that

$$b = \times_{C_1,\dots,C_{k(b)}}(s_1, \dots, s_{k(b)}) = \times_{A_1,\dots,A_{k(a)}}(l_1, \dots, l_{k(a)}) = a \,,$$

as desired. □

4.3. Simplicial complexes

4.4. Generalised permutahedra

5. Hopf algebra structure on the marked permutations

In this section we consider the algebra structure of $\mathcal{A}(\texttt{MPer})$. We show that the pattern algebra on marked permutations is freely generated.

Our strategy is as follows: we describe a unique factorization of marked permutations with the inflation product, in Corollary 43. We also construct a set of generators of the type of Lyndon words, as introduced in [CFL58], leading to a notion of Lyndon marked permutations $\mathcal{L}_{SL}$, in Definition 50.

Finally, we reach the following result, which is the main theorem of this section:

Theorem 36. The algebra $\mathcal{A}(\texttt{MPat})$ is freely generated by $\{\mathbf{p}_{l^*} \,|\, l^* \in \mathcal{L}_{SL}\}$.

Furthermore, in the end of this section we enumerate the primitive elements of the Hopf algebra on marked permutations, which correspond to the simple marked permutations introduced below.

This section is organized as follows: we start in Section 5.1 and in Section 5.2 by establishing unique factorization theorems in marked permutations, by building an isomorphism between marked permutations and words of simple marked permutations. The proof that this construction is indeed an isomorphism is postponed to Section 5.4 In Section 5.3 we state and prove the main theorems with the help of some lemmas, whose proof is postponed to Section 5.5. Finally, in Section 5.6, we enumerate all the primitive elements of the pattern Hopf algebra $\mathcal{A}(\texttt{MPer})$.

5.1. Unique factorizations

We say that a permutation (resp. marked permutation) is $\oplus$-indecomposable if it does not result from an inflation of 12 with non-empty permutations (resp. if it has no non-trivial decomposition of the form $\tau_1 \oplus \tau_2^*$ or $\tau_1^* \oplus \tau_2$). Similarly, we say that a permutation (resp. a marked permutation) is $\ominus$-indecomposable if it does not result from an inflation of 21 with non-empty permutations (resp. if it has no non-trivial decomposition of the form $\tau_1 \ominus \tau_2^*$ or $\tau_1^* \ominus \tau_2$).

We recall now the definition of a simple permutation and of a simple marked permutation

Definition 37 (Simple permutations and simple marked permutations)**.** A permutation π is called simple if any expression $\tau[\tau_1, \dots] = \pi$ has $\tau = \pi$ and $\tau_1 = \cdots = 1$. A marked permutation π^* is called *simple* if any factorization $\pi^* = \tau_1^* * \tau_2^*$ has either $\tau_1^* = \bar{1}$ or $\tau_2^* = \bar{1}$.

Example 38. Examples of simple marked permutations include $\bar{1}423, 23\bar{1}$ and $31\bar{4}2$, see Fig. 1.

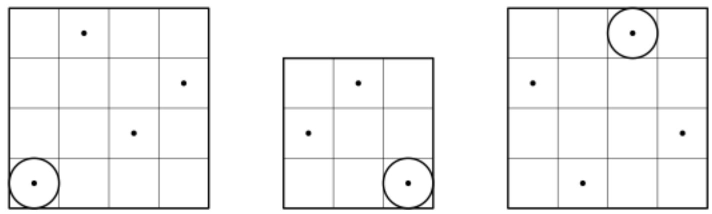


Figure 1: A $\oplus$-decomposable, $\ominus$-decomposable and an indecomposable simple marked permutations

If π is an $\oplus$-indecomposable permutation, then $\bar{1} \oplus \pi$ and $\pi \oplus \bar{1}$ are simple marked permutations. Similarly, if τ is an $\ominus$-indecomposable permutation, then $\bar{1} \ominus \tau$ and $\tau \ominus \bar{1}$ are simple marked permutations.

These decomposable simple marked permutations play an important role, because they are the only ones that get in the way of a unique factorization theorem for the inflation product. In the following we carefully unravel all these issues. We discuss how to enumerate simple marked permutations in Section 5.6.

Remark 39 ($\oplus$-relations and $\ominus$-relations)**.** Consider τ_1, τ_2 permutations that are $\oplus$-indecomposable. Then we have the following relations, called $\oplus$*-relations*

$$(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) * (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1 \, .$$

Consider now π_1, π_2 permutations that are $\ominus$-indecomposable. Then we have the following relation, called $\ominus$*-relations*

$$(\bar{1} \ominus \pi_1) * (\pi_2 \ominus \bar{1}) = (\pi_2 \ominus \bar{1}) * (\bar{1} \ominus \pi_1) = \pi_2 \ominus \bar{1} \ominus \pi_1 \, .$$

We wish to establish that these are precisely all the inflation relations that hold for the simple marked permutations.

We define the alphabet $\Omega := \{\text{simple marked permutations}\}$, and consider the set $\mathcal{W}(\Omega)$ of words on Ω, that forms a monoid under the usual concatenation of words. When $w \in \mathcal{W}(\Omega)$, we write w^* for the consecutive inflation of its letters. So for instance $(\bar{1}2, \bar{2}1)^* = \bar{1}2 * \bar{2}1 = \bar{2}13$. We use the convention that the inflation of the empty word is $\bar{1}$. This defines a morphism of monoids $* : \mathcal{W}(\Omega) \to \mathcal{G}(\texttt{MPer})$.

Definition 40 (Monoidal equivalence relation on $\mathcal{W}(\Omega)$)**.** We now define an equivalence relation on $\mathcal{W}(\Omega)$. Take a word $\omega = (s_1^*, \dots, s_k^*)$ in $\mathcal{W}(\Omega)$. We say that $\omega \sim (s_1^*, \dots s_{i-1}^*, s_{i+1}^*, s_i^*, \dots, s_k^*)$ whenever $s_i^* * s_{i+1}^* = s_{i+1}^* * s_i^*$ is an $\oplus$-relation or an $\ominus$-relation.

We further take the transitive closure to obtain an equivalence relation on $\mathcal{W}(\Omega)$.

We trivially have that if $\omega_1 \sim \omega_2$ and $\zeta_1 \sim \zeta_2$, then $\omega_1 \cdot \zeta_1 \sim \omega_2 \sim \zeta_2$. This means that the quotient $\mathcal{W}(\Omega)/_\sim$ is a monoid.

Theorem 41. The star map $* : w \mapsto w^*$ defines an isomorphism from $\mathcal{W}(\Omega)/_\sim$ to the monoid of marked permutations with the inflation product.

We postpone the proof of Theorem 41 to Section 5.4, and explore its consequences here.

As a consequence of this theorem, any two factorization of a marked permutation π^* into simple marked permutations, when seen as words in Ω, are related by $\sim$. The following corollary is immediate:

Corollary 42. If α^* is a marked permutation and we have simple factorizations $\alpha^* = s_1^* * \dots * s_k^* = r_1^* * \dots * r_j^*$, then $k = j$ and $\{s_1^*, \dots, s_k^*\} = \{r_1^*, \dots, r_j^*\}$ as multisets.

We define $j(\alpha^*)$ to be the number of simple factors in any simple factorization. Further, define $\mathfrak{fac}(\alpha^*)$ as the multiset of simple factors in any simple factorization. These are well defined as a consequence of Corollary 42.

A factorizations of a marked permutation α^*,

$$\alpha^* = s_1^* * \dots * s_j^*,$$

is said to be *stable* if the sequence $(s_1^*, \dots, s_j^*)$ of marked permutations satisfies the *stability conditions*, which are the following:

- **$\oplus$-stability condition** - There is no i integer and π, τ $\oplus$-indecomposable permutations such that

$$s_i^* = \bar{1} \oplus \pi \text{ and } s_{i+1}^* = \tau \oplus \bar{1};$$

- **$\ominus$-stability condition** - There is no i integer and π, τ $\ominus$-indecomposable permutations such that

$$s_i^* = \tau \ominus \bar{1} \text{ and } s_{i+1}^* = \bar{1} \ominus \pi.$$

Because it is clear that any equivalence class in $\mathcal{W}(\Omega)/_\sim$ admits a unique stable word, the following corollary is immediate.

Corollary 43 (Unique stable factorization)**.** Let α^* be a marked permutation. Then, there is a unique factorization in the inflation product into simple marked permutations that is stable.

5.2. Lyndon factorization on marked permutation

We introduce an order on permutations and two orders on marked permutations.

Definition 44 (Orders on marked permutations)**.** The *lexicographic order on permutations* is the lexicographic order when reading the one-line notation of permutations, and is written $\pi \leq_{per} \tau$.

Recall that, for a marked permutation $\alpha^* = (\leq_P, \leq_V)$ in I, we define its rank $pm(\alpha^*)$ as the rank of $*$ in $I \sqcup \{*\}$ with respect to the order $\leq_P$. We define the *lexicographic order on marked permutations*, also denoted $\leq_{per}$, as follows: we say that $\pi^* <_{per} \tau^*$ if $\pi \leq_{per} \tau$ or if $\pi = \tau$ and $pm(\pi^*) \leq pm(\tau^*)$.

This in particular endows our alphabet Ω of simple marked permutations with an order. When we compare words on $\mathcal{W}(\Omega)$ we extend the lexicographic order $\leq_{per}$ and denote it simply as $\leq$.

We define the *factorization order* $\leq_{fac}$ on marked permutations as follows: Let $\pi^* = s_1^* * \cdots * s_k^*$ and $\tau^* = t_1^* * \cdots * t_j^*$ be the respective stable factorizations. Then, we say that $\pi^* \leq_{fac} \tau^*$ if $(s_1^*, \dots, s_k^*) \leq (t_1^*, \dots, t_j^*)$ in $\mathcal{W}(\Omega)$.

Example 45. On permutations we have $132 \leq_{per} 231 \leq_{per} 4123$. In particular, the empty permutation is the smallest permutation.

On marked permutations, we have $1\bar{3}2 \leq_{per} 13\bar{2} \leq_{per} \bar{2}31 \leq_{per} 412\bar{3}$. In particular, the trivial marked permutation $\bar{1}$ is the smallest marked permutation.

On words, we have the following examples: $(24\bar{1}3, 31\bar{4}2) \leq (31\bar{4}2, 24\bar{1}3)$, $(\bar{1}432) \leq (21\bar{3}, \bar{1}32)$ and $(\bar{1}423, 2\bar{4}13, 2\bar{4}13) \leq (2\bar{4}13, \bar{1}423, 2\bar{4}13)$.

On marked permutations, because the $(24\bar{1}3, 31\bar{4}2)$ and $(31\bar{4}2, 24\bar{1}3)$ are stable, we have that $24\bar{1}3 * 31\bar{4}2 \leq_{fac} 31\bar{4}2 * 24\bar{1}3$. Notice however, that $(21\bar{3}, \bar{1}32)$ is not a stable word, as it violates the $\oplus$-stability condition for $i = 1$. Instead, we have $(21\bar{3}, \bar{1}32)^* = \bar{1}32 * 21\bar{3} \leq_{fac} \bar{1}432$.

Remark 46. If $\omega_1 \geq \omega_2$ are stable words in $\mathcal{W}(\Omega)$, then it follows from Corollary 43 that these are precisely the simple factorizations of ω_1^*, ω_2^*. Thus, $\omega_1^* \geq_{fac} \omega_2^*$.

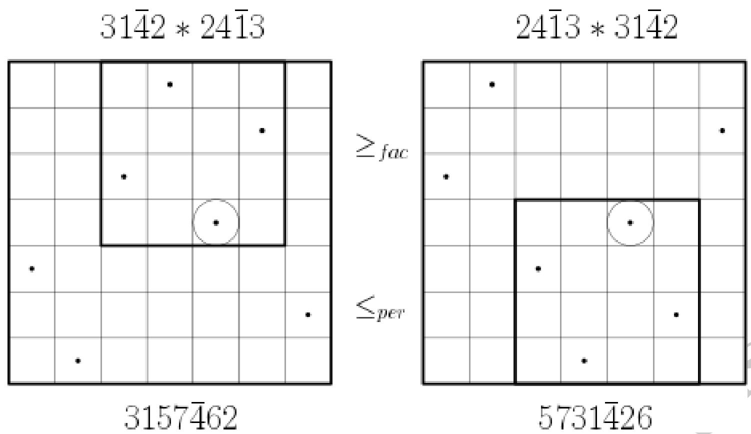


Figure 2: Two marked permutations and their order relations.

Remark 47. As a consequence of Theorem 41, among all words of simple marked permutations that inflate to α^*, the stable factorization is the smallest one in the lexicographical order in $\mathcal{W}(\Omega)$.

We shift the discussion to the topic of Lyndon words. This is because the unique factorization theorem obtained in Corollary 43 for marked permutations is not enough to establish the freeness of $\mathcal{A}(\texttt{MPer})$ and, as in [Var14], the study of Lyndon words allows us to obtain an improved unique factorization theorem in Theorem 52.

Definition 48 (Lyndon words)**.** Given an alphabet A with a total order, a word $l \in \mathcal{W}(A)$ is said to be a *Lyndon word* if, for any concatenation factorization $l = a_1 \cdot a_2$ into non-empty words we have that $a_2 \geq l$.

Example 49 (Examples of Lyndon words)**.** Consider the usual alphabet $\Omega = \{\bar{1} <_{per} \bar{1}2 <_{per} \bar{1}32 <_{per} \cdots <_{per} 23\bar{1} <_{per} 24\bar{1}3 <_{per} \dots\}$, then a Lyndon word in this alphabet is, for instance, $(\bar{1}2, \bar{1}32, \bar{1}2, 24\bar{1}3)$. Note, however, that $(\bar{1}, \bar{1})$ is **not** a Lyndon word.

Definition 50 (Stable Lyndon marked permutations)**.** A word on simple marked permutations $\omega = (s_1^*, \dots, s_j^*) \in \mathcal{W}(\Omega)$ is called *stable Lyndon*, or SL for short, if it is a Lyndon word and satisfies the *stability* conditions introduced above.

A marked permutation l^* is called *stable Lyndon*, or SL for short, if it is the inflation of an SL word $l = (s_1^*, \dots, s_j^*)$. We write $\mathcal{L}_{SL}$ for the set of SL marked permutations.

We see latter in Theorem 53 that $\mathcal{L}_{SL}$ is precisely the set that indexes the free basis of $\mathcal{A}(\texttt{MPer})$. To establish the corresponding unique factorization theorem, we first recall a classical fact in Lyndon words.

Theorem 51 (Unique Lyndon factorization theorem, [CFL58])**.** Consider a finite alphabet A with a total order.

Then any word has a unique factorization, in the concatenation product, into Lyndon words $l_1, \dots, l_k$ such that $l_1 \geq \cdots \geq l_k$ for the lexicographical order in $\mathcal{W}(A)$.

This theorem is central in establishing the freeness of $\mathcal{K}\langle\Omega\rangle$. So is the unique factorization into Lyndon marked permutations below, in establishing the freeness of $\mathcal{A}(\texttt{MPer})$.

Theorem 52 (Unique stable Lyndon factorization theorem)**.** Let α^* be a marked permutation. Then there is a unique sequence of SL words on marked permutations $(l_1, \dots, l_k)$ such that $l_i \geq l_{i+1}$ and $\alpha^* = l_1^* * \cdots * l_k^*$.

Proof. The existence follows from Corollary 43. Indeed, for any marked permutation α^*, there is a unique stable factorization $s_1^*, \dots, s_{j(\alpha^*)}^*$, and from the Lyndon factorization theorem, the word $(s_1^*, \dots, s_{j(\alpha^*)}^*) \in \mathcal{W}(\Omega)$ factors into Lyndon words $l_1, \dots, l_k$ such that $l_i \geq l_{i+1}$. These words are stable because $(s_1^*, \dots, s_{j(\alpha^*)}^*)$ is stable, and we clearly have that $\alpha^* = l_1^* * \cdots * l_k^*$. We then obtain the desired sequence of SL words $(l_1, \dots, l_k)$.

For the uniqueness, suppose we have SL words $m_1 \geq \cdots \geq m_{k'}$ that satisfy the required properties. We wish to show that this is precisely the sequence $(l_1, \dots, l_k)$ obtained above. Write $m_k = (r_{k,1}^*, \dots, r_{k,z_k}^*)$ and, for readability purposes, consider as well the re-indexing $m_1 \cdots m_{k'} = (r_1^*, \dots, r_z^*)$.

First observe that from $\alpha^* = m_1^* * \cdots * m_{k'}^*$ we get that $r_1^* * \cdots * r_z^*$ is a simple factorization of α^*. Further, the stability property in this factorization is a given for any i that is not of the form $z_1 + \cdots + z_v$, $v = 1, \dots, k' - 1$. On the other hand, we have that $m_k \geq m_{k+1}$ so $r_{k,z_k}^* \geq_{per} r_{k,1}^* \geq_{per} r_{k+1,1}^*$. this implies the stability condition for any i that is of the form $z_1 + \cdots + z_v$, $v = 1, \dots, k' - 1$.

Thus, $r_1^* * \cdots * r_z^*$ is the stable factorization of α^*, so that $(r_1^*, \dots, r_z^*) = (s_1^*, \dots, s_k^*)$. Further, $m_1 \geq \cdots \geq m_{k'}$ is the Lyndon factorization of $(r_1^*, \dots, r_z^*)$, so $(m_1, \dots, m_{k'}) = (l_1, \dots, l_k)$ by the uniqueness in Theorem 51. $\square$

We define $k(\alpha^*)$ to be the number of factors in its stable Lyndon factorization. Note that $k(\alpha^*) \leq j(\alpha^*)$ holds for any marked permutation α^*, where we recall that $j(\alpha^*)$ is the number of simple factors in a simple factorsation of α^*.

5.3. Freeness of the pattern algebra in marked permutations

In this section we state the main lemmas that help us proving the freeness of the algebra $\mathcal{A}(\texttt{MPer})$ in Theorem 36.

We consider the set of SL marked permutations $\mathcal{L}_{SL}$, which play the role as free generators, and consider a multiset of SL marked permutations $\{l_1^* \geq_{fac} \cdots \geq_{fac} l_k^*\}$ and $\alpha^* = l_1^* * \cdots * l_k^*$.

Then, all the terms that occur in the right hand side of

$$\prod_{i=1}^{k} \mathbf{p}_{l_i^*} = \sum_{\beta^*} \binom{\beta^*}{l_1^*, \ldots, l_k^*} \mathbf{p}_{\beta^*} \,, \tag{15}$$

correspond to *quasi-shuffles* of $l_1^*, \ldots, l_k^*$. Below we establish that the marked permutation $\alpha^* = l_1^* * \cdots * l_k^*$ is the biggest such marked permutation occurring in the right hand side of (15), with respect to a suitable total order related to $\leq_{fac}$. With this, the linear independence of all products of the form $\prod_{i=1}^{k} \mathbf{p}_{l_i^*}$ follows from the linear independence of $\{\mathbf{p}_{\alpha^*} \,|\alpha^* \in \mathcal{G}(\texttt{MPer})\}$, established earlier in Remark 21:

Theorem 53. Let α^* be a marked permutation, and suppose that $(l_1^*, \ldots, l_k^*)$ is its Lyndon factorization. Then there are coefficients $c_{\beta^*} \geq 0$ such that

$$\prod_{i=1}^{k} \mathbf{p}_{l_i^*} = \sum_{|\beta^*|<|\alpha^*|} c_{\beta^*} \, \mathbf{p}_{\beta^*} + \sum_{\substack{|\beta^*|=|\alpha^*| \\ j(\beta^*)<j(\alpha^*)}} c_{\beta^*} \, \mathbf{p}_{\beta^*} + \sum_{\substack{|\beta^*|=|\alpha^*| \\ j(\beta^*)=j(\alpha^*) \\ \beta^* \leq_{fac} \alpha^*}} c_{\beta^*} \, \mathbf{p}_{\beta^*} \,, \tag{16}$$

where $c_{\alpha^*} \geq 0$.

The proof of the main theorem, Theorem 53, is the corollary of three lemmas that describe the structure of generic quasi-shuffles of Lyndon marked permutations. We postpone the proofs of these lemmas to Section 5.5.

Lemma 54. Let β^* be a quasi-shuffle of the marked permutations $l_1^*, \ldots, l_k^*$. Then, $|\beta^*| \leq |\alpha^*|$. Further, if $|\beta^*| = |\alpha^*|$, then $j(\beta^*) \leq j(\alpha^*)$.

Lemma 55 (Block breaking lemma)**.** Let β^* be a quasi-shuffle of the Lyndon marked permutations $l_1^*, \dots, l_k^*$, such that $|\beta^*| = |\alpha^*|$ and $j(\beta^*) = j(\alpha^*)$. Then $\mathfrak{fac}(\beta^*) = \mathfrak{fac}(\alpha^*)$.

Further, there is a marked permutation γ^* with a stable factorization $(r_1^*, \dots, r_{j(\gamma^*)}^*)$ such that

- we have $\mathfrak{fac}(\gamma^*) = \mathfrak{fac}(\beta^*)$. In particular, $|\gamma^*| = |\beta^*| = |\alpha^*|$ and $j(\gamma^*) = j(\beta^*) = j(\alpha^*)$;
- we have $\gamma^* \geq_{fac} \beta^*$;
- we can split the indexing set $[j(\gamma^*)]$ into k many disjoint increasing sequences $q_1^{(i)} < \dots < q_{j(l_i^*)}^{(i)}$ for $i = 1, \dots, k$ such that

$$l_i^* = r_{q_1^{(i)}}^* * \dots * r_{q_{j(l_i^*)}^{(i)}}^* , \tag{17}$$

  is precisely the stable factorization of each l_i^*.

These lemmas will be proven simultaneously below.

We remark that, here, the chosen ordering $\leq_{per}$ on the simple marked permutations plays a role. That is, in proving that $\gamma^* \geq_{fac} \beta^*$ we use properties of the order $\leq_{per}$ introduced above. This is unlike the work in [Var14], where any order in the indecomposable permutations give a set of free generators.

Lemma 56 (Block preserving lemma)**.** Let γ^* be a marked permutation such that $[j(\gamma^*)]$ splits into disjoint increasing sequences $q_1^{(i)} < \dots < q_{j(l_i^*)}^{(i)}$ for $i = 1, \dots, k$, where

$$l_i^* = r_{q_1^{(i)}}^* * \dots * r_{q_{j(l_i^*)}^{(i)}}^* ,$$

is precisely the stable factorization of each l_i^*. Then, $\gamma^* \leq_{fac} l_1^* * \dots * l_k^*$.

In the remaining of the section we assume the previous lemmas and conclude the proof of the freeness of the pattern algebra. The proof of these lemmas can be found in Section 5.5.

Proof of Theorem 53. It follows from Remark 18 that $c_{\alpha^*} \geq 0$. Define the *product order* $\leq_{prod}$ on marked permutations as follows: Let $\pi^* = s_1^* * \dots * s_k^*$ and $\tau^* = t_1^* * \dots * t_j^*$ be the respective stable factorizations. Then we say that $\pi^* \leq_{prod} \tau^*$ if

- $|\pi^*| < |\tau^*|$ or if $|\pi^*| = |\tau^*|$ and $j(\pi^*) < j(\tau^*)$;
- or if $|\pi^*| = |\tau^*|$, $j(\pi^*) = j(\tau^*)$ and $(s_1^*, \dots, s_k^*) \leq (t_1^*, \dots, t_j^*)$ in the lexicographic order in $\mathcal{W}(\Omega)$.

This order is the appropriate one to establish a *triangularity argument*. We conclude the proof if we show that any β^* that is a *quasi-shuffle* of $l_1^* \dots, l_k^*$ has $\beta^* \leq_{prod} \alpha^*$.

Indeed, suppose that β^* is a quasi-shuffle of $l_1^* \dots l_k^*$. Then, from Lemma 54 we have that $|\beta^*| = |\alpha^*|$ and $j(\beta^*) = j(\alpha^*)$. From Lemma 55 we have that $\mathfrak{fac}(\beta^*) = \mathfrak{fac}(\alpha^*)$, and that there is a marked permutation γ^* that satisfies $\mathfrak{fac}(\gamma^*) = \mathfrak{fac}(\beta^*)$ and $\gamma^* \geq_{fac}$, so we have that $\gamma^* \geq_{prod} \beta^*$.

Further, the given marked permutation γ^* also satisfies the assumptions on Lemma 56, so it follows that $\gamma^* \leq_{fac} \alpha^*$. Again, because $\mathfrak{fac}(\gamma^*) = \mathfrak{fac}(\alpha^*)$, we have that $\gamma^* \leq_{prod} \alpha^*$. It follows that $\beta^* \leq_{prod} \gamma^* \leq_{prod} \alpha^*$, as desired. $\square$

Proof of Theorem 36. We first prove that the given set is algebraically free. Suppose otherwise, that

$$\{ \prod_{l^* \in L} \mathbf{p}_{l^*} \, | L \text{ is a multiset of SL marked permutations } \}, \tag{18}$$

is not linear dependent, and there is some relation $\sum_L \alpha_L \prod_{l^* \in L} \mathbf{p}_{l^*} = 0$, where only finitely many α_L are non-zero. We wish to establish a contradiction.

Write $\alpha(L) = l_1^* * \dots l_k^*$, and take M_L to be the multiset of SL marked permutations that maximizes $\alpha(M_L)$ with respect to the order $\leq_{prod}$, subject to $\alpha_{M_L} \neq 0$.

Then, from Theorem 53 applied to each L, we have:

$$0 = \sum_L \alpha_L \prod_{l^* \in L} \mathbf{p}_{l^*} = \sum_L \alpha_L \sum_{\beta^* \leq_{prod} \alpha(L)} c_{\beta^*}^{(L)} \, \mathbf{p}_{\beta^*} \tag{19}$$

$$= \sum_{\beta^* \leq_{prod} \alpha(M_L)} \mathbf{p}_{\beta^*} \sum_L \alpha_L c_{\beta^*}^{(L)} \Rightarrow \tag{20}$$

$$\sum_L \alpha_L c_{\beta^*}^{(L)} = 0, \;\; \beta^* \text{ marked permutation}, \tag{21}$$

where we recall that $\{\mathbf{p}_{\beta^*}\}$ is linearly independent, from Remark 21.

Note that for any multiset $L \neq M_L$ with $\alpha_L \neq 0$ satisfies $\alpha(M_L) >_{prod} \alpha(L)$, so we have that $c_{\alpha(M_L)}^{(L)} = 0$. Hence, for $\beta^* = \alpha(M_L)$ in (19), we get

$0 = \sum_L \alpha_L c^{(L)}_{\alpha(M_L)} = \alpha_{M_L} c^{(M_L)}_{\alpha(M_L)}$, which is not zero by assumption. We obtain a contradiction with the fact that the set in (18) is not linearly independent, as desired.

To show that the desired set spans $\mathcal{A}(\mathtt{MPer})$, we argue again by contradiction. Suppose that there is some $\mathbf{p} = \sum_{\beta^*} \alpha_{\beta^*} \mathbf{p}_{\beta^*}$ that is not spanned by the elements in (18), and let γ^* be the maximal marked permutation with respect to $\leq_{prod}$ such that $\alpha_{\gamma^*} \neq 0$. Further, take such an element $\mathbf{p}$ that is minimizes γ^*.

Now, from Theorem 52, γ^* has an SL factorization given by SL marked permutations $l_1^* \geq_{fac} \cdots \geq_{fac} l_k^*$. Then, observe that $\mathbf{p}' := \mathbf{p} - \frac{\alpha_{\gamma^*}}{c_{\gamma^*}} \prod_{i=1}^k \mathbf{p}_{l_i^*}$ only has terms $\mathbf{p}_{\beta^*}$ for $\beta^* <_{prod} \gamma^*$.

However, because we assumed that $\mathbf{p}$ is not spanned by the set in (18), $\mathbf{p}'$ is also not spanned by this set, which is a contradiction with the minimality of $\mathbf{p}$. $\square$

5.4. Proofs of unique factorizations

We start with some basic principles regarding marked permutations.

Definition 57. Let β^* be a marked permutation on the set X. That is, $\beta^* = (\geq_P, \geq_V)$ is a pair of orders in the set $X \sqcup \{*\}$. A *doubly connected interval* on β^*, or a *DC interval* for short, is a non-empty set $I \subseteq X$ such that $I \sqcup \{*\}$ is an interval on both orders, and we denote it as $I \trianglelefteq \beta^*$.

Note that X is always a DC interval, called the trivial DC interval. We convention that $\emptyset \trianglelefteq \beta^*$.

If β^* is a marked permutation in X, and $I \subseteq X$, we denote $I^* := I \sqcup \{*\}$ for simplicity of notation.

Remark 58. Suppose that $\beta^* = (\leq_P, \leq_V)$ is a marked permutation, and $I \trianglelefteq \beta^*$ is a DC interval. Consider $m_P, M_P \in X \cup \{*\}$ the minimal, respectively maximal, element under the order $\leq_P$. If $m_P, M_P \in I^*$, then $I = X$.

Symmetrically, if $m_V, M_V \in X \cup \{*\}$ are the minimal, respectively maximal, elements under the order $\leq_V$, and $m_V, M_V \in I^*$, then $I = X$.

For a marked permutation α^*, we will use the notation m_P, M_P, m_V, M_V in the remaining of the section to refer to its respective extremes.

Remark 59. If $\alpha^* = a_1^* * a_2^*$, then $\mathbb{X}(a_2^*)$ is a DC interval in α^*. On the other hand, if $I \trianglelefteq \alpha^*$, then $\alpha^* = \alpha^*|_{I^c} * \alpha^*|_I$, so right factors of α^* are in bijection with DC intervals of α^*. In particular, I is a maximal proper DC interval of α^* if and only if $\alpha^*|_{I^c}$ is simple.

Further, α^* is $\oplus$-decomposable if and only if there is a DC interval $I \trianglelefteq \alpha^*$ such that I^* contains both m_P, m_V, or contains both M_P, M_V. In the first case, α^* factors as $\beta_1^* \oplus \beta_2$, and in the second factors as $\beta_1 \oplus \beta_2^*$.

Similarly, α^* is $\ominus$-decomposable if and only if there is a DC interval $I \trianglelefteq \alpha^*$ such that I^* contains both M_P, m_V, or contains both m_P, M_V. In the first case, α^* factors as $\beta_1 \ominus \beta_2^*$, and in the second factors as $\beta_1^* \ominus \beta_2$.

Remark 60. For a DC interval I, we define $\alpha^*\backslash_{I^*}$ to be $\alpha^*|_{I^c}$ with the marked element removed.

Observation 61 ($\oplus$ factorization and $\ominus$ factorization in permutations)**.** Let α be a permutation. Then there are unique $\oplus$-indecomposable permutations $a_1, \dots, a_k$ and $\ominus$-indecomposable permutations $b_1, \dots, b_j$ such that

$$\alpha = a_1 \oplus \cdots \oplus a_k = b_1 \ominus \cdots \ominus b_j \, .$$

The following is a direct consequence of Observation 61.

Corollary 62 ($\oplus$ factorization and $\ominus$ factorization in marked permutations)**.** Let α^* be a marked permutation. Then there are unique $\oplus$-indecomposable permutations $a_1, \dots, a_k, b_1, \dots, b_j$ and β^* a $\oplus$-indecomposable marked permutation such that

$$\alpha^* = a_1 \oplus \cdots \oplus a_k \oplus \beta^* \oplus b_j \oplus \cdots \oplus b_1 \, .$$

To β^* we call the $\oplus$-kernel of α^*, write $\ker^{\oplus}(\alpha^*) = \beta^*$, and we say that α^* is $(k+j) - \oplus$ decomposable.

Further, there are unique $\ominus$-indecomposable permutations $c_1, \dots, c_{k'}$ and $d_1, \dots, d_{j'}$, and a unique $\ominus$-indecomposable marked permutation γ^* such that

$$\alpha^* = c_1 \ominus \cdots \ominus c_{k'} \ominus \gamma^* \ominus d_{j'} \ominus \cdots \ominus d_1 \, .$$

To γ^* we call the $\ominus$-kernel of α^*, and write $\ker^{\ominus}(\alpha^*) = \gamma^*$, and we say that α^* is $(k` + j`) - \ominus$-decomposable.

Example 63 ($(k+j)$-decomposition)**.** Consider the marked permutation $21\bar{3}54$. This is a 2-decomposable marked permutation, as it admits the factorization

$$21\bar{3}54 = 21 \oplus \bar{1} \oplus 21 \, .$$

We remark that a marked permutation is $\oplus$-indecomposable if it is 0-$\oplus$ decomposable. Similarly, a marked permutation is $\ominus$-indecomposable if it is 0-$\ominus$ decomposable.

Lemma 64 (Simple $\oplus$ factor lemma)**.** Suppose that α^* is a marked permutation that has the following $\oplus$ factorization

$$\alpha^* = a_1 \oplus \cdots \oplus a_u \oplus \beta^* \oplus b_v \oplus \cdots \oplus b_1 \,,$$

where $u + v > 0$, that is, α is $\oplus$-decomposable. Consider any factorization $\alpha^* = s^* * \gamma^*$, where s^* is a simple marked permutation.

- If both $u > 0$ and $v > 0$, then either $s^* = \bar{1} \oplus b_1$ or $s^* = a_1 \oplus \bar{1}$.
- If $u = 0$, then $s^* = \bar{1} \oplus b_1$ and there is a unique simple left factor of α^*.
- If $v = 0$, then $s^* = a_1 \oplus \bar{1}$ and there is a unique simple left factor of α^*.

Proof. Let us deal with the case where both $u \neq 0$ and $v \neq 0$ first. Assume that neither $s^* = \bar{1} \oplus b_1$ nor $s^* = a_1 \oplus \bar{1}$. Consider the following DC intervals, that are all distinct

$$Y_1 = \mathbb{X}(a_2 \oplus \cdots \oplus a_u \oplus \beta^* \oplus b_v \oplus \cdots \oplus b_1) \,,$$

$$Y_2 = \mathbb{X}(a_1 \oplus \cdots \oplus a_u \oplus \beta^* \oplus b_v \oplus \cdots \oplus b_2) \,,$$

$$Y = \mathbb{X}(\gamma^*) \,.$$

Note that $m_P \in Y_2^*$ and $M_P \in Y_1^*$.

Note that $\alpha^*|_{Y_1^c} = a_1 \oplus \bar{1}$ and $\alpha^*|_{Y_2^c} = \bar{1} \oplus b_1$ are simple marked permutations, so both Y_1, Y_2 are **maximal proper** DC intervals. Further, note that because Y comes from a simple factorization, it is a maximal proper DC interval. The maximality gives us $Y \cup Y_1 = X = Y \cup Y_2$.

From $Y \cup Y_1 = X$ we have that $m_P \in Y$, and from $Y \cup Y_2 = X$ we get that $M_P \in Y$. This contradicts Remark 58, and the assumption that neither $s^* = \bar{1} \oplus b_1$ nor $s^* = a_1 \oplus \bar{1}$ is wrong, as desired.

Now suppose without any loss of generality, that $u = 0$, and $v > 0$. Suppose for sake of contradiction that $s^* \neq \bar{1} \oplus b_1$, and define the following distinct DC intervals

$$Y_2 = \mathbb{X}(\beta^* \oplus b_v \oplus \cdots \oplus b_2) \text{ and } Y = \mathbb{X}(\gamma^*) \,.$$

Because both s^* and $\bar{1} \oplus b_1$ are simple, the DC intervals Y_2, Y are both **maximal proper**, and $X = Y \cup Y_2$. Further, $m_P, m_V \in Y_2$ by construction, so $M_P, M_V \in Y$ by Remark 58. Consider the DC interval $I = \mathbb{X}(\beta^*)$, and notice that $m_P, m_V \in I$ by construction. Now both Y^* and I^* are intervals in $\leq_P$ that intersect and contain both maximum and minimum, so $I \cup Y = X$.

We claim that β^* is $\oplus$ decomposable, contradicting the construction of β^*. In particular, we have the following decomposition:

$$\beta^* = \alpha^* \setminus_{Y^*} \oplus \alpha^*|_{I \cap Y} .$$

Notice that neither $\alpha^* \setminus_{Y^*}$ is empty, because Y is a proper DC interval, nor is $\alpha^*|_{I \cap Y}$ empty because it contains the marked element.

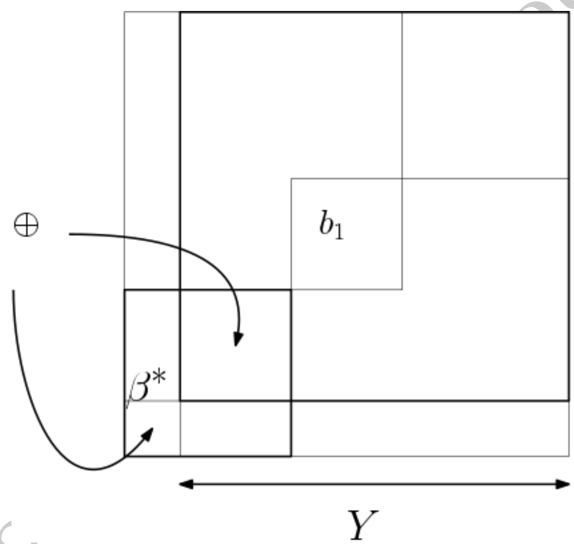


Figure 3: The DC interval Y has to cover β^* awful figure

The case where $v = 0$ and $u > 0$ is similar, and this concludes the proof. □

Corollary 65. If α^* is a marked permutation that is $n - \oplus$-decomposable, and has a simple factorization $s_1^* * \cdots * s_j^*$, then

1. $s_{n+1}^* * \cdots * s_j^* = \ker^{\oplus}(\alpha^*)$.
2. The word $(s_1^*, \ldots, s_n^*)$ is $\sim$-equivalent to $(\bar{1} \oplus b_1, \ldots, \bar{1} \oplus b_v, a_1 \oplus \bar{1}, \ldots, a_u \oplus \bar{1})$, where $\alpha^* = a_1 \oplus \cdots \oplus a_k \oplus \beta^* \oplus b_j \oplus \cdots \oplus b_1$ is the $\oplus$ decomposition of α^*.

Proof. We will use induction on n. The base case is $n = 0$, where there is nothing to establish in 2. and we need only to claim that

$$s_1^* * \cdots * s_j^* = \ker^{\oplus}(\alpha^*)\,,$$

However, because $n = 0$, α^* is $\oplus$- indecomposable, so $\ker^{\oplus}(\alpha^*) = \alpha^*$, as desired.

Now for the induction step, we assume that $n \geq 1$ and that the result is valid for any $m-\oplus$-decomposable marked permutation, where $m < n$. Write

$$\alpha^* = a_1 \oplus \cdots \oplus a_k \oplus \beta^* \oplus b_j \oplus \cdots \oplus b_1\,.$$

From Lemma 64, s_1^* is either $a_1 \oplus \bar{1}$ or $\bar{1} \oplus b_1$, thus $\zeta^* := s_2^* * \cdots * s_k^*$ is either

$$a_2 \oplus \cdots \oplus a_k \oplus \beta^* \oplus b_j \oplus \cdots \oplus b_1\,,$$

or

$$a_1 \oplus \cdots \oplus a_k \oplus \beta^* \oplus b_j \oplus \cdots \oplus b_2.$$

Without loss of generality, assume the first case, that $\zeta^* = a_2 \oplus \cdots \oplus a_k \oplus \beta^* \oplus b_j \oplus \cdots \oplus b_1$

Then, $\zeta^* = s_2^* * \cdots * s_j^*$ is $(n-1)$-$\oplus$-decomposable, so the induction hypothesis applies to the factorization of ζ^*, that is, we have that

$$s_{n+1}^* * \cdots * s_j^* = \ker^{\oplus}(\zeta^*) = \beta^*\,.$$

Further, the induction case gives us that $(\bar{1} \oplus b_1, \ldots, \bar{1} \oplus b_v, a_2 \oplus \bar{1}, \ldots, a_u \oplus \bar{1})$ can be obtained $(s_2^*, \ldots, s_k^*)$ by a series of $\oplus$ -relations.

So

$$\begin{aligned}
&(\bar{1} \oplus b_1, \ldots, \bar{1} \oplus b_v, a_1 \oplus \bar{1}, a_2 \oplus \bar{1}, \ldots, a_u \oplus \bar{1}) && (22)\\
\sim&(a_1 \oplus \bar{1}, \bar{1} \oplus b_1, \ldots, \bar{1} \oplus b_v, a_2 \oplus \bar{1}, \ldots, a_u \oplus \bar{1}) && (23)\\
\sim_{ind.hip.}&(s_1^*, s_2^*, \ldots, s_k^*) && (24)
\end{aligned}$$

concluding the desired property 2, and the proof. □

Lemma 66 (Simple $\ominus$-factor lemma)**.** Suppose that α^* is a marked permutation that has the following $\ominus$ factorization

$$\alpha^* = a_1 \ominus \cdots \ominus a_u \ominus \beta^* \ominus b_v \ominus \cdots \ominus b_1\,,$$

where $u + v > 0$, that is, α is $\ominus$-decomposable. Consider any factorization $\alpha^* = s^* * \gamma^*$, where s^* is a simple marked permutation.

- If both $u > 0$ and $v > 0$, then either $s^* = \bar{1} \ominus b_1$ or $s^* = a_1 \ominus \bar{1}$.
- If $u = 0$, then $s^* = \bar{1} \ominus b_1$ and there is a unique simple left factor of α^*.
- If $v = 0$, then $s^* = a_1 \ominus \bar{1}$ and there is a unique simple left factor of α^*.

Proof. The proof of this result follows the lines of Lemma 64 □

Corollary 67. If α^* is a marked permutation that is $n - \ominus$-decomposable, and has a simple factorization $s_1^* * \cdots * s_j^*$, then

1. $s_{n+1}^* * \cdots * s_j^* = \ker^{\ominus}(\alpha^*)$.
2. By applying $\ominus$-relations to $(s_1^*, \ldots, s_n^*)$ we can get the sequence $(\bar{1} \ominus b_1, \ldots, \bar{1} \ominus b_v, a_1 \ominus \bar{1}, \ldots, a_u \ominus \bar{1})$, where $\alpha^* = a_1 \ominus \cdots \ominus a_k \ominus \beta^* \ominus b_j \ominus \cdots \ominus b_1$ is the $\ominus$ decomposition of α^*.

Proof. This proof is similar to the proof of Corollary 65. □

Lemma 68 (Simple factor lemma)**.** Suppose that α^* is a marked permutation that is both $\oplus$- indecomposable and $\ominus$- indecomposable.

Consider two factorizations of α^*: $\alpha^* = s_1^* * \beta_1^* = s_2^* * \beta_2^*$, where s_1^*, s_2^* are simple marked permutations. Then $s_1^* = s_2^*$.

Proof. Recall that a simple left factor of α^* corresponds to a maximal proper DC interval, according to Remark 59. It suffices then to see that α^* can only have one maximal proper DC interval.

Consider the following DC intervals $\mathbb{X}(\beta_1^*) = Y_1$ and $\mathbb{X}(\beta_2^*) = Y_2$, which are proper maximal DC intervals according to Remark 59, and suppose that they are distinct, for sake of contradiction.

Further, consider m_P, M_P, m_V, M_V the usual maximal and minimal elements in $\mathbb{X}(\alpha^*) \sqcup \{*\}$.

By maximality we have that $Y_1 \cup Y_2 = X$, so $m_P, M_P, m_V, M_V \in Y_1^* \cup Y_2^*$.

First, observe that if a DC interval of α^* contains both m_V, M_V, then this is $\mathbb{X}(\alpha^*)$. Similarly, if a DC interval of α^* contains both m_P, M_P, then this is $\mathbb{X}(\alpha^*)$.

Without loss of generality suppose that $m_P \in Y_1$, there are only two cases to consider:

- We have that $m_P, m_V \in Y_1^*$ and $M_P, M_V \in Y_2^*$. Then we claim that that α^* is $\oplus$-decomposable. Indeed, if $x \in Y_1^*, y \notin Y_1^*$, the DC interval condition implies that $m_P \leq_P x <_P y$ and $m_V \leq_V x <_V y$, so we have that
$$\alpha^* = \alpha^*|_{Y_1} \oplus \alpha^* \setminus_{Y_1^*} .$$

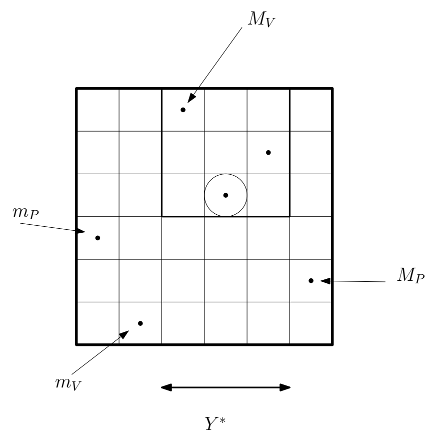


Figure 4: An indecomposable marked permutation and a maximal proper ideal

- We have that $m_P, M_V \in Y_1^*$ and $m_P, M_V \in Y_2^*$. Then we claim that that α^* is $\ominus$-decomposable. Indeed, if $x \in Y_1, y \notin Y_1$, the DC interval condition implies that $m_P \leq_P x <_P y$ and $M_V \geq_V x >_V y$, so we have that

$$\alpha^* = \underbrace{\alpha^*|_{Y_1}}_{\text{A marked permutation}} \ominus \underbrace{\alpha^* \backslash_{Y_1^*}}_{\text{A non-marked permutation}} .$$

In either case, we reach a contradiction, so we must have that $Y_1 = Y_2$, that is $\beta_1^* = \beta_2^*$ and $s_1^* = s_2^*$. ☐

Proof of Theorem 41. Consider the map $* : \mathcal{W}(\Omega) \to \mathcal{G}(\texttt{MPer})$, that sends a word of simple marked permutations to its inflation. We will describe the fibres of this map, that is precisely which words are mapped to the same marked permutation α^*. Naturally, this entails a classification of all simple factorization of α^*.

Let us suppose that we have a relation

$$s_1^* * \cdots * s_k^* = r_1^* * \cdots * r_j^* =: \alpha^* , \tag{25}$$

that does not result from the application of the relations in Remark 39, that is

$$(s_1^*, \dots, s_k^*) \not\sim (r_1^*, \dots, r_j^*) . \tag{26}$$

Further choose such factorizations minimizing $k+j$. We consider three cases:

The marked permutation α^* is both $\oplus$-indecomposable and $\ominus$-indecomposable: in this case, from Lemma 68 we know that $s_1^* = r_1^*$ and that $s_2^* * \cdots * s_k^* = r_2^* * \cdots * r_j^*$. By minimality, $(s_2^*, \ldots, s_k^*)$ and $(r_2^*, \ldots, r_j^*)$ are by $\oplus$-relations and $\ominus$-relations, which contradicts the fact that $(s_1^*, \ldots, s_k^*)$ and $(r_1^*, \ldots, r_j^*)$ are not related by $\oplus$-relations and $\ominus$-relations.

The marked permutation α^* is $\oplus$ -decomposable: Assume now that α^* is $n - \oplus$-decomposable, for some $n > 0$. That is, there are $\oplus$-indecomposables such that

$$\alpha^* = a_1 \oplus \cdots \oplus a_u \oplus \beta^* \oplus b_v \oplus \cdots \oplus b_1 .$$

Then Corollary 65 tells us that

$$s_{n+1}^* * \cdots * s_k^* = r_{n+1}^* \cdots * r_j^* = \ker^{\oplus}(\alpha^*) , \tag{27}$$

where naturally $\ker^{\oplus}(\alpha^*) = \beta^*$, and also that

$$s_1^* * \cdots * s_n^* = r_1^* * \cdots * r_n^* , \tag{28}$$

where $s_1^*, \ldots, s_n^*$ and $r_1^*, \ldots, r_n^*$ are all $\oplus$ -decomposable simple marked permutations that are related by $\oplus$ relations to

$$(\bar{1} \oplus b_1, \ldots, \bar{1} \oplus b_v, a_1 \oplus \bar{1}, \ldots, a_u \oplus \bar{1}) . \tag{29}$$

Similarly, the word $(r_1^*, \ldots, r_n^*)$ is related to (29). It follows that $(s_1^*, \ldots, s_n^*) \sim (r_1^*, \ldots, r_n^*)$. Further, by the minimality assumption, and because $n > 0$, we have that

$$(r_{n+1}^*, \ldots, r_j^*) \sim (s_{n+1}^*, \ldots, s_k^*) ,$$

These facts contradict (26), obtaining a contradiction.

The marked permutation α^* is $\ominus$ -decomposable: This case is similar to the previous one.

We conclude that the simple factorization of a marked permutation is unique up to the relations in Remark 39. $\square$

An immediate corollary of the previous discussion pertains DC interval chains. We observe in **??** that to a DC interval of β^* corresponds a left factor of β^*. Naturally, to a DC interval chain $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k$ it corresponds a factorization of β^* of size k

Corollary 69 (DC interval chains)**.** There is a

5.5. Proof of main Lemmas

We start by fixing some notation that will be assume for this section. Consider a multiset $\{l_1^* \geq \cdots \geq l_k^*\}$ of Lyndon marked permutations, and let α^* be the marked permutation with Lyndon factorization $(l_1^*, \dots, l_k^*)$ which exists and is unique by Theorem 52, and has $\alpha^* = l_1^* * \cdots * l_k^*$.

Note that $\mathcal{S} := \biguplus_{i=1}^k \mathfrak{fac}(l_i^*) = \mathfrak{fac}(\alpha^*)$, from Corollary 42.

Proof of Lemmas 54 and 55. First we assume that $\mathbb{X}(\beta^*) = X$, that is, β is a permutation in $X \sqcup \{*\}$, and set $j = j(\beta^*)$. By **??**, the marked permutation β^* is a quasi-shuffle of $l_1^*, \dots, l_k^*$, that is, there are sets $I_1, \dots, I_k$ such that:

$$I_1 \cup \cdots \cup I_k = X \text{ and } \beta^*|_{I_i} = l_i^* \;\; \forall l_i^* \,. \tag{30}$$

Suppose that β^* has the following stable factorization

$$\beta^* = r_1^* * \cdots * r_j^* \,,$$

with a corresponding DC interval chain

$$\emptyset = J_{j+1} \subsetneq J_j \subsetneq \cdots \subsetneq J_1 = X \,,$$

as given in Corollary 69, so that $\beta^*|_{J_p \setminus J_{p+1}} = r_p^*$ for $p = 1, \dots, j$.

We first establish that $|\beta^*| = |\alpha^*|$ and that $j = j(\alpha^*)$. Indeed, from $\beta^* \geq_{prod} \alpha^*$ we have that $|\beta^*| \geq |\alpha^*|$. Since β^* is a quasi-shuffle of $l_1^*, \dots, l_k^*$, the sets $I_1, \dots, I_k$ cover X and so

$$|\beta^*| = \#X \leq \sum_{i=1}^k \#I_i = \sum_{i=1}^k |l_i^*| = |\alpha^*| \,. \tag{31}$$

Hence, we conclude that $|\beta^*| = |\alpha^*|$. Also, with equality in (31), we get that the sets I_i are mutually disjoint. From $\beta^* \geq_{prod} \alpha^*$ and $|\beta^*| = |\alpha^*|$, we have that $j \geq j(\alpha^*)$. It suffices then to show the opposite inequality, $j \leq j(\alpha^*)$.

Recall that $j(\alpha^*) = \sum_i j(l_i^*)$, as established in Corollary 42. The following is a DC interval chain of l_i^*:

$$\emptyset = J_{j+1} \cap I_i \subseteq J_j \cap I_i \subseteq \cdots \subseteq J_1 \cap I_i = I_i \,, \tag{32}$$

so let us consider the set $U_p := \{i \in [k] | J_{p+1} \cap I_i \neq J_p \cap I_i\}$. Note that the length of the DC chain is given precisely by the number of inequalities in (32), that is $\#\{p \in [j] | i \in U_p\}$.

We claim that each U_p is a singleton. First it is clear that each U_p is non empty, as otherwise we would have

$$J_{p+1} = \bigcup_i J_{p+1} \cap I_i = \bigcup_i J_p \cap I_i = J_p \, .$$

On the other hand, recall that $j \geq j(\alpha^*)$, and the length of the DC chain in (32) is at most $j(l_i^*)$, from Corollary 69, so

$$\sum_i j(l_i^*) \geq \sum_i \#\{p \in [j] | i \in U_p\} = \sum_{p=1}^{j} \#U_p \geq j \, . \tag{33}$$

However, $j(\alpha^*) = \sum_i j(l_i^*)$, so we conclude that $j(\alpha^*) = j$ and we obtain equality all through (33). This concludes the proof of Lemma 54.

Further, we obtain that each U_p is a singleton and that $j(l_i^*) = \#\{p \in [j] | i \in U_p\}$. This gives us a map $\zeta : [j] \to [k]$ that sends p to the unique element in U_p. This map that gives the desired increasing sequences, and satisfies

$$\#\zeta^{-1}(i) = \#\{p \in [j] | i \in U_p\} = j(l_i^*) \text{ for } i = 1, \ldots, k \, .$$

Write $\zeta^{-1}(i) = \{p_1^{(i)} < \cdots < p_{j(l_i^*)}^{(i)}\} \subseteq [j]$, so the restriction of the DC interval chain (32), is

$$\emptyset = J_{1+p^{(i)}_{j(l_i^*)}} \cap I_i \subsetneq J_{p^{(i)}_{j(l_i^*)}} \cap I_i \subsetneq \cdots \subsetneq J_{p_1^{(i)}} \cap I_i = I_i \, ,$$

then we can compute the factors of the factorization corresponding to the previous DC interval chain.

First note that $J_{p+1} \setminus J_p \subseteq I_{\gamma(p)}$, since we have the wonderful set theory relation:

$$J_p \setminus J_{p+1} = \biguplus_i (J_p \cap I_i) \setminus (J_{p+1} \cap I_i) = (J_p \cap I_{\gamma(p)}) \setminus (J_{p+1} \cap I_{\zeta(p)}) \subseteq I_{\zeta(p)} \, ,$$

If $\zeta(p) = i$, we can write $p = p_o^{(i)}$ for some o, so we have that

$$(J_{p_o^{(i)}} \cap I_i) \setminus (J_{p_{o+1}^{(i)}} \cap I_i) = \bigcup_{x = p_o^{(i)}}^{p_{o+1}^{(i)} - 1} (J_x \setminus J_{x+1}) \cap I_i = J_{p_o^{(i)}} \setminus J_{p_o^{(i)}+1} \, .$$

With this, we have

$$\begin{aligned} l_i^*|_{(J_{p_o^{(i)}}\cap I_i)\setminus(J_{p_{o+1}^{(i)}}\cap I_i)} &= \left(\beta^*|_{J_{p_o^{(i)}}\setminus J_{p_{o+1}^{(i)}}}\right)|_{(J_{p_o^{(i)}}\cap I_i)\setminus(J_{p_{o+1}^{(i)}}\cap I_i)} \\ &= \beta^*|_{(J_{p_o^{(i)}}\cap I_i)\setminus(J_{p_{o+1}^{(i)}}\cap I_i)} = \beta^*|_{J_{p_o^{(i)}}\setminus J_{p_o^{(i)}+1}} = r^*_{p_o^{(i)}}\,, \end{aligned}$$

and so

$$l_i^* = r^*_{p_1} * \cdots * r^*_{p_{j(l_i^*)}}\,,$$

is a simple factorization of l_i^*, and $\mathfrak{fac}(l_i^*) = \{r^*_{p_1^{(i)}}, \ldots, r^*_{p^{(i)}_{j(l_i^*)}}\}$.

Then, we indeed have that $\mathfrak{fac}(\beta^*) = \biguplus_{i=1}^k \mathfrak{fac}(l_i^*)$, and the desired sequences are precisely $\{p_o^{(i)} | o = 1, \ldots, j(l_i^*)\}$. □

Proof of Lemma 56. Let us reformulate what is given as follows: we have a bijection ζ between the set $\mathcal{I} = \{(a,b) | a \in [k], 1 \le b \le j(l_a^*)\}$ and $[j]$ given as $\zeta(a,b) = p_b^{(a)}$, that satisfies $\zeta(a,b) < \zeta(a,b+1)$ and

$$l_a^* = s^*_{\zeta(a,1)} * \cdots * s^*_{\zeta(a,j(l_a^*))}\,. \tag{34}$$

The idea from here on is to force the occurrences of the simple factors of each l_i^* to be in the stable order. The resulting marked permutation, γ^*, is in general distinct from β^*.

Namely, the marked permutation l_a^* has a stable factorization $l_a^* = s^*_{\pi(\zeta(a,1))} * \cdots * s^*_{\pi(\zeta(a,j(l_a^*)))}$ that results from commuting some factors in (34), according to **??**. This gives rise to a permutation π of the set $\mathcal{I}$.

We identify π with its conjugate permutation on $[j]$ via the bijection ζ. Then we define

$$\gamma^* = r^*_{\pi(1)} * \cdots * r^*_{\pi(j)}\,, \tag{35}$$

a pictorial example is described in Fig. 5, where we assume that the following are stable factorizations: $l_1 = r_5^* r_2 * r^8$, $l_2^* = r_1^* * r_6^*$ and $l_3^* = r_3^* * r_4^* * r_7^*$.

Because each r_i^* is simple, we have $\mathfrak{fac}(\gamma^*) = \mathfrak{fac}(\beta^*) = \mathcal{S}$. We remark however that the factorization (35) need not be stable. So if $(q_1^*, \ldots, q_j^*)$ is the stable factorization of γ^*, from **??** we have that $(r^*_{\pi(1)}, \ldots, r^*_{\pi(j)}) \le (q_1^*, \ldots, q_j^*)$, that is:

$$\gamma^* \ge_{fac} \beta^*\,,$$

and because $|\gamma^*| = |\beta^*|$ and $j(\gamma^*) = j(\beta^*)$, we have that $\gamma^* \ge_{prod} \beta^*$. Finally, □

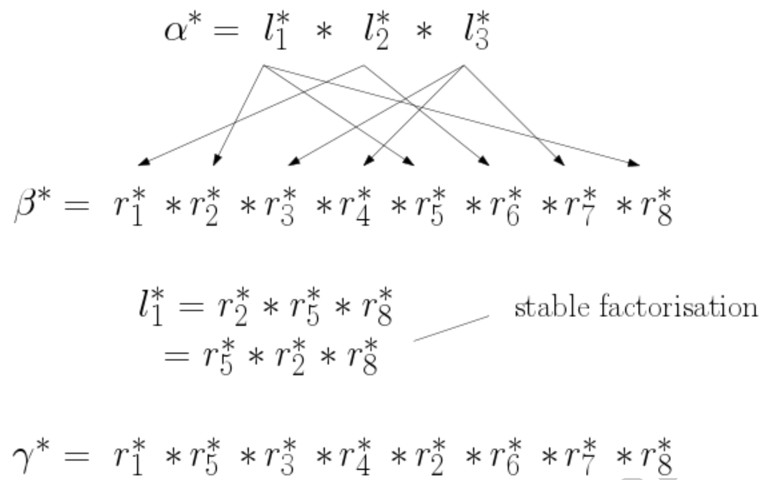


Figure 5: Example of construction of γ^*.

Proof of Lemma 56. This is a classic technique in the combinatorics of Lyndon words, and is also used in [Var14]. Let $\omega = (r_1^*, \dots, r_{(}\gamma^*)^*)$ be the stable factorization of γ^*, and denote ω_i for the stable factorizations of l_i^*.

Note that ω is a shuffle of $\omega_1, \dots, \omega_k$. Note further that each of the words ω_i is, by definition, a Lyndon word. Then, by [] we have that $\omega \leq \omega_1 \cdots \omega_k$ in the lexicographic order, where we compare the letters with $\leq_{per}$.

Finally, note that $\omega_1 \cdots \omega_k$ is the stable factorization of α^*, so we obtain that $\gamma^* \leq_{fac} \alpha^*$, and because $|\gamma^*| = |\alpha^*|$ and $j(\gamma^*) = j(\alpha^*)$, we have that $\gamma^* \geq_{prod} \alpha^*$, as desired. $\square$

5.6. Primitive elements, growth rates and asymptotic analysis

The space of primitive elements of a Hopf algebra H, $P(H)$, is the subspace of H given by $\{a \in H | \Delta a = a \otimes 1 + 1 \otimes a\}$. When H is a graded coalgebra, enumerating the dimension of $H_n \cap P(H)$ is a classical problem in Hopf algebras and may have applications, for instance in proving that two Hopf algebra H_1, H_2 are not related by H_1 being a Hopf subalgebra of H_2.

Proposition 70. Let h be a connected combinatorial presheaf with a monoid structure, and let $\mathcal{I} \subseteq \mathcal{G}(h)$ be the set of irreducible elements of $\mathcal{G}(h)$. Then, the space of primitive elements of $\mathcal{A}(h)$ is spanned by $\left\{ \mathbf{p}_f \,|f \in \mathcal{I} \right\}$.

Proof. Let $a = \sum_f c_f \, \mathbf{p}_f$ be a generic element from $\mathcal{A}(h)$. Then, the equation

$\Delta a = a \otimes 1 + 1 \otimes a$ becomes

$$\sum_{g_1, g_2 \in \mathcal{G}(h)} c_{g_1 \cdot g_2} \, \mathbf{p}_{g_1} \otimes \mathbf{p}_{g_2} = \sum_f c_f (\mathbf{p}_f \otimes \mathbf{p}_{\bar{1}} + \mathbf{p}_{\bar{1}} \otimes \mathbf{p}_f) \, .$$

From **??**, the elements of the form $\mathbf{p}_{g_1} \otimes \mathbf{p}_{g_2}$ form a basis of $\mathcal{A}(h)^{\otimes 2}$, so we have that for any $g_1 \neq 1, g_2 \neq 1$,

$$c_{g_1 \cdot g_2} = 0 \, .$$

Thus we conclude that a is spanned by $\left\{ \mathbf{p}_f \, | f \in \mathcal{I} \right\}$, as desired. $\square$

In particular, the space of primitive elements of $\mathcal{A}(\mathtt{MPer})$ is spanned by $\{ \mathbf{p}_{\pi^*} \, | \pi^* \text{ is simple } \}$, and so we are interested in enumerating the simple marked permutations. We saw that if π is an $\oplus$-indecomposable permutation, then $\bar{1} \oplus \pi$ and $\pi \oplus \bar{1}$ are simple marked permutations. Similarly, if τ is a $\ominus$-indecomposable permutation, then $\bar{1} \ominus \tau$ and $\tau \ominus \bar{1}$ are simple marked permutations.

Consider the following power series

- The power series $P^*(x) = \sum_{\pi^* \text{ marked permutation}} x^{|\pi^*|} = \sum_{n \geq 1} n \cdot n! \, x^{n-1}$ counts marked permutations.
- The power series $P(x) = \sum_{\pi \text{ permutation}} x^{|\pi|} = \sum_n n! \, x^n$ counts permutations.
- The power series $S^*(x) = \sum_{k \geq 0} s_k x^k = \sum_{\pi^* \text{ simple marked permutations}} x^{|\pi^*|}$ counts simple marked permutations. This is the main generating function that we aim to enumerate here.
- The power series $S^*_o(x) = \sum_{k \geq 0} so_k x^k = \sum_{\substack{\pi^* \text{ simple marked permutations} \\ \text{indecomposable}}} x^{|\pi^*|}$ counts simple marked permutations that are neither $\oplus$-decomposable nor $\ominus$-decomposable.
- The power series $P^{\oplus}(x) = \sum_{\pi : \oplus - \text{ decomposable}} x^{|\pi|}$ counts permutations that are $\oplus$-indecomposable. This also counts the marked permutations that are $\ominus$-indecomposable.

Observation 71. Because any $\oplus$-decomposable simple marked permutation is either of the form $\bar{1} \oplus \pi$ or $\pi \oplus \bar{1}$ for π a $\oplus$-indecomposable permutation, and symmetrically for $\ominus$ simple marked permutations, we have that

$$S^*(x) = S^*_o(x) + 4 P^{\oplus}(x) \, . \tag{36}$$

n	0	1	2	3	4	5	6	7	8	9
so_n	0	0	0	8	78	756	7782	85904	1016626	12865852
s_n	0	4	4	20	130	1040	9626	99692	1132998	13959224

Table 1: First elements of the sequences so_n and s_n.

Naturally, the power series $P^{\oplus}(x)$ can be directly computed from Observation 61, giving

$$\frac{1}{1-P^{\oplus}(x)} = P(x) = \sum_{n\geq 0} n!x^n \,.$$

Proposition 72. Let π^* be a marked permutation. Then, there are four cases

- There is a unique simple marked permutation s^* and a marked permutation α^* such that $\pi^* = s^* * \alpha^*$ and this simple marked permutation s^* is not $\oplus$-decomposable nor $\ominus$-decomposable.
- The marked permutation π^* is $\oplus$-decomposable.
- The marked permutation π^* is $\ominus$-decomposable.
- $\pi^* = \bar{1}$.

In particular, we have the following equation

$$P^*(x) = S_o^*(x)P^*(x) + 2\frac{\partial(P(x) - P^{\oplus}(x))}{\partial x} + 1 \,. \tag{37}$$

This allows us to enumerate easily the simple marked permutations, and the simple indecomposable marked permutations. See Table 1.

Proof. Suppose that π^* is neither $\oplus$-decomposable nor $\ominus$-decomposable. Then, either $\pi^* = \bar{1}$, or there is a simple factorization of π^*. According to Lemma 68, the first factor is unique, and so we fall precisely in the first case, as desired.

Finally, we observe that

$$\frac{\partial(P(x) - P^{\oplus}(x))}{\partial x}$$

counts the marked permutations that are $\oplus$-decomposable (and by symmetry, the ones that are $\ominus$-decomposable). $\square$

From (37) along with the fact that $P^*(x) = \frac{\partial P(x)}{\partial x}$, we have that

$$S_o^*(x) = -1 + \frac{2}{P(x)^2} - \frac{1}{P^*}.$$

$$S^*(x) = 3 + \frac{2}{P(x)^2} - \frac{1}{P'(x)} - \frac{4}{P(x)}.$$

6. Conjectures, open problems and issues

6.1. •

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