Probability 2

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Due until: 1st October at 5 p.m.

Exercises marked with * should be easier after attending the lecture on Thursday.

Exercise 1 (2 points). Let X, Y be random variables in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and $\mathcal{B} \subseteq \mathcal{A}$ a σ-subalgebra. Define $\mathcal{C} = \sigma(\mathcal{B}, Y)$ to be the smallest σ-algebra containing $\mathcal{B}$ such that Y is $\mathcal{C}$-measurable.

Assuming that X is independent from the σ-algebra $\mathcal{C}$, show that

$$\mathbb{E}[XY|\mathcal{B}] = \mathbb{E}[X]\mathbb{E}[Y|\mathcal{B}]\,.$$

Exercise 2 (3 points). Let $Y \geq 0$ be a random variable in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and $\mathcal{B} \subseteq \mathcal{A}$ a σ-subalgebra. We write $X = \mathbb{E}(Y|\mathcal{B})$. Prove that, there exists a set $S \in \mathcal{A}$ of probability 0 such that

$$\big\{X = 0\big\} \subseteq \{Y = 0\} \cup S; \tag{1}$$

$$\{Y = +\infty\} \subseteq \big\{X = +\infty\big\} \cup S. \tag{2}$$

(Hint: for the second one, first show that $\mathbb{E}[Y \mathbb{1}[X < n]]$ is finite for any $n \geq 0$.)

Exercise 3 (3 points). Let X, Y be random variables in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that $\mathbb{E}[X^2|Y] = Y^2$ and that $\mathbb{E}[X|Y] = Y$. Show that $X = Y$ a.s.

(Hint: Consider $\mathbb{E}[(X - Y)^2|Y]$ and use Exercise 2.)

Exercise 4 (2 points *). We say that a random variable X has finite exponential moments if there exists $s > 0$ such that both $\mathbb{E}[\exp(sX)]$ and $\mathbb{E}[\exp(-sX)]$ are finite.

Show that if X has finite exponential moments, then, for any σ-subalgebra $\mathcal{B}$, the r.v. $\mathbb{E}[X|\mathcal{B}]$ has finite exponential moments as well.