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The kernel of chromatic quasisymmetric functions on graphs and nestohedra

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Abstract. We study canonical Hopf algebra morphisms from generalised permutahedra and from graphs to *QSym* by finding generators for their kernel and image.

Keywords: chromatic symmetric function, quasisymmetric functions, Hopf algebras, generalized permutahedra

This is an extended abstract, of which the full version [11] is yet to be published.

1 Introduction

Chromatic function on graphs

For a graph G with vertex set $V(G)$, a colouring f of the graph G is a map $f : V(G) \to \mathbb{N}$. A colouring is *proper* if no edge is monochromatic. Stanley defines in [15] the *chromatic symmetric function* of G in commuting variables $\{x_i\}_{i\geq 1}$ as

$$\Psi_{\mathbf{G}}(G) = \sum\nolimits_f x_f\,,$$

where we write $x_f = \prod_{v\in V(G)} x_{f(v)}$, and the sum runs over proper colourings of the graph G. Note that $\Psi_{\mathbf{G}}(G)$ is in the ring *Sym* of symmetric functions. The ring *Sym* is a Hopf subalgebra of *QSym*, the ring of quasisymmetric functions. A long standing conjecture in this subject, commonly referred to as the *tree conjecture*, is that if two trees T_1, T_2 are not isomorphic, then $\Psi_{\mathbf{G}}(T_1) \neq \Psi_{\mathbf{G}}(T_2)$.

When $V(G) = [n]$, the natural ordering on the vertices allows us to consider a non-commutative analogue of $\Psi_{\mathbf{G}}$, as done by Gebhard and Sagan in [6]. They define the chromatic symmetric function on non-commutative variables $\{\mathbf{a}_i\}_{i\geq 1}$ as

$$\boldsymbol{\Psi}_{\mathbf{G}}(G) = \sum\nolimits_f \mathbf{a}_f\,,$$

where we write $\mathbf{a}_f = \prod_{v=1}^n \mathbf{a}_{f(v)}$, and we sum over the proper colourings f of G.

Note that $\boldsymbol{\Psi}_{\mathbf{G}}(G)$ is also symmetric in the variables $\{\mathbf{a}_i\}_{i\geq 1}$. Such functions are *word symmetric function*. The ring of word symmetric functions, **WSym** for short, was

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introduced in [13], and is sometimes called the ring of symmetric functions in non-commutative variables.

We consider graphs whose vertex sets are of the form $[n]$ for some $n \geq 0$, and write $\mathbf{G}$ for the free linear space generated by such graphs. This can be endowed with a Hopf algebra structure, as described by Schmitt in [14].

In this paper we describe generators for $\ker \Psi_{\mathbf{G}}$ and $\ker \boldsymbol{\Psi}_{\mathbf{G}}$. A similar problem was already considered for posets. In [5], Féray studies $\Psi_{\mathbf{Pos}}$, the Gessel quasisymmetric function defined on the poset Hopf algebra, and describes a set of generators of its kernel.

Some elements of the kernel of $\Psi_{\mathbf{G}}$ have already been constructed in [8] by Guay-Paquet and independently in [10] by Orellana and Scott. These relations, called *modular relations*, extend naturally to the non-commutative case. We introduce them now.

Given a graph G and an edge set E that is disjoint from $E(G)$, let $G \cup E$ denote the graph G with the edges in E added. In [8] and [10], it was observed that for a graph G, if we have edges $e_3 \in G$ and $e_1, e_2 \notin G$ such that $\{e_1, e_2, e_3\}$ forms a triangle, then

$$\boldsymbol{\Psi}_{\mathbf{G}}(G) - \boldsymbol{\Psi}_{\mathbf{G}}(G \cup \{e_1\}) - \boldsymbol{\Psi}_{\mathbf{G}}(G \cup \{e_2\}) + \boldsymbol{\Psi}_{\mathbf{G}}(G \cup \{e_1, e_2\}) = 0\,. \tag{1.1}$$

For such a graph G, we call the formal sum $G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ in $\mathbf{G}$ a *modular relation on graphs*. An example is given in Figure 1. Our goal is to show that these modular relations span the kernel of the chromatic symmetric function.

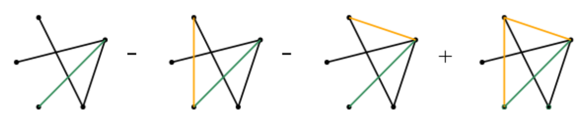

Figure 1: Example of a modular relation.

Theorem 1 (Kernel and image of $\boldsymbol{\Psi}_{\mathbf{G}} : \mathbf{G} \to \mathbf{WSym}$)**.** *The modular relations span* $\ker \boldsymbol{\Psi}_{\mathbf{G}}$. *The image of* $\boldsymbol{\Psi}_{\mathbf{G}}$ *is* ***WSym***.

Two graphs G_1, G_2 are said to be isomorphic if there is a bijection between the vertices that preserves edges. For the commutative version of the symmetric function, if two isomorphic graphs G_1, G_2 are given, we know that $\Psi_{\mathbf{G}}(G_1)$ and $\Psi_{\mathbf{G}}(G_2)$ are the same. The formal sum in $\mathbf{G}$ given by $G_1 - G_2$ is called an *isomorphism relation on graphs*.

Theorem 2 (Kernel and image of $\Psi_{\mathbf{G}} : \mathbf{G} \to \mathbf{Sym}$)**.** *The modular relations and the isomorphism relations generate the kernel of the commutative chromatic symmetric function* $\Psi_{\mathbf{G}}$. *The image of* $\Psi_{\mathbf{G}}$ *is* ***Sym***.

The second part of this theorem follows from previous work. For instance, in [4], several bases of *Sym* are constructed that are of the form $\{\Psi_{\mathbf{G}}(G_\lambda) | \lambda \vdash n\}$.

In the last section of this paper we introduce a new graph invariant $\tilde{\Psi}(G)$. That modular relations on graphs are in the kernel of $\tilde{\Psi}$ is easy to see. It will follow from

Theorem 2 that $\ker \Psi_{\mathbf{G}} \subseteq \ker \tilde{\Psi}$. This reduces the tree conjecture in $\Psi_{\mathbf{G}}$ to this new invariant $\tilde{\Psi}_{\mathbf{G}}$.

The maps $\boldsymbol{\Psi}_{\mathbf{G}}$ and $\Psi_{\mathbf{G}}$ arise as a more general construction in Hopf algebras. For a Hopf algebra $\mathbf{H}$, a *character* η of $\mathbf{H}$ is a linear map $\eta : \mathbf{H} \to \mathbb{K}$ that preserves the multiplicative structure and the unit of $\mathbf{H}$. In [2], Aguiar, Bergeron, and Sotille define a *combinatorial Hopf algebra* as a pair $(\mathbf{H}, \eta)$ where $\mathbf{H}$ is a Hopf algebra and $\eta : \mathbf{H} \to \mathbb{K}$ a character of $\mathbf{H}$. For any combinatorial Hopf algebra $(\mathbf{H}, \eta)$, a canonical Hopf algebra morphism to *QSym* is constructed in [2]. The maps $\Psi_{\mathbf{G}} : \mathbf{G} \to Sym$ and $\boldsymbol{\Psi}_{\mathbf{G}} : \mathbf{G} \to$ **WSym** are Hopf algebra morphisms that can be obtained in such a manner: If we take the character $\eta(G) = \mathbb{1}[G \text{ has no edges}]$, the canonical Hopf algebra morphism for $(\mathbf{G}, \eta)$ is exactly the map $\Psi_{\mathbf{G}}$. The map $\boldsymbol{\Psi}_{\mathbf{G}}$ arises from a parallel result in Hopf monoids, as presented in [11]. The Gessel quasisymmetric function $\Psi_{\mathbf{Pos}}$ on posets arises similarly.

We establish similar results to Theorem 1 and Theorem 2 in the combinatorial Hopf algebra of nestohedra, which is a Hopf subalgebra of generalised permutahedra.

Generalised Permutahedra

Generalised permutahedra are particular polytopes that include permutahedra, associahedra and graph zonotopes. The reader can see some results in the topic in [12].

The Minkowsky sum of two polytopes $\mathfrak{a}, \mathfrak{b}$ is set as $\mathfrak{a} +_M \mathfrak{b} = \{a + b | a \in \mathfrak{a}, b \in \mathfrak{b}\}$. The Minkowsky difference $\mathfrak{a} -_M \mathfrak{b}$ is defined as the unique polytope $\mathfrak{c}$ that satisfies $\mathfrak{b} +_M \mathfrak{c} = \mathfrak{a}$, if it exists. We denote the Minkowsky sum of several polytopes as ${}^M\sum_i \mathfrak{a}_i$.

If we let $\{e_i | i \in I\}$ be the canonical basis of $\mathbb{R}^I$, a *simplex* is a polytope of the form $\mathfrak{s}_J = conv\{e_j | j \in J\}$ for non-empty $J \subseteq I$, and a generalised permutahedron in $\mathbb{R}^I$ is a polytope of the form

$$\mathfrak{q} = \left({}^M\sum_{\substack{J \neq \emptyset \\ a_J > 0}} a_J \mathfrak{s}_J \right) -_M \left({}^M\sum_{\substack{J \neq \emptyset \\ a_J < 0}} |a_J| \mathfrak{s}_J \right) ,$$

for reals $\mathcal{L}(\mathfrak{q}) = \{a_J\}_{\emptyset \neq J \subseteq I}$ that can be either positive, negative or zero. We identify a generalised permutahedron $\mathfrak{q}$ with the list $\mathcal{L}(\mathfrak{q})$. Note that not every list of reals will give us a generalised permutahedron, since the Minkowsky difference is not always defined.

A *nestohedron* is a generalised permutahedron where the coefficients a_J are in $\{0, 1\}$. We identify a nestohedron $\mathfrak{q}$ with the family $\mathcal{F}(\mathfrak{q}) \subseteq 2^I \setminus \emptyset$ corresponding to those J such that $a_J = 1$. Finally, for a set $A \subseteq 2^I \setminus \emptyset$, we write $\mathcal{F}^{-1}(A)$ for the unique nestohedron $\mathfrak{q}$ that satisfies $\mathcal{F}(\mathfrak{q}) = A$. Note that in this case, every subset $A \subseteq 2^I \setminus \emptyset$ will give rise to a nestohedron.

In [1], Aguiar and Ardila define **GP**, a Hopf algebra structure on the linear space generated by generalised permutahedra in $\mathbb{R}^n$ for $n \geq 0$. The Hopf subalgebra **Nesto** is the linear space generated by nestohedra. In [7], Grujić introduced a quasisymmetric map in generalised permutahedra $\Psi_{\mathbf{GP}} : \mathbf{GP} \to QSym$ that we will recall now.

For a polytope $\mathfrak{q} \subseteq \mathbb{R}^I$, Grujić defines a function $f : I \to \mathbb{N}$ as $\mathfrak{q}$-generic if the face of $\mathfrak{q}$ that minimises $\sum_{i \in I} f(i) x_i$, denoted $\mathfrak{q}_f$, is a point. Equivalently, f is $\mathfrak{q}$-generic if it lies in the interior of the normal cone of some vertex.

Then Grujić defines for $\{x_i\}_{i \geq 1}$ commutative variables, the quasisymmetric function:

$$\Psi_{\mathbf{GP}}(\mathfrak{q}) = \sum_{f \text{ is } \mathfrak{q}\text{-generic}} x_f \,. \tag{1.2}$$

If we consider the character $\eta(\mathfrak{q}) = \mathbb{1}[\mathfrak{q} \text{ is a point}]$, then $\Psi_{\mathbf{GP}}$ is the canonical Hopf algebra morphism associated with the combinatorial Hopf algebra $(\mathbf{GP}, \eta)$.

In [1], Aguiar and Ardila define the graph zonotope $Z : \mathbf{G} \to \mathbf{GP}$, a Hopf algebra morphism that is injective and maps $\Psi_{\mathbf{G}}$ to $\Psi_{\mathbf{GP}}$. They also define other maps from other combinatorial Hopf algebras, like matroids, to $\mathbf{GP}$, that preserve the canonical Hopf algebra morphisms. If we are able to describe $\ker \Psi_{\mathbf{GP}}$, then such maps $Z : \mathbf{H} \to \mathbf{GP}$ give us some information on $\ker \Psi_{\mathbf{H}}$ using that $Z(\ker \Psi_{\mathbf{H}}) = \ker \Psi_{\mathbf{GP}} \cap Z(\mathbf{H})$.

We discuss now a non-commutative version of $\Psi_{\mathbf{GP}}$, where we will establish an analogue of Theorem 1 to nestohedra. Consider the Hopf algebra of word quasisymmetric functions **WQSym**, a version of *QSym* in non-commutative variables introduced in [9].

For a generalised permutahedron $\mathfrak{q}$ and non-commutative variables $\{\mathbf{a}_i\}_{i \geq 1}$, we set

$$\mathbf{\Psi}_{\mathbf{GP}}(\mathfrak{q}) = \sum_{f \text{ is } \mathfrak{q}\text{-generic}} \mathbf{a}_f \,.$$

It is easily seen (and shown in [11]) that $\mathbf{\Psi}_{\mathbf{GP}}(\mathfrak{q})$ is a word quasisymmetric function. This defines a Hopf algebra morphism between **GP** and **WQSym**. Let us call $\Psi_{\mathbf{Nesto}}$ and $\mathbf{\Psi}_{\mathbf{Nesto}}$ to the restrictions of $\Psi_{\mathbf{GP}}$ and $\mathbf{\Psi}_{\mathbf{GP}}$ to **Nesto**, respectively.

Our next theorems describe the kernel of the maps $\Psi_{\mathbf{Nesto}}$ and $\mathbf{\Psi}_{\mathbf{Nesto}}$, using some modular relations that we will present latter, in Theorem 14. In fact, these modular relations generalise the ones for graphs, presented in [8] and mentioned earlier in (1.1), in the sense that the graph zonotope embedding $Z : \mathbf{G} \to \mathbf{GP}$, presented in [1], maps modular relation on graphs to modular relations on nestohedra.

Theorem 3 (Kernel and image of of $\mathbf{\Psi}_{\mathbf{Nesto}} : \mathbf{Nesto} \to \mathbf{WQSym}$)**.** *The space* $\ker \mathbf{\Psi}_{\mathbf{Nesto}}$ *is generated by the modular relations in nestohedra. The image of* $\mathbf{\Psi}_{\mathbf{Nesto}}$ *is* **SC**, *a proper subspace of* ***WQSym*** *introduced in Definition 9 below.*

Let us denote by $\mathbf{WQSym}_n$ the linear space of homogeneous word quasisymmetric functions of degree n, and let $\mathbf{SC}_n = \mathbf{SC} \cap \mathbf{WQSym}_n$. A monomial basis for **SC** is presented in Definition 9. The dimension of $\mathbf{SC}_n$ is computed in [11], where in particular it is shown that it is exponentially smaller than the dimension of $\mathbf{WQSym}_n$.

Two generalised permutahedra $\mathfrak{q}_1, \mathfrak{q}_2$ are isomorphic if one can be obtained from the other by permuting the coordinates. If $\mathfrak{q}_1, \mathfrak{q}_2$ are isomorphic, the commutative chromatic quasisymmetric functions $\Psi_{\mathbf{GP}}(\mathfrak{q}_1)$ and $\Psi_{\mathbf{GP}}(\mathfrak{q}_2)$ are the same. We call to $\mathfrak{q}_1 - \mathfrak{q}_2$ an **isomorphism relation on nestohedra**.

Theorem 4 (Kernel and image of $\Psi_{\mathbf{Nesto}} : \mathbf{Nesto} \to \mathbf{QSym}$)**.** *The space* $\ker \Psi_{\mathbf{Nesto}}$ *is generated by the modular relations and the isomorphism relations. The image of* $\Psi_{\mathbf{Nesto}}$ *is* ***QSym****.*

A description of $\ker \mathbf{\Psi}_{\mathbf{Nesto}}$ is less general than a description of $\ker \mathbf{\Psi}_{\mathbf{GP}}$. Nevertheless, most of the combinatorial objects embedded in **GP** are also in **Nesto**, such as graphs and matroids, so the result in the **Nesto** Hopf subalgebra can already be used to help us on other kernel problems.

We will use boldface for non-commutative Hopf algebras, their elements, and the associated combinatorial objects, like word symmetric functions, for sake of clarity.

2 Preliminaries

For an equivalence relation $\sim$ on a set A, we call $[x]_\sim$ to the equivalence class of x in $\sim$, and we write $[x]$ when $\sim$ is clear from context. We write both $\mathcal{E}(\sim)$ and $A/\sim$ for the set of equivalence classes of $\sim$.

2.1 Linear algebra preliminaries

The following easy linear algebra lemmas will be useful to compute generators of the kernels and the images of Ψ and $\mathbf{\Psi}$. These lemmas describe a sufficient condition for a set $\mathcal{B}$ to span the kernel of a linear map $\phi : V \to W$. The proofs of these lemmas are basic linear algebra and can be found in [11].

Lemma 5. *Let V be a finite dimensional vector space with a basis $\{a_i | i \in I\}$ indexed by $I = [m]$, $\phi : V \to W$ be a linear map, and $\mathcal{B} = \{b_j | j \in J\} \subseteq \ker \phi$ be a family of relations.*

Assume that there exists $\overline{I} \subseteq I$ such that:

- *the elements $(\phi(a_i))_{i \in \overline{I}}$ form a linearly independent family in W,*
- *for $i \in I \setminus \overline{I}$ we have $a_i = b + \sum_{k=i+1}^{m} \lambda_k a_k$ for some $b \in \mathcal{B}$ and some scalars λ_k;*

Then $\mathcal{B}$ spans $\ker \phi$. Additionally, we have that $(\phi(a_i))_{i \in \overline{I}}$ is a basis of the image of ϕ.

The following lemma will help us dealing with the composition $\Psi = \mathrm{comu} \circ \mathbf{\Psi}$: we give a sufficient condition for a natural enlargement of the set $\mathcal{B}$ to generate $\ker \Psi$.

Lemma 6. *We will use the same notation as in Lemma 5. Let $\phi_1 : W \to W'$ be a linear map and call $\phi' = \phi_1 \circ \phi$. Take an equivalence relation $\sim$ in $\{a_i\}_{i \in \overline{I}}$ that satisfies $\phi'(a_i) = \phi'(a_j)$ whenever $a_i \sim a_j$. Define $\mathcal{C} = \{a_i - a_j | \, a_i \sim a_j\}$ and write $\phi'([a_i]) = \phi'(a_i)$ with no ambiguity.*

Assume the hypothesis in Lemma 5 and, additionally, suppose that $(\phi'([a_i]))_{[a_i] \in \mathcal{E}(\sim)}$ is linearly independent.

Then $\ker \phi'$ is generated by $\mathcal{B} \cup \mathcal{C}$. Furthermore, $(\phi'([a_i]))_{[a_i] \in \mathcal{E}(\sim)}$ is a basis of $\operatorname{im} \phi'$.

2.2 Hopf algebras and associated combinatorial objects

In the following, all the Hopf algebras $\mathbf{H}$ have a grading, denoted as $\mathbf{H} = \oplus_{n \geq 0} \mathbf{H}_n$.

An *integer composition*, or simply a composition, of n, is a list $\alpha = (\alpha_1, \cdots, \alpha_k)$ of positive integers which sum is n. We write $\alpha \models n$. We denote $l(\alpha)$ for the length of the list and we denote as $\mathcal{C}_n$ the set of compositions of size n.

An *integer partition*, or simply a partition, of n is a non-increasing list $\lambda = (\lambda_1, \cdots, \lambda_k)$ of positive integers which sum is n. We denote $\lambda \vdash n$. We write $l(\lambda)$ for the length of the list and we denote as $\mathcal{P}_n$ the set of partitions of size n. By disregarding the order of the parts on a composition α we obtain a partition denoted $\lambda(\alpha)$.

A *set partition* $\boldsymbol{\pi} = \{\boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_k\}$ of a set I is a collection of non-empty disjoint subsets of I, called *blocks*, that cover I. We write $\boldsymbol{\pi} \vdash I$. We denote $l(\boldsymbol{\pi})$ for the size of the set partition. We write $\mathbf{P}_I$ for the family of set partitions of I, or simply $\mathbf{P}_n$ if $I = [n]$. By counting the elements on each block we obtain an integer partition denoted $\lambda(\boldsymbol{\pi}) \vdash \#I$. We identify a set partition $\boldsymbol{\pi} \in \mathbf{P}_I$ with an equivalence relation $\sim_{\boldsymbol{\pi}}$ on I, where $x \sim_{\boldsymbol{\pi}} y$ if $x, y \in I$ are on the same block of $\boldsymbol{\pi}$.

A *set composition* $\vec{\boldsymbol{\pi}} = S_1 | \cdots | S_l$ of I is a list of non-empty disjoint subsets of I that cover I. We write $\vec{\boldsymbol{\pi}} \models S$. We denote $l(\vec{\boldsymbol{\pi}})$ for the size of the set composition. We call $\mathbf{C}_I$ to the family of set compositions of I, or simply $\mathbf{C}_n$ if $I = [n]$. By disregarding the order of a set composition $\vec{\boldsymbol{\pi}}$, we obtain a set partition $\boldsymbol{\lambda}(\vec{\boldsymbol{\pi}}) \vdash I$. By counting the elements on each block we obtain a composition denoted $\alpha(\vec{\boldsymbol{\pi}}) \models \#I$. A set composition is naturally identified with a total preorder $P_{\vec{\boldsymbol{\pi}}}$ on I, where $x P_{\vec{\boldsymbol{\pi}}} y$ if $x \in S_i, y \in S_j$ for $i \leq j$.

A *colouring* of the set I is a function $f : I \to \mathbb{N}$. The set composition type $\vec{\boldsymbol{\pi}}(f)$ of a colouring $f : I \to \mathbb{N}$ is the set composition obtained after deleting the empty sets of $f^{-1}(1) | f^{-1}(2) | \cdots$.

We recall that in partitions and in set partitions, it is defined a classical *coarsening order* $\leq$, where we say that $\lambda \leq \tau$ ($\boldsymbol{\pi} \leq \boldsymbol{\tau}$, resp.) if τ is obtained from π by adding some parts (resp. if $\boldsymbol{\tau}$ is obtained from $\boldsymbol{\pi}$ by merging some blocks).

Recall that the homogeneous component $QSym_n$ (resp. Sym_n, $\mathbf{WSym}_n$, $\mathbf{WQSym}_n$) of the Hopf algebra $QSym$ (resp. Sym, $\mathbf{WSym}$, $\mathbf{WQSym}$) has a monomial basis indexed by compositions (resp. partitions, set partitions, set compositions). We will denote this basis by $\{M_\alpha\}_{\alpha \in \mathcal{C}_n}$ (resp. $\{m_\lambda\}_{\lambda \in \mathcal{P}_n}$, $\{\mathbf{m}_{\boldsymbol{\pi}}\}_{\boldsymbol{\pi} \in \mathbf{P}_n}$, $\{\mathbf{M}_{\vec{\boldsymbol{\pi}}}\}_{\vec{\boldsymbol{\pi}} \in \mathbf{C}_n}$).

2.3 Monomial basis and graph Hopf algebra

We now discuss the monomial expansion of chromatic symmetric function on graphs:

Lemma 7 ([6, Proposition 3.2])**.** *For a graph G we say that a set partition $\boldsymbol{\tau}$ of $V(G)$ is proper if no block of $\boldsymbol{\tau}$ contains an edge. Then have that $\boldsymbol{\Psi}_{\mathbf{G}}(G) = \sum_{\boldsymbol{\tau}} \mathbf{m}_{\boldsymbol{\tau}}$, where the sum runs over all proper set partitions of $V(G)$.*

For a set partition $\boldsymbol{\pi}$, we define the graph $K_{\boldsymbol{\pi}}$ where $\{i,j\} \in E(K_{\boldsymbol{\pi}})$ if $i \sim_{\boldsymbol{\pi}} j$. A set partition $\boldsymbol{\tau}$ is proper in $K^c_{\boldsymbol{\pi}}$ if and only if $\boldsymbol{\tau} \leq \boldsymbol{\pi}$. Hence, as a consequence of Lemma 7,

$$\boldsymbol{\Psi}_{\mathbf{G}}(K^c_{\boldsymbol{\pi}}) = \sum\nolimits_{\boldsymbol{\tau} \leq \boldsymbol{\pi}} \mathbf{m}_{\boldsymbol{\tau}}. \tag{2.1}$$

2.4 Monomial basis and nestohedra Hopf algebra

We define, for a non-empty set $A \subseteq [n]$, the set $A_{\vec{\boldsymbol{\pi}}} = \{\text{minima of A in } P_{\vec{\boldsymbol{\pi}}}\}$, where we recall that $P_{\vec{\boldsymbol{\pi}}}$ is a total preorder on $[n]$. We say that $A_{\vec{\boldsymbol{\pi}}} = pt$ if $A_{\vec{\boldsymbol{\pi}}}$ is a singleton. The following lemma is well known in the folklore of generalised permutahedra and is shown, for instance, in [11].

Lemma 8 (Vertex normal cone characterization)**.** *Let $\mathfrak{q}$ be a nestohedron. A colouring f is $\mathfrak{q}$-generic if and only if $A_{\vec{\boldsymbol{\pi}}(f)} = pt$ for every $A \in \mathcal{F}(\mathfrak{q})$. Furthermore, the face $\mathfrak{q}_f$ that minimizes $\sum_i f(i)x_i$ only depends on the set composition $\vec{\boldsymbol{\pi}}(f)$.*

We write $\mathfrak{q}_{\vec{\boldsymbol{\pi}}}$ for the face $\mathfrak{q}_f$ for any f of set composition type $\vec{\boldsymbol{\pi}}$, without ambiguity. For $\vec{\boldsymbol{\pi}} \in \mathbf{C}_n$, we define the *fundamental nestohedron* as $\mathfrak{p}^{\vec{\boldsymbol{\pi}}} = \mathcal{F}^{-1}\{A \subseteq [n] \,|\, A_{\vec{\boldsymbol{\pi}}} = pt\}$.

On set compositions, we write that $\vec{\boldsymbol{\pi}}_1 \preceq \vec{\boldsymbol{\pi}}_2$ whenever $\mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}_1}) \subseteq \mathcal{F}(\mathfrak{p}^{\vec{\boldsymbol{\pi}}_2})$. Equivalently, $\vec{\boldsymbol{\pi}}_1 \preceq \vec{\boldsymbol{\pi}}_2$ if for any non-empty $A \subseteq [n]$ we have $A_{\vec{\boldsymbol{\pi}}_1} = pt \Rightarrow A_{\vec{\boldsymbol{\pi}}_2} = pt$. With this, $\preceq$ is a preorder, called *singleton commuting preorder* or *SC preorder*.

Additionally, we define the equivalence relation $\sim$ in $\mathbf{C}_n$ as $\vec{\boldsymbol{\pi}} \sim \vec{\boldsymbol{\tau}}$ if $\mathfrak{p}^{\vec{\boldsymbol{\pi}}} = \mathfrak{p}^{\vec{\boldsymbol{\tau}}}$. A combinatorial interpretation of this equivalence relation can be found below in Proposition 10, which also motivates the name of the preorder defined above.

It is natural to consider $\mathbf{N}_{[\vec{\boldsymbol{\pi}}]} = \sum_{\vec{\boldsymbol{\tau}} \sim \vec{\boldsymbol{\pi}}} \mathbf{M}_{\vec{\boldsymbol{\tau}}}$, which forms a linear independent family.

Definition 9. *We define the singleton commuting Hopf algebra, or **SC** for short, as the graded vector subspace $\oplus_{n \geq 0} \mathbf{SC}_n$ of **WQSym**, where $\{N_{[\vec{\boldsymbol{\pi}}]} : [\vec{\boldsymbol{\pi}}] \in \mathbf{C}_n / \sim\}$ is a basis of each $\mathbf{SC}_n$. As a consequence of Theorem 3, **SC** is a Hopf algebra.*

The following proposition will not be used in the proof of the main theorems, but gives us a way to describe the equivalence classes of $\sim$. In particular, in [11], it allows us to compute the dimensions of $\mathbf{SC}_n$. The proof of Proposition 10 can be found in [11].

Proposition 10. *For $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\tau}} \in \mathbf{C}_I$, the following are equivalent.*

- *We have that* $\mathfrak{p}^{\vec{\pi}} = \mathfrak{p}^{\vec{\tau}}$.
- *We have* $\boldsymbol{\lambda}(\vec{\pi}) = \boldsymbol{\lambda}(\vec{\tau})$ *and each* $a, b \in I$ *that satisfies both* $a\, P_{\vec{\pi}}\, b$ *and* $b\, P_{\vec{\tau}}\, a$ *are either singletons or in the same block in* $\boldsymbol{\lambda}(\vec{\pi})$.

From the definition of $\preceq$, we have the following consequence of Lemma 8.

$$\mathbf{\Psi}_{\mathbf{GP}}(\mathfrak{p}^{\vec{\pi}}) = \sum_{\vec{\pi} \preceq \vec{\tau}} \mathbf{M}_{\vec{\tau}}\,. \tag{2.2}$$

As presented, (2.2) seems to shows that $(\mathbf{\Psi}_{\mathbf{GP}}(\mathfrak{p}^{\vec{\pi}}))_{\vec{\pi} \in \mathbf{C}_n}$ writes triangularly with respect to the monomial basis. Since $\preceq$ is not an order, that is not the case, but we obtain a related result with this reasoning:

Lemma 11. *The family* $(\mathbf{\Psi}(\mathfrak{p}^{[\vec{\pi}]}))_{[\vec{\pi}] \in \mathbf{C}_n/\sim}$ *forms a basis of* ***SC***.

The following lemma is helpful to show Theorem 4 and is shown in [11].

Lemma 12. *There is an order* $\leq'$ *on* $\mathcal{C}_n$ *that satisfies* $\vec{\pi} \preceq \vec{\tau} \Rightarrow \alpha(\vec{\pi}) \leq' \alpha(\vec{\tau})$.

3 Main theorems on graphs

With Lemma 5, we will show that the kernel of $\mathbf{\Psi}_{\mathbf{G}}$ is spanned by the modular relations.

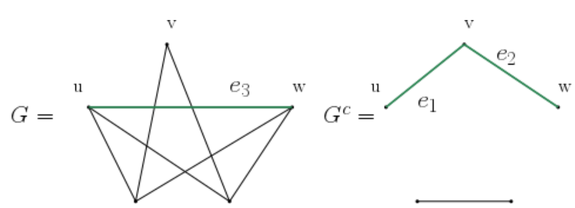

Figure 2: Choice of edges in proof of Theorem 1

Proof of Theorem 1. Recall that $\mathbf{G}_n$ is spanned by graphs with vertex set $[n]$. We choose an order $\geq$ in this family of graphs in a way that the number of edges is non-decreasing.

Recall that for a set partition π of the vertex set $[n]$, we define K_π as the graph where $\{i, j\} \in E(K_\pi)$ if $i \sim_\pi j$. Then, from (2.1), we know that $\{\mathbf{\Psi}_{\mathbf{G}}(K_\pi^c) | \pi \in \mathbf{P}_n\}$ writes as a triangular matrix over the monomial basis of **WSym**, hence forms a linearly independent set in **WSym**.

In order to apply Lemma 5 to the set of modular relations on graphs, it suffices to show the following: if a graph G is not of the form K_π^c, then we can find a formal sum $G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ that is a modular relation. Indeed, G is the graph with least edges in that expression, so it is the smallest in the order $\geq$. If the above holds, Lemma 5 implies that the modular relations generate the space $\ker \mathbf{\Psi}_{\mathbf{G}}$ and $\{\mathbf{\Psi}_{\mathbf{G}}(K_\pi^c) | \pi \in \mathbf{P}_n\}$ forms a basis of $\operatorname{im} \mathbf{\Psi}_{\mathbf{G}}$, so, from (2.1), $\operatorname{im} \mathbf{\Psi}_{\mathbf{G}} = \mathbf{WSym}$.

To find the desired modular relation, it is enough to find a triangle $\{e_1, e_2, e_3\}$ such that $e_1, e_2 \notin E(G)$ and $e_3 \in E(G)$. Consider τ, the set partition given by the connected components of G^c. By hypothesis, $G \neq K^c_\pi$, so there are vertices v, w in the same block of $\boldsymbol{\tau}$ that are not neighbours in G^c. Without loss of generality we can take such u, w that are at distance 2 in G^c, so they have a common neighbour v in G^c. The edges $e_1 = \{v, u\}$, $e_2 = \{v, w\}$ and $e_3 = \{u, w\}$ form the desired triangle, concluding the proof. □

Proof of Theorem 2. Our goal is to apply Lemma 6 to the map $\Psi_{\mathbf{G}} = \text{comu} \circ \boldsymbol{\Psi}_{\mathbf{G}}$ for the equivalence relation corresponding to graph isomorphism. First, if $\lambda(\boldsymbol{\pi}) = \lambda(\boldsymbol{\tau})$ then $K^c_{\boldsymbol{\pi}}$ and $K^c_{\boldsymbol{\tau}}$ are isomorphic graphs. Define without ambiguity $r_{\lambda(\boldsymbol{\pi})} = \Psi_{\mathbf{G}}(K^c_{\boldsymbol{\pi}})$.

From the proof of Theorem 1, to apply Lemma 6 it is enough to establish that the family $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ is linearly independent. Indeed, it would follow that $\ker \Psi_{\mathbf{G}}$ is generated by the modular relations and the isomorphism relations, and $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ is a basis of $\operatorname{im} \Psi_{\mathbf{G}}$ concluding the proof.

The linear independence of $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ follows from the fact that it writes as an upper triangular matrix under the coarsening order in integer partitions. Indeed, from (2.1), if we let $\boldsymbol{\tau}$ run over set partitions and σ run over integer partitions, we have

$$r_{\lambda(\boldsymbol{\pi})} = \Psi_{\mathbf{G}}(K^c_{\boldsymbol{\pi}}) = \sum_{\boldsymbol{\tau} \leq \boldsymbol{\pi}} m_{\lambda(\boldsymbol{\tau})} = \sum_{\sigma \leq \lambda(\boldsymbol{\pi})} a_{\boldsymbol{\pi},\sigma}\, m_\sigma\,,$$

where $a_{\boldsymbol{\pi},\sigma} = \#\{\boldsymbol{\tau} \vdash [n] | \lambda(\boldsymbol{\tau}) = \sigma,\, \boldsymbol{\tau} \leq \boldsymbol{\pi}\}$. Note that $a_{\boldsymbol{\pi},\lambda(\boldsymbol{\pi})} = 1$, so $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ is linearly independent. From a dimention argument, $(r_\lambda)_{\lambda \in \mathcal{P}_n}$ spans Sym_n, so $\operatorname{im} \Psi_{\mathbf{G}} = Sym$. □

Remark 13. *We have obtained in the proof of Theorem 2 that $(r_\lambda)_{\lambda \vdash n}$ is a basis for Sym_n. This basis is different from other "chromatic bases" proposed in [4]. The proof gives us a recursive way to compute the coefficients ζ_λ on the span $\Psi_{\mathbf{G}}(G) = \sum_\lambda \zeta_\lambda r_\lambda$. It is then natural to ask if combinatorial properties can be obtained for these coefficients, which are isomorphic invariants.*

Similarly in the non-commutative case, we obtain that $\mathbf{WSym}_n$ is spanned by $(\boldsymbol{\Psi}_{\mathbf{G}}(K^c_{\boldsymbol{\pi}}))_{\boldsymbol{\pi} \vdash [n]}$, and so other coefficients arise. We can again ask for combinatorial properties of these coefficients. The same can be asked in the next section for the nestohedra case.

4 Main theorems on nestohedra

For non-empty sets $A \subseteq [n]$, we define $\text{Orth}\, A = \{\vec{\boldsymbol{\pi}} \in \mathbf{C}_n | A_{\vec{\boldsymbol{\pi}}} = pt\}$. We have:

Theorem 14 (A modular relation for $\boldsymbol{\Psi}_{\mathbf{Nesto}}$). *Let $\{A_k | k \in K\}$ and $\{B_j | j \in J\}$ be two disjoint families of non-empty subsets of $[n]$. Let us write $\mathcal{K} = \cup_{k \in K} (\text{Orth}\, A_k)^c$, and $\mathcal{J} = \cup_{j \in J} \text{Orth}\, B_j$. Consider the nestohedron $\mathfrak{q} = \mathcal{F}^{-1}\{A_k | k \in K\}$.*

Suppose that $\mathcal{K} \cup \mathcal{J} = \mathbf{C}_n$. Then,

$$\textstyle\sum_{T \subseteq J} (-1)^{\#T}\, \boldsymbol{\Psi}_{\mathbf{GP}} \left[\mathfrak{q} +_M \mathcal{F}^{-1}\{B_j | j \in T\} \right] = 0\,.$$

The proof of this result is done combinatorially, and is presented in [11].

The sum $\sum_{T\subseteq J}(-1)^{\#T}\left[\mathfrak{q} +_M \mathcal{F}^{-1}\{B_j|j\in T\}\right]$ is called a *modular relation on nestohedra*.

It can be noted that, if $l = G - G\cup\{e_1\} - G\cup\{e_2\} + G\cup\{e_1, e_2\}$ is a modular relation on graphs, then the graph zonotope $Z(l)$ is the modular relation on nestohedra corresponding to $\mathfrak{q} = Z(G)$ (i.e. $\{A_k|k\in K\} = E(G)$), $B_1 = e_1$ and $B_2 = e_2$. In this case, the condition $\mathcal{K}\cup\mathcal{J} = \mathbf{C}_n$ follows from the fact that no proper colouring of G is monochromatic in both e_1 and e_2, which is imposed by $e_3\in G$.

Recall that we set $\mathfrak{p}^{\vec{\pi}} = \mathcal{F}^{-1}\{A\subseteq[n]|A_{\vec{\pi}} = pt\}$, which depends only on the SC-equivalence class of $\vec{\pi}$ (by definition of $\sim$) and are called the *fundamental nestohedra*. Write, without ambiguity, $\mathfrak{p}^{[\vec{\pi}]} = \mathfrak{p}^{\vec{\pi}}$.

We follow here roughly the same idea as in the graph case: We use the family of nestohedra $(\mathfrak{p}^{[\vec{\pi}]})_{[\vec{\pi}]\in\mathbf{C}_n/\sim}$, constructed above, whose image by $\mathbf{\Psi}_{\mathbf{GP}}$ is linearly independent and is rich enough to span the image, to apply Lemma 5.

Proof of Theorem 3. We will apply Lemma 5 with the modular relations from Theorem 14.

First recall that $\mathbf{Nesto}_n$ is a linear space generated by the nestohedra in $\mathbb{R}^n$. We choose a total order $\geq$ on the nestohedra so that $\#\mathcal{F}(\mathfrak{q})$ is non decreasing.

We have seen in Lemma 11 that $(\mathbf{\Psi}_{\mathbf{GP}}(\mathfrak{p}^{[\vec{\pi}]}))_{[\vec{\pi}]\in\mathbf{C}_n/\sim}$ is linearly independent. Therefore, it suffices to show that for any $\mathfrak{q}$ that is not a fundamental nestohedron, we can write some modular relation b as $b = \mathfrak{q} + \sum_i \lambda_i\mathfrak{q}_i$, where $\#\mathcal{F}(\mathfrak{q}) < \#\mathcal{F}(\mathfrak{q}_i)\ \forall i$. So $\mathfrak{q} < \mathfrak{q}_i\ \forall i$.

Indeed, it would follow from Lemma 5 that the modular relations on nestohedra span $\ker\mathbf{\Psi}_{\mathbf{Nesto}}$. As a consequence, $\operatorname{im}\mathbf{\Psi}_{\mathbf{Nesto}}$ is spanned by the sets $\{\mathbf{\Psi}_{\mathbf{GP}}(\mathfrak{p}^{[\vec{\pi}]})|[\vec{\pi}]\in\mathbf{C}_n/\sim\}$ for each $n\geq 0$. From Lemma 11, this image is $\mathbf{SC}_n$.

To obtain the desired modular relation, we invoke Theorem 14 on $\{A\in\mathcal{F}(\mathfrak{q})\}$ and $\{B\notin\mathcal{F}(\mathfrak{q})\}$. Let us write $\mathcal{K} = \cup_{A\in\mathcal{F}(\mathfrak{q})}(\operatorname{Orth} A)^c$ and $\mathcal{J} = \cup_{B\notin\mathcal{F}(\mathfrak{q})}\operatorname{Orth} B$. We will first show that we have $\mathcal{K}\cup\mathcal{J} = \mathbf{C}_n$.

Take, for sake of contradiction, some $\vec{\pi}\notin\mathcal{K}\cup\mathcal{J}$. Note that $\vec{\pi}\notin\mathcal{K}$ is equivalent to $A_{\vec{\pi}} = pt$ for every $A\in\mathcal{F}(\mathfrak{q})$. Note as well that $\vec{\pi}\notin\mathcal{J}$ is equivalent to $B_{\vec{\pi}}\neq pt$ for every $B\notin\mathcal{F}(\mathfrak{q})$. Therefore, if $\vec{\pi}\notin\mathcal{K}\cup\mathcal{J}$, then $\mathfrak{q} = \mathfrak{p}^{\vec{\pi}}$, contradicting the assumption that $\mathfrak{q}$ is not a fundamental nestohedron. We obtain that $\mathcal{K}\cup\mathcal{J} = \mathbf{C}_n$. Finally, note that

$$\mathfrak{q} + \sum_{\substack{T\subseteq\mathcal{F}(\mathfrak{q})^c\\ T\neq\emptyset}}(-1)^{\#T}\left[\mathfrak{q} +_M \mathcal{F}^{-1}(T)\right],$$

is a modular relation of the desired form, concluding the hypothesis of Lemma 5. □

For the commutative case we will apply Lemma 6. Note that we already have a generator set of $\ker\mathbf{\Psi}_{\mathbf{Nesto}}$, so similarly to the proof of Theorem 2, we just need to establish some linear independence.

Recall that two nestohedra $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are isomorphic if there is a permutation matrix P such that $x\in\mathfrak{q}_2 \Leftrightarrow Px\in\mathfrak{q}_1$. Since we are in the commutative case now, if $\vec{\pi}_1$ and

$\vec{\pi}_2$ share the same composition type, then $\mathfrak{p}^{\vec{\pi}_1}$ and $\mathfrak{p}^{\vec{\pi}_2}$ are isomorphic, and so we have $\Psi_{\mathbf{GP}}(\mathfrak{p}^{\vec{\pi}_1}) = \Psi_{\mathbf{GP}}(\mathfrak{p}^{\vec{\pi}_2})$. Set $R_{\alpha(\vec{\pi})} := \Psi_{\mathbf{GP}}(\mathfrak{p}^{\vec{\pi}})$ without ambiguity.

Proof of Theorem 4. We will apply Lemma 6 to the map $\Psi_{\mathbf{GP}} = \mathrm{comu} \circ \boldsymbol{\Psi}_{\mathbf{GP}}$ on the equivalence relation corresponding to the isomorphism of nestohedra.

From the proof of Theorem 3, to apply Lemma 6 it is enough to establish that the family $(R_\alpha)_{\alpha \in \mathcal{C}_n}$ is linearly independent. It would follow that $\ker \Psi_{\mathbf{GP}}$ is generated by the modular relations and the isomorphism relations, and $(R_\alpha)_{\alpha \in \mathcal{C}_n}$ is a basis of $\operatorname{im} \Psi_{\mathbf{G}}$, concluding the proof.

To show the linear independence of $(R_\alpha)_{\alpha \in \mathcal{C}_n}$, we write R_α on the monomial basis of $QSym$, and use the order $\leq'$ mentioned in Lemma 12.

As a consequence of (2.2), if we write $A_{\vec{\pi},\beta} = \#\{\vec{\tau} \in \mathbf{C}_n | \vec{\pi} \preceq \vec{\tau},\, \alpha(\vec{\tau}) = \beta\}$, we have:

$$R_{\alpha(\vec{\pi})} = \Psi_{\mathbf{GP}}(\mathfrak{p}^{\vec{\pi}}) = \sum_{\vec{\pi} \preceq \vec{\tau}} M_{\alpha(\vec{\tau})} = A_{\vec{\pi},\alpha(\vec{\pi})} M_{\alpha(\vec{\pi})} + \sum_{\alpha(\vec{\pi}) <' \beta} A_{\vec{\pi},\beta} M_\beta \,, \tag{4.1}$$

It is clear that $A_{\vec{\pi},\alpha(\vec{\pi})} > 0$, so independence follows, which completes the proof. $\square$

5 A word on graph invariants

Consider the ring $\mathbb{K}[q_1, q_2, \cdots]$ on countable many commuting variables, and let R be such ring modulo the relations $q_i(q_i - 1)^2 = 0$. Let $Sym(R)$ be the ring of symmetric functions with coefficients in R.

Consider the graph invariant $\tilde{\Psi}(G) = \sum_f x_f q_i^{c_G(f,i)}$ in $Sym(R)$, where the sum runs over **all** colourings f, and $c_G(f,i)$ stands for the number of monochromatic edges of colour i in the colouring f (i.e. edges $\{v_1, v_2\}$ such that $f(v_1) = f(v_2) = i$).

Let $l = G - G \cup \{e_1\} - G \cup \{e_2\} + G \cup \{e_1, e_2\}$ be a modular relation on graphs, i.e. $\{e_1, e_2, e_3\}$ are edges that form a triangle, and for a colouring f, set

$$s_f = q_i^{c_G(f,i)} - q_i^{c_{G\cup\{e_1\}}(f,i)} - q_i^{c_{G\cup\{e_2\}}(f,i)} + q_i^{c_{G\cup\{e_1,e_2\}}(f,i)} \,.$$

It is easy to see that s_f is always zero in R, so $\tilde{\Psi}(l) = \sum_f x_f s_f = 0$.

It follows that any modular relation is in $\ker \tilde{\Psi}$. From Theorem 2 we have that $\ker \Psi_{\mathbf{G}} \subseteq \ker \tilde{\Psi}$, so we obtain the following proposition.

Proposition 15. *For any graphs G_1, G_2, we have $\Psi_{\mathbf{G}}(G_1) = \Psi_{\mathbf{G}}(G_2) \Rightarrow \tilde{\Psi}(G_1) = \tilde{\Psi}(G_2)$.*

If we find a graph invariant satisfying Proposition 15 that takes different values for non-isomorphic trees, we obtain a proof of the tree conjecture. This is in line with what has been done in [3], where it was shown that non-isomorphic proper caterpillars have different chromatic symmetric functions. We wish to use Theorem 2 to prove Proposition 15 for other invariants.

Now we have $\tilde{\Psi}(G)|_{q_i=0} = \Psi_{\mathbf{G}}$. So $\ker \tilde{\Psi} = \ker \Psi_{\mathbf{G}}$. We note that other specialisations are also alowed, like $\tilde{\Psi}(G)|_{q_i=1}$ and $\frac{d}{dq_i} \tilde{\Psi}(G)|_{q_i=1}$.

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