Probability 2

Exercise sheet nb. 6

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Due until: 29th October at 5 p.m.

Exercises marked with * should be easier after attending the lecture on Thursday.

Exercise 1 (2 points). The goal of the exercise is to give a martingale-based proof of Kolmogorov's 0-1 law. Let $(Y_i)_{i\geq 1}$ be a sequence of independent random variables. We set

$$\mathcal{F}_n = \sigma(Y_1, \dots, Y_n), \qquad \mathcal{F}_\infty = \sigma\left(\bigcup_{n\geq 1} \mathcal{F}_n\right),$$
$$\mathcal{G}_n = \sigma(Y_n, Y_{n+1}, \dots), \qquad \mathcal{G}_\infty = \bigcap_{n\geq 1} \mathcal{G}_n.$$

Consider $A \in \mathcal{G}_\infty$.

1. Show that $X_n := \mathbb{E}[1_A|\mathcal{F}_n]$ is a martingale that converges a.s. and in L^1. Compute X_n (Hint: $A \in \mathcal{G}_{n+1}$). Conclude that $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

2. Suppose that Z is a real $\mathcal{G}_\infty$-measurable random variable. Show that $Z = a$ a.s. for some constant a.

Exercise 2 (4 points). Let $\{Y_n\}_{n\geq 1}$ be a sequence of independent positive random variables such that for all $n \geq 1$ we have that $\mathbb{E}[Y_n] = 1$, and consider $X_n = \prod_{k=1}^n Y_k$. We use the filtration $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

1. Show that $\{X_n\}_{n\geq 0}$ is a martingale. and that $\{Z_n\}_{n\geq 0}$ defined as $Z_n = \sqrt{X_n}$ is a supermartingale. Show that both converge a.s.

2. Suppose that $\prod_{k=1}^\infty \mathbb{E}[\sqrt{Y_k}] = 0$. Compute the a.s. limit of Z_n and X_n. Show that $\{X_n\}_{n\geq 0}$ is not u.i. (Hint: Use Fatou's Lemma on $\mathbb{E}[Z_n]$)

3. Suppose that $\prod_{k=1}^\infty \mathbb{E}[\sqrt{Y_k}] > 0$. Show that $\{Z_n\}_{n\geq 0}$ is a Cauchy sequence in $\mathcal{L}^2$. Show that $||X_n - X_m||_1 \leq 2||Z_n - Z_m||_2$ and show that X_n is a closed martingale.

   (Hint: Write $||X_n - X_m||_1 = \mathbb{E}[|(Z_n - Z_m)(Z_n + Z_m)|]$ and use Cauchy-Schwarz inequality in $\mathcal{L}^2$, recall that $\mathcal{L}^1$ is a complete metric space)

Exercise 3 (3 points *). Let $(X_n)_{n\geq 1}$ be a sequence of random variables such that for some $a \in [0,1]$, $X_0 = a$ a.s. and, for each $n \geq 0$,

$$\mathbb{P}\big(X_{n+1} = \tfrac{X_n}{2}\big|\mathcal{F}_n\big) = 1 - X_n, \quad \mathbb{P}\big(X_{n+1} = \tfrac{1+X_n}{2}\big|\mathcal{F}_n\big) = X_n,$$

where $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, and $\mathbb{P}(A|\mathcal{F}) = \mathbb{E}[\mathbb{1}_A|\mathcal{F}]$.

1. Show that $(X_n)_{n\geq 0}$ is a martingale that converges a.s. and in L^2 to some random variable X_∞. Compute $\mathbb{E}(X_\infty)$.
2. Prove that $\mathbb{E}[(X_{n+1} - X_n)^2] = \frac{1}{4}\mathbb{E}[X_n(1 - X_n)]$.
3. Compute $\mathbb{E}[X_\infty(1 - X_\infty)]$. What is the distribution of X_∞?