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Chromatic invariants over graphs

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Preface

This is a Semester Project developed at the ETH Zürich as part of the Master Program in Pure Mathematics, where I was supervised by Prof. Dr. Valentin Féray from Zürich University.

In this project, during the autumn semester of 2015, I worked in some topics in chromatic invariants of graphs as part of a Masters in Pure Mathematics in ETH Zürich. We consider a paper from Richard Stanley that introduces a new algebraic invariant and some conjectures on the field that have already been tackled in the literature. We are supposed to go over several notions that trail from colouring of graphs to posets, and we will gather some of the usual methods to explore the new invariants proposed.

All relevant definitions that are not well known for a final year Bachelor student are present.

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Acknowledgment

I would like to thank the ETH University for all the conditions provided for my work and success in this Semester Paper, where I tested my limits regarding the research experience and writing abilities.

I would like to thank my supervisor from ETH University, Prof. Dr. Angelika Steger, for giving me the opportunity to do a Semester Paper in my Masters program.

I would like also to thank my supervisor from Zürich University, Prof. Dr. Valentin Féray, for all the insights and ideas for the project that led me through the path that I show here. With his help I managed find a very interesting topic and motivation to pursue and investigate particular cases and reasonings in this project. For that contribution to my academic work, I'm with no doubt grateful.

I would like to thank all my friends for the support and motivation, for teaching me inumerable lessons of friendship and thoughtfulness either from giving the example or through appealing to my senses when I didn't have it. In particular, my dear friend Ana Borges has never let me down and shared with me all her knowledge to my best interest: for such altruism I will carry the uttermost respect and honour. Also, I think that what Beatrice da Costa has done for me is of remarkable value and I can't help to address many thanks. I also would thank my family for all support, with special regards to my mother that always motivated me to work outside my confort zone and challenged me to tackle the hard problems with fearless courage.

R.P.

Summary and Conclusions

All in all, the work done in this project covers two distinct paths given by two conjectures from R. Stanley, regarding the chromatic symmetric functions. A short work about chromatic functions also got some attention during the project, as well as some general facts from the literature on symmetric functions.

The first conjecture asks if the chromatic symmetric function is a complete invariant for trees. The fact that it is not an invariant for general graphs can be readily proven, as Stanley's paper provided with two simple counterexamples, (non-isomorphic) graphs with the same chromatic symmetric function. From that, the simpler task to describe well known properties of trees from it's chromatic symmetric function was set and we are able to find the number of leaves of a tree, for instance. In the literature, more extensions to this were found, so we are able to devise the degree sequence of a tree knowing only its chromatic symmetric function , and the same happens to the girth and number of paths of a certain length.

Some personal work was done following this lines. By interpreting the coefficients of the chromatic symmetric function in a suitable way we can find the number of tails of certain shapes of trees. From this steams a possible path towards the main conjecture on this work. If we are able to devise a general formula in a concrete way that we will explain later, inverting said formula gives uniqueness of chromatic symmetric function as desired.

Some general families of trees (like proper caterpillars and forks) are also dealt with and fully distinguished, and some theory regarding $\mathscr{L}$ polynomials from compositions is developed for that goal. Finally, if we enrich the coefficients of the chromatic symmetric function with more information, we are able to distinguish all trees via an algorithmic construction, as done in 2013 by Rosa Orellana and Geoffrey Scott.

On the second conjecture we try to write the chromatic symmetric function in some different basis of the symmetric functions, and ask about the combinatorial meaning of the coefficients, the more difficult conjecture asks if a certain family of graphs, called indifference graphs of (3+1)-free posets, has non-negative coefficients in the $\{e_\lambda\}$ basis.

For this conjecture a natural flow of work has been established: in the literature we are able to find ways to reduce the problem to a smaller family of graphs, but it is still an unanswered

question today.

The task of the second conjecture is tackled via the so called modular relations. By expressing the chromatic symmetric function of a graph as a convex combination of a set S of symmetric functions with positive coefficients in the $\{e_\lambda\}$ basis, we reduce the problem of e-positivity to the symmetric functions in S. For that, we interpret the coefficients of a linear combination as probabilities of certain events.

In the literature, we can find a reduction of the problem to the so called (3+1 and 2+2)-free posets which have a precise geometric interpretation.

Although the work is not exhaustive, the goal is to go over the main results approaching both conjectures, as well as to provide a solid foundation to the theory of symmetric functions and it's applications to graph theory.

Contents

0.1 Background

In this chapter we will deal with the main definitions that lay the foundations to our work henceforth. We will start with a quick definition of a graph and its main properties and examples, nomenclature and notation. Then, we will talk about partitions and main definitions that we want to set. Afterwards, we deal with symmetric functions, indicate some properties such as its main basis. Finally we introduce the chromatic symmetric function of a graph and some of its properties.

We will assume knowledge in basic algebra and some familiarity with graphs.

0.1.1 Background in Graphs

A graph is a pair $G = (V(G), E(G))$, we assume that $V(G)$ is a finite set, where each edge connects two vertices, so each edge is a set of two vertices. We also reserve the variable n for $\#V(G)$ whenever G is clear without further notice. Throughout the work we assume that graphs have neither loops nor parallel edges without further notice (called *simple graphs* in the literature), and reserve the name *multigraph* for graphs that may have parallel edges and loops. We can regard $E(G) \subseteq \binom{V(G)}{2}$ as a family whose elements are 2-element sets of vertices. Throughout the script we can refer to an edge between vertices u and v by $\{u, v\}$, $u - v$ or $u \sim v$.

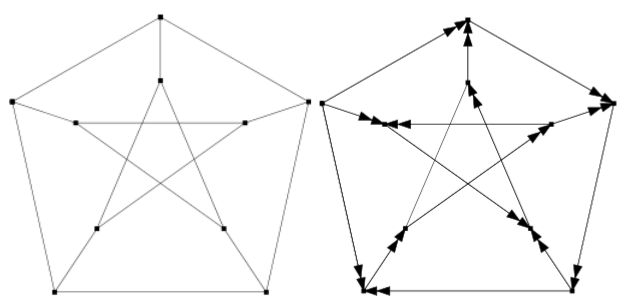


Figure 1: The Petersen Graph: simple and one orientation

We can encode a graph by specifying the set of neighbours of each vertex, $N(v) := \{u \in V(G) : u - v \in E(G)\}$.

Sometimes we appeal to the notion of *oriented graph*, where we regard that for each edge the two endpoints are distinguishable - one is the source and the other is the sink - and in such case we consider that loops and parallel edges should not exist. We regard the edge set as a subset of

$V(G)^2$ that is disjoint from the diagonal and, if $(u,v) \in E(G)$, then $(v,u) \notin E(G)$

There is a clear underlying map from oriented graphs to simple graphs, that disregards orientation and preserves vertices. An *orientation of a graph* G is a pre-image of G through this map, and the set of orientations of G, $\mathcal{O}(G)$ is the set of pre-images.

There are some special graphs that we often want to refer. The *complete graph* is the graph with n vertices and all $\binom{n}{2}$ possible edges, and we write it as K_n. The *complete bipartite graph* has $m+n$ vertices that split into two sets in such a way that the set of edges is exactly those that go from one set to another, and we write it as $K_{m,n}$.

☞ Definition: Trees

A *tree* is an acyclic connected graph. It is a theorem (Diestel, 2000, Cor. 1.5.3) that a tree must have $n-1$ edges.

A rooted tree, or generally a rooted graph, is just a graph with a distinct vertex regarded as its root

A path in a graph is a finite sequence of distinct vertices $P = v_1 v_2 \dots v_n$ such that $\{v_k, v_{k+1}\} \in E(G)$ for each $k = 1, \dots, n-1$. We say that P is a path between v_1 and v_n, and a graph is said to be *connected* if for each pair of vertices there is a path between them.

There is an order relation among the graphs supported in a fixed set of vertices, we say that $G \subseteq H$, or G is a subgraph of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

☞ Definition: Stable partitions and proper colourings

A set partition $\biguplus_i V_i = V(G)$ of the vertex set is said to be a *stable partition* if there are no two connected vertices in the same part V_i. A proper vertex-colouring of G, or a colouring for short, is a function from the graph vertex set $V(G)$ to the natural numbers such that two neighbouring vertices have a different image, called colour.

There are several atomic operations on graphs. We consider two right now: Given an edge e of the graph G, we define the operation *edge deletion* resulting in the graph $G \setminus e$, where the vertex set is preserved and we delete e from the edge set.

We define as well the operation *edge contraction* resulting in the graph G/e, where we collapse the endpoint of the edge $e = \{u,v\}$ into one vertex w, with $N(w) = N(v) \cup N(v)$ without creating parallel edges nor loops (so some edges might merge into one). All the remaining vertices and edges are preserved. We can contract edges in any given order, so we can write $(G/e_1)/e_2$ as $G/\{e_1, e_2\}$, which is exactly what happens in edge deletion, as we would have written $G \setminus \{e_1, e_2\}$ for the deletion of two edges e_1, e_2.

In multigraphs, the edge contraction G/e may create parallel edges and loops, preserving all edges in G except from e.

We define the *chromatic number* $\chi(G)$ of the graph G as the smallest number N such that a colouring exists with no colour exceeding N and we define the chromatic polynomial $\chi_G(k)$ as the number of colourings with colour set $\{1, \dots, k\}$

So the chromatic polynomial is k^n for the discrete graph (no edges) and we have a *deletion-contraction relation*: for a given edge $e \in G$,

Proposition 0.1.1. For a graph G we have

$$\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k)\,.$$

Note, however, that this implies that $\chi_G(k)$ is indeed a polynomial of degree n.

0.1.2 Background in Partitions

An integer partition of a positive integer $n \in \mathbb{N}$ is a multi-set of positive integers that sum up to n. So any finite multi-set λ of positive integers is a partition for some n. We say that $\lambda \vdash n$. We represent partitions as a tuple (4, 1, 1) in non decreasing order, or explicitly saying that it is a partition of n by $(4, 1, 1) \vdash 6$, or by an explicit sum $4 + 1 + 1 = 6$. Since the main notation may be misleading, we also interpret partitions as a tuple $(\lambda_1, \dots, \lambda_k)$ and we make the additional assumption that the numbers are in non-increasing order, however we can identify any tuple with a partition by ordering its elements: for instance we can argue about the partition $(\lambda_1 - 73, \dots, \lambda_k)$ which, although it might not be in its ordered form, should be regarded as a partition in any case. We can also represent a partition in a multiplicative fashion $\langle 1^{\lambda^{(1)}} 2^{\lambda^{(2)}} \dots k^{\lambda^{(k)}} \rangle$, which means that λ has $\lambda^{(j)}$ parts of size j, and we reserve the upper indices of the form $\lambda^{(j)}$ for this notation so there are no confusion with λ_j. So $\langle 1^2 4^1 \rangle$ and $4+1+1$ and $(1, 4, 1)$ are representations of the same partition. We write that $k \in \lambda$ whenever it has a part of size k.

Partitions are very hard to count, but for smaller cases we can find all permutations of an integer n by hand, for instance

$$\begin{aligned}5 &= 5\\ &= 4+1\\ &= 3+2\\ &= 3+1+1\\ &= 2+2+1\\ &= 2+1+1+1\\ &= 1+1+1+1+1 \qquad (1)\end{aligned}$$

are all partitions of size 5.

If $p(n) := \#\{$ partitions of $n\}$ then $p(5) = 7$. In the same way we can find that $p(1) = 1$, $p(2) = 2$ and $p(4) = 5$, but any intuitive general formula cannot come from this.

We can draw graphically a partition λ through a young tableau, which is an arrangement of squares in a matrix in such a way that in the i-th line there are λ_i blocks.

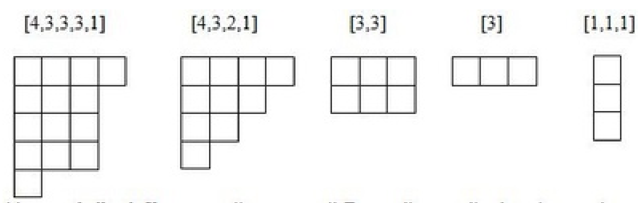


Figure 2: The young tableau of some partitions.

Some of the work towards finding a formula for $p(n)$ was done via its generating function. In fact it is clear that

$$\sum_k p(k)x^k = \prod_k (1 + x^k + (x^k)^2 + (x^k)^3 + \dots) = \prod_k \frac{1}{1-x^k}.$$

In particular, if we want to count the number of partitions whose parts are all distinct, $p_d(n)$, we get

$$\sum_k p_d(k)x^k = \prod_k (1+x^k).$$

And if we want to count the number of partitions only in odd parts, $p_o(n)$ we get

$$\sum_k p_o(k)x^k = \prod_{k\text{ odd}} (1+x^k+(x^k)^2+(x^k)^3+\dots) = \prod_{k\text{ odd}} \frac{1}{1-x^k}.$$

It turns out that $\prod_k(1+x^k) = \prod_k \frac{1-x^{2k}}{1-x^k} = \prod_{k\text{ odd}} \frac{1}{1-x^k}$, so by using generation functions we are able to show that $p_d(n) = p_o(n)$, in fact this reasoning will lead us to more general results, the so called Rogers-Ramanujan identities (e.g. see Guerreiro, 2009).

In general, we refer to the set of integer partitions of n as $\mathscr{P}(n)$ or $\mathscr{P}_n$, and $\mathscr{P}$ is the set of partitions of any non-negative integer. Some structures in combinatorics lead to partition structures, and we usually use λ for the underlying number partition. For instance, a *compositions* of n is a list of positive integers with finite length $\alpha = (\alpha_1,\dots,\alpha_j)$ that sum n. A *weak composition* of n is an infinite list of non-negative integers $\alpha = (\alpha_1,\dots,\alpha_j,\dots)$ that sum n and, by disregarding all zeros, we have a mapping from the weak compositions to the compositions that we will assume that it's clear by context. The underlying integer partition of a composition α is simply the partition $(\alpha_1,\dots,\alpha_j)$ which we denote as $\lambda(\alpha)$. For a *weak* composition the operation λ should disregard all zeros, so $\lambda((1,0,1,0,0,\dots) = (1,1)$.

We can invert the λ map in a canonical way: if μ is a partition, we denote $\mu' = (\mu_1,\mu_2,\dots)$ as the unique weak composition of type μ in non-increasing order.

The length of a partition λ i.e., the number of parcels of a partition, is denoted it by $l(\lambda)$. Also, given a partition $\lambda \vdash n$ there is a dual partition $\lambda^T = (\lambda_1^T,\dots) \vdash n$ where $\lambda_i^T = \#\{j|\lambda_j \geq i\}$. So, for instance if $\lambda = 5+3+3+2+1 \vdash 14$ then $\lambda^T = 5+4+3+1+1 \vdash 14$. With this, we have $l(\lambda) = \lambda_1^T$. It is a proposition that the number of partitions of n that $\lambda = \lambda^T$ is exactly the number of partitions of n in odd parts.

There are several ways to order partitions, we will focus on two of them defined in (Stanley, 1999, Ch. 7). The first one we will call *containment order*. It orders the global set $\mathscr{P}$ of partitions and we say that $\lambda = (\lambda_1,\dots,\lambda_{l(\lambda)}) \subseteq \mu = (\mu_1,\dots,\mu_{l(\mu)})$ if $\lambda_i \leq \mu_i$ for each i, where the entries of the partition occur in non-increasing order. In particular it must be the case that $(\lambda) \leq l(\mu)$. So $(3,3,2,1,1) \leq (3,2,1,1)$ but $(3,3,3,1) \not\leq (3,3,2,1,1)$. This means that, in a graphical reasoning, the young tableau of λ can fit inside the young tableau of μ without rotations.

The second ordering is called *dominance order*, is defined within the set $\mathscr{P}_n$ and $(\mathscr{P}_n, \leq)$ is a lattice. We then say that $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)}) \leq \mu = (\mu_1, \ldots, \mu_{l(\mu)})$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)}$ and $\sum_{i=0}^{k} \lambda_i \leq \sum_{i=0}^{k} \mu_i$ for each k. This is the same notion of majoration in the Muirhead's inequality, for those familiar with it.

For instance, in $\mathscr{P}_5$ we have $(5) \geq (4,1) \geq (3,2) \geq (3,1,1) \geq (2,1,1,1)$, but $(3,1,1)$ and $(2,2,1)$ are not comparable.

It is a fact that, whenever we have $\mu \leq \lambda$, then $\lambda^T \leq \mu^T$.

We can also define set partitions, i.e. the family of partition of a set into disjoint subsets. Those are counted by the Stirling numbers of the second kind, which count the number of set partitions of an n-set into k blocks. The number of set partitions of a given set is enumerated by the Bell numbers, which is as mysterious as the number of partitions.

We denote a set partition of A as a list of subsets of A, $\pi = \{\pi^1, \ldots, \pi^{l(\pi)}\}$, or by an union $\pi \uplus_i \pi^i$ whenever it's clear that π is not a set, where each $\pi^i \subseteq A$. We write $l(\pi)$ for the number of blocks in π. We denote the family of set partitions of a ground set A as Π_A, the family of set partitions of the ground canonical set $\{1, \ldots, n\}$ by Π_n, and $\Pi = \cup_n \Pi_n$. We define as well the *type* of a set partition π by its underlying integer partition $\lambda(\pi)$ and write $\Pi_\lambda = \{\pi \in \Pi : \lambda(\pi) = \lambda\}$ for the family of set partitions of a fixed type λ in Π_n.

Given a subset of edges of a graph E, the connected components of the graph $(V(G), E)$ yield a set partition of the set of vertices. The underlying set partition will be denoted by $\pi(E)$ and the underlying integer partition $\lambda(\pi(E))$ is simply $\lambda(E)$.

Given a function $f : A \to B$, it is common to define the kernel of f, as a set partition $\ker f$ of A, as well as $\operatorname{im} f$ as a multiset over B. Through this work we will use the definition of ker so we will make it explicit: Set $\ker f = \biguplus_{b \in B} f^{-1}(b)$. So, for instance, if f is a proper colouring of a graph G, then $\ker f$ is a stable partition of G.

Opposed to the dominance order, the containment order has sort of an extension to set partitions, the *coarsening order*. We say that two set partitions $\pi_1 \subseteq \pi_2$, or π_2 is coarser than π_1, if for any part $A \in \pi_1$ there is a part $B \in \pi_2$ such that $A \subseteq B$. We say as well that π_1 is finer than π_2, because it separates the set into smaller pieces. So, for instance, the set partitions of $A = \{1,2,3,4,5\}$ that follow $\pi_1 = \{\{1,2,3\},\{4,5\}\}$, $\pi_2 = \{\{1,2\},\{3\},\{4,5\}\}$ and $\pi_3 = \{\{1,2\},\{3,4,5\}\}$ satisfy $\pi_2 \subseteq \pi_1$ but

$\pi_3 \not\subseteq \pi_1$.

0.1.3 Background in Symmetric Functions

Henceforth, for a weak composition $\alpha = (\alpha_1, \dots)$ we adopt the following monomial notation:

$$x^\alpha = \prod_i x_i^{\alpha_i}$$

The main algebraic tool that we will use here are the *symmetric functions of homogeneous degree n* over enumerable many variables $x_1, x_2, \dots$

A symmetric function is given by an infinite sum $f = \sum_\alpha c_\alpha x^\alpha$ over all weak compositions of n, where the coefficients are invariant under permutation, so if $\lambda(\alpha) = \lambda(\beta)$ then $c_\alpha = c_\beta$. Consequentially, the series are symmetric in the same sense as functions are symmetric: if you switch the role of two variables the function remains the same. We call the space of homogeneous symmetric functions of degree n space Λ_n, and the space of symmetric functions is $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ is, in fact, a ring.

So, for instance, an homogeneous symmetric function is something like 0, $x_1 + x_2 + \dots$ or $x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 \dots$ and has necessarily an infinite number of parcels, except $0 \in \Lambda$.

Take the symmetric functions $m_\mu = \sum_{\lambda(\alpha)=\mu} x^\alpha$ as the *monomial symmetric functions.* For instance, $m_{(2)} = x_1^2 + x_2^2 + \dots$ and $m_{(1,1,1)} = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 + \dots$

In fact, it's clear that any homogeneous symmetric function f of degree n can be uniquely written as a linear span of $\{m_\mu\}_{\mu \vdash n}$ by the condition on symmetry. As a consequence, the space of homogeneous symmetric functions of degree n has dimension enumerated by the integer partitions on n. This reveals that there is some combinatorial interest in this algebraic structure.

There are some other basis that we should consider, and we will introduce here the power-sum basis and the elementary symmetric function basis.

☞ Definition: Power-sum basis

Let $p_n = m_{(n)}$ and define for a partition $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$

$$p_\lambda = \prod_{i=1}^{l(\lambda)} p_{\lambda_i}$$

For instance $p_3 = x_1^3 + x_2^3 + \dots$ while $p_{(2,1)} = (x_1 + x_2 + \dots)(x_1^2 + x_2^2 + \dots) = (x_1^3 + x_2^3 + \dots) + \sum_{i \neq j} x_i^2 x_j = m_3 + m_{(2,1)}$.

It is clear that each p_n is a symmetric function so each p_λ is a symmetric function. In fact p_n provides us with a ring isomorphism between the polynomials in enumerable many variables $\mathbb{K}[x_1, x_2, \dots]$ and the symmetric functions, via $x_i \to p_i$, though such map does not preserve degree.

Equivalently, the symmetric functions $\{p_\lambda\}_{\lambda \vdash n}$ form a basis for Λ_n, which we should prove in the next chapter.

☞ Definition: Elementary symmetric function basis

For a fixed integer n we define

$$e_n = m_{(1,\dots,1)} = \sum_{i_1 < i_2 < \dots < i_n} \left(\prod_{j=1}^{n} x_{i_j} \right).$$

Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we define $e_\lambda = \prod_{j=1}^{k} e_{\lambda_k}$.

These are the elementary symmetric functions and mimic the elementary symmetric polynomials that span the symmetric polynomials in finitely many variables.

So, for instance, $e_1 = x_1 + x_2 + \dots = p_1 = m_{(1)}$, $e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 + \dots = m_{(1,1)}$ and $e_{(2,1)} = e_2 e_1 = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + \dots = m_{(2,1)} + 3m_{(1,1,1)}$.

The fact that the elementary symmetric functions form a basis will be proven afterwards but, what is remarkable is that not all coefficients β_μ from the span $m_\lambda = \sum_\mu \beta_\mu e_\mu$ are non-negative as, for instance, is the case of $\lambda = (2)$: we have $m_{(2)} = e_{(1,1)} - 2e_{(2)}$. This is a key property that will allow us to ask when certain symmetric functions have positive coefficients over the e-basis and whether its coefficients have a combinatorial meaning.

0.1.4 Symmetric Functions in graphs and main conjectures

☞ Definition: Chromatic symmetric function of a graph

In the ring of symmetric functions we define the chromatic symmetric function of a graph G

with n vertices as

$$\chi_G = \sum_{\rho} x_{\rho(1)} \dots x_{\rho(n)}\,.$$

Where the sum runs over all proper colourings $\rho : V(G) \to \mathbb{N}$. It's clear that such sum is a symmetric function.

The chromatic symmetric function of a graph was introduced in Stanley (1995), and indeed Richard Stanley is the author of many chromatic invariants, for instance we introduced as well a chromatic invariant in sets in Stanley (1970), or to introduce some results on acyclic invariants in Stanley (2009). We should point out that there are several duality like connections between colourings and acyclic orientations, pointed out in Stanley (2009) as well.

Example 0.1.2. To have a simple idea, for the graph $G = K_1$ with only one vertex, any function $\rho : V(G) \to \mathbb{N}$ is proper, so $\chi_G = x_1 + x_2 + \cdots = m_{(1)}$.

For the graph $G = K_2$ with two vertices and an edge connecting them, the proper colourings $\rho : V(G) \to \mathbb{N}$ are the injective ones, so $\chi_G = \sum_{i \neq j} x_i x_j = \sum_{i<j} 2 x_i x_j = 2 m_{(1,1)}$.

Proposition 0.1.3 (Coefficients over m basis)**.** Given a composition α of type $\lambda(\alpha) = \lambda \vdash n$ where $\lambda = \langle 1^{\lambda^{(1)}} 2^{\lambda^{(2)}} \dots k^{\lambda^{(k)}} \rangle = (\lambda_1, \dots, \lambda_{l(\lambda)})$, the coefficient in x^α of the chromatic symmetric function of a given graph is exactly the number of *stable partitions* of type λ times $\lambda^{(1)}! \lambda^{(2)}! \dots \lambda^{(k)}!$.

In particular, the coefficient of a chromatic symmetric function on x^α depends only on the type of α, showing the symmetry of the chromatic symmetric function .

We will adopt the notation $\lambda! = \lambda^{(1)}! \lambda^{(2)}! \dots \lambda^{(k)}!$.

A small proof is in order:

Proof. The coefficient $\chi_G[x^\alpha]$ is simply

$$\#\{\text{ proper colourings that uses } \alpha_i \text{ times the colour } i\,\}\,.$$

We can relate each colouring f with a stable partition π by considering the kernel: $\ker f$ is a stable partition.

We will count the set $\mathscr{C}^\alpha_\pi$ of colourings f with $\ker f = \pi$ that contribute to the coefficient x^α, or equivalently that use α_i times the colour i. Set $C_\pi^\alpha = \#\mathscr{C}_\pi^\alpha$. Then

$$\#\{\text{ proper colourings that uses } \alpha_i \text{ times the colour } i\,\} = \sum_\pi C_\pi^\alpha .$$

A first observation is that, in order for a colouring f to contribute to the coefficient x^α, the number of blocks of the stable partition $\ker f = \pi = \{\pi^1, \dots\}$ of size j must be exactly the number of parts of α of size j, which is $\lambda^{(j)}$.

Hence, $\lambda(\ker f) = \lambda(\alpha)$. So we only need to compute

$$\sum_{\pi:\lambda(\pi)=\lambda(\alpha)} C_\pi^\alpha .$$

We will show that, given that $\lambda(\pi) = \lambda(\alpha)$ holds, the number C_π^α does not depend on π. Fix a stable partition π of type λ. The set of colourings f such that $\ker f = \pi$ can be constructed as follows: suppose that $\pi = \{\pi^1, \dots \pi^{l(\lambda)}\}$, then a colouring f such that $\ker f = \pi$ is just an injective function $\bar{f} : \pi \to \mathbb{N}$ that assigns to each block π_i a colour $\bar{f}(\pi_i)$.

Since the colouring f contributes to the coefficient x^α, it must be the case that $\# f^{-1}(i) = \alpha_i$. This means that $\bar{f}$ must map a block π_k of size j to a colour i that satisfies $\alpha_i = j$. Since there are $\lambda^{(j)}$ such colours and $\lambda^{(j)}$ such blocks (since $\lambda(\pi) = \lambda$), we should simply pick, for each size j, a bijection between the blocks of size j and the colours i that satisfy $\alpha_i = j$, which are $\lambda^{(j)}!$ many.

On the other hand, we easily see that the condition $\# f^{-1}(i) = \alpha_i$ is sufficient for the underlying colouring f to satisfy $\ker f = \pi$ and to contribute to the coefficient of x^α.

Then, counting $\mathscr{C}_\pi^\alpha$ amounts to, for each size j, pick a bijection between the blocks of size j and the colours i that satisfy $\alpha_i = j$, thus

$$C_\pi^\alpha = \prod_j \lambda^{(j)}! = \lambda!.$$

and does not depend on the stable partition π.

So, the coefficient of x^α in $\sum_k x_{k(v_1)} x_{k(v_2)} \dots x_{k(v_n)}$ is exactly the number of stable partitions of G of type λ times the constant $\lambda!$. □

It will be shown in Proposition 2.3.7 that, given χ_G, we can decide whether G is a tree or not, but it is conjectured that, within the family of trees, χ_G classifies all isomorphism classes:

Conjecture 0.1.4 (Stanley's conjecture)**.** The algebraic invariant χ_G classifies trees.

The main tool to approach this conjecture is from the expansion of the chromatic symmetric function in the p-basis, shown in Stanley (1995).

Theorem 0.1.5 (P-Basis span)**.** We have

$$\chi_G = \sum_{S\subseteq E(G)} (-1)^{\#S} p_{\lambda(S)}\,.$$

The proof of 0.1.5 can be found in 2.1.3.

In the case of a tree, all the edge-sets that contribute for p_λ, contribute with the same sign: there are no cycles, each edge that we take from a graph disconnects a connected component, so the number of connected components of $E(G)\setminus S$ is exactly $n-1-\#S$. Hence we only need to consider the number $\theta_\mu = \#\{S \subseteq E(G) | \lambda(E(G)\setminus S) = \mu\}$ and find the relevant sign.

When computing θ_μ, we only need to search for the subsets S of size such that $l(\mu) = n - \#(E(G)\setminus S) = \#S + 1$, hence the formula becomes:

Corollary 0.1.6. Let T be a tree, and let $\theta_\mu = \#\{S \in \binom{E(T)}{l(\mu)-1} | \lambda(E(G)\setminus S) = \mu\}$, then we have:

$$\chi_T = \sum_{S\subseteq E(T)} (-1)^{\#S} p_{\lambda(S)} = (-1)^n \sum_{\mu} \theta_\mu (-1)^{l(\mu)} p_\mu\,.$$

Example 0.1.7. Take, for instance, the following tree T_1

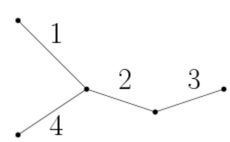


Figure 3: Tree T_1 with 5 vertices and 4 edges.

Recall that $\theta_\mu = \#\{S \subseteq E(G) | \lambda(E(G)\setminus S) = \mu\}$. Then, for any tree, we have $\theta_{(5)} = 1$ due to $S = \emptyset$, we have as well $\theta_{(1,1,1,1,1)} = 1$ due to $S = E(G)$.

For the remaining ones we have

$$
\begin{aligned}
\theta_{(4,1)} &= 3 \\
\theta_{(3,2)} &= 1 \;\; \text{due to set } \{2\} \\
\theta_{(3,1,1)} &= 4 \\
\theta_{(2,2,1)} &= 2 \;\; \text{due to sets } \{1,2\},\, \{2,4\} \\
\theta_{(2,1,1,1)} &= 4\,.
\end{aligned}
$$

So we get

$$\chi_G = p_{(5)} - 3p_{(4,1)} - p_{(3,2)} + 4p_{(3,1,1)} + 2p_{(2,2,1)} - 4p_{(2,1,1,1)} + p_{(1,1,1,1,1)}\,.$$

This closes the introduction on the tree classification conjecture. Now we ask about the e-basis, and the coefficients of the span of chromatic symmetric function in such basis.

Sometimes, the coefficients don't have any special combinatorial formula behind, or such isn't expected to be found because the coefficients aren't even integer or positive. So when facing the problem of combinatorially interpret the coefficients in a given basis, we first ask for positiveness.

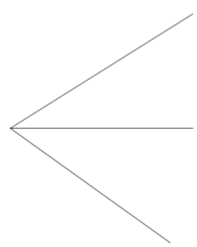


Figure 4: A non e-positive graph since $\chi_G = m_{(1,1,1,1)} + 6m_{(2,1,1)} + 3m_{(3,1)} = 3e_{(2,1,1)} - 12e_{(2,2)} + 9e_{(3,1)} + 36e_{(4)}$

Some graphs are not e-positive, as the so called claw in figure 0.1.4.

So, we can see that not all chromatic symmetric function of graphs G have non-positive coefficients on the e-basis, unlike the m-basis. However, we may conjecture which graphs have non-negative coefficients on the e-basis, and that is what our next conjecture is about.

☞ Definition: Poset

A poset is a set with a partial order $(P, \geq)$ that must be transitive, antisymmetric and reflexive. It is thoroughly studied in combinatorics and arises in many cases: for instance the poset of partitions, with any of the orders introduced in the background.

Another example of a poset is the family of parts of a set A, $\mathscr{P}(A)$ ordered with inclusion $\subseteq$. So, for instance, if $A = \{1,2\}$ then $\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$. This is called the *boolean poset*.

The most common way to represent a poset is via an acyclic oriented graph or Hasse diagram. In both we represent the elements from P as points, and draw an edge between two dots if the underlying elements are comparable.

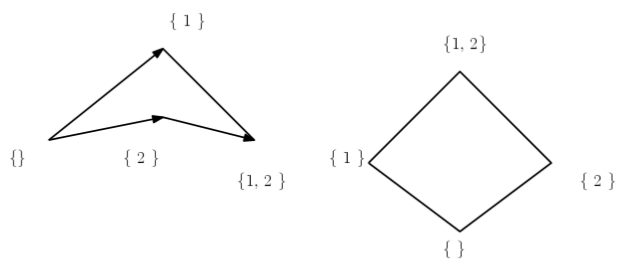


Figure 5: The Boolean poset of $A = \{1,2\}$ represented as an acyclic graph (left) and as a Hasse diagram (right)

In the acyclic orientation, the edge must be an arrow from the lower element to the higher element. In the Hasse diagram, the position of the elements must be such that the direction of the arrows is understandable, for instance the higher element must be represented in a higher position in the diagram, as in figure 5.

We also ignore edges that follow from transitivity and also edges between the same element.

☞ Definition: Incomparability graph, (3+1)-free graphs and other free graphs

Given a partial ordered set $P = (V, \geq)$ we draw the graph $G(P)$ with vertex set V and connect $x - y$ if neither $x \geq y$ nor $y \geq x$, i.e. if they are incomparable.

These graphs are here identified because they compose a very broad family of graphs. Incidentally they will provide us with cases of counterexamples to e-positiveness in symmetric functions.

A poset is said to be *(3+1)-free* if there are no three elements $x < y < z$ and a fourth one w such that w is incomparable to all x, y, z. Similarly, a poset is said to be *(2+2)-free* if there are no four elements $x < y$ and $z < w$ such that x and y are incomparable to both z, w. A 3-free poset has no three element chains $x < y < z$.

The incomparability graphs that arise from (3+1)-free poset is called *(3+1)-free incomparability graph*, and similarly for the (2+2)-free and 3-free posets.

Conjecture 0.1.8 (e-positivity conjecture)**.** A (3+1)-free incomparability graph has non-negative coefficients over the e-basis.

Some partial results do exist for this conjecture. The following one can be found in the literature from Stanley (1995). We will present a different proof of this fact in Chapter 3.

Theorem 0.1.9. A 3-free incomparability graph has positive coefficients over the e-basis.

The following theorem will be the centre of our discussion in chapter 3, due to the work from Mathieu Guay-Paquet in Guay-Paquet (2013).

Theorem 0.1.10. If all (2+2 and 3+1)-free posets are e-positive, then all (3+1)-free posets are e-positive.

0.2 Objectives

The main objectives of this Master's project are

1. To get a broad comprehension over a topic in Algebraic Combinatorics.
2. Be able to develop a small presentation over the search during the semester.
3. Get in touch with mathematical research questions and test my ideas on the topic.

Through the conjecture here exposed we expect do go over a lot of material that regards Algebraic Combinatorics, in particular poset theory as well as to identify some algebraic invariants in graphs closely related to the chromatic symmetric function of graphs.

0.3 Structure of the Report

The rest of the report is organized as follows. Chapter 1 gives an introduction to symmetric functions, its basis and relation between the coefficients, algebraic operations and further propositions in the topic.

In the following chapters we will discuss each of the conjectures 0.1.8 and 0.1.4. In chapter 2 we not only study some strategies written in the literature to tackle the problem, but also provide some partial solutions to the conjecture, showing how we can distinguish between certain families of trees.

Chapter 3 is dedicated to study the Conjecture 0.1.8. Here we will deal with modular relations, like 0.1.1 and part-listings, notions that pave the way to the reduction of (3+1)-free posets to (3+1 and 2+2)-free posets concluding in the Theorem 0.1.10. With the methods there developed, we conclude as well Theorem 0.1.9.

Chapter 1

Basics of the ring of symmetric functions

In this chapter we give a light introduction to the theory of symmetric functions. We deal with change of basis matrix results in Proposition 1.1.5 and Proposition 1.1.1, that provide the coefficients of some of the most common basis on the symmetric functions in the monomial basis, most of which are of interest in the combinatorics realm. We also provide some algebraic tools in the linear space that will endow the symmetric functions with a solid structure. In the end of this chapter, we will be able to show that the symmetric function bases previously defined are indeed basis sets.

There are several other basis in the symmetric functions, such as the Schur functions, that could be introduced in this work, though have no application henceforth. For that, we send the reader to Stanley (2003) or Stanley (1999).

1.1 Basis of the symmetric functions

We have already mentioned that Λ has several bases, one of them is the monomial basis. In this chapter we will recall other bases and devise some change of basis relations, following the work on Stanley (1999).

We have already seen the p-basis and the e-basis, where $p_\lambda = \prod_i p_{\lambda_i} = \prod_i \left[\sum_{k\geq 1} x_k^{\lambda_i}\right]$ and $e_\lambda = \prod_i e_{\lambda_i} = \prod_i \sum_{i_1<i_2<\cdots<i_{\lambda_i}} \left(\prod_{j=1}^{\lambda_i} x_{i_j}\right)$. We will introduce the h basis as well.

☞ Definition: Complete homogeneous symmetric functions

Given an integer n, we define $h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \le i_2 \le \cdots \le i_n} \prod_{j=1}^{n} x_{i_j}$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$ we define $h_\lambda = \prod_{j=1}^{l(\lambda)} h_{\lambda_j}$.

Given a set $S \subseteq \mathbb{N}_0$ containing 0, an S-matrix is a matrix with infinitely many rows and columns with finitely many entries different from 0. We define $\mathrm{row}(A)$ and $\mathrm{col}(A)$ to be weak compositions such that $\mathrm{row}(A)_i = \sum_j a_{i,j}$ and $\mathrm{col}(A)_j = \sum_i a_{i,j}$.

We have promised to show that the the e-basis is indeed a basis for each Λ_n, and we will need the following proposition:

Proposition 1.1.1. For a partition $\lambda \vdash n$ and α a weak composition of n, let the coefficient of x^α in $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_{l(\lambda)}}$ be $M_{\lambda,\alpha}$.

Then $M_{\lambda,\alpha}$ is number of $\{0,1\}$-matrices A with $\mathrm{row}(A) = \lambda$ [1] and $\mathrm{col}(A) = \alpha$.

Besides, the coefficient $M_{\lambda,\alpha}$ only depends on the type of α, so we write $M_{\lambda,\mu} := M_{\lambda,\alpha}$ whenever $\lambda(\alpha) = \mu$.

Proof. By analysing the product $e_{\lambda_1} \cdots e_{\lambda_{l(\lambda)}}$, the coefficient of x^α is the number of ways to choose sets $S_1, \cdots, S_{l(\lambda)}$ of positive integers such that $\#S_i = \lambda_i$ and j occurs a total of α_j times in said sets.

Now, for each choice of sets $S_1, \cdots, S_{l(\lambda)}$ of positive integers, we find bijectively a $\{0,1\}$-matrix A with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$ such that $\#S_i = \lambda_i$ and j occurs a total of α_j times and this will imply the first part of the proposition.

In fact, it is enough to set a $\{0,1\}$-matrix $A = f(S_1, \dots,)$, for the sets $S_1, S_2, \dots$, such that the characteristic vector of S_i is the i-th row of A, i.e. $j \in S_i$ iff $a_{i,j} = 1$.

In this manner, it is easy to see that any $\{0,1\}$-matrix, arising from a choice of sets $S_1, \dots$ of positive integers, satisfies $\mathrm{row}(A) = \lambda$ (because $\#S_i = \lambda_i$) and $\mathrm{col}(A) = \alpha$ (because j occurs a total of α_j times).

On the other hand, for any $\{0,1\}$-matrix with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \alpha$ we can construct a sequence of sets $S_1, \cdots, S_{l(\lambda)}$ of positive integers such that $\#S_i = \lambda_i$ and j occurs a total of α_j times in said sets, inverting the function f and proving the bijection property.

Since e_λ is a symmetric function, $M_{\lambda,\alpha}$ only depends on the type of α, so we write $M_{\lambda,\mu} := M_{\lambda,\alpha}$ whenever $\lambda(\alpha) = \mu$. □

Proposition 1.1.2. If $M_{\lambda,\mu} \neq 0$ then $\lambda^T \geq \mu$. Moreover, $M_{\lambda,\lambda^T} = 1$

[1] here we force the notation and see λ as a week composition, adding zeros to the end

Proof. We only have to show that if there is a $\{0,1\}$-matrix A with $\text{row}(A) = \lambda$ and $\text{col}(A) = \mu$ then $\lambda^T \geq \mu$, and this would conclude with the first part of the proposition.

Construct now the $\{0,1\}$-matrix A' from A by shifting each entry with 1 as much as possible to the left, preserving the number of 1's in a row. Then we obtain $\text{row}(A) = \lambda$ and $\text{col}(A') = \lambda^T$.

On the other hand, we see that $\text{col}(A') \geq \text{col}(A)$. Indeed, by counting the number of ones in the first k columns in both $\{0,1\}$-matrices we get $\mu_1 + \cdots + \mu_k$ in A, and $\lambda_1^T + \cdots + \lambda_k^T$ in A'. It is obvious that there are at least as many ones in the first columns of A' than in the first columns of A, so $\mu_1 + \cdots + \mu_k \leq \lambda_1^T + \cdots + \lambda_k^T$ concluding $\text{col}(A') \geq \text{col}(A)$, hence $\lambda^T \geq \mu$.

If $\mu = \lambda$, it is easy to see that there is only one $\{0,1\}$-matrix A such that $\text{row}(A) = \lambda$ and $\text{col}(A) = \lambda^T$, namely the $\{0,1\}$-matrix with all the 1's shifted to the left, and there is only one such $\{0,1\}$-matrix that satisfies $\text{row}(A) = \lambda$' and $\text{col}(A) = \lambda^T$. □

Corollary 1.1.3. The elementary symmetric functions of degree n form a basis of Λ_n.

Proof. We order the e-basis according to some completion of $\geq$ and the m-basis according to the same completion but in reverse order.

Then, recall that if $\lambda \geq \mu$ then $\mu^T \geq \lambda^T$. So the transition matrix is upper triangular and the diagonal has only ones since $M_{\lambda,\lambda^T} = 1$. □

Corollary 1.1.4. The elementary symmetric functions ring is isomorphic to a polynomial ring $\mathbb{K}[x_1, x_2, \ldots]$.

Proof. The homomorphism mapping e_n to the free variable x_n is an isomorphism because the e-basis is algebraic independent, i.e., the symmetric functions $\{e_\lambda\}$ are linearly independent. □

The same thing happens with the p-basis defined in Chapter 1 in 0.1.3 and with the h-basis defined in this chapter in 1.1.

We will briefly compute the coefficients of the h-basis when written in the monomial basis.

Proposition 1.1.5. Write the symmetric function h_λ in the monomial basis, obtaining

$$h_\lambda = \sum_{\mu \vdash n} N_{\lambda,\mu} m_\mu .$$

Then $N_{\lambda,\alpha}$ is number of $\mathbb{N}_0$-matrices A with $\text{row}(A) = \lambda$ and $\text{col}(A) = \alpha$.

Besides, the coefficient $N_{\lambda,\alpha}$ only depends on the type of α, so we write $N_{\lambda,\mu} := N_{\lambda,\alpha}$ whenever $\lambda(\alpha) = \mu$.

Proof. It goes exactly in the same way as Proposition 1.1.1. □

1.2 Algebraic operations on symmetric functions

Although the ring of symmetric functions Λ is very similar to a polynomial ring, we can extract some algebraic machinery that will turn out very helpful and brings unique applications.

So in this section we will develop the understanding of some algebraic operations on Λ: a ring isomorphism ω that will turn out to be an involution, some generation function identities on symmetric functions and the so called Hall's inner product.

1.2.1 The ω involution

☞ Definition: The involution ω

We define the involution ω by fixing values for an algebraically independent set e_n and extending as an homomorphism. This is possible because of Corollary 1.1.4.

We set $\omega(e_n) = h_n$.

Theorem 1.2.1. The mapping is indeed an involution, i.e. ω^2 =id.

Proof. Again it is only enough to establish the theorem for the symmetric functions e_n, i.e. we have to prove that $\omega(h_n) = e_n$.

Define the formal power series

$$H(t) = \sum_{n\geq 0} h_n t^n \in \Lambda[[t]];$$

$$E(t) = \sum_{n\geq 0} e_n t_n \in \Lambda[[t]]\,.$$

In fact, we note that $H(t) = \prod_n \sum_k (x_n t)^k = \prod_n (1 - x_n t)^{-1}$ and $E(t) = \prod_n (1 + x_n t)$ hence

$$H(-t)E(t) = 1.$$

By analysing this product coefficient-wise we get $h_0 e_0 = 1$ and for $n \geq 1$,

$$\sum_{k=0}^{n} (-1)^n (-1)^k e_k h_{n-k} = 0 \tag{1.1}$$

or, substituting $k \mapsto n-k$, we have

$$h_0 e_n + \sum_{k=1}^{n} (-1)^k h_k e_{n-k} = 0. \tag{1.2}$$

Now apply ω to the equation (1.1) to get

$$(-1)^n \sum_{k=0}^{n} (-1)^k h_k \omega(h_{n-k}) = 0. \tag{1.3}$$

Now we use strong induction: for $n = 0$ we have $\omega(e_0) = h_0 = e_0$ so we get $\omega(h_0) = e_0$.

For the induction step, if $n \geq 1$, we assume that $\omega(h_i) = e_i$ for every $i < n$ and, from (1.3), we have

$$h_0 \omega(h_n) + \sum_{k=1}^{n} (-1)^k h_k e_{n-k} = 0\,.$$

But from equation (1.2) and since $h_0 = 1$, we must have $\omega(h_n) = e_n$ concluding the induction step. □

Corollary 1.2.2. Since ω^2 =id, ω is an isomorphism, so preserves basis and so the h-basis is a basis of Λ.

So this is one short example of what the involution ω can give us.

1.2.2 Generating functions identities

There are several equalities in the generating function setting that are of our interest and we are going to develop this section over one powerful such theorem, based on Stanley (1999).

From here on we denote the symmetric function basis with a letter, for instance $m_\lambda(x)$ means that the symmetric function is taken in the realm of the variables $\{x_1, x_2, \dots\}$.

Proposition 1.2.3. We have

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda} m_\lambda(x) e_\lambda(y) \tag{1.4}$$

Proof. First note that $\sum_\lambda m_\lambda(x)e_\lambda(y) = \sum_{\lambda,\mu} M_{\lambda,\mu} m_\lambda(x)m_\mu(x)$ from Proposition 1.1.1 and so $x^\alpha y^\beta$ has coefficient $M_{\lambda(\alpha),\lambda(\beta)}$ in the right hand side of (1.4).

For the left hand side, we see that each monomial appearing in the expansion of $\prod_{i,j}(1+x_i y_j)$ corresponds to a $\{0,1\}$-matrix - with 1's in the entry (i,j) if the parcel $x_i y_j$ is picked - and the $\{0,1\}$-matrices A that contribute to the monomial $x^\alpha y^\beta$ are the ones that satisfy $\mathrm{row}(A) = \alpha$ and $\mathrm{col}(A) = \beta$.

By 1.1.1 there are exactly $M_{\lambda(\alpha),\lambda(\beta)}$ such $\{0,1\}$-matrices, so the coefficient of $x^\alpha y^\beta$ in the left hand side of (1.4) is $M_{\lambda(\alpha),\lambda(\beta)}$ as well. □

In a similar manner we get the proposition in the realm of complete homogeneous symmetric functions.

Proposition 1.2.4. We have

$$\prod_{i,j}(1-x_i y_j)^{-1} = \sum_\lambda m_\lambda(x)h_\lambda(y) \tag{1.5}$$

Proof. On the right hand side of (1.5) we get from 1.1.5 that the coefficient of $x^\alpha y^\beta$ is $N_{\lambda(\alpha),\lambda(\beta)}$.

On the left side we have the product $\prod_{i,j}(1-x_i y_j)^{-1} = \prod_{i,j}\sum_k (x_i y_j)^k$ so, after expansion, each summand comes from a function $f:(i,j)\mapsto k$ such that 0 is picked all but finitely many times. We can represent said function f as a $\mathbb{N}_0$-matrix.

Given $\mathbb{N}_0$-matrix A, the underlying monomial for which A contributes is $x^\alpha y^\beta$ where $\mathrm{row}(A) = \alpha$ and $\mathrm{col}(A) = \beta$, hence the coefficient of $x^\alpha y^\beta$ on the left hand side of (1.5) is $N_{\lambda(\alpha),\lambda(\beta)}$ by proposition 1.1.5. □

We write $z_\lambda = \prod_k \lambda^{(k)}!k^{\lambda^{(k)}}$. The number of permutations of cyclic type $\lambda = \langle 1^{\lambda^{(1)}} 2^{\lambda^{(2)}} \dots\rangle$ is computed in (Proposition 1.3.2 Stanley, 1999) and is given by

$$n!\prod_k \frac{1}{\lambda^{(k)}!k^{\lambda^{(k)}}} = n! z_\lambda^{-1}\,.$$

We write $z_\lambda = \prod_k \lambda^{(k)}!k^{\lambda^{(k)}}$ for the proportion of permutations of type λ. As it turns out, this number reveals itself in the realm of symmetric functions.

Proposition 1.2.5. We have

$$\prod_{i,j}(1-x_i y_j)^{-1} = \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(x) p_{\lambda}(y)\,. \tag{1.6}$$

Proof. We first show that $\prod_{i,j}(1-x_i y_j)^{-1} = \exp\sum_{n\geq 1}\frac{1}{n}p_n(x)p_n(y)$. For that, take

$$\begin{aligned}\log\prod_{i,j}(1-x_i y_j)^{-1} &= \sum_{i,j} -\log(1-x_i y_j)\\ &= \sum_{i,j}\sum_{n\geq 1}\frac{1}{n}x_i^n y_j^n\\ &= \sum_{n\geq 1}\frac{1}{n}p_n(x)p_n(y)\,.\end{aligned}$$

On the other hand we can apply the definition of the exponential, obtaining

$$\exp\sum_{n\geq 1}\frac{1}{n}p_n(x)p_n(y) = \prod_{n\geq 1}\sum_{k\geq 0}\frac{1}{k!n^k}\big(p_n(x)p_n(y)\big)^k = \sum_{k_1,k_2,\ldots}\prod_{n\geq 1}\frac{1}{k_n!n^{k_n}}(p_n(x)p_n(y))^{k_n}\,,$$

where in the last part we sum over all weak compositions $\vec{k}$.

Write $\lambda = \langle 1^{\lambda^{(1)}}2^{\lambda^{(2)}}\ldots\rangle$. So the coefficient of $p_\lambda(x)p_\lambda(y)$ in (1.6) is

$$\prod_{n\geq 1}\frac{1}{\lambda^{(n)}!n^{\lambda^{(n)}}} = z_\lambda^{-1}\,. \quad \square$$

1.2.3 Hall's inner product

We now introduce our third and last algebraic tool in Λ.

☞ Definition: Hall's inner product

We know that both the m-basis and the h-basis are basis of Λ, so we can define the bilinear product

$$\langle m_\lambda, h_\mu\rangle = \delta_{\lambda,\mu}\,.$$

This yields a symmetric bilinear product, since $\langle h_\lambda, h_\mu\rangle = \sum_\phi N_{\lambda,\phi}\langle m_\phi, h_\mu\rangle = N_{\lambda,\mu}$ and symmetry follows from the symmetry of the matrix N, trivial from Proposition 1.1.5.

The following criterion comes in handy to determine dual basis of this inner product and is an adaptation form the one introduced in Stanley (1999).

Proposition 1.2.6. Let $p(n)$ denote the number of integer partitions of n. Given, for each n, two families of $p(n)$ symmetric functions $\{u_\lambda\}_{\lambda\vdash n}, \{v_\lambda\}_{\lambda\vdash n}$ of Λ_n, they form a dual basis with respect to Hall's inner product if and only if

$$\prod_{i,j}(1-x_i y_j)^{-1} = \sum_\lambda u_\lambda(x) v_\lambda(y)\,. \tag{1.7}$$

Proof. Write $u_\lambda = \sum_\rho A_{\lambda,\rho} m_\rho$ and $v_\lambda = \sum_\rho B_{\lambda,\rho} v_\rho$.

A natural observation is that the matrices A and B are block matrices, i.e. the only non-zero entries occur in finite square matrices along the diagonal, the ones indexed by partitions of the same size. This happens because for $\lambda \vdash n$, both u_λ and v_λ are homogeneous of degree n and so, have a span of the form $u_\lambda = \sum_{\rho\vdash n} A_{\lambda,\rho} m_\rho$ and $v_\lambda = \sum_{\rho\vdash n} B_{\lambda,\rho} m_\rho$.

Then, since $\langle m_\rho, h_\psi\rangle = \delta_{\rho,\psi}$, we have

$$\begin{aligned}
\langle u_\phi, v_\gamma\rangle &= \sum_{\rho,\psi} A_{\phi,\rho} B_{\gamma,\psi}\langle m_\rho, h_\psi\rangle \\
&= \sum_\rho A_{\phi,\rho} B_{\gamma,\rho} \\
&= (AB^T)_{\phi,\gamma}\,.
\end{aligned}$$

So, we get that $\{u_\lambda, v_\lambda\}$ are dual basis iff $AB^T = I$.

However, we have from 1.5 that

$$\sum_\lambda m_\lambda(x) h_\lambda(y) = \prod_i (1-x_i y_i)^{-1}\,.$$

And as well

$$\begin{aligned}
\sum_\lambda u_\lambda(x) v_\lambda(y) &= \sum_{\rho,\psi}\left[\sum_\lambda A_{\lambda,\rho} B_{\lambda,\psi}\right] m_\rho(x) h_\psi(y) \\
&= \sum_{\rho,\psi} (B^T A)_{\psi,\rho} m_\rho(x) h_\psi(y)\,.
\end{aligned}$$

Since $\{m_\rho(x)h_\phi(y)\}$ is linearly independent, we get that $\{u_\lambda, v_\lambda\}$ satisfy 1.7 iff $(B^T A)_{\rho,\psi} = \delta_{\rho,\psi}$, or $B^T A = I$.

So, to obtain the equivalence of theorem, under the cardinality condition, we will show that

$$B^T A = I \Leftrightarrow AB^T = I. \tag{1.8}$$

Recall that the only non-zero entries occur in square finite matrices along the diagonal, the ones indexed by partitions of the same size. So if we restrict the matrices B and A to B_n, A_n indexed with the entries $\{\lambda \vdash n\}$ we clearly maintain $B_n^T A_n = I$. so $\det(B_n \neq 0 \neq \det A_n$ and we can invert these matrices, so $A_n^{-1} = B_n^T$ since we have square matrices of finite size.

Then $A_n B_n^T = I$ and extending to the infinite matrices we get $AB^T = I$ as desired. The reverse implication is dealt with in the same way and using the equivalence 1.8. □

With this and Proposition 1.2.5 we get that $\{p_\lambda\}$ is an orthogonal basis, and in particular it is a basis that satisfies $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$.

Chapter 2

Classification of trees by the chromatic symmetric function

In this chapter we introduce the advances towards conjecture 0.1.4,

- We start by studying the conjecture in the context of a small family of trees, studied in Fougere (2003), to get acquainted with the methods and tricks to tackle the conjecture.
- We introduce as well the *tailings* of a tree, that will allow us to obtain some algebraic relations between the coefficients of the chromatic symmetric function for trees and pave a possible way towards the conjecture.
- We continue with collecting some broad basic results and general invariants that we can compute (such as the degree sequence) as in Martin et al. (2008).
- Then, we dig into the work developed in Aliste-Prieto and Zamora (2014) where it is shown that proper caterpillars are distinguished, providing a solution to a solid partial form of the conjecture.
- Afterwards, we show that, by having some more information besides the chromatic symmetric function of a tree, we can indeed distinguish trees, according to Orellana and Scott (2014) and Smith et al. (2015).

We now set some notation. We say that, given two algebraic invariants of graphs A_G and B_G, A_G *determines* B_G if we can obtain the invariant $B_G = f(A_G)$ through some function f and write that A_G is a *stronger invariant*. We say as well that A_G, an algebraic invariant of graphs, *classifies* a family of graphs $\mathscr{F}$ if whenever G and H are two nonisomorphic graphs in $\mathscr{F}$, then $A_G \neq A_H$. Finally, an algebraic invariant of graphs A_G *separates* a family of graphs $\mathscr{F}$ (from its complement) if $A_{\mathscr{F}} := \{A_G | G \in \mathscr{F}\}$ and $A_{\mathscr{F}^c}$ are disjoint. We also say that A_G solves the membership problem of $\mathscr{F}$.

2.1 Tools towards the conjecture

We start by recalling the Proposition 0.1.3.

Proposition 2.1.1 (Coefficients over m basis)**.** Given $\alpha = \langle 1^{\alpha^{(1)}} 2^{\alpha^{(2)}} \dots k^{\alpha^{(k)}} \rangle = (\alpha_1, \dots, \alpha_{l(\alpha)}) \vdash n$, the coefficient in x^α of the chromatic symmetric function of a given graph is exactly the number of *stable partitions* of type $\lambda(\alpha)$ times $\alpha^{(1)}!\alpha^{(2)}! \dots \alpha^{(k)}!$.

With this, we can do a little detour on random graphs and the following is simply an application of the previous proposition.

Proposition 2.1.2. Denote by $G_{n,p}$ the random graph with n vertices and each edge is in the graph with probability p independently from the other edges.[1]

Then

$$[m_\lambda]\mathbb{E}[\chi_{G_{n,p}}] = \binom{n}{\lambda_1, \dots, \lambda_{l(\lambda)}} (1-p)^{\sum_{j=1}^{l(\lambda)} \binom{\lambda_j}{2}}$$

Proof. It's a classical application of the linearity of expectation. Let's call

$$X_{\lambda,G} = \#\{\text{ stable parttitions of type } \lambda \text{ in } G_{n,p}\}.$$

From Proposition 0.1.3 we get that $\mathbb{E}[\chi_{G_{n,p}}] = \sum_\lambda m_\lambda \lambda! \mathbb{E}[X_{\lambda,G_{n,p}}]$ so it only remains to be shown that:

$$\mathbb{E}[X_{\lambda,G_{n,p}}] = \frac{1}{\lambda!}\binom{n}{\lambda_1, \dots, \lambda_{l(\lambda)}} (1-p)^{\sum_{j=1}^{l(\lambda)} \binom{\lambda_j}{2}}.$$

For that, let's denote Π_λ as the set of set partitions of the vertex set of type λ. For each $\pi \in \Pi_\lambda$, we write X_π as the random variable that takes value 1 if π is stable in $G_{n,p}$ and takes value 0 otherwise.

[1] This is a common model on random graphs, and has many motivations in its study throughout the literature.

We know as well that $X_{\lambda,G_{n,p}} = \sum_{\pi\in\Pi_\lambda} X_\pi$, since $X_{\lambda,G_{n,p}}$ is the number of stable partitions of type λ. What we want to compute is $\mathbb{E}[X_{\lambda,G_{n,p}}] = \sum_{\pi\in\Pi_\lambda}\mathbb{E}[X_\pi] = \sum_{\pi\in\Pi_\lambda}\mathbb{P}[X_\pi = 1]$.

Now $X_\pi = 1$ if and only if π is stable in $G_{n,p}$ if and only if all the edges within a block aren't in $G_{n,p}$. By independence it follows:

$$\mathbb{P}[X_\pi = 1] = \prod_{j=1}^{l(\lambda)} (1-p)^{\binom{\lambda_i}{2}}.$$

On the other hand, the number of set partitions in Π_λ is $\binom{n}{\lambda_1,\dots,\lambda_{l(\lambda)}}\frac{1}{\lambda!}$ so

$$\sum_{\pi\in\Pi_\lambda}\mathbb{P}[X_\pi = 1] = \frac{1}{\lambda!}\binom{n}{\lambda_1,\dots,\lambda_{l(\lambda)}}(1-p)^{\sum_{j=1}^{l(\lambda)}\binom{\lambda_i}{2}}. \quad \square$$

Our goal in this chapter is to obtain information from the coefficients of the chromatic symmetric function of a given graph.

Most of it goes through the corollary 2.1.4, copied and proved here.

Theorem 2.1.3 (P-Basis span)**.** We have

$$\chi_G = \sum_{S\subseteq E(G)} (-1)^{\#S} p_{\lambda(S)}.$$

Or

$$\chi_G = \sum_{\lambda\vdash n} p_\lambda \sum_{S\subseteq E(G)|\lambda(S)=\lambda} (-1)^{\#S}.$$

Proof. This proof follows the lines of (Stanley, 1995, Theorem 2.5).

Write, for short, for a colouring k of the graph G, $x_k = x_{k(v_1)}x_{k(v_2)}\cdots$.

First note that

$$p_{\lambda(S)} = \prod_i \left(x_1^{\#V_i} + x_2^{\#V_i} + \dots\right) = \sum_{k\in K_S} x_k,$$

where K_S is the set of colourings of G monochromatic in all connected components of the graph spanned by the edges S, and $V_1, V_2, \dots,$ is the partition of the vertex set into connected components.

A colouring k lies in K_S if and only if all the edges of S connect vertices of the same colour. Let $\bar{E}_k = \{\{v,w\}\in E(G)|k(v)=k(w)\}$.

Summing up we get

$$\begin{aligned}\sum_{S\subseteq E(G)}(-1)^{\#S}p_{\lambda(S)} &= \sum_{S\subseteq E(G)}(-1)^{\#S}\sum_{k\in K_S}x_k\\ &=\sum_k x_k\sum_{S\subseteq \bar{E}_k}(-1)^{\#S}\\ &=\sum_k x_k\sum_{S\subseteq \bar{E}_k}(-1)^{\#S}1^{\#(\bar{E}_k\setminus S)}.\end{aligned}$$

Now if $\bar{E}_k=\emptyset$, $\sum_{S\subseteq \bar{E}_k}(-1)^{\#S}1^{\#(\bar{E}_k\setminus S)}=(-1)^{\#\emptyset}1^{\#\emptyset}=1$, and all the colourings k that satisfy $\bar{E}_k=\emptyset$ are exactly the proper ones by definition.

However, if $\bar{E}_k\neq\emptyset$, we can use the binomial formula and get

$$\begin{aligned}\sum_{S\subseteq E(G)}(-1)^{\#S}p_{\lambda(S)} =\sum_k x_k\sum_{S\subseteq \bar{E}_k}(-1)^{\#S}1^{\#(\bar{E}_k\setminus S)} &= \sum_{k\text{ proper colouring}}x_k+\sum_{k\text{ non-proper colouring}}(1+(-1))^{\#\bar{E}_k}\\ &=\sum_{k\text{ proper colouring}}x_k=\chi_G,\end{aligned}$$

due to the fact that $0^k=0$. □

For trees we have a simpler expression, which we recover here from corollary 2.1.4. This comes from the fact that all the edge-sets that contribute for p_λ, contribute with the same sign: there are no cycles, each edge that we take from a graph disconnects a connected component, so the number of connected components of $E(G)\setminus S$ is exactly $n-1-\#S$.

Corollary 2.1.4. Let T be a tree, and let $\theta_\mu=\#\{S\in\binom{E(T)}{l(\mu)-1}|\lambda(E(G)\setminus S)=\mu\}$, then we have:

$$\chi_T=\sum_{S\subseteq E(T)}(-1)^{\#S}p_{\lambda(S)}=(-1)^n\sum_\mu\theta_\mu(-1)^{l(\mu)}p_\mu.$$

For sake of simplicity, we write $\chi_G=\sum_{\lambda\vdash n}a_\lambda p_\lambda$ and henceforth, whenever the graph G is clear in the context, we address a_λ for the coefficients of p_λ in χ_G, and $a_\lambda(G)$ if the graph is not clear from the context.

2.2 Some work on particular types of trees

In this chapter we will deal with some simple cases treated in the literature.

The goal here is to explore in the literature the different subfamilies of trees that have been studied and shown to be distinguished by the chromatic symmetric function on trees. We will focus on the case of forks.

☞ Definition: Fork

A *fork* is a tree T with $a+b=n$ vertices and $a+1\geq 3$ leaves and the remaining vertices are at least three and form a path with the degree sequence $(2,\dots,2,a+1)$, i.e. the inner vertices of the path have no leaf attached and one of the tips has only one leaf. Such graph is said to be a fork of type (b,a) or, as it is unique up to isomorphism, **the fork** of type (b,a) and is denoted by $\mathscr{F}_{b,a}$.

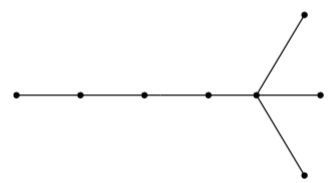


Figure 2.1: A fork of type $(b,a)=(5,3)$

In Fougere's thesis (Fougere, 2003, Section 2.1) this case was studied following the methods presented henceforth, which enable us to distinguish among the several graphs $\mathscr{F}_{b,a}$ by looking to its chromatic symmetric function coefficients.

☞ Definition: Unique stable bipartition

A given tree T, since it has no cycles, has no odd cycles, hence T is bipartite $V(T)=V_1\uplus V_2$ and this set partition is trivially unique by connectedness, i.e. there is no other stable set partition of $V(G)$ into two blocks. The underlying integer partition $B(T):=\lambda(V_1\uplus V_2)=(\lambda_1,\lambda_2)$ is an isomorphism invariant of a tree.

We will first see that the chromatic symmetric function uniquely determines such partition, i.e. two trees T_1,T_2 with $B(T_1)\neq B(T_2)$ have different chromatic symmetric function span over the m_λ basis. In fact, chromatic symmetric function gives a lot of information regarding stable partitions, as a consequence of proposition 0.1.3. The ideas in this proposition will be widely used in the remaining of this work.

Proposition 2.2.1. Given the chromatic symmetric function of a tree T, we can compute $B(T)$.

Proof. This serves as an example for the general method that we will exploit. We appeal to Proposition 0.1.3 to get that if we write $\chi_G = \sum_{\lambda \vdash n} a_\lambda m_\lambda$ then $a_\lambda = \#\{$ stable partitions of type $\lambda\} \times \lambda!$.

Since we know that there is only one stable partition with two blocks, there is only one integer partition λ into two parts that has $a_\lambda > 0$, and by definition this partition is $B(T) = \lambda$ □

We can actually compute $B(T)$ for the simple case of forks: For a fork of type (b, a),

- If we have $2|b$, then $(\lambda_1, \lambda_2) = (\frac{b}{2} + a, \frac{b}{2})$
- If $2 \not| b$, then $(\lambda_1, \lambda_2) = (\frac{b-1}{2} + a, \frac{b+1}{2})$.

Proposition 2.2.2. For a fixed fork of type (b, a), there is only one other fork that has the same integer partition $B(T) = (\lambda_1, \lambda_2)$, namely, if $2|b$ then $(b-1, a+1) = \vec{k}$ is the only solution to $B(\mathscr{F}_{\vec{k}}) = B(\mathscr{F}_{b-1,a+1})$.

Proof. Fix a pair (b', a') with $B(\mathscr{F}_{(b',a')}) = (\lambda'_1, \lambda'_2)$ such that $(\lambda_1, \lambda_2) = (\lambda'_1, \lambda'_2)$. Clearly if $2|b$ and $2|b'$ that would mean $b = b'$ and $a = a'$, and the same if $2 \not| b$ and $2 \not| b'$ so we have wlog $2|b$ and $2 \not| b'$. Then $(\frac{b}{2} + a, \frac{b}{2}) = (\lambda_1, \lambda_2) = (\lambda'_1, \lambda'_2) = (\frac{b'-1}{2} + a', \frac{b'+1}{2})$.

Note that, since $a \geq 2$, from the equality of partitions $(\frac{b}{2} + a, \frac{b}{2}) = (\frac{b'-1}{2} + a', \frac{b'+1}{2})$ we know that the bigger parts should be equal and the same for the smaller parts implies the equations $\frac{b}{2} + a = \frac{b'-1}{2} + a'$ and $\frac{b'+1}{2} = \frac{b}{2}$, so $(b, a) = (b'+1, a'-1)$, which concludes $(b-1, a+1) = \vec{k}$. □

So given the chromatic symmetric function it only remains to be shown that $\mathscr{F}_{b,a}$ and $\mathscr{F}_{b-1,a+1}$ can somehow be distinguished, given $2|b$. We know that their unique stable bipartition (λ_1, λ_2) is the same.

We can do it by looking to the stable partitions of type $(\lambda_1 - 1, 1, \ldots, 1)$, which amounts to compute the number of stable partitions of this type by Proposition 0.1.3, which is developed in Fougere (2003).

Equivalently, we count the number of stable sets of size $\lambda_1 - 1$. Note that intuitively there should be just a few of them because there are either only one or two stable sets of size λ_1, however, there should be more freedom in choosing such an independent set in the tree with more leaves.

The following Lemma was proven in Fougere (2003).

Lemma 2.2.3. The number of stable sets of size k in a path with m vertices is exactly

$$y(m,k) = \binom{m-k+1}{k}.$$

Proof. Let $Y(m,k)$ be the number of stable sets of size k in a path with m vertices. Note that $Y(m,1) = m$ and $Y(m,0) = 1$ is clear for any $m \geq 1$. Note as well that $y(m,1) = m$ and $y(m,0) = 1$.

We just observe now that both functions obey to the following recurrence relation:

$$Y(m+1,k+1) = Y(m,k+1) + Y(m-1,k).$$

In fact, for the function Y, this equation separates the stable sets among those who contain the last element of the path and those who don't.

For the function y we have:

$$\begin{aligned} y(m,k+1) + y(m-1,k) &= \binom{m-k}{k+1} + \binom{m-k}{k} \\ &= \binom{m-k+1}{k+1} \\ &= y(m+1,k+1) \end{aligned}$$

And this completes the proof. $\square$

Proposition 2.2.4. Suppose $2|b$, and let $a \geq 2$, then the number of stable partitions of size $\lambda_1 - 1 = a + \frac{b}{2} - 1$ in $\mathscr{F}_{b,a}$ is

$$\binom{\frac{b}{2}+1}{\frac{b}{2}-1} + a.$$

The number of stable partitions of size $\lambda_1 - 1 = a + \frac{b}{2} - 1$ in $\mathscr{F}_{b-1,a+1}$ is

$$\binom{\frac{b}{2}+1}{\frac{b}{2}-2} + (a+1)\frac{b}{2}.$$

Consequently, the chromatic symmetric function of both graphs distinguishes them, through the coefficient in $m_{(\lambda_1-1,1,\ldots,1)}$.

Proof. Let

$$C_1 = \#\{\text{ stable sets with } \lambda_1 - 1 \text{ elements in } \mathscr{F}_{(b,a)}\}$$

and

$$C_2 = \#\{\text{ stable sets with } \lambda_1 - 1 \text{ elements in } \mathscr{F}_{(b-1,a+1)}\}.$$

We count the number of stable sets with $\lambda_1 - 1$ elements by counting how many vertices of the a leaves are in the set, and note that $y(m,k) = 0$ whenever $m - k + 1 < k$, or $m + 1 < 2k$, from 2.2.3, implying that we have to take a big amount of leaves into our stable set. It follows that

$$\begin{aligned} C_1 &= y(b, \lambda_1 - 1) + \sum_{k=1}^{a} y(b-1, \lambda_1 - 1 - k)\binom{a}{k} \\ &= y(b, \frac{b}{2} + a - 1) + \sum_{k=1}^{a} y(b-1, \frac{b}{2} + a - k - 1)\binom{a}{k}. \end{aligned}$$

But since $2\left(\frac{b}{2} + a - 1\right) = b+2a-2 \geq b+2 > b+1$ we have $y(b, \frac{b}{2}+a-1) = 0$, and $2\left(\frac{b}{2} + a - k - 1\right) = b + 2(a-k) - 2 > b - 1 + 1$ for $k < a - 1$ we have:

$$\begin{aligned} C_1 =& y(b-1, \frac{b}{2} + a - a - 1)\binom{a}{a} + y(b-1, \frac{b}{2} + a - a + 1 - 1)\binom{a}{a-1} \\ =& \binom{b - 1 - \frac{b}{2} + 1 + 1}{\frac{b}{2} - 1} + \binom{b - 1 - \frac{b}{2} + 1}{\frac{b}{2}} a \\ =& \binom{\frac{b}{2} + 1}{\frac{b}{2} - 1} + a. \end{aligned}$$

To compute C_2 we apply the same method and note that

$$\begin{aligned} C_2 &= y(b-1, \lambda_1 - 1) + \sum_{k=1}^{a+1} y(b-2, \lambda_1 - 1 - k)\binom{a+1}{k} \\ &= y(b-1, \frac{b}{2} + a - 1) + \sum_{k=1}^{a+1} y(b-2, \frac{b}{2} + a - k - 1)\binom{a+1}{k}. \end{aligned}$$

But since $2\left(\frac{b}{2} + a - 1\right) = b+2a-2 \geq b+2 > b$ we have $y(b, \frac{b}{2}+a-1) = 0$, and $2\left(\frac{b}{2} + a - k - 1\right) =$

$b+2(a-k)-2>=b-2$ for $k<a$ we have:

$$\begin{aligned}
C_2 &= y(b-1,\frac{b}{2}+a-1)+\sum_{k=1}^{a+1} y(b-2,\frac{b}{2}+a-k-1)\binom{a+1}{k}\\
&= y(b-2,\frac{b}{2}+a-a-1-1)\binom{a+1}{a+1}+y(b-2,\frac{b}{2}+a-a-1)\binom{a+1}{a}\\
&= \binom{b-2-\frac{b}{2}+2+1}{\frac{b}{2}-2}+\binom{b-2-\frac{b}{2}+1+1}{\frac{b}{2}-1}(a+1)\\
&= \binom{\frac{b}{2}+1}{\frac{b}{2}-2}+(a+1)\frac{b}{2}.
\end{aligned}$$

As a consequence, $C_1 \neq C_2$ and we can distinguish these two graphs by the coefficient of their chromatic symmetric function on $m_{(\lambda_1-1,1,\ldots,1)}$. □

Corollary 2.2.5. The chromatic symmetric function distinguishes forks.

This method is now clear: we find an algebraic invariant, in this case $B(T)$ that can be computed from the chromatic symmetric function and almost solves the problem, simplifying it to a more computable level. Then we only need to distinguish between a smaller set of trees and we can use different properties of the chromatic symmetric function like trying to see some other coefficients.

It would be interesting, though not a big advance towards Conjecture 0.1.4, if we could distinguish between forks and non-fork trees with the chromatic symmetric function of graphs. In that case, for the initial conjecture 0.1.4, we would only need to focus on non-fork trees.

2.3 Retrieving some simple properties

In this section we will try to see some simple properties of the coefficients of the chromatic symmetric function of a given graph G.

Recall that, for sake of simplicity, we write $\chi_G = \sum_{\lambda\vdash n} a_\lambda p_\lambda$ and henceforth, whenever the graph G is clear in the context, we write a_λ for the coefficients of p_λ in χ_G, and $a_\lambda(G)$ if the graph is not clear from the context.

☞ Leaves

We will also call a *vertex-leaf* for a vertex of degree one, and an *edge-leaf* for an edge that connects to a vertex-leaf.

The first proposition resulting from this project addresses the combinatorial interpretation of the coefficients a_λ in trees.

Proposition 2.3.1. If a graph G is a tree, then the number of edge-leaves is given by $\theta_{(1,n-1)} = (-1)^n a_{(n-1,1)}$, equivalently, the number of vertex-leaves is given by $(-1)^n a_{(n-1,1)}$ for $n \neq 2$, and $2a_{(1,1)}$ for $n = 2$.

Proof. From the formula on Corollary 2.1.4, we get that the coefficient on $p_{(n-1,1)}$ is

$$(-1)^{n-2}\theta_{(n-1,1)} = (-1)^n \#\{S \subseteq \binom{E(G)}{1} | \lambda(E(G) \setminus S) = (n-1,1)\}.$$

Or $(-1)^n a_{(n-1,1)} = \#\{e \in E(G) | \lambda(E(G) \setminus \{e\}) = (n-1,1)\}$. However, an edge e satisfies $\lambda(E(G) \setminus \{e\}) = (n-1,1)$ if it connects an isolated vertex to the tree i.e. it is an edge-leaf.

So $\theta_{(n-1,1)} = (-1)^n a_{(n-1,1)}$ is exactly the number of edge-leaves, as desired. □

We have as well a natural relation between the chromatic polynomial and the chromatic symmetric function of a graph. Recall that the chromatic polynomial is $\chi_G(k) = \#\{\rho : V(G) \to \{1,\ldots,k\} : \rho$ is a proper coloring $\}$. The connection between $\chi_G(k)$ and the chromatic symmetric function is given in the next Proposition and uses the notion of an evaluation map on symmetric functions.

In fact, an evaluation map, which is to evaluate each of the variables to a real number on a symmetric function, always makes sense when the evaluation takes all but finitely many variables to zero. In the case of chromatic symmetric function of graphs we have:

Proposition 2.3.2. The evaluation map $x_i \to 1$ for $1 \leq i \leq k$ and $x_i \to 0$ otherwise in χ_G yields $\chi_G(k)$, the chromatic polynomial defined in the first chapter.

Proof. Simply note that in the sum $\chi_G = \sum_{k \text{ proper}} x_k$ after the evaluation $x_i \to 1$ for $1 \leq i \leq k$ and $x_i \to 0$ all monomials with indices bigger than k vanish.

Then we end up with $\sum_{k:V(G)\to\{1,\ldots,k\} \text{ proper}} x_k \Big|_{x_i=1} = \sum_{k:V(G)\to\{1,\ldots,k\} \text{ proper}} 1$ which is exactly the number of proper colourings with k colours, counted by $\chi_G(k)$ by definition. □

Lemma 2.3.3. Given the chromatic symmetric function of a graph χ_G, we can compute the number of edges $\#E(G)$ of the graph.

Proof. We can do it in several ways, here we will appeal to Proposition 0.1.3, and claim that $\#E(G) = \binom{n}{2} - \frac{1}{(n-2)!}\chi_G[m_{(2,1,\dots,1)}]$.

Clearly, $[m_{(2,1,\dots,1)}]\frac{1}{(n-2)!}\chi_G$ is the number of stable partitions of G of type $(2,1,\dots,1)$ by 0.1.3, as $(2,1\dots,1)! = \langle 1^{n-2}2^1\rangle! = (n-2)!$

However, a partition π of type $(2,1,\dots,1)$ is stable if the block with two vertices has no edge between them, so the number of stable partitions in G of type $(2,1,\dots,1)$ is exactly the number of pairs of distinct vertices without an edge, hence it is $\binom{n}{2} - \#E(G)$. □

In the particular case of trees we have more general results, for instance the one presented in (Martin et al., 2008):

Proposition 2.3.4. Given a tree G and an integer k between 1 and n, we have:

$$(-1)^{n-k}\sum_{\lambda:l(\lambda)=k} a_\lambda = \binom{n-1}{k}.$$

Proof. We have,

$$\begin{aligned}
\sum_{\lambda:l(\lambda)=k}(-1)^{n-l(\lambda)}a_\lambda &= \sum_{\lambda:l(\lambda)=k}\theta_\lambda \\
&= \sum_{\lambda:l(\lambda)=k}\#\{S\in\binom{E(T)}{n-l(\lambda)}|\lambda(S)=\lambda\} \\
&=\#\cup_{\lambda:l(\lambda)=k}\{S\in\binom{E(T)}{n-k}|\lambda(S)=\lambda\} \\
&=\#\{S\in\binom{E(T)}{n-k}\} \\
&=\binom{n-1}{n-k}
\end{aligned}$$

□

Write $c(H)$ for the number of connected components of a graph H.

Recall that $a_\mu = \sum_{\substack{A\subseteq E(G)\\ \lambda(A)=\mu}}(-1)^{\#A}$. We are interested in establishing that certain coefficients are non-zero in order to find the number of connected components of a graph. In fact, we have that $a_{\lambda(E(G))}$ satisfies a deletion-contraction relation:

Lemma 2.3.5. For a multigraph H, (a non-necessarily simple graph), let $F_H = (-1)^{n-c(H)} a_{\lambda(E(H))}$, where $c(H)$ is the number of connected components of the graph H.

If e is an edge such that $c(H \setminus e) = c(H)$ (i.e., by erasing the edge e no connected component of H gets disconnected) and is not a loop, then we have

$$F_H = F_{H\setminus e} + F_{H/e}$$

If H has a loop e, then $F_H = 0$.

Besides, if H is a forest, then $F_H = 1$.

This has been proven long ago in the much stronger realm of Tutte polynomials, for instance in Bollobás (2013), but we introduce and independent proof for sake of completeness.

Proof. Call $\lambda(E(H)) = \mu$.

Let's first suppose that H is a forest. Then, there is only one edge set $A \subseteq E(H)$ such that $\lambda(A) = \lambda(E(H))$, which is $E(H)$ itself, which has exactly $n - c(H)$ edges.

Then
$$F_H = (-1)^{n-c(H)}(-1)^{\#E(H)} = 1\,.$$

Now, suppose H is a multigraph and e does not disconnect any component of H and it is not a loop, then we can write

$$F_H = (-1)^{n-c(H)} \sum_{\substack{e\notin A\subseteq E(H)\\ \lambda(A)=\mu}} (-1)^{\#A} + (-1)^{n-c(H)} \sum_{\substack{e\in A\subseteq E(H)\\ \lambda(A)=\mu}} (-1)^{\#A}.$$

We claim that $F_{H\setminus e} = (-1)^{n-c(H)} \sum_{\substack{e\notin A\subseteq E(H)\\ \lambda(A)=\mu}} (-1)^{\#A}$ and $F_{H/e} = (-1)^{n-c(H)} \sum_{\substack{e\in A\subseteq E(H)\\ \lambda(A)=\mu}} (-1)^{\#A}$.
For the first one, note that $c(H) = c(H \setminus e)$ is given, and $E(H \setminus e) = E(H) \setminus e$ so

$$F_{H\setminus e} = (-1)^{n-c(H)} \sum_{\substack{A\subseteq E(H)\setminus e\\ \lambda(A)=\mu}} (-1)^{\#A}.$$

For the latter one, it's clear that $c(H) = c(H/e)$ and call $\mu' = \lambda(E(H/e)) \vdash n-1$, then

$$F_{H/e} = (-1)^{n-1-c(H)} \sum_{\substack{A\subseteq E(H/e)\\ \lambda(A)=\mu'}} (-1)^{\#A} = (-1)^{n-c(H)} \sum_{\substack{A\subseteq E(H/e)\\ \lambda(A)=\mu'}} (-1)^{1+\#A}$$

Now there is a clear bijection between edge sets $e \in A \subseteq E(H)$ such that $\lambda(A) = \mu$ and $A' \subseteq E(H/e) = E(H) \setminus \{e\}$ such that $\lambda(A') = \mu'$, given by $A = A' \cup \{e\}$, that satisfy $\#A = \#A' + 1$.

Then our formula becomes

$$F_{H/e} = (-1)^{n-c(H)} \sum_{\substack{e\in A\subseteq E(H)\\ \lambda(A)=\mu}} (-1)^{\#A}. \tag{2.1}$$

Now, for the case that e is a loop, we still have $F_{H\setminus e} = (-1)^{n-c(H)} \sum_{\substack{e\notin A\subseteq E(H)\\ \lambda(A)=\mu}} (-1)^{\#A}$, by the same reasoning. However we now have that H/e has still n vertices, so, instead of holding the equation 2.1, we have $F_{H/e} = (-1)^{1+n-c(H)} \sum_{\substack{e\in A\subseteq E(H)\\ \lambda(A)=\mu}} (-1)^{\#A}$ and

$$F_H = F_{H\setminus e} - F_{H/e} = 0$$

because $H \setminus e = H/e$.

This concludes the deletion contraction lemma. □

The next result was given by (Martin et al., 2008).

Lemma 2.3.6. Given the chromatic symmetric function of a graph χ_G, we can count the number of connected components of G, denoted by $c(G)$. Indeed $c(G) = \min\{l(\lambda) | a_\lambda(G) \neq 0\}$.

Proof. From Theorem 2.1.3, the only p_λ with non-zero coefficients are the ones that come from $p_{\lambda(S)}$ for sets $S \subseteq E(G)$.

However, it is impossible for the graph spanned by the edges $S \subseteq E(G)$ to have less connected components than G, i.e. $l(\lambda(S)) = c((V(G), S)) \geq c(G)$.

So

$$\min\{l(\lambda) | a_\lambda(G) \neq 0\} \geq c(G)\,.$$

Now call $\mu = \lambda(E(G))$, we will show that $a_\mu \neq 0$, concluding $\min\{l(\lambda)|a_\lambda(G) \neq 0\} \leq l(\mu) = c(G)$.

We have that $a_\mu = (-1)^{n-l(\mu)} \sum_{\substack{A\subseteq E(G)\\ \lambda(A)=\mu}} (-1)^{\#A} = (-1)^{l(\mu)-c(G)} F_G$. However, it is clear from Lemma 2.3.5 that we have $F_G > 0$ for any simple graph G. Then, $a_\mu \neq 0$, concluding that $c(G) \geq \min\{l(\lambda) | a_\lambda \neq 0\}$. □

A tree is a connected graph with $n-1$ edges and since we can test both these conditions we get:

Corollary 2.3.7. The chromatic symmetric function separates trees.

2.3.1 Retrieving the degree sequence directly and Tailings of trees

In this section we develop some ideas to find the number of vertices with a certain degree, by finding explicit formulas for small cases, with the information that the graph at hand is a tree. The work here presented was the result of the work of the author during the Project, as well as the results and ideas obtained

We have already found a way to obtain the number of vertices of degree 1, the leaf-vertices, in Proposition 2.3.1. However we can improve this result by counting the number of vertices with degree 2:

Proposition 2.3.8. The number of vertices with degree 2 of a tree T with $n > 4$ is given by

$$\left(\sum_{\substack{j+k=n-1\\ j\geq k}} \theta_{(j,k,1)}\right) - \binom{n-1}{2} + \binom{n-1-\theta_{(n-1,1)}}{2} + \theta_{(n-2,2)}$$

Recall the definition of the coefficients $\theta_\lambda(T)$ from Corollary 2.1.4, where we have $\theta_\mu(T) = \#\{S \in \binom{E(T)}{l(\mu)-1} | \lambda(E(T)\setminus S) = \mu\}$.

Proof. Consider the set $S_2 = \binom{E(T)}{2}$ and $P = \{S \in S_2 | 1 \in \lambda(E(T)\setminus S)\}$ which satisfies, by definition,

$$\#P = \sum_{\substack{j+k=n-1\\ j\geq k}} \theta_{(j,k,1)}\,.$$

The number of vertices with degree 2 is exactly the cardinality of the set

$$V = \{\{e_1, e_2\} \in S_2 | \text{ edges share a vertex of degree 2}\} \subseteq P\,.$$

Consider as well the set $S_l = \{S \in S_2 | S \text{ has a leaf-edge } \} \subseteq P$.

We claim that $P = S_l \cup V$. Indeed if $S \in P$ then there is an isolated vertex v in $E(T) \setminus S$ so it must have, at most, two incident edges in T: if it has one, it's a leaf and $S \in S_l$, if it has two then $S \in V$.

Consequently $\#V = \#P - \#S_l + \#(S_l \cap V) - \#(P \setminus (S_l \cup V)) = \#P - \#S_l + \#(S_l \cap V)$.

We have already noted that $\#P$ is not hard to compute with the chromatic symmetric function so we only need to compute $\#S_l$ and $\#(S_l \cap V)$.

Note that S_l is not hard to compute as well, since is simply, according to Proposition 2.3.1,

$$\#\binom{E(T)}{2} \setminus \binom{E(T) \setminus L(T)}{2} = \binom{n-1}{2} - \binom{n-1-\theta_{(n-1,1)}}{2}.$$

To compute $\#(S_l \cap V)$, we draw a bijection $\phi : S_l \cap V \to S_{(2,n-2)} = \{e \in E(T) | \lambda(E(T) \setminus \{e\}) = (n-2,2)\}$. For $S \in S_l \cap V$, there is a vertex v of degree two with an incident leaf-edge e_1 and another edge e_2 (which is not a leaf-edge, since $n > 4$) so define $\phi(S) = e_2$.

It is clear that e_2 satisfies $\lambda(\{e_2\}^c) = (n-2,2)$, separating the edge e_1 from the remaining of the tree, and that any edge satisfying $\lambda(\{e\}^c) = (n-2,2)$ is of the form $\phi(\{e, e_1\})$ for some unique e_1, exactly the edge in the block with two vertices (which is unique, from $n > 4$).

Thus we obtain that $\#(S_l \cap V) = \theta_{(n-2,2)}$ and we conclude the proof. □

A natural path here to follow is to devise a general way to obtain the number of vertices of a certain degree. Though the methods of the proof of Proposition 2.3.8 are somehow *ad hoc*, and only works for n big enough, the construction of some sets S_l, P and V is quite intuitive and their intersections count structures in the form of "tails" of the original tree.

A natural idea of a partial result on the distinction of trees using the chromatic symmetric function is to count the number of vertex-leaves and go on and count the number of structures that occur on the tip of the tree.

Conjecture 2.3.9. There is an algebraic formula, only dependent on the coefficients of the chromatic symmetric function on the p-basis, for the degree sequence.

Latter in this work it will be shown that the chromatic symmetric function actually determines the degree sequence, but in a different way, showing that there is an algebraic invariant S_T classified by the chromatic symmetric function that classifies the degree sequence.

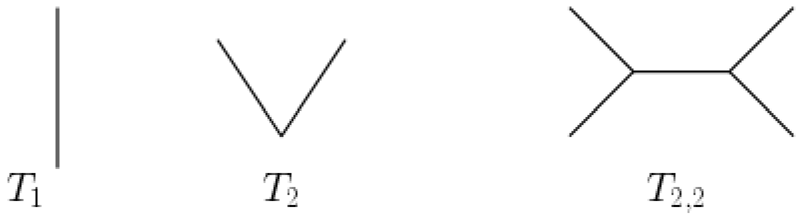


Figure 2.2: Some examples of trees

The Conjecture 2.3.9 motivates the following definition.

☞ Definition: Tailings

Given a rooted tree T (see Definition 0.1.1), the number of *tails of the form* (T, v) on a graph G is the number of rooted subtree (T', v) embedded in G, isomorphic to (T, v), such that there is an edge $e \in G$ to which v is incident to e and $G \setminus e$ has T' as one of its connected components.

Given a rooted tree T, the number of *tailings of the form* (T, v) on a graph G is the number of edges $e \in G$ such that $G \setminus e$ has one of its connected components isomorphic to (T, v), where e is incident to the root v.

Note that if a tree T is "symmetric" over an edge e, then there is a tree T' which is isomorphic to both connected components of $T \setminus e$ - such case counts only as one tailing, but as two different tails of the form (T, v).

The number of tails of the form T on the graph G is denoted as $\epsilon_G(T)$ and the number of tailings of G is written as $\mu_G(T)$.

We will compute the tailings of some trees, then devise a general formula for some such numbers from the chromatic symmetric function of a tree. The goal will be to obtain a family of formulas for each $\mu_G(T, v)$ with fixed rooted tree (T, v), depending only on the coefficients of the chromatic symmetric function of the graph G.

Example 2.3.10. Take, for instance, the graphs in figure 2.2. For the graph $G = T_{2,2}$, we have $\mu_G\left(\overset{r}{\bullet}\!-\!\!-\!\!-\!\bullet\right) = \epsilon_G\left(\overset{r}{\bullet}\!-\!\!-\!\!-\!\bullet\right) = 0$.

The distinction between these two notions is quite small, and it is noticed only if T has nearly half of the edges of G, among other conditions. For instance, still considering $G = T_{2,2}$ we have

$$\epsilon_G\left(\overset{r}{\bullet}\!<\!\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\right) = 2$$

but

$$\mu_G\left(\overset{r}{\bullet}\!<\!\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\right) = 1,$$

since the same tailing yields two different tails of the same shape.

On the other hand, $\mu_{T_1}\left(\overset{r}{\bullet}\right) = 1$ whereas $\epsilon_{T_1}\left(\overset{r}{\bullet}\right) = 2$.

Finally, we have both $\mu_{T_2}\left(\overset{r}{\bullet}\right) = \epsilon_{T_2}\left(\overset{r}{\bullet}\right) = 2$.

We will now interpret the coefficients of the chromatic symmetric function of trees, and in particular the coefficients θ_λ, as tailings of the original tree T of certain forms. Related to Proposition 2.3.1 we have the following claim, which generalizes the interpretation of the coefficients θ_λ for trees:

Proposition 2.3.11. If T is a tree with $n > 2$ vertices, then

$$\theta_{(n-1,1)} = \mu_T\left(\overset{r}{\bullet}\right).$$

Additionally, if we assume that n is big enough, then,

$$\theta_{(n-2,2)} = \mu_T\left(\overset{r}{\bullet}\!-\!\bullet\right)$$

$$\theta_{(n-2,1,1)} = \binom{\mu_T\left(\overset{r}{\bullet}\right)}{2} + \mu_T\left(\overset{r}{\bullet}\!-\!\bullet\right)$$

$$\theta_{(n-3,2,1)} = \left(\mu_T\left(\overset{r}{\bullet}\right) - 1\right)\mu_T\left(\overset{r}{\bullet}\!-\!\bullet\right) + 2\mu_T\left(\overset{r}{\bullet}\!<\!\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\right) + 2\mu_T\left(\overset{r}{\bullet}\!-\!\bullet\right)$$

$$\theta_{(n-3,3)} = \mu_T\left(\overset{r}{\bullet}\!-\!\bullet\!-\!\bullet\right) + \mu_T\left(\overset{r}{\bullet}\!<\!\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\right).$$

There are other simpler cases, and possibly the following one would be to compute $\theta_{(n-3,1,1,1)}$ which we leave to a later occasion.

With this, we obtain some algebraic equations on the coefficients of the chromatic symmetric function of trees. Indeed, from proposition 2.3.11 we obtain:

Corollary 2.3.12. A given tree T with a big enough number of vertices n satisfies

$$\theta_{(n-2,1,1)} = \theta_{(n-2,2)} + \binom{\theta_{(n-1,1)}}{2}.$$

Our goal in this project went through finding the most possible formulas as in Proposition 2.3.11, and with those formulas obtain algebraic relations like the ones in Proposition 2.3.11.

If enough formulas were settled with a broad generality (which means to describe how big n should be for the formulas to hold), we could invert the equations in order to obtain the number of *tailings* as a function of the coefficients of the chromatic symmetric function on trees.

With a solid theory on *tailings*, this would enable us to show that the chromatic symmetric function distinguishes among a broad class of trees (the ones distinguished by some tailings).

However, since we obtain algebraic relations with the equations of the form 2.3.12, either we get more algebraic relations, which means less independent equations, or more algebraically independent relations which means less algebraic relations between the coefficients.

2.3.2 Retrieving the degree sequence indirectly

In this section we follow the work in Martin et al. (2008) where some algebraic invariants are determined from the chromatic symmetric function of graphs, namely the so called connector polynomial and the subtree polynomial are determined from the chromatic symmetric function of a tree.

In this section, we restrict ourselves to the study of trees and these algebraic invariants are only defined for trees. For the classification problem it makes no difference in light of Corollary 2.3.7.

☞ Definition: Connector Polynomial, Subtree polynomial

Given a tree T, recall that a *subtree* is a tree S that satisfies $V(S) \subseteq V(T)$ and $E(S) \subseteq E(T)$. For a subtree $S \subseteq T$ let $L(S)$ be the set of edge-leaves of S, defined in Definition 2.3.

For a nonempty set of edges $A \subseteq E(T)$, there is a minimal set $K(A) \subseteq E(T)$ of edges such that the graph with $A \cup K(A)$ as edges is connected, disregarding isolated vertices.

The *subtree polynomial* S_T of a tree T is given by

$$S_T(p, r) = \sum_{\text{subtrees } S} q^{\#E(S)} r^{\#L(S)} \tag{2.2}$$

The *connector polynomial* K_T of a tree T is given by

$$K_T(x, y) = \sum_{\emptyset \neq A \subseteq E(T)} x^{\#A} y^{\#K(A)} \tag{2.3}$$

Let's also define *stars* as trees where only one vertex has degree bigger than one and note that there is, up to isomorphism, only one star with k edges.

Some simple properties can be obtain for these algebraic invariants, namely the number of edges and vertices. For instance, we can compute

$$\begin{aligned} K_S(1,1) = &\sum_{\emptyset \neq A \subseteq E(T)} 1^{\#A} 1^{\#K(A)} \\ =&\#\{A | \emptyset \neq A \subseteq E(T)\} \\ =&2^{\binom{n-1}{2}} - 1\,. \end{aligned}$$

Obtaining our first proposition regarding the connector polynomial.

Proposition 2.3.13. $K_T(1,1) = 2^{\binom{n-1}{2}} - 1$.

Example 2.3.14. We will start with some simple examples of trees to get the feeling of these polynomial invariants.

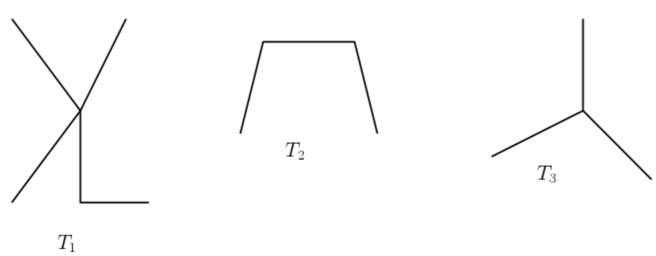


Figure 2.3: Examples of trees and their polynomial invariants.

Take for instance the trees in figure 2.3, which we will compute the subtree polynomial.

			# Copies of S			
Subtree S	$\#E(S)$	$\#L(S)$	In T_1	In T_2	In T_3	In a generic tree
	0	0	6	4	4	$\#V(T)$
	1	1	5	3	3	$\#E(T)$
	2	2	7	2	3	$\sum_v \binom{\deg(v)}{2}$
	3	2	3	1	0	
	3	3	4	0	1	$\sum_v \binom{\deg(v)}{3}$
	4	4	1	0	0	$\sum_v \binom{\deg(v)}{4}$
	4	3	3	0	0	
	4	2	0	0	0	
	5	4	1	0	0	

Table 2.1: Subtrees of trees T_1, T_2 and T_3

To obtain the subtree polynomial, we compute the occurrences of subtrees through the information in the table 2.1, obtaining

$$\begin{aligned} S_{T_1}(q,r) &= 6+5qr+7q^2r^2+3q^3r^2+4q^3r^3+q^4r^4+3q^4r^3+q^5r^4 \\ S_{T_2}(q,r) &= 4+3qr+2q^2r^2+q^3r^2 \\ S_{T_3}(q,r) &= 4+3qr+3q^2r^2+q^3r^3\,. \end{aligned}$$

We now see why it is so important that such invariants are determined by the chromatic symmetric function of trees. Incidentally, we can determine the degree sequence from said invariants, namely from the subtree polynomial, as we are going to see in the next example.

The motivation for the next example is to find the maximal degree $d_{\max}$ and how many vertices have degree $d_{\max}$, which can be counted by how many stars with $d_{\max}$ edges occur as a subtree of T.

Example 2.3.15. Observe that the degree sequence of T_3 is $(3,1,1,1)$ and the number of vertices of degree 3 is exactly the number of subtrees of the form of a star with three edges. Incidentally, 3 is the biggest number k to which there is a star with k edges in T_3, and this is a simple way to find the number of vertices with the highest degree through S_T.

In general, the method just described behaves like this: If we draw our attention to the Table 2.1, the formula for generic trees tells us that the number of stars with 2, 3 and 4 edges has a simple formula depending only on the degree of the vertices and, hence, on the degree sequence.

This is because a subtree with the shape of a star with k leaves has a special vertex, v, of which we choose k neighbouring edges in $\binom{\deg v}{k}$ possible ways. So we can actually compute the number of star subtrees with a certain number of leaves knowing the degree sequence.

With this, knowing the number of star subtrees we can actually determine the degree sequence, through inverting a matrix.

Theorem 2.3.16. The subtree polynomial determines the degree sequence.

Proof. First, note that knowing the degree sequence is equivalent to know the degree incidence sequence $\vec{d} = (d_0, d_1, d_2, \cdots)$ where d_i is the number of vertices with degree i.

Define as well the star incidence sequence $\vec{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \cdots)$ where σ_i is the number of subtrees with the form of a star with i edges, as we have discussed in the example 2.3.15.

A star subtree with k edges is a choice of a vertex v with $\deg v \geq k$ and a choice of k of it's incident edges. Then it is clear that $\sigma_n = \sum_{j \geq n} \binom{j}{n} d_j$, which is clearly invertible - as the system is upper triangular with 1's in all diagonal entries - so $\vec{\sigma}$ determines $\vec{d}$.

On the other hand, σ_k is exactly the coefficient on $q^k r^k$ of S_T, as the star with k edges is the only tree with exactly k edges and k leaf-edges. $\square$

We will now see that in fact the polynomials just defined (connector and subtree polynomials) provide, indeed, equivalent information. In particular, the connector polynomial determines as well the degree sequence.

Lemma 2.3.17 (Connector's Lemma)**.** Given a subtree S of T and an edge set $A \subseteq S$, then

$$L(S) \subseteq A \Leftrightarrow A \cup K(A) = S.$$

This means, that the leaves of a subtree generate the said subtree in the sense that the subtree S is exactly the minimal connected subgraph of T that contains all leaves of S.

Proof. For one implication, suppose that $A \cup K(A) = S$ but for sake of contradiction $l \in L(S)$ such that $l \notin A$.

Then $l \in K(A)$. However, let $K' = K(A) \setminus \{l\}$, which satisfies that $A \cup K' = S \setminus \{l\}$ is connected. But this contradicts the minimality of $K(A)$. We conclude that $L(S) \subseteq A$, as desired.

For the other implication, suppose that $L(S) \subseteq A$. Since $A \subseteq S$ is given and S is connected (being a tree), we get by minimality of $K(A)$ that $K(A) \subseteq S \setminus A$. We claim that $K(A) = S \setminus A$.

Suppose, for sake of contradiction, that there is some $l \in S \setminus (A \cup K(A))$, then $S \setminus \{l\}$ is disconnected, as l cannot be a leaf, i.e. $l \notin L(S) \subseteq A$. Consequently, $S \setminus \{l\}$ has two connected components which are both trees, so $A \cup K(A)$ sits in only one of the connected components, since it is connected.

However, as $L(S) \subseteq A$, and since both connected components of $S \setminus \{l\}$ have leaves (of the original subtree, S), we know that A sits in both connected components, a contradiction.

We conclude that there is no edge $l \in S \setminus (A \cup K(A))$ and then $S = A \cup K(A)$. □

Theorem 2.3.18. We can obtain the polynomial K_T from the subtree polynomial, and symmetrically obtain S_T from K_T via

$$\begin{aligned} S_T(q,r) =& K_T(qr, q(1-r)); \\ K_T(x,y) =& S_T\left(x+y, \frac{x}{x+y}\right). \end{aligned}$$

Incidentally, both polynomials give exactly the same information

Proof. Just compute $K_T(qr, q(1-r))$ according with (2.3) in Definition 2.3.2 to obtain:

$$\begin{aligned} K_T(qr, q(1-r)) &= \sum_{\emptyset \neq A \subseteq E(T)} (qr)^{\#A}(q(1-r))^{\#K(A)} \\ &= \sum_{\emptyset \neq A \subseteq E(T)} q^{\#A+\#K(A)} r^{\#A}(1-r)^{\#K(A)} \\ &= \sum_{\text{subtree } S} \sum_{\substack{A \subseteq E(S) \\ A \cup K(A) = E(S)}} q^{\#E(S)} r^{\#A}(1-r)^{\#K(A)} \\ &= \sum_{\text{subtree } S} q^{\#E(S)} \sum_{\substack{A \subseteq E(S) \\ A \cup K(A) = E(S)}} r^{\#A}(1-r)^{\#K(A)} \end{aligned}$$

Now, according to the definition of the subtree polynomial in 2.2, to obtain that $S_T(q,r) =$

$K_T(qr, q(1-r))$ we need only to establish that

$$\sum_{\substack{A\subseteq E(S)\\ A\cup K(A)=E(S)}} r^{\#A}(1-r)^{\#K(A)} = r^{\#L(S)}.$$

But, according to Lemma 2.3.17 we have that if $A \subseteq E(S)$ then $A \cup K(A) = E(S)$ is equivalent to $L(S) \subseteq A \subseteq E(S)$.

Hence

$$\begin{aligned}
\sum_{\substack{A\subseteq E(S)\\ A\cup K(A)=E(S)}} r^{\#A}(1-r)^{\#K(A)} &= \sum_{L(S)\subseteq A\subseteq E(S)} r^{\#A}(1-r)^{\#K(A)}\\
&= \sum_{k=\#L(S)}^{\#E(S)} \binom{\#E(S)-\#L(S)}{k-\#L(S)} r^k(1-r)^{\#E(S)-k}\\
&= \sum_{k=0}^{\#E(S)-\#L(S)} \binom{\#E(S)-\#L(S)}{k} r^k r^{\#L(S)}(1-r)^{\#E(S)-\#L(S)-k}\\
&= r^{\#L(S)}(r+(1-r))^{\#E(S)-\#L(S)}\\
&= r^{\#L(S)}.
\end{aligned}$$

This completes the first equality. For the second one, since we have $S_T(q,r) = K_T(qr, q(1-r))$ just write $x = qr$ and $y = q(1-r)$ obtaining

$$\begin{aligned}
K_T(x,y) =& S_T(q,r)\\
=& S_T\left(x+y, \frac{x}{x+y}\right),
\end{aligned}$$

which completes the proof. □

Now to complete the objective that we are tasked for in this subsection, we will introduce the concluding part, where we provide the ideas to obtain the connector polynomial from the subtree polynomial.

In fact, in (Martin et al., 2008) it is presented, with a lot of intuition, the exact formula for the coefficients of K_S in function of the coefficients of the chromatic symmetric function of the tree T. As before, denote by $a_\lambda(T)$ the coefficient of the chromatic symmetric function of the graph

T in p_λ, and simply write a_λ whenever the graph T is given by context.

Theorem 2.3.19. For a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ and integers $a, b, i, j, l = l(\lambda)$ define

$$\psi(\lambda, a, b) = (-1)^{a+b} \binom{l-1}{l-n+a+b} \sum_{k=1}^{l} \binom{\lambda_k - 1}{a}.$$

Then we have

$$K_T(x, y) = \sum_{a \geq 0} \sum_{b \geq 0} x^a y^b \sum_{\lambda \vdash n} \psi(\lambda, a, b) a_\lambda(T).$$

Where n is the number of vertices of T, which can be obtained from K_T by Proposition 2.3.13. Consequently, χ_G determines K_T and S_T.

Proof. Our goal in this theorem is to show that

$$\#\{\emptyset \neq A \subseteq E(T) | \#A = a, \#K(A) = b\} = \sum_{\lambda \vdash n} a_\lambda \psi(\lambda, a, b).$$

Now, though this demonstration is quite heavy, it is very neat and the method resembles a lot a discrete version of Fubini's Theorem for multivariate integrals. The interested reader can find the original proof in the original paper from (Theorem 1 Martin et al., 2008).

For our first observation, let $\mathscr{I}[P]$ be an indicator variable that is 1 whenever P holds, and 0 otherwise. Then, if X and Y are sets, a general principle that we can apply is

$$\mathscr{I}[X = Y] = (-1)^{\#Y} \sum_{X \subseteq Z \subseteq Y} (-1)^{\#Z},$$

because:

- For the case that $X = Y$ the sum simplifies to one term.
- If $X \not\subseteq Y$ then the sum is empty.
- For the remaining cases we use the binomial formula and get $(-1)^{2\#X}(1 + (-1))^{-\#X+\#Y} = 0$.

With this we now compute:

$$\begin{aligned}\#\{\emptyset \neq A \subseteq E(T) | \#A = a, \#K(A) = b\} &= \sum_{\substack{A\subseteq E(T)\\ \#A=a}} \sum_{\substack{B\subseteq E(T)\setminus A\\ \#B=b}} \mathscr{I}[K(A) = B]\\ &= \sum_{\substack{A\subseteq E(T)\\ \#A=a}} \sum_{\substack{B\subseteq E(T)\setminus A\\ \#B=b}} (-1)^b \sum_{K(A)\subseteq C\subseteq B} (-1)^{\#C}.\end{aligned}$$

From here we just change the order of the sums, write $D = B \setminus C$ and $F = C \cup A$:

$$\begin{aligned}\#\{\emptyset \neq A \subseteq E(T) | \#A = a, \#K(A) = b\} &= \sum_{\substack{\emptyset\neq A\subseteq E(T)\\ \#A=a}} \sum_{\substack{B\subseteq E(T)\setminus A\\ \#B=b}} (-1)^b \sum_{K(A)\subseteq C\subseteq B} (-1)^{\#C}\\ &=(-1)^b \sum_{\substack{\emptyset\neq A\subseteq E(T)\\ \#A=a}} \sum_{\substack{K(A)\subseteq C\\ C\subseteq E(T)\setminus A}} \sum_{\substack{C\subseteq B\subseteq E(T)\setminus A\\ \#B=b}} (-1)^{\#C}\\ &=(-1)^b \sum_{\substack{\emptyset\neq A\subseteq E(T)\\ \#A=a}} \sum_{\substack{K(A)\subseteq C\\ C\subseteq E(T)\setminus A}} (-1)^{\#C} \binom{\#(E(T)\setminus(A\cup C))}{b-\#C}\\ &=(-1)^b \sum_{\substack{\emptyset\neq A\subseteq E(T)\\ \#A=a}} \left[\sum_{\substack{K(A)\cup A\subseteq F\\ F\subseteq E(T)}} (-1)^{\#F-a} \binom{\#(E(T)\setminus F)}{b-\#F+a}\right]\\ &=(-1)^{a+b} \sum_{F\subseteq E(T)} (-1)^{\#F} \binom{\#(E(T)\setminus F)}{b-\#F+a} \sum_{\substack{A\in\binom{F}{a}\\ K(A)\subseteq F}} 1\,.\end{aligned}$$

Now we will simplify $\sum_{\substack{A\in\binom{F}{a}\\ K(A)\subseteq F}} 1 = \#\{A \in \binom{F}{a} | K(A) \subseteq F\}$. Suppose that $\lambda(F) = \lambda$, so given that $A \cup K(A) \subseteq F$ means that A sits inside only one connected component spanned by F, and such set can be chosen in $\binom{k-1}{a}$ different ways from a connected component with k vertices.

We obtain that, if $\lambda(F) = \lambda$ then summing up over all connected components we get

$$\sum_{\substack{A\in\binom{F}{a}\\ K(A)\subseteq F}} 1 = \sum_{k=1}^{l(\lambda)} \binom{\lambda_k - 1}{a}.$$

Incidentally, the assumption $\lambda(F) = \lambda$ implies $\#F = n - l(\lambda)$ and $\#(E(T) \setminus F) = l(\lambda) - 1$.

With that we get:

$$\begin{aligned}
\#\{\emptyset \neq A \subseteq E(T) | \#A = a, \#K(A) = b\} &= (-1)^{a+b} \sum_{F \subseteq E(T)} (-1)^{\#F} \binom{\#(E(T) \setminus F)}{b - \#F + a} \sum_{\substack{A \in \binom{F}{a} \\ K(A) \subseteq F}} 1 \\
&= (-1)^{a+b} \sum_{\lambda \vdash n} \sum_{\substack{F \subseteq E(T) \\ \lambda(F) = \lambda}} (-1)^{\#F} \binom{\#(E(T) \setminus F)}{b - \#F + a} \sum_{\substack{A \in \binom{F}{a} \\ K(A) \subseteq F}} 1 \\
&= (-1)^{a+b} \sum_{\lambda \vdash n} \sum_{\substack{F \subseteq E(T) \\ \lambda(F) = \lambda}} (-1)^{\#F} \binom{\#(E(T) \setminus F)}{b - \#F + a} \sum_{k=1}^{l(\lambda)} \binom{\lambda_k - 1}{a} \\
&= \sum_{\lambda \vdash n} (-1)^{a+b} \binom{l(\lambda) - 1}{a + b - n + l(\lambda)} \left[\sum_{k=1}^{l(\lambda)} \binom{\lambda_k - 1}{a} \right] \left[\sum_{\substack{F \subseteq E(T) \\ \lambda(F) = \lambda}} (-1)^{\#F} \right] \\
&= \sum_{\lambda \vdash n} \psi(\lambda, a, b) \left[\sum_{\substack{F \subseteq E(T) \\ \lambda(F) = \lambda}} (-1)^{\#F} \right]
\end{aligned}$$

According to Theorem 2.1.3, we have that $\sum_{\substack{F \subseteq E(T) \\ \lambda(F) = \lambda}} (-1)^{\#F} = a_\lambda(T)$, thus obtaining a simpler expression

$$\#\{\emptyset \neq A \subseteq E(T) | \#A = a, \#K(A) = b\} = \sum_{\lambda \vdash n} \psi(\lambda, a, b) a_\lambda .$$

Thus concluding the proof. □

Corollary 2.3.20. The chromatic symmetric function of a tree T classifies the degree sequence.

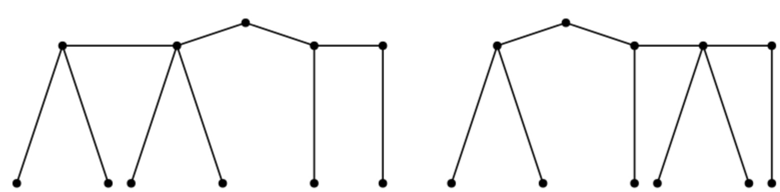


Figure 2.4: Two trees with the same subtree polynomial, but with different chromatic symmetric function

It should be pointed out that the subtree polynomial, and the connector polynomial, are a strictly weaker graph invariant than the chromatic symmetric function of trees, since in (Martin et al., 2008) an example of two trees is given with the same subtree polynomial, in figure 2.4, where their chromatic symmetric functions differ.

2.4 Proper Caterpillars

In this section, we present a work done in Aliste-Prieto and Zamora (2014) where it is shown that proper caterpillars are indeed distinguished by the chromatic symmetric function of graphs.

We start by defining the notion of caterpillar and proper caterpillar:

☞ Definition: Caterpillar, Proper caterpillar

A caterpillar is a tree T such that the graph $T \setminus L_v$ - where L_v is the seat of leaf-vertices is a path. Such path $P(T) = v_1 - v_2 - \cdots - v_k$ is called the spine of the caterpillar.

A caterpillar is called proper if every vertex of its spine is neighbour to one leaf. We call the *clusters* of a caterpillar to the set of leaf-vertices adjacent to one vertex in the spine. The clusters, then, form a partition of the leaf-vertices.

So, in particular, a fork is a caterpillar, and a fork is proper if and only if $b = 3$.

Figure 2.5: Proper caterpillars T_1 with $n = 12$ and spine of size $k = 4$ and T_2 with $n = 12$, $k = 3$

We will define, for each caterpillar, a polynomial, called U-polynomial. We will as well define to each composition a polynomial, called $\mathscr{L}$-polynomial. Finally, we will define a map from the compositions to the caterpillars Ψ, that sends the $\mathscr{L}$-polynomial to the U-polynomial.

Our goal is to use the equation from Theorem 0.1.5 to show that we can distinguish proper caterpillars via the U-polynomials, by using the uniqueness theorems of $\mathscr{L}$-polynomials of the underlying composition of a caterpillar, shown in the section "Algebraic Factorization of Compositions".

Recall that, for a graph G and a set of edges $A \subseteq E(G)$, the notation $\lambda(A)$ is the partition in connected components of the graph $(V(T), A)$ with only A as edges, and

☞ Definition: U-polynomials

We will adopt the short notation $x_{(\lambda_1,\ldots,\lambda_k)} = \prod_{j=1}^{k} x_{\lambda_j}$. Given a caterpillar T with n vertices, for a partition $\lambda \vdash n$ recall that $\theta_\lambda(T) = \#\{A \subseteq E(T) | \lambda(E(G) \setminus A) = \lambda\}$ and write

$$U_T(x) = \sum_{\lambda \vdash n} \theta_\lambda(T) x_\lambda .$$

Note that the coefficients θ_λ can be obtained from the chromatic symmetric function via Theorem 0.1.5, namely

$$\theta_\lambda = (-1)^{l(\lambda)-1} [p_\lambda] \chi_G .$$

Given a caterpillar T with leaf-edges L_e, we define the weighted polynomial

$$U_T^L(x) = \sum_{L_e \subseteq A \subseteq E(T)} x_{\lambda(A)} ,$$

So, for instance, for the caterpillar T_2 in figure 2.5 we have four different subsets A of edges that satisfy $L_e \subseteq A \subseteq E(T)$ which lead to:

$$U_T^L = x_3 x_4 x_5 + x_3 x_9 + x_5 x_7 + x_{12}$$

A natural consequence pointed in (Aliste-Prieto and Zamora, 2014, Proposition 2.1) is that U_T^L does not depend on x_1 in proper caterpillars T. Indeed, an x_1 only occurs if $\lambda(A)$ has a one, i.e., if there is an isolated vertex, but such vertex cannot be neither a leaf because $L_e \subseteq A$, nor in the spine because T is a proper caterpillar.

Theorem 2.4.1. If we write $\theta_\mu(T) = \{A \subseteq E(T) | \lambda(E(G) \setminus A) = \mu\}$, then we get

$$U_T(x) = \sum_{\mu \vdash n} \theta_\mu x_\mu = \sum_{A \subseteq E(T)} x_{\lambda(A)} .$$

Also, U_T is classified from the chromatic symmetric function of the caterpillar T.

Proof. The formula holds trivially, by counting the coefficient of x_μ in U_T. Moreover, note that θ_μ is the absolute value of $a_\mu(T)$, the coefficient of p_λ in χ_G, according to Corollary 2.1.4. □

We have as well that the U-polynomial classifies the U^L-polynomial for a proper caterpillar:

Proposition 2.4.2. If T is a caterpillar we have

$$U_T(x)\big|_{x_1=0} = U_T^L(x)\big|_{x_1=0}$$

Proof. First, let's note that

$$U_T(x)\big|_{x_1=0} = \sum_{A\subseteq E(T)} x_{\lambda(A)}\big|_{x_1=0} = \sum_{\substack{A\subseteq E(T)\\ 1\notin\lambda(A)}} x_{\lambda(A)}\,.$$

On the other hand we have that $U_T^L(x)\big|_{x_1=0} = \sum_{L_e\subseteq A\subseteq E(T)} x_{\lambda(A)}\big|_{x_1=0} = \sum_{\substack{L_e\subseteq A\subseteq E(T)\\ 1\notin\lambda(A)}} x_{\lambda(A)}$.

Now we argue that, whenever $1\notin\lambda$, it holds that $\{A\subseteq E(T)|\lambda(A)=\lambda\} = \{L_e\subseteq A\subseteq E(T)|\lambda(A)=\lambda\}$. Indeed, from $1\notin\lambda$ we get, in caterpillars, that $L_e\subseteq A$.

Then the sets $\{A\subseteq E(T)|\lambda(A)=\lambda\}$ and $\{L_e\subseteq A\subseteq E(T)|\lambda(A)=\lambda\}$ are the same, concluding that

$$\sum_{\substack{A\subseteq E(T)\\ 1\notin\lambda(A)}} x_{\lambda(A)} = \sum_{\substack{L_e\subseteq A\subseteq E(T)\\ 1\notin\lambda(A)}} x_{\lambda(A)}\,.$$

□

Corollary 2.4.3. If T is a proper caterpillar, then

$$U_T(x)\big|_{x_1=0} = U_T^L(x)\big|_{x_1=0} = U_T^L(x)\,.$$

In particular, the chromatic symmetric function classifies U_T^L.

Algebraic factorization of compositions

We should set preliminary definitions beforehand. A composition $\beta = (\beta_1,\ldots,\beta_k)\models n$ of n is a list of positive integers that sum up to n, and the length is denoted as $k=l(\beta)$. The set of compositions of n is $\mathscr{C}_n$ and the set of all compositions will be denoted as $\mathscr{C} = \bigcup_n \mathscr{C}_n$.

We have two operations of interest on compositions, both map $\mathscr{C}_n\times\mathscr{C}_m\to\mathscr{C}_{n+m}$.

The *concatenation* of two compositions α,β is the natural definition which obeys $l(\alpha\cdot\beta) = l(\alpha)+l(\beta)$

The *near-concatenation* of two compositions α,β collapses the last element of α with the first one of β in order to get $\alpha \odot \beta = (\alpha_1, \ldots, \alpha_{l(\alpha)-1}, \alpha_{l(\alpha)} + \beta_1, \beta_2, \ldots, \beta_{l(\beta)})$ so $l(\alpha \odot \beta) = l(\alpha) + l(\beta) - 1$. The near-concatenation is not defined for the empty composition.

So, for instance, $(1,1,1) \odot (1,2) = (1,1,2,2) \models 6$, and $(2,3,1) \odot (2) = (2,3,3) \models 8$.

There is a notion of partial order in $\mathscr{C}_n$, called the *coarsening order*, and we say that $\alpha \leq \beta$ if there are compositions $\gamma_1, \gamma_2, \ldots, \gamma_j$ compositions such that $\gamma_1 \cdot \ldots \cdot \gamma_j = \alpha$ but $\gamma_1 \odot \cdots \odot \gamma_j = \beta$.

So, for instance, $(2,3,3) \geq (2,3,1,2)$, or $(2,3,3)$ is coarser than $(2,3,1,2)$

Said order is equivalent to the coarsening order in the set $\vec{\Pi}$, the family of interval partitions of a totally ordered set. We usually represent these partitions as a diagram, for instance $\gamma_1 \simeq \cdot\cdot|\cdots|\cdot|\cdot\cdot$ and $\gamma_2 \simeq \cdot\cdot|\cdots|\cdots$. The natural connection from interval partitions to compositions gives $\gamma_1 \to (2,3,1,2)$ and $\gamma_2 \to (2,3,3)$, and a notion of coarsening would be to remove separations, i.e. we say that γ is coarser than θ is we can transform γ into θ by removing bars, and we write $\gamma \leq \theta$. Such order relation translates to the coarsening order in the compositions.

So, we have a maximal element (n) and a minimal element $(1,1,\ldots,1)$ and, for instance, $(1,2,4,3) \leq (1,6,3)$ as $(1,2,4,3) = (1,2) \cdot (4,3)$ and $(1,6,3) = (1,2) \odot (4,3)$ In particular, if $\alpha \leq \beta$ then $l(\alpha) \geq l(\beta)$. The interval partition poset is isomorphic to the so called boolean poset of a set with $n-1$ elements, hence the composition poset just defined is as well isomorphic to the boolean poset.

☞ Definition: $\mathscr{L}$-polynomial

Given a composition β, we define its $\mathscr{L}$-polynomial as

$$\mathscr{L}_\beta(x) = \sum_{\alpha \leq \beta} x_\alpha .$$

For instance, if we take $\beta = (3)$, all the compositions on $\mathscr{C}_3$ are smaller or equal to β, so $\mathscr{L}_\beta = x_1^3 + 2x_1x_2 + x_3$.

On the other hand, if we take $\beta = (1,1,\ldots,1)$, which is the minimal element in $\mathscr{C}_n$, we get $\mathscr{L}_\beta = x_1^n$

We state that two compositions have the same $\mathscr{L}$-polynomial by writing $\alpha \sim_{\mathscr{L}} \beta$. The class of compositions with the same $\mathscr{L}$-polynomial as α is denoted as $[\alpha]_{\mathscr{L}}$.

Compositions are the sorted version of partitions, and they naturally help us encode the structure of a caterpillar, since we can correspond a unique caterpillar for each composition in the following way: given a composition β consider a path $v_1 - v_2 - \cdots - v_{l(\beta)}$ and attach to the vertex v_i exactly $\beta_i - 1$ leaves, obtaining thus the caterpillar $\Psi(\beta)$. Note that the condition that $\Psi(\beta)$ is a proper caterpillar is equivalent to that β has no ones. Note as well that theβ_i's are the size of the clusters.

We have the flip map $*$ that, for a composition α yields its reverse α^* where the coefficients are read from last to first. This leads to the equivalence relation in the set of compositions given by $\alpha_1 \sim_* \alpha_2$ if either $\alpha_1 = \alpha_2$ or $\alpha_1 = \alpha_2^*$. The class equivalences on such equivalence relation are called reverse-classes and written $[\alpha]_*$, and naturally may have either one (for the so called *palindromes*) or two elements.

The natural observation here is that Ψ is surjective to the proper caterpillars, and we have $\Psi(\alpha) = \Psi(\beta)$ only when α is in the same reverse-class as β. Hence, Ψ reduces to a bijection from the reverse-classes of compositions to caterpillars. We call Φ to the inverse of Ψ. Latter, we will show that such bijection transforms U^L polynomials in $\mathscr{L}$ polynomials.

For now we will present the results that lead up to the classification of the compositions with the same $\mathscr{L}$-polynomial.

For that, we understand that the goal here is to identify, for each $\beta \in \mathscr{C}_n$, the class $[\beta]_{\mathscr{L}}$ of all compositions α such that $\mathscr{L}_\beta = \mathscr{L}_\alpha$, i.e. all α which are not distinguished from β by the $\mathscr{L}$-polynomial.

Equivalently, we are looking for the α's that are not distinguished by the multiset $M(\alpha) := \{\lambda(\alpha) | \gamma \leq \alpha\}$ which is a property of the poset of compositions.

☞ Definition: Multiplicative notation and operations

Let we just define the notion of "power" of a composition with the aid of the near-concatenation notion absorbed beforehand. So we consider $\alpha^{\odot k} = \underbrace{\alpha \odot \cdots \odot \alpha}_{k \text{ times}}$.

A third operation of interest on compositions is as follows: given $\alpha \in \mathscr{C}_n$ and $\beta \in \mathscr{C}_m$ with $m, n \geq 1$, we define $\alpha \circ \beta = \beta^{\odot \alpha_1} \cdot \beta^{\odot \alpha_2} \cdots \beta^{\odot \alpha_{l(\alpha)}}$.

This mapping, though being quite unexpected at first, behaves very naturally. Indeed, this operation on $\mathscr{C}$ is a monoid, meaning that the operation is associative and has an identity

$$\alpha = (1) \in \mathscr{C}_1 ,$$

as shown in (Proposition 3.3 L. Billera and Willgenburg, 2006). It is a simpler fact that this operation commutes with the reverse operation, i.e., $(\alpha \circ \beta)^* = \alpha^* \circ \beta^*$.

☞ Factorization, Irreducibles

We say that a composition α admits a *factorisation* if it can be written as $\alpha = \alpha_1 \circ \cdots \circ \alpha_k$ where all of the compositions α_i have size at least one.

If α does not have a factorization, we say that α is *irreducible*. Every composition admits a factorization into irreducibles, and such factorization is unique.

Theorem 2.4.4. Every composition α admits a unique factorization into irreducibles.

Proof. The proof can be found in (L. Billera and Willgenburg, 2006, Theorem 3.6). □

Theorem 2.4.5. If $\alpha = \alpha_1 \circ \cdots \circ \alpha_k$ is an irreducible factorization, then:

$$[\alpha]_{\mathscr{L}} = \{[\alpha_1]_* \circ \cdots \circ [\alpha_k]_*\} \tag{2.4}$$

I.e, $\beta \sim_{\mathscr{L}} \alpha$ whenever $\beta = \beta_1 \circ \cdots \circ \beta_k$ is an irreducible factorization and either $\beta_i = \alpha_i$ or $\beta_i = \alpha_i^*$ for all $1 \leq i \leq k$.

Proof. The proof can be found in (L. Billera and Willgenburg, 2006, Theorem 4.2). □

2.4.1 Relation between U^L and $\mathscr{L}$ polynomials and classification of caterpillars

This section has a taste of conclusion, since what we will show here unifies the previous two subsections.

We prove that a U^L-polynomial is in fact a $\mathscr{L}$-polynomial of the underlying composition, and then the classification of the sets $[\alpha]_{\mathscr{L}}$ is closely related to the classification of the caterpillars with the same U^L-polynomial.

Finally, by using the fact that we can obtain the U^L-polynomial from the chromatic symmetric function we conclude the uniqueness of chromatic symmetric function on proper caterpillars.

Theorem 2.4.6. If β is a composition of n, we have:

$$U^L_{\Psi(\beta)} = \mathscr{L}_\beta. \tag{2.5}$$

Proof. Fix β and write $T = \Psi(\beta)$. We want to show that, for a fixed composition, we have

$$\sum_{L(T)\subseteq A\subseteq E(T)} x_{\lambda(A)} = \sum_{\alpha\geq\beta} x_\alpha .$$

We will establish a one-to-one correspondence t between set compositions $\alpha \geq \beta$ and edge sets A such that $L(T) \subseteq A \subseteq E(T)$ in such a way that $\lambda(t(\alpha)) = \lambda(\alpha)$, so $x_{\lambda(t(\alpha))} = x_\alpha$, concluding the proof.

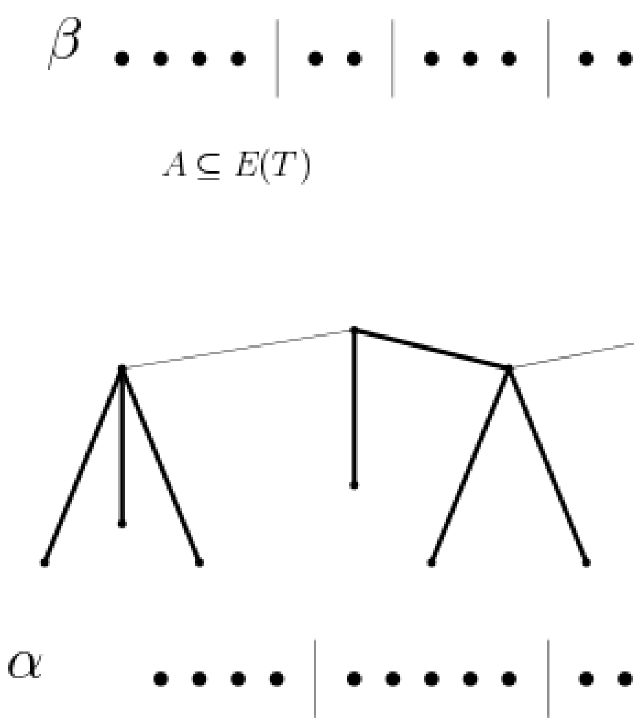


Figure 2.6: Relation between caterpillar $\Psi(\beta)$ and $\alpha \geq \beta$, with $t(\alpha) = A$ a boldface.

For that, label the vertices in the spine of T as $v_1 - v_2 - \cdots - v_k$ in the order given by $\beta = (\beta_1, \cdots, \beta_k)$. The fact that $\alpha \geq \beta$ implies that there are numbers $i_0 = 1, i_1, \ldots, i_j = k+1$ such that

$\alpha_s = \sum_{r=i_{s-1}}^{i_s-1} \beta_r$ for $1 \leq s \leq j$. Then we write $t(\alpha) = L_v \cup \{\{v_{i-1}, v_i\} | i \notin \{i_1, i_2, \ldots i_j\}, 2 \leq i \leq k\}$.

Note that, if $A = t(\alpha)$, the edges in $E(T) \setminus A$ are all edges in the spine and separate the cluster of vertices exactly between the i_j-th cluster and the next one, so $\lambda(A)_s = \sum_{r=i_{s-1}}^{i_s-1} \beta_r = \alpha_s$.

Then we conclude that $x_{\lambda(A)} = x_\alpha$ and it remains to show that t is indeed a bijection to conclude (2.5).

Indeed, we can invert t through the following construction: given a set of vertices $A \subseteq E(T)$ with $L(T) \subseteq A$, the connected components of the graph with edges A have a canonical order (given by the spine, which is ordered by β) so we can regard $\lambda(A)$ as a composition, preserving the order of the clusters and not in the usual non-increasing display. It is now obvious that $\lambda \circ t = id$ and $t \circ \lambda = id$. □

The following lemma serves to distinguish proper caterpillars that are not distinguished by the $\mathscr{L}$-polynomial / U^L-polynomial.

Lemma 2.4.7. Suppose that S and T are two proper caterpillars such that $\Phi(S) = [\alpha \circ \gamma]^*$ and $\Phi(T) = [\beta \circ \gamma]^*$, where α, β have the same size.

Then, if γ is not a palindrome and $\alpha \neq \beta$, we can distinguish S and T by their chromatic symmetric function. In fact, we can do it using the U-polynomial.

Sketch of proof. This is a highly technical Lemma that we want to postpone for another version of this work. The main idea is to evaluate the coefficient $x_{(1,\delta-1,n-\delta)}$, which in fact counts the number of leaves of some subtree, where δ is defined as follows:

Denote by $\alpha_{i,j}$ the truncated composition $(\alpha_i, \alpha_{i+1}, \ldots, \alpha_j)$ from α. Denote as well by $k(\sigma, \omega) = i$, for compositions $\omega \neq \sigma$, the smallest coordinate i such that $\sigma_i \neq \omega_i$.

Set $\gamma \models m$, $\alpha_{1,k(\alpha,\beta)} \models a$ and $\gamma_{1,k(\gamma,\gamma^*)} \models b$. Then $\delta := am + b$.

Such δ represents the first index in which $\alpha \circ \gamma$ and $\beta \circ \gamma$ differ and that simply reflects on the number of leaf-vertices before such index in the caterpillar, which can be counted with the coefficient on x_1 in the U polynomial.

The full proof can be found in (Aliste-Prieto and Zamora, 2014, Theorem 3.3). □

Theorem 2.4.8. Given two different proper caterpillars S and T, with $\alpha \in \Phi(S)$ and $\beta \in \Phi(T)$, then the U-polynomials of S and T are different.

Proof. Write α and β in their irreducible forms:

$$\alpha = \alpha_1 \circ \cdots \circ \alpha_k$$
$$\beta = \beta_1 \circ \cdots \circ \beta_j$$

By Corollary 2.4.3, we can use the $\mathscr{L}$-polynomials of α and β to distinguish the U-polynomial and, so, for the case that $[\alpha]_{\mathscr{L}} \neq [\beta]_{\mathscr{L}}$ there is nothing left to prove. Hence, from Theorem 2.4.5 we can assume that $\beta \in [\alpha]_{\mathscr{L}}$. This means that $k = j$ and $\alpha_i \sim \beta_i$ for every i.

Recall that the composition commutes with the reverse operation $*$, so if β is a palindrome we would have $\beta = \beta^* = \beta_1^* \circ \cdots \circ \beta_k^*$ which would mean, by uniqueness of factorization, that all β_i are palindromes and, necessarily, that $\alpha = \beta$. So we can assume henceforth that α and β are not palindromes and, so, there is some l such that α_l is not a palindrome, and α_i is a palindrome for all $i > l$. Consequently, $\alpha_i = \beta_i$ for any $i > l$.

With no loss of generality, assume that $\alpha_l = \beta_l$. We can force that to happen by setting $\alpha \to \alpha^*$, preserving the original caterpillar and its U-polynomial.

Call $\gamma = \alpha_l \circ \cdots \circ \alpha_k = \beta_l \circ \cdots \circ \beta_k$, which is not a palindrome since α_l is not a palindrome.

Now, since $\alpha \neq \beta$, we have $\alpha_1 \circ \cdots \circ \alpha_{l-1} \neq \beta_1 \circ \cdots \circ \beta_{l-1}$ and by Lemma 2.4.7 we conclude that the U polynomials of $T = \Psi(\alpha)$ and $S = \Psi(\beta)$ are different. □

This concludes the discussion on caterpillars, as we have established that proper caterpillars are either distinguished by their U^L polynomial (i.e., through the $\mathscr{L}$-polynomial of an underlying composition) or, if that doesn't happen, the stronger invariant U-polynomial distinguished them.

For a further work we ask if the same methods presented lead to the separation of the caterpillars with the chromatic symmetric function of graphs as invariant.

2.5 Reconstruction a tree

In this section we follow the work from Orellana and Scott (2014), as well as from Smith et al. (2015). The goal is to reconstruct a given tree knowing only limited information besides its chromatic symmetric function. We have, for each set of edges, access to the resulting sizes of the connected components after deletion of said edges, given by a function Θ.

In fact, we will describe an algorithm. The proof that the knowledge of Θ is enough to distinguish trees is highly constructive and the outline of the construction is the following: The function Θ will provide us with an ordering of the edges $e_1, e_2, \ldots$ via ordering the resulting number partitions $\Theta(\{e_1\}), \ldots$ in the dominance order. We will show that by introducing the edges in such order in the tree, we always maintain one connected component. Moreover, to find where to attach the new edge, we will introduce the notion of *attraction*, which can be defined by the function $\Theta(\{e_1, e_2\})$.

Afterwards we discuss some extremal examples that will show that we are using the least information possible, in the sense that accesses Θ just a few times times. These examples were constructed in Orellana and Scott (2014), as well as in Smith et al. (2015).

2.5.1 Labelled sets, ordering edges and an algorithm

Let T be a tree and $E(T) = \{e_1, \ldots, e_{n-1}\}$. We define a function $\Theta_T : 2^{E(T)} \to \mathscr{P}(n)$ as $\Theta_T(A) = \pi(E(T) \setminus A)$ to be the integer partition of the vertices given by the connected components after erasing the edges A from T.

The information given by the chromatic symmetric function is equivalent to the knowledge of the numbers $\{\theta_\lambda | \lambda \in \mathscr{P}\}$ where $\theta_\lambda = \#\{A \subseteq E(T) | \Theta(E(G) \setminus A) = \lambda\}$, according to the formula in Corollary 2.1.4.

In the end of the chapter we will prove the following theorem:

Theorem 2.5.1. The function Θ distinguishes trees. Besides, we can do such distinction by using only $O(n^2)$ many values of the Θ function.

Note that, for singletons $\{e\}$, since T is a tree, $l(\Theta(\{e\})) = 2$ so we get $\Theta(\{e\}) = (a, n-a)$ for some a.

We will consider that our edges are sorted such that $\Theta(\{e_1\}) \leq \Theta(\{e_2\}) \leq \ldots \Theta(\{e_{n-1}\})$ in the order given in the background, the dominance order - where we have that $(\frac{n}{2}, \frac{n}{2}) \leq \cdots \leq (n-2, 2) \leq (n-1, 1)$ - and denote T_i to be the graph spanned by the edges $\{e_1, \ldots, e_i\}$, ignoring isolated vertices. We should note that to obtain our goal, Theorem 2.5.1, we are allowed to evaluate Θ in singletons and sort the edges accordingly, to obtain the $O(n^2)$ bound.

We will need some preliminary definitions:

☞ Definition: Centroid

We call the weight of $v \in V(T)$, $w(v)$, the size of the biggest connected component of $G \setminus v$.

A vertex that minimizes the weight is called a *centroid*.

Proposition 2.5.2. There are at most two centroids in any tree.

Proof. To show that there cannot exist three centroids we first show that two given centroids must have an edge between them. Since a tree is acyclic, there are now thre centroids in a tree.

Suppose $n \geq 3$, since $n < 3$ is trivial. Take then two centroids v, x, and note that none of them should be leafs. There is a path between them $v - v_1 - v_2 - \cdots - v_j - x$ with j vertices between v and x. Our contradiction assumption will be that $j > 0$.

By erasing x, the graph will have $\deg x$ components, where we know that $\deg x \geq 2$ components, and we claim that the component containing v should be the only biggest one. Indeed, if there is some component C_i of $T \setminus x$ not containing v with $\#C_i = w(x)$, then the component of x in $T \setminus v$ contains $C_i \cup \{x\}$ and so $w(v) \geq \#C_i \cup \{x\} > w(x)$, contradicting the fact that v, x are both centroids.

By symmetry, the component containing x is the biggest one in $T \setminus v$.

Suppose by sake of contradiction that $j > 0$, i.e. if the number of vertices between v and x is non-zero, set C_1 as the biggest component of $T \setminus v_1$. Either $v \not\in C_1$ or $x \not\in C_1$ (or both).

If $v \not\in C_1$ then C_1 and v_1 and x are in the same component of $T \setminus v$, the biggest one, then $w(v) \geq 2 + \#C_1 > \#C_1 = w(v_1)$, contradicting that v is a centroid.

If, on the other side, $x \not\in C_1$ we get analogously that $w(x) > w(v_1)$, contradicting that x is a centroid.

So there is no vertex v_1 between w and v. □

From the information on $\Theta(\{e\})$ we can find whether T has one or two centroids: simply, if there is an edge $\Theta(\{e\}) = (\frac{n}{2}, \frac{n}{2})$ then T has two centroids, otherwise it has only one.

Example 2.5.3. We will stop here the explanation of our method and focus on an example, take for instance the tree in figure 2.7. Note that we have $\Theta(\{e_1\}) = \Theta(\{e_2\}) = (5,3)$ and $\Theta(\{e_i\}) = (7,1)$

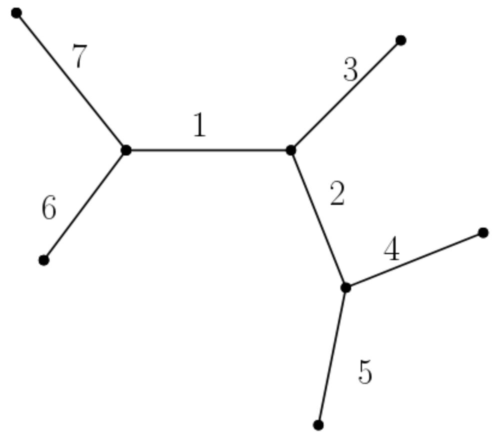


Figure 2.7: A tree with labelled edges according to dominance order of $\lambda(\{e\})$.

for the remaining edges, so the labels of the edges are already in the correct order. We also have that the vertex shared by edges e_1, e_2, e_3 is the unique centroid, with weight 3.

It is as well easy to observe that the trees T_i are always connected.

We can find this proposition in (Smith et al., 2015, Lemma 1.2):

Proposition 2.5.4. T_i is a tree for each i. In other words, the graph spanned by $\{e_1, \dots, e_i\}$ is connected.

Proof. To show Proposition 2.5.4, we will use induction, and as clearly T_1 is connected we only have to show that e_i has one endpoint in T_{i-1} for $i > 1$ to conclude the induction step. Besides, it is clear that one of the vertices of e_1 is a centroid (or both, if there are two of them).

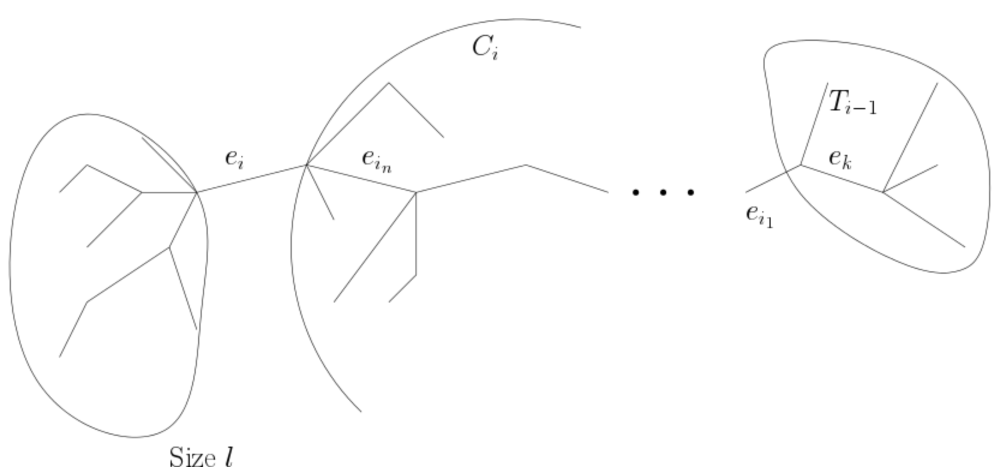


Figure 2.8: Diagram of the proof of Proposition 2.5.4

Suppose that there is a non-empty (unique) path $(e_i, e_{i_1}, \dots, e_{i_j}, e_k)$ between e_i and $e_k \in T_{i-1}$ with $k < i$ and $i_j > i$ for any j. This means that, if $\Theta(\{e_i\}) = (n-l, l)$ and $\Theta(\{e_k\}) = (n-m, m)$ with $n-l \geq l$ and $n-m \geq m$, from the ordering property we have $l \leq m$. But all $e_{i_1}, \dots, e_{i_k}$ and the tree T_{i-1} sit inside the same connected component C_i of $T \setminus e_i$, hence that connected component contains one of the connected components of $T \setminus e_k$ which has size at least $m+1$. Hence, C_i doesn't have size $l \leq m$ and so, has size $n-l$.

Hence, both sides of e_{i_1} contain, at least, $l+1$ vertices: one of the sides contains $C_i' = T \setminus C_i$ plus, at least, one vertex, whereas the other side contains one full block of the partition $\Theta(\{e_k\})$ plus, at least, one vertex.

So $\Theta(\{e_{i_1}\}) < \Theta(\{e_i\})$, contradicting the fact that $i_1 > i$. □

Picking up where we left of, the order of the labels gives us a way to attach sequentially the edges that preserves connectedness. It remains to show that the function Θ gives enough information on where e_i connects to T_{i-1}.

In fact, we have the following proposition, which we are going to prove after a required preliminary Lemma, that says exactly what information we need to go from T_{i-1} to T_i.:

Proposition 2.5.5. Let T be a tree and let be given the set of edges $E(T)$, as well as, for each centroid c, the set N_c of edges that are incident to c. Suppose that $i > 1$.

Given also the partitions $\Theta(\{e_p\})$, $\Theta(\{e_p, e_q\})$ and $\Theta(\{e_p, e_q, e_r\})$, as well as which edges in T_{i-1} share a vertex, we can find the edges $e_k \in T_{i-1}$ that share a vertex with e_i.

Note that if we know the neighbouring edges of the edge e_i in T_{i-1} then the tree is determined up to isomorphism by an induction argument, which is our goal, so the demonstration of such proposition leads immediately to the first part of Theorem 2.5.1, and in fact that theorem will be a corollary of the upcoming proof.

A notion will be of interest throughout the construction of our tree, which we introduce before we prove Proposition 2.5.5.

☞ Definition: Attraction of edges

The edge e_i is said to *attract* e_j if there is a path that begins in a centroid, contains e_j and ends in e_i. If none of the edges e_i and e_j attract each other, we say that those edges repel.

So, in the example 2.5.3, the edges e_6 and e_7 attract e_1 and the other ones repel e_1. If a tree has two centroids, then the edge that connects them is attracted by every other edge.

This definition allows us to claim the following Lemma, introduced in Smith et al. (2015):

Lemma 2.5.6. Let $t_1 \leq t_2$, and $\Theta(\{e_{t_1}\}) = (n - i_1, i_1)$, $\Theta(\{e_{t_2}\}) = (n - i_2, i_2)$. Suppose that $\frac{n}{2} \geq i_1 \geq i_2$, then we have several cases:

- It is impossible for e_{t_1} to attract e_{t_2}.
- If e_{t_1} attracts e_{t_2} we have $\Theta(\{e_{t_1}, e_{t_2}\}) = (n - i_1, i_1 - i_2, i_2)$
- If e_{t_1} and e_{t_2} repel then $\Theta(\{e_{t_1}, e_{t_2}\}) = (n - i_1 - i_2, i_1, i_2)$

Proof. This is shown in (Orellana and Scott, 2014, Prop. 5.2). □

Proof of proposition 2.5.5. So, now we must show two properties: first, that we can decide attractiveness only using the information provided, and second, to construct the tree using the property of attraction.

In fact, we can decide whether e_p attracts e_q from Lemma 2.5.6, by only consulting the values $\Theta(\{e_p, e_q\})$, $\Theta(\{e_p\})$ and $\Theta(\{e_q\})$.

Now, if we know exactly which edges in T_{i-1} are attracted from e_i, these form a path with $f(e_i) = f$ edges between e_i and a centroid, let $e_{j_1}, \ldots, e_{j_f}$ be the edges of such path.

There are some cases to consider, and note at this point that, by induction hypothesis, we know, in T_{i-1}, which edges connect to the same vertex. For each case, we will see which edges in T_{i-1} share an edge with e_i

- **There are at least two attracted edges, i.e. $f > 1$.**

  Then e_i connects to e_{j_f} and all edges that connect with e_{j_f} except $e_{j_{f-1}}$ and those that connect with $e_{j_{f-1}}$.

- **The tree has only one centroid, and $f = 1$ or $f = 0$.**

  If $f = 0$ then e_i is incident to the unique centroid, so all edges $e \in T_{i-1}$ that are incident to the centroid (so the set N_c) connect with e_i (and only those).

If $f = 1$, then e_{j_1} connects with e_i as well as all edges that connect with e_{j_1} but are not incident to the centroid should be connected to e_i, which we can compute by consulting which edges are adjacent to e_{j_1} in $E(T) \setminus N_c$.

- **T has two centroids and only one edge attracts e_i, and $f \leq 1$.**

  Then, such only edge should be the one connecting the centroids and our task resumes to find to which edges attach to the same centroid, and we can do it by consulting the sets N_{c_1} and N_{c_2}. □

Example 2.5.7. We will now illustrate the demonstration of Proposition 2.5.5 by constructing all the steps in some examples.

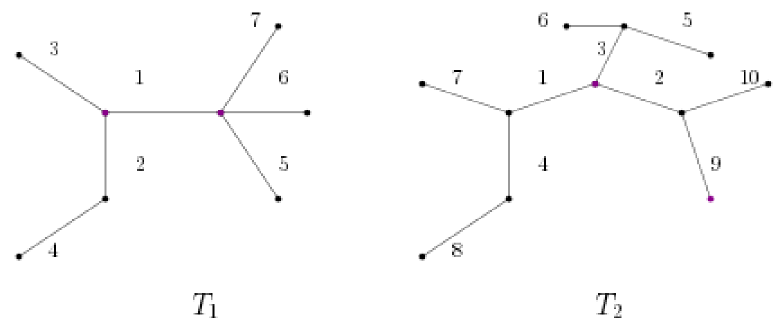


Figure 2.9: Labeled edges according to dominance order

In figure 2.9, there are two trees.

For T_1 we have two centroids and the sets $N_{c_1} = \{e_1, e_2, e_3\}$ and $N_{c_2} = \{e_1, e_5, e_6, e_7\}$.

The attaching sequence starts by connecting e_2, e_3 to some endpoint of e_1. Then we see that e_4 attracts edges e_1, e_2, by Lemma 2.5.6, since $\Theta(\{e_4\}) = (7,1)$, $\Theta(\{e_2\}) = (6,2)$ and $\Theta(\{e_4, e_2\}) = (6,1,1)$, for instance, which is of the form $(n - t_2, t_2 - t_1, t_1)$ rather than $(n - t_2 - t_1, t_2, t_1)$.

So $f(e_4) > 1$ and we attach e_4 to e_2 and all edges that connect with e_2 so far (e_1 and e_3) except e_1 and those that connect with e_1, so we attach e_4 only to e_2.

For the edges e_5, e_6, e_7 we observe that e_1 is the only edge that is attracted to them, so $f = 1$ and we consult the sets N_c to see to which centroid they attach.

Now, for the tree T_2, which has one centroid, we have $N_c = \{e_1, e_2, e_3\}$. So, for these three edges connect to the centroid.

For e_4 we see that $\Theta(\{e_4\}) = (9,2)$, $\Theta(\{e_2\}) = (8,3)$, $\Theta(\{e_1\}) = (7,4)$, $\Theta(\{e_1, e_4\}) = (7,2,2)$ and $\Theta(\{e_2, e_4\}) = (6,3,2)$. So e_4 attracts e_1 but not e_2. By checking for e_3 as well, we can see that e_1 is the only edge that is attracted to e_4, so e_4 connects to e_1 and all edges connected to e_1 except all those edges that connect to the centroid. So it only connects with e_1.

Now we have the following: e_3 is the only edge attracted to e_5 and to e_6, e_2 is the only edge attracted to e_9 and to e_{10}, e_8 attracts e_4 and e_1 and e_7 only attracts e_1.

So by applying the case $f = 1$ we get that the edge that connect to e_5 is only e_3, the edges that connect to e_6 are e_5 and e_3.

Again for $f = 1$, the edges that connect to e_7 are e_1 and e_4. By applying $f > 1$ to e_8, the edges that connect to e_8 are e_4 and no other.

The edge that connects to e_9 is only e_2 and the edges that connect to e_{10} are e_2 and e_9.

2.5.2 Conclusion

To conclude the main theorem 2.5.1, we only have to find a way to compute the sets N_c given the value of the function Θ on $O(n^2)$ sets.

Proof of Theorem 2.5.1. By applying Proposition 2.5.5 inductively we are able to construct the information $E_e^{(i)} = \{e' \in E(T_i) | e \cap e' \neq \emptyset\}$ iteratively, if we are provided with the information $N_c = \{e \in E(T) | c \in e\}$ for centroids c (we don't need, though, to distinguish between the centroids).

To obtain the latter, recall that in the proof of the Proposition 2.5.5 the number $f = f(e)$ represent the number of edges attracted by e, then note that, if T has one centroid, then $f = 0$ is equivalent to e_i connecting to the centroid.

Indeed because (for the direct implication) any path from the centroid to the edge e_i cannot go through any other edge so it must connect directly to the centroid, and (for the inverse implication) the path between e_i and a vertex in a tree is unique. If T has two centroids, then $f \leq 1$ is equivalent to "e_i connects to a centroid", because T/e_1 has only one centroid and we can apply the previous claim.

For the one centroid case, we can find all edges incident to the unique centroid by searching for the edges that don't attract any other edge. We can adapt the claim for the two centroid case: we can find $N_{c_1} \cup N_{c_2}$ by finding all edges e that satisfy $f(e) \leq 1$.

So it's clear that, if T has one centroid, $N_c = \{e \in E(T) | f(e) = 0\}$ whereas if T has two centroids $N_{c_1} \cup N_{c_2} = \{e \in E(T) | f(e) \leq 1\}$. We distinguish between N_{c_1} and N_{c_2} via $\Theta(\{e_1, e_2, e\})$ in the following way:

We know that e_1 connects both centroids, and we can suppose wlog that e_2 attaches to, say, the left side of e_1 and call the centroid on that side c_1, and c_2 to the remaining centroid. Then considering $\Theta(\{e_1, e_2, e\})$, if e is incident to c_1, then one of the parts of $\Theta(\{e_1, e_2, e\})$ is $\frac{n}{2}$, if e is incident to c_2 then all parts are smaller than $\frac{n}{2}$. □

Remark 2.5.8. The idea for two centroids is exactly where the paper Smith et al. (2015) improves the ideas from Orellana and Scott (2014) and reconstructs every tree in general.

In fact, we only use the information of the form $\Theta(\{e, f, g\})$ in these special cases, when the notion of attraction is not enough, specifically, where the new edge to add connects to a centroid in a tree T with two centroids and e, f are fixed. So we consult the values of Θ quadratically many times in total.

Example 2.5.9. Going back to the example 2.5.7, and to figure 2.9, we would like to compute the sets N_{c_1} and N_{c_2} in T_1 by only consulting the function Θ.

We start by identifying all edges that satisfy $f(e) \leq 1$ in T_1 which are $N_{c_1} \cup N_{c_2} = \{e_1, e_2, e_3, e_5, e_6, e_7\}$.

For those, we know that $N_{c_1} \cap N_{c_2} = e_1$ and we assume wlog that $e_2 \in N_{c_1}$. For the remaining edges $e \in N_{c_1} \cup N_{c_2}$ we decide whether $e \in N_{c_1}$ by checking if $\Theta(\{e_1, e_2, e\})$ has a part with size $\frac{n}{2} = 4$.

In fact, we have $\Theta(\{e_1, e_2, e_3\}) = (4, 2, 1, 1)$, whereas $\Theta(\{e_1, e_2, e_5\}) = \Theta(\{e_1, e_2, e_6\}) = \Theta(\{e_1, e_2, e_7\}) = (3, 2, 2, 1)$. So $N_{c_1} = \{e_1, e_2, e_3\}$ and $N_{c_2} = \{e_1, e_5, e_6, e_7\}$.

There are some cases of trees that have the same information of the Θ function for sets of fixed size.

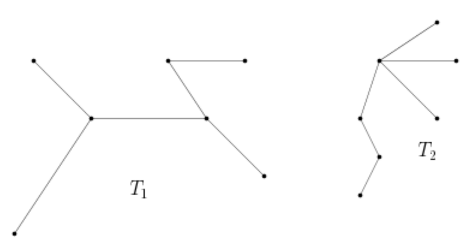


Figure 2.10: Two trees with the same values of $\Theta(\{e\})$

For instance, in Orellana and Scott (2014) it is given the example in Figure 2.10, where there are four edges such that $\Theta(\{e\}) = (6, 1)$, and in the remaining two edges we have $(5, 2)$ and $(4, 3)$.

Naturally, there is no pair of non-isomorphic trees with the same Θ for sets of size one, two and three, according to Theorem 2.5.1.

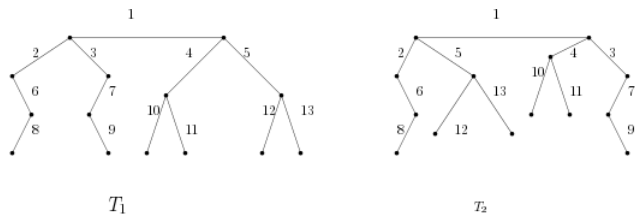


Figure 2.11: Two trees with the same values of $\Theta(\{e\})$ and $\Theta(\{e_p, e_q\})$

For an example of a tree T that requires the information regarding edge sets of size three, there is an other example from Orellana and Scott (2014), recovered here in Figure 2.11. In that paper, all values of $\Theta(\{e_p, e_q\})$ are compared and checked that, indeed, are equal.

On the other hand, $\Theta(\{e_1, e_2, e_3\}) = (7, 6, 1)$ in T_1 whereas $\Theta(\{e_1, e_2, e_3\}) = (4, 4, 3, 3)$ in T_2.

Note that both trees have two centroids. Our proof shows that we don't need to check for information of type $\Theta(\{e_p, e_q, e_r\})$ for the trees with only one centroid, so no such counterexample could arise.

Additional work can be considered here in this topic. For instance, it is of interest to find if there are examples of non-isomorphic trees T, R with $\Theta_T(A) = \Theta_R(A)$ for all sets of size $\#A = k$ fixed, where $2 < k < n$. This would generalize the discussion started in Orellana and Scott (2014) where they find two trees T, R with $\Theta_T(A) = \Theta_R(A)$ for all sets of size $\#A = 2$.

Chapter 3

e-positivity conjecture

In this chapter we introduce some advances towards Conjecture 0.1.8, starting by collecting a fundamental lemma that substitutes the notion of "deletion-contraction" presented in the Proposition 0.1.1.

We also present a proof of the reduction of the conjecture to the (2+2)-free posets through the modular relations, explaining the work done in Guay-Paquet (2013).

3.1 Modular law

In this section, we aim to find a substitution of the deletion-contraction law for a chromatic polynomial, mentioned in Proposition 0.1.1. Such substitution is not simple to erase vertices: an erasure of a vertex lowers the degree of the homogeneous chromatic symmetric function of the graph. Incidentally, since $G\backslash e$ has $n-1$ vertices, $\chi_{G\backslash e}$ is homogeneous of degree $n-1$, unlike χ_G and $\chi_{G\backslash e}$. So, it is not hard to believe that $\chi_G = \chi_{G\backslash e} - \chi_{G/e}$ does not hold in general.

Fortunately, we have a suitable theorem that resembles Proposition 0.1.1 in the chromatic symmetric function context, which writes as:

Theorem 3.1.1. Given that the edges $e_1, e_2.e_3$ form a triangle in the graph G, we have

$$\chi_G + \chi_{G/\{e_1,e_2\}} = \chi_{G/e_1} + \chi_{G/e_2}.$$

Proof. This is mainly a case analysis, so that in the end we will see that the coefficients on the

m-basis are the same, via Proposition 0.1.3. Namely, for a given partition λ we want to show that the number of stable set partitions of type λ in the different graphs have the same sum.

For a given graph H and a partition of its vertex set π, denote as X_H^π the indicator variable for "π is stable in H ", i.e. $X_H^\pi = 1$ if π is stable in H and $X_H^\pi = 0$ otherwise. Recall as well that, if $\lambda = \langle 1^{\lambda^{(1)}} 2^{\lambda^{(2)}} \dots \rangle$ then we write $\lambda! = \prod_k \lambda^{(k)}!$.

Then, from 0.1.3, we can write the different chromatic functions as:

$$\chi_G = \sum_{\lambda \vdash n} m_\lambda \lambda! \sum_{\pi \in \Pi_\lambda} X_G^\pi;$$

$$\chi_{G/e_1} = \sum_{\lambda \vdash n} m_\lambda \lambda! \sum_{\pi \in \Pi_\lambda} X_{G/e_1}^\pi;$$

$$\chi_{G/e_2} = \sum_{\lambda \vdash n} m_\lambda \lambda! \sum_{\pi \in \Pi_\lambda} X_{G/e_2}^\pi;$$

$$\chi_{G/\{e_1,e_2\}} = \sum_{\lambda \vdash n} m_\lambda \lambda! \sum_{\pi \in \Pi_\lambda} X_{G/\{e_1,e_2\}}^\pi.$$

Then, it is enough to prove that

$$X_G^\pi + X_{G/\{e_1,e_2\}}^\pi = X_{G/e_2}^\pi + X_{G/e_1}^\pi.$$

Consider that e_1 connects the vertices u, v, e_2 links the vertices v, w and e_3 links the vertices w, u.

There is a general rule that we want to apply: if π is stable in the graph H then it is also stable in a graph with less edges, namely $H \setminus e$. Naturally, the counter-reciprocal holds: if π is not stable in the graph $H \setminus e$ then it is also not stable in a graph with more edges, namely H.

So there are four cases:

Case 1: $X_{G/e_2}^\pi = 1$ and $X_{G/e_1}^\pi = 1$.

So the partition π is proper in G/e_2 and hence it is also proper in $G/\{e_2, e_1\}$.

It remains to establish that $X_G^\pi = 1$, indeed suppose that there is an edge e' in G, that connects vertices in the same block in π. By properness of π in both G/e_1 and G/e_2, we get that $e' \notin G/e_1$ and that $e' \notin G/e_2$, which is a contradiction with $e' \in G$.

So we conclude that $X_G^\pi + X_{G/\{e_1,e_2\}}^\pi = 2 = X_{G/e_2}^\pi + X_{G/e_1}^\pi$.

Case 2: $X_{G/e_2}^\pi = 1$ and $X_{G/e_1}^\pi = 0$, or the other way around.

With no loss of generality, let's assume that $X^{\pi}_{G/e_2} = 1$ and $X^{\pi}_{G/e_1} = 0$.

So the partition π is proper in G/e_2 and hence it is also proper in $G/\{e_2, e_1\}$. On the other hand, π is not proper in G/e_1 so it is also non-proper in G.

Then we conclude that

$$X^{\pi}_{G} + X^{\pi}_{G/\{e_1,e_2\}} = 0 + 1 = X^{\pi}_{G/e_2} + X^{\pi}_{G/e_1}.$$

Case 3: $X^{\pi}_{G/e_2} = 0$ and $X^{\pi}_{G/e_1} = 0$.

We have that π is not proper in G/e_1, so it is also non-proper in G.

It remains to establish that $X^{\pi}_{G/\{e_1,e_2\}} = 0$. Indeed, suppose that π is proper in $G/\{e_1, e_2\}$, but since π is not proper in G/e_1, then e_2 must connect two vertices in the same block of π, i.e. w and v are on the same block. The same holds for G/e_2 so v and u are on the same block, which means that the edge e_3 connects two vertices on the same block (v and w).

However, the edge e_3 is in $G/\{e_1, e_2\}$ so π is not proper in $G/\{e_1, e_2\}$, a contradiction with the fact that $X^{\pi}_{G/\{e_1,e_2\}} = 1$. □

3.2 Reduction to (2+2)-free posets

Several work has been done in Conjecture 0.1.8, for instance Proposition 0.1.9. We will follow the work from Guay-Paquet (2013) in this chapter. Our goal is to set the preliminaries for Theorem 0.1.10, which states that the chromatic symmetric function of any (3+1)-free poset incompatibility graph is a convex combination of certain chromatic symmetric function of (3+1 and 2+2)-free poset's incompatibility graphs.

We will also establish Proposition 0.1.9 from the methods here developed, following the lines on Guay-Paquet (2013).

We advise the reader to consult the definitions in chapter 1, namely Definition 0.1.4, where we have introduced $G(P)$, which is the indifference graph of the poset P.

3.2.1 Part-Listings

In this section we will present an encoding of (3+1)-free posets. Informally, the encoding works as follows: we add sequentially vertices to our poset at certain levels and, regarding the level at which we attach, we decide the order relation between the new point and the remaining ones. We can represent this encoding of posets by a finite list of positive integers, representing the levels of the introduced vertices. We call such lists *weakened part-listings*.

Let x be the newly introduced vertex and y another vertex already in P, then it is set:

- **(Rule 1)** $x < y$ whenever y is two levels above x.
- **(Rule 2)** $y < x$ if x is at least one level above y.

Example 3.2.1. Suppose we are given the list $(1, 3, 4, 2, 4, 3)$ as a weakened part-listing, then we can construct the relevant poset by adding successively vertices in the specified levels.

Start by adding z_1 to our poset, representing the vertex in the first position on the part-listing, which is on the first level. Then we add z_2 at the level three and, since z_1 is two levels below it, we set $z_2 > z_1$ according to Rule 2. Adding z_3 to the level four gives us that $z_1 < z_3$ and $z_2 < z_3$ since they all occur in lower levels. Adding z_4 in the level two amounts to $z_1 < z_4$ by rule 2 and $z_4 < z_3$ by rule 1. Note that z_4 and z_2 aren't comparable in this poset.

Figure 3.1: A weakened part-listing and it's poset represented as a Hasse diagram

Only few posets can be formed with these rules, so we allow ourselves to add bicoloured graphs, instead of an isolated vertex, at once. So a *part-listing* is a weakened part-listening empowered by this special capability of injecting a bicoloured graph in two levels. We represent said graphs as $(i, G, i+1)$ where $G = G_1 \uplus G_2$ is a bicoloured graph (with G_1 coloured **bottom** or **blue** and G_2 coloured **top** or **red**), and i and $i+1$ stands for the levels of the vertices of the bicoloured graph.

☞ Definition: Part-Listings

An ordered list **L** of parts, called *blocks*, of the type

$$\{n|n\in\mathbb{N}\} \text{ or } \{(i,G,i+1)|i\in\mathbb{N},\ G \text{ is a bicoloured graph }\},$$

where the graphs G are considered up to isomorphism of bicoloured graphs, is a *part-listing* and represents a poset P constructed as follows:

The vertices of the poset P are $\{j|L_j\in\mathbb{N}\}\uplus\biguplus_{(i,G,i+1)\in L}V(G)$, which are the positions of the numbers on the list plus the vertices of the graphs G for each part of type $(i,G,i+1)$. The vertices of the poset are partitioned into layers $P=\biguplus P_i$, the levels, so each vertex arising from a number i is said to be in the i-th layer P_i, for each part $(i,G,i+1)$, any blue vertex in G arises in P_i, whereas the red vertices are in P_{i+1}.

Given two elements x,y of P, whenever we say that x occurs *strictly before* y it means that the part in the list **L** from which x arises strictly precedes the part where y arises. In particular, they can't arise from the same bicoloured graph. If we say that it occurs *before* then it can be the case that they arise in the same bicoloured graph and, so, it is possible to draw the part-listing in a way that x occurs after y.

So, for distinct $x,y\in P$ we say that $x\leq y$ if either:

- x is at least two layers below y.
- x is one layer below y and occurs strictly before in the list.
- x and y occur in the same block $(i,G,i+1)$, x is in the i-th layer and y is in the $i+1$-th layer, and are connected in the graph G.

Throughout this essay we will deal with a part-listing **L** and its underlying poset interchangeably, so we can talk about a minimum of the poset **L**, for instance.

For technical reasons, we don't consider G as a bicoloured graph but rather as a class of isomorphic bicoloured graphs (so we don't distinguish among vertices with the same colour). Of course the underlying poset does not depend on the graph G itself, but rather on its bicoloured graph isomorphism type, so there is no harm on this assumption.

Normally, we can talk about the part-listing A,i,B which is the concatenation of the part-listings A, (i) and B. We might even write AB for a part-listing, whenever it is clear.

Figure 3.2: A part-listing and it's poset represented as a Hasse diagram

Example 3.2.2. The part-listing in Figure 3.2 is given as the list $(2,1,3,1,3,(2,G,3),1)$, where the graph G is a path with three vertices coloured "bottom-top-bottom".

The resulting poset, on the right, is (3+1)-free, and it should be natural to accept that any poset arising from a part-listing is (3+1)-free, as we are going to show in the next proposition.

Proposition 3.2.3. Any poset arising from a part-listing is (3+1)-free.

Sketch of Proof. The proof follows these ideas.

Suppose that we have three points $x < y < z$ that are all not comparable to another point w, for sake of contradiction.

- We have $x < y$ only if the layer of x is below the layer of y, then the layer of x, y and z occur in an increasing fashion.
- Since w is not comparable with both x and z, they must occur within one layer from w, so we are forced to assume that x occurs in the layer, say, $i-1$, whereas y, w occur in the same layer i and z in the $(i+1)$-th layer.
- Also, for rule 2 to apply, x, y and z have to occur in this order (in the broad sense discussed earlier).
- Besides we have $x \not< w \not< z$, then z should occur before w, and w should occur before x (in the broad sense discussed earlier) in order to rule 1 not to apply.
- Since w occurs between z and x to avoid comparability with those, it is either between z and y or between y and x (in this order).
- If (with no loss of generality) w occurs between z and y we conclude that, for z and y to be comparable, they should be part of a same block $(i, G, i+1)$. Then the same should happen to w.

- The vertex x does not occur in the block $(i, G, i+1)$, as $(i, G, i+1)$ can only span two layers, so x occurs strictly after y to avoid comparability with w, and so neither rule 1 nor 2 applies and x and y aren't comparable as well, which is a contradiction. □

The interesting claim here is that any (3+1)-free poset is indeed the poset of a part-listing. Naturally there are some repetitions in this encoding, in the sense that one poset may have many part-listings, and we will list some possibilities afterwards. The following theorem is from Guay-Paquet (2013).

Theorem 3.2.4 (Existence of part-listings)**.** Any (3+1)-free poset is given by a part-listing.

Proof. By (Theorem 3.3 Guay-Paquet et al., 2013), every (3+1)-free poset has a *compatible listing* which can be translated to a part-listing by (Proposition 2.5 Guay-Paquet, 2013). □

If we talk about only weakened part-listings (so, again, ignoring the bicoloured blocks), we preserve the (3+1)-freeness, and improve to a (3+1 and 2+2)-free poset

Proposition 3.2.5. The underlying poset P of a weakened part-listing L is (3+1 and 2+2)-free.

Proof. It is clear that P is (3+1)-free, being the underlying poset of a part-listing. Now suppose that there are four distinct elements $x < y$ and $w < z$ in P that are not comparable aside from the given inequalities, i.e. a (2+2) subposet in P, for sake of contradiction.

So, if l is the level function, we have that $l(x) \leq l(y) - 1$ and $l(w) \leq l(z) - 1$. By incomparability we have as well $l(y) \leq l(w) + 1$ and $l(z) \leq l(x) + 1$ so we get

$$l(x) \leq l(y) - 1 \leq l(w) \leq l(z) - 1 \leq l(x)\,.$$

So equality occurs across the previous equation and we get $l(y) = l(z) = l(x) - 1 = l(w) - 1$.

Note that without bicoloured blocks, we don't have to distinguish between "occurring before of strictly before in the part-listing". In this case, in order to have $x < y$ and $w < z$, so x has to occur before y, and w before than z. However, since y and w are incomparable, as well as z and x are incomparable, x has to occur after z and w has to occur after y.

Let p be the function that gives the position in which each vertex occurs in the part-listing L. Then $p(x) < p(y) < p(w) < p(z) < p(x)$, which is a contradiction. □

3.2.2 Operations on Part-Listings

We will establish latter that there are several operations on Part-Listings that preserve the underlying (3+1)-free poset and, incidentally, the chromatic symmetric function of the incomparability graph.

The next relation draws our attention to the fact that, although the posets might change, the underlying chromatic symmetric function of the incomparability graph are preserved nevertheless.

☞ Definition: $\sim_m$ and $\sim_P$ relations

Given several part-listings $L_1, L_2, \dots, L_k$, and $\chi_1, \dots, \chi_k$ the chromatic symmetric functions of the incomparability graphs of the underlying (3+1)-posets of the part-listings, we write

$$\alpha_1 L_1 + \alpha_2 L_2 \cdots \sim_m 0$$

whenever

$$\sum_i \alpha_i \chi_i = 0\,.$$

More formally, we define an equivalence relation on the free $\mathbb{R}$-space generated by the part-listings via equality of the chromatic symmetric functions of the incomparability graphs of the underlying (3+1)-free posets of the part-listings extended linearly.

We have as well the relation $\sim_P$ between part listings, which is the minimal linear relation that satisfies $L_1 \sim_P L_2$ whenever L_1 and L_2 are part-listings that represent the same poset. It is intuitive that the relation $\sim_P$ is stronger than $\sim_m$ If we denote by ker $\sim$ the subspace of points v that $v \sim 0$ then we note that $\ker \sim_P \subseteq \ker \sim_m$ and so $\sim_P$ is finer than $\sim_m$.

We will write A^+, for a part-listing A, to specify the same part listing where each vertex goes up one level (which yield the same (3+1)-free poset, i.e. $A \sim_P A^+$).

In the following paragraphs we will study the relation $\sim_P$ and some of its properties.

$A, i, j, B \sim_P A, j, i, B$ whenever $|i-j| \geq 2$, since two vertices at least two levels apart are always in relation to each other. By the same reasoning we have the following:

- $A, i, (j, G, j+1), B \sim_P A, (j, G, j+1), i, B$ whenever $i \geq j+3$ or $i \leq j-2$.

- $A,(i,G,i+1),(j,G,j+1),B \sim_P A,(j,G,j+1),(i,G,i+1),B$ whenever $|i-j| \geq 3$.

Example 3.2.6. We copy here figure 3.2 from a previous example 3.2.2.

Figure 3.3: A part-listing and it's poset represented as a Hasse diagram

In here, we can say that the second and the third block can be interchanged, because the levels differ by, at least, two levels, yielding the same (3+1)-free poset.

Additionally, we have as well the *circulation* property, i.e. $A^+B \sim_P BA$. Indeed, when comparing two vertices v_i, v_j in different blocks that occur in this order in the part-listing A^+B, with levels $l(v_i)+1, l(v_j)$, are comparable iff the vertices are comparable when occurred in inverse order with levels $l(v_i), l(v_j)$, respectively.

We introduce now *bipartization*. Whenever two consecutive blocks are within the two same levels, we can collapse both blocks into one bipartite graph and still yield the same (3+1)-free poset. Say that $ABCD$ is a part listing where blocks B and C all lie inside levels i and $i+1$, then clearly there is a bipartite graph G such that

$$ABCD \sim_P A(i,G,i+1)D$$

If a part-listing B spans over only two levels, call $G(B)$ to its bipartization, the bicoloured graph G such that $B \sim_P (1,G,2)$.

Example 3.2.7. Going back to the part-listing in figure 3.2, there is a bipartite graph G that collapses the 5-th block (vertex in level 3) and 6-th block (bipartite graph in levels 2 and 3).

The following theorem translates Theorem 3.1.1 to the realm of part-listings.

Theorem 3.2.8. Suppose that e_1, e_2 are two edges that share a vertex in the bicoloured graph G. Denote $G_1 = G \backslash e_1$, $G_2 = G \backslash e_2$ and $G_{1,2} = G \backslash \{e_1, e_2\}$. Denote as well the part listings $L = (i,G,i+1)$, $L_1 = (i,G_1,i+1)$, $L_2 = (i,G_2,i+1)$, $L_{1,2} = (i,G_{1,2},i+1)$.

Then we have the modular relation, using the notation $\sim_m$ introduced in Definition 3.2.2:

$$ALB + AL_{1,2}B \sim_m AL_1B + AL_2B\,.$$

Proof. By removing edges from the poset, we are adding edges to the indifference graph, so it is clear that the indifference graphs of $(A(i, G_{1,2}, i+1)B)$, $(A(i, G_2, i+1)B)$, $(A(i, G_1, i+1)B)$ and $(A(i, G, i+1)B)$ play a role of G, G_1, G_2 and $G_{1,2}$ in Theorem 3.1.1, respectively, concluding the proof. □

This theorem motivates a stronger relation on the usual $\mathbb{R}$-space generated by part-listings.

☞ Definition: $\sim_M$ relation

The relation $\sim_M$ is generated by a set of equations, in the sense that it is the minimal linear relation that satisfies the following:

Whenever we have part-listings of the form given in the theorem 3.2.8, namely $L = (A(i, G, i+1)B)$, $L_{1,2} = (A(i, G_{1,2}, i+1)B)$, $L_1 = (A(i, G_1, i+1)B)$ and $L_2 = (A(i, G_2, i+1)B)$ we have:

$$L + L_{1,2} \sim_M L_1 + L_2\,.$$

Note that this relation is finer than $\sim_m$, since it's clear that $\sum_i \alpha_i L_1 \sim_M 0 \Rightarrow \sum_i \alpha_i L_i \sim_m 0$, i.e. we have $\ker \sim_M \subseteq \ker \sim_m$. However, it is not always the case that $\sum_i \alpha_i L_1 \sim_M 0 \Rightarrow \sum_i \alpha_i L_i \sim_P 0$, and one should think of $\sim_M$ only as an equality in the chromatic symmetric function level.

3.2.3 A dual basis on the Part-Listings

Here we will devise some preliminary definitions and work with some examples of posets that will play a strong role in the main theorem of this chapter, Theorem 0.1.10.

We will define a probability functional F_k on bicoloured blocks of part listings. Afterwards, we will be able to prove that a (3+1)-free poset incomparability graph can be written as a linear combination of simpler graphs, through relations obtained in theorem 3.2.8, where the coefficients are given by the functionals F_k, hence the linear relations are, in fact, convex relations.

☞ Definition: V_r^s, *udu* and *dud* vectors, probability functionals

First let's note that the Definition 3.2.2 of relation $\sim_M$ can be easily extended to bicoloured graphs (regarding those bicoloured graphs as a part listing with only one block), i.e. $G + G_{1,2} \sim_M G_1 + G_2$ whenever $(1, G, 2) + (1, G_{1,2}, 2) \sim_M (1, G_1, 2) + (1, G_2, 2)$, using the notation from the theorem 3.2.8.

Consider now the $\mathbb{R}$-space V_r^s generated by all **bicoloured graphs** G with r vertices coloured "bottom" and s vertices coloured "top", modulo the linear extension of the relation $\sim_M$. Indeed there is no danger in identifying the relation $\sim_M$ in bicoloured graphs and in part-listings because a deletion of an edge in G translates to an addition of an edge in the incomparability graph.

In such vector space there are some relevant vectors, for instance, there is a bicoloured graph $D_k = G(L_d)$ where L_d is the following part-listing $(\underbrace{1,\dots,1}_{k \text{ times}}, \underbrace{2,\dots,2}_{s \text{ times}}, \underbrace{1,\dots,1}_{r-k \text{ times}})$ for $0 \le k \le r$. There is as well a bicoloured graph $U_k = G(L_u)$ where L_u is the following part-listing $(\underbrace{2,\dots,2}_{k \text{ times}}, \underbrace{1,\dots,1}_{s \text{ times}}, \underbrace{2,\dots,2}_{r-k \text{ times}})$ for $0 \le k \le r$.

We address to the vectors U_k and D_k as, respectively, udu and dud vectors, a mnemonic for up-down-up and down-up-down.

We define as well the *probabilistic functionals* F_k having the following value in the spanning set of V_r^s: given a bicoloured graph G, $F_k(1, G, 2) = F_k(G)$ is the probability of a uniformly random maximal matching of $K_{s,r}$ to have exactly k edges from G.

It remains to show that this definition makes sense, i.e. satisfy the relations in the space V_r^s that are given by $\sim_M$, which we will do shortly

Proposition 3.2.9. Let $G, G_1, G_2, G_{1,2}$ be given bicoloured graphs as in theorem 3.2.8 satisfying $G + G_{1,2} \sim_M G_1 + G_2$, then

$$F_k(G) + F_k(G_{1,2}) = F_k(G_1) + F_k(G_2).$$

Of course, F_k does only depend on the isomorphism type of bicoloured graphs.

As a consequence, the definition of the functionals F_k makes sense in the vector spaces V_r^s.

Proof. Consider the two edges e_1, e_2 as in Theorem 3.2.8 and the random matching M. We condition on the two events $e_1 \not\in M$ and $e_2 \not\in M$ that cover the whole probability space, since it is impossible that both e_1, e_2 are in M, because these edges share a vertex.

Conditioning to $e_1 \not\in M$ we get

$$\mathbb{P}[\#M \cap G_1 = k | e_1 \not\in M] = \mathbb{P}[\#M \cap G = k | e_1 \not\in M]$$

and

$$\mathbb{P}[\#M \cap G_{1,2} = k | e_1 \not\in M] = \mathbb{P}[\#M \cap G_2 = k | e_1 \not\in M]1,.$$

Then clearly

$$\mathbb{P}[\#M \cap G_1 = k | e_1 \not\in M] + \mathbb{P}[\#M \cap G_{1,2} = k | e_1 \not\in M] = \mathbb{P}[\#M \cap G = k | e_1 \not\in M] + \mathbb{P}[\#M \cap G_2 = k | e_1 \not\in M]\,.$$

In the same manner we get

$$\mathbb{P}[\#M \cap G_1 = k | e_2 \not\in M] + \mathbb{P}[\#M \cap G_{1,2} = k | e_2 \not\in M] = \mathbb{P}[\#M \cap G = k | e_2 \not\in M] + \mathbb{P}[\#M \cap G_2 = k | e_2 \not\in M]\,.$$

Finally, conditioning to $e_1, e_2 \not\in M$ yield in the same manner

$$\begin{aligned} \mathbb{P}[\#M \cap G_1 = k | e_1, e_2 \not\in M] &= \mathbb{P}[\#M \cap G_{1,2} = k | e_1, e_2 \not\in M] \\ &= \mathbb{P}[\#M \cap G = k | e_1, e_2 \not\in M] \\ &= \mathbb{P}[\#M \cap G_2 = k | e_1, e_2 \not\in M]\,. \end{aligned}$$

Now

$$\begin{aligned} F_k(G) &= \mathbb{P}[\#G \cap M = k] \\ &= \mathbb{P}[(\#G \cap M = k) \cap (e_1 \not\in M)] + \mathbb{P}[(\#G \cap M = k) \cap (e_2 \not\in M)] - \mathbb{P}[(\#G \cap M = k) \cap (e_1, e_2 \not\in M)] \\ &= \mathbb{P}[\#G \cap M = k | e_1 \not\in M]\mathbb{P}[e_1 \not\in M] + \mathbb{P}[\#G \cap M = k | e_2 \not\in M]\mathbb{P}[e_2 \not\in M] \\ &\qquad - \mathbb{P}[\#G \cap M = k | e_1, e_2 \not\in M]\mathbb{P}[e_1, e_2 \not\in M]\,. \end{aligned}$$

And similar equalities for G_1, G_2 and $G_{1,2}$.

Developing both sides of the equality, we get the desired equality. $\square$

Before we stroll on to the concluding theorem, we will discuss a bit some important features of the *udu* and *dud* vectors, for instance that they form a basis of the space V_r^s, and what we can deduce from the "local" description of bipartite graphs to extend to the full generality of part-listings. Also, we want to know why they are simple enough to work with afterwards.

The modular relation $\sim_M$ is quite general, in the sense that if we have in a local level $G \sim_M \sum_i c_i U_i$ then we have in the global level the same relation $A(j,G,j+1)B \sim_M \sum_i c_i A(j,U_i,j+1)B$.

So, if we can write all bipartite graphs G as convex combinations of the U_k vectors (or the D_k vectors) then any question regarding the chromatic symmetric function of a (3+1)-free poset reduces to a convex combination of chromatic symmetric function of posets arising from simpler part-listings. In fact, since the udu and dud vectors are given by bipartization of weakened part-listings, we now know that we are talking about convex combination of weakened part-listings. Said weakened part-listings turn out to give strong properties to their underlying posets (such as (2+2)-freeness).

In this sense it is important for us to make sure that the U_k vectors are minimal possible, in the sense that they are a minimal spanning set for the V_r^s space. For that the functionals are going to help us. In fact, the relation between the functionals and the *udu* and *dud* vectors is very tight:

Proposition 3.2.10. In the vector space V_r^s defined above, we have the following:

- If $r \geq s$ then $F_j(U_k) = \delta_{j,k}$.
- If $s \geq r$ then $F_j(D_k) = \delta_{j,k}$.

Proof. Let M be a uniformly taken maximal random matching.

For $r \geq s$, a maximal matching has s edges, so all s vertices from the top have an incident edge and we see exactly which vertices on top have an incident edge that occurs in both G and M. Note that a vertex on top is either connected to all vertices on the bottom or connected to none of them.

Namely, U_k has k vertices that have all possible incident edges and the remaining top vertices are isolated, so it's clear that any maximal matching has k edges in common with U_k, so $F_j(U_k) = \delta_{k,j}$.

The $s \geq r$ case is similar. □

Theorem 3.2.11. The probability functionals are a basis for the dual space of V_r^s.

Proof. Let M_k be a matching with k vertices, i.e. a set of k independent edges of the maximal bicoloured graph $K_{s,r}$. Note that such matchings with k edges are all isomorphic bicoloured graphs, so we can take any of these matchings with no loss of generality.

We claim that for any bicoloured graph G, where $G \in V_r^s$, we have $G \sim_M \sum_i c_i M_i$ for some coefficients c_i. Indeed, by contradiction assumption take the bicoloured graph G with fewest edges that doesn't write as $G \sim_M \sum_{c_i} M_i$ for c_i real numbers. If G is a matching, we are already done, so let e_1, e_2 be two edges in V_r^s that share a vertex in G.

By Theorem 3.2.8 we have $G \sim_M G_1 + G_2 - G_{1,2}$. and by minimality, all G_1, G_2 and $G_{1,2}$ are in the span of the matchings M_k in V_r^s, concluding a contradiction with the fact that G is minimal that doesn't write as $\sum_i c_i M_i$.

Now we know that V_r^s is spanned by $\{M_k\}_{k=0}^{\min\{r,s\}}$. Note that the matrix A with $A_{i,j} = F_i(M_j)$ is upper triangular. Indeed, for $j < i$ we clearly have $F_i(M_j) = \mathbb{P}[\#M_j \cap M = i] = 0$. Additionally, for the diagonal, we have $F_i(M_i) = \mathbb{P}[\#M \cap M_i = i] = \mathbb{P}[M_i \subseteq M] = \frac{|r-s|!}{\max\{r,s\}!} > 0$.

So A is non-singular, and $\{M_k\}_{k=0}^{\min\{r,s\}}$ is indeed a basis of V_r^s (being linearly independent). Consequently, the functionals are a basis of the dual of V_r^s. □

Corollary 3.2.12. If $r \geq s$, then the *udu* vectors form a dual basis of the probability functionals F_k, whereas if $r \leq s$ the vectors *dud* form a dual basis of the probability functionals F_k.

As a consequence, every vector $G \in V_r^s$ is a linear combination of the *udu* vectors or the *dud* vectors, accordingly, and the coefficients are exactly $F_k(G)$ which are non-negative and have sum one.

Proof. From 3.2.11, we ought just to do a simple computation to show the duality, which has been done in Proposition 3.2.10, so we are done. □

3.3 Conclusions

Here we establish the transformation of the local notion of $\sim_M$ to the global notion of part-listings, and show in full generality that a chromatic symmetric function of an incomparability graph of a (3+1)-free poset arises as a convex combination of chromatic symmetric function of incomparability graphs of the underlying posets of weakened part-listings.

Theorem 3.3.1. The chromatic symmetric function of an incomparability graphs of (3+1)-free poset is a convex combination of the chromatic symmetric function of incomparability graphs of (3+1 and 2+2)-free posets.

Proof. For any part-listing, we can reduce every bicoloured block to convex combination of weakened part-listings, from Corollary 3.2.12, preserving its chromatic symmetric function and erasing all the bicoloured blocks. So we wrte $L \sim_m \sum_i p_i L_i$ where each L_i has only bicoloured blocks of type U_k, so can be regarded, by bipartization, as weakened part-listings.

Then, the chromatic symmetric function of an incomparability graphs of (3+1)-free poset is a convex combination of the chromatic symmetric function of incomparability graphs of underlying posets of weakened part-listings. From Proposition 3.2.5 we have that all underlying posets of weakened part-listings are (3+1 and 2+2)-free. □

The methods used in the proof of Theorem 3.3.1 are actually quite strong, in the sense that we have established a simpler way to visualize part-listings. As a consequence, with the theory developed we are now in condition to provide a proof of Theorem 0.1.9, from Guay-Paquet (2013).

To obtain such, we will need to reduce every 3-free poset to some basic cases, that amount to the following Lemma.

Lemma 3.3.2. The chromatic symmetric function of the complete graph K_n in the e-basis is $\chi_{K_n} = n! e_n$.

Proof. It is clear that $\chi_{K_n} = n! m_{(1,\ldots,1)}$ by counting stable partitions and from Theorem 0.1.3. But $e_n = \sum_{i_1 < i_2 < \cdots < i_n} \prod_j x_{i_j} = m_{(1,\ldots,1)}$. □

Theorem 3.3.3. The incomparability graph of a 3-free poset has non-negative coefficients over the e-basis.

Proof. First we prove that, any 3-free poset P can be given as a part listing with only one block, which is a bicoloured graph, of the form $(1, G, 2)$. Define, for $y \in P$, the set $L_y := \{x \in P \setminus \{y\} | x \leq y\}$ and $G_1 = \{y \in P | L_y = \emptyset\}$.

Indeed, there are no two different comparable points on $G_2 = P \setminus G_1$ (and the same holds for G_1, which is trivial). Incidentally, if $x, y \in G_2$ are different such that $x \leq y$, let $z \in L_x \neq \emptyset$, then $z \leq x \leq y$, contradicting that P is 3-free.

So the vertices of P split into two sets, G_1 and G_2, which we can represent as a part-listing with only one block, a bicoloured graph $G = G_1 \uplus G_2$, so we conclude that P is given as a part-listing with only one block.

Write $E_{r_0}^{s_0}$ for the complete bicoloured graph U_{s_0} in $V_{r_0}^{s_0}$, which has the form "down-up" and corresponds to the graph with maximal number of edges in $V_{r_0}^{s_0}$. We will now prove that, by circulation, we can write G as a convex combination of vectors of the form $E_{r_0}^{s_0}$ under the equivalence $\sim_m$ (and not $\sim_M$, since circulation only holds in the poset level). With no loss of generality assume that $r \geq s$, and we will act inductively on s, with base case $s = 0$ being obvious (the only part-listing spanning V_r^0 is the empty one which is exactly E_r^0).

So, for the induction step, by Proposition 3.2.11, we can write

$$G \sim_M \sum_i c_i U_k$$

for non-negative coefficients $c_i = F_i(G)$ and $U_s = E_r^s$. For the remaining udu vectors, we can use circulation to see that

$$(1, U_k, 2) \sim_P (\underbrace{2,\ldots,2}_{k \text{ times}}, \underbrace{1,\ldots,1}_{r \text{ times}}, \underbrace{2,\ldots,2}_{s-k \text{ times}}) \sim_P (\underbrace{2,\ldots,2}_{k-1 \text{ times}}, \underbrace{1,\ldots,1}_{r \text{ times}}, \underbrace{2,\ldots,2}_{s-k \text{ times}}, 1) \in V_{r+1}^{s-1}.$$

We have that indeed by bipartization there is a graph G' such that

$$(1, G', 2) \sim_P (\underbrace{2,\ldots,2}_{k-1 \text{ times}}, \underbrace{1,\ldots,1}_{s \text{ times}}, \underbrace{2,\ldots,2}_{r-k \text{ times}}, 1)$$

and, by induction hypothesis, is given as a convex combination of the vectors of type $E_{r_0}^{s_0}$.

So we get that we can write $(1, G, 2)$ as a convex combination of vectors of the form $E_{r_0}^{s_0}$.

Now, the vectors $E_{r_0}^{s_0}$ are all of the form down-up and so, the incomparability graph is the disjoint union of two complete graphs, K_{r_0} and K_{s_0}. So, by Lemma 3.3.2, we have that the chromatic symmetric function of the incomparability graph of the poset $E_{r_0}^{s_0}$ is $r_0! e_{r_0} s_0! e_{s_0} = r_0! s_0! e_{(r_0, s_0)}$ and since we have $(1, G, 2) \sim_m \sum_i c_i' E_{r_0}^{s_0}$ for non-negative coefficients c_i', the theorem follows. □

Example 3.3.4. Consider the bipartite graph G in figure 3.4.

Note that $G \in V_4^3$. Let's write $U_k^{(s)}$ for the udu vectors in V_r^s and so we don't make confusion

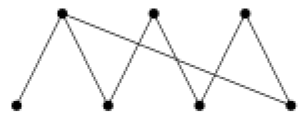


Figure 3.4: A bipartite graph G, with 3 vertices on top and 4 on bottom

between the udu vectors in different spaces throughout this example. Let's also identify any bipartite graph H with its underlying part-listings $(1,H,2)$.

First, we compute the probabilities $F_k(G)$ to determine its span in the V_4^3 space.

$$
\begin{aligned}
F_0(G) &= \frac{3}{4!/1!} = \frac{3}{24}\\
F_1(G) &= \frac{5}{4!/1!} = \frac{5}{24}\\
F_2(G) &= \frac{11}{4!/1!} = \frac{11}{24}\\
F_3(G) &= \frac{5}{4!/1!} = \frac{5}{24}.
\end{aligned}
$$

Then we get, according to 3.2.12, that

$$
\begin{aligned}
G &\sim_M F_0(G)\left(\quad\right) + F_1(G)\left(\quad\right) + F_2(G)\left(\quad\right) + F_3(G)\left(\quad\right)\\
&\sim_M \frac{3}{24}U_0^{(3)} + \frac{5}{24}U_1^{(3)} + \frac{11}{24}U_2^{(3)} + \frac{5}{24}U_3^{(3)}
\end{aligned}
$$

A first observation is that we have in fact $U_3^{(3)} = E_4^3$ so we don't want to change this factor. We have as well, by circulation, that

$$U_0^{(3)} = \left(\quad\right) \sim_P \left(\quad\right) = U_0^{(2)}.$$

In a similar manner we get that $U_0^{(2)} \sim_P U_0^{(1)} \sim_P U_0^{(0)} = E_7^0$ so we have so far

$$G \sim_m \frac{3}{24}E_7^0 + \frac{5}{24}U_1^{(3)} + \frac{11}{24}U_2^{(3)} + \frac{5}{24}E_4^3.$$

The proof of theorem 3.3.3 asks us, then, to circulate both $U_1^{(3)}$ and $U_2^{(3)}$. So we get:

$U_1^{(3)} = \Big($  $\Big) \sim_P \Big($  $\Big) =: H_1\,.$

$U_2^{(3)} = \Big($  $\Big) \sim_P \Big($  $\Big) =: H_2\,.$

So we have to compute the values of $F_k(H_1)$ and $F_k(H_2)$ to span H_1 and H_2 in the basis of V_5^2 from Corollary 3.2.12. Indeed, we have:

$$\begin{aligned} F_0(H_1) &= \frac{4}{5!/3!} = \frac{1}{5} \\ F_1(H_1) &= \frac{16}{5!/3!} = \frac{4}{5} \\ F_2(H_1) &= \frac{0}{5!/3!} = 0 \end{aligned}$$

$$\begin{aligned} F_0(H_2) &= \frac{0}{5!/3!} = 0 \\ F_1(H_2) &= \frac{8}{5!/3!} = \frac{3}{5} \\ F_2(H_2) &= \frac{12}{5!/3!} = \frac{2}{5}\,. \end{aligned}$$

So, if we recall that $U_0^{(2)} \sim_P E_7^0$ we get

$$H_1 \sim_M \frac{1}{5} U_0^{(2)} + \frac{4}{5} U_1^{(2)} \sim_P \frac{1}{5} E_7^0 + \frac{4}{5} U_1^{(2)}\,,$$

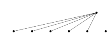
whereas

$$H_2 \sim_M \frac{3}{5} U_1^{(2)} + \frac{2}{5} U_2^{(2)}\,.$$

Adding up everything and noting that $U_2^{(2)} = E_5^2$ we get, so far, that

$$G \sim_m \frac{5}{24} E_4^3 + E_7^0 \left[\frac{3}{24} + \frac{5}{24}\frac{1}{5} \right] + U_1^{(2)} \left[\frac{5}{24}\frac{4}{5} + \frac{11}{24}\frac{2}{5} \right] + E_5^2 \frac{11}{24}\frac{3}{5} = \frac{5}{24} E_4^3 + E_5^2 \frac{33}{120} + E_7^0 \frac{4}{24} + U_1^{(2)} \frac{42}{120}$$

And we have now to write $U_1^{(2)}$ as a convex combination of vectors of the form $E_{r_0}^{s_0}$. We circulate to get

$U_1^{(2)} = \Big($  $\Big) \sim_P \Big($  $\Big) =: H_3\,.$

Again, we compute the coefficients $F_k(H_3)$ to obtain its convex span in the space V_6^1.

$$
\begin{aligned}
F_0(H_3) =& \frac{1}{6!/5!} = \frac{1}{6} \\
F_1(H_3) =& \frac{5}{6!/5!} = \frac{5}{6}\,.
\end{aligned}
$$

And we note that $U_0^{(1)} = E_7^0$ and $U_1^{(1)} = E_6^1$ to get

$$
H \sim_M \frac{1}{6} U_0^{(1)} + \frac{5}{6} U_1^{(1)} = \frac{1}{6} E_7^0 + \frac{5}{6} E_6^1\,.
$$

So our expansion of the graph G over the vectors $E_{r_0}^{s_0}$ becomes

$$
G \sim_m \frac{5}{24} E_4^3 + \frac{33}{120} E_5^2 + E_7^0 \frac{4}{24} + \frac{42}{120}\frac{5}{6} E_6^1 + \frac{42}{120}\frac{1}{6} E_7^0 = \frac{5}{24} E_4^3 + \frac{33}{120} E_5^2 + \frac{35}{120} E_6^1 + E_7^0 \frac{27}{120}\,.
$$

With this, we can compute the chromatic symmetric function of the underlying poset of $(1, G, 2)$ (which is, by chance, G^c) in the following way, by appealing to Lemma 3.3.2:

$$
\begin{aligned}
\chi_{G^c} =& \frac{5}{24} \chi_{E_4^3} + \frac{33}{120} \chi_{E_5^2} + \frac{35}{120} \chi_{E_6^1} + \frac{27}{120} \chi_{E_7^0} \\
=& \frac{5\;4!3!}{24} e_{(3,4)} + \frac{33\;2!5!}{120} e_{(2,5)} + \frac{35\;6!}{120} e_{(1,6)} + \frac{27\;7!}{120} e_{(7)}\,,
\end{aligned}
$$

which is clearly e-positive.

There are some open questions in the realm of this chapter. Note that the original Conjecture 0.1.8 reduces to the following: if we start with any weakened part-listing **L**, its underlying poset $P = P(L)$ and its incomparability graph $G = G(P)$, we should obtain that the chromatic symmetric function of the graph G is e-positive.

We must, though, clarify that there are several other graphs that are e-positive that are not enumerated by part-listings so it is also a further work to be done to classify all graphs that are e-positive.

Chapter 4

Summary and Recommendations for Further Work

We will, in this chapter, work a little bit over what we have done in this project, its main conclusions and further work in the topic.

4.1 Summary and Conclusions

In this work we have covered a lot of results in the literature regarding the chromatic symmetric function of graphs: we have written the proof of weaker forms of the main conjectures in the topic, Conjecture 0.1.4 and Conjecture 0.1.4, from Richard Stanley, in several families of trees; we have as well studied simple properties that one can extract from the chromatic symmetric function on trees, and also studied how the modular relation behaves in incomparability graphs of posets.

In particular some other algebraic invariants were studied, which closely relate to the chromatic symmetric function of trees, such as the subtree polynomial in Section 2.3.2. We always try to write the main demonstrations in a coherent way: it should be clear that the particular algebraic invariants to be used, for instance the Unique Bipartition of a Tree in the forks, in Corollary 2.2.5, though not strong enough to solve the problem, helps us to narrow the problem of distinction between types of trees to some simple cases. The chromatic symmetric function is stronger and can take care of the remaining simple cases. Some of the main results in this

Project were applications of this method, for instance Corollary 2.2.5 and Theorem 2.4.8.

We have met our objectives, which were to get acquaintance with the general methods to attack Conjecture 0.1.4.

We have as well talked about Part-Listings that encode posets and incomparability graphs of posets. We established some connections of the structure of part-listings to the underlying posets (for instance, in Proposition 3.2.3 and Proposition 3.2.5) and observed the results of the modular relation on part-listings. As a consequence, we were able to write the proof of the main theorem on Chapter 3, Theorem 3.3.1, regarding the reduction of Conjecture 0.1.8 to the (3+1 and 2+2)-free poset case. Another strong consequence is that the theory of Part-Listings is strong enough to show a simpler case of our main Conjecture 0.1.8, namely Theorem 3.3.3.

4.2 Findings and Goals

We now recover the objectives of this work.

1. To get a broad comprehension over a topic in Algebraic Combinatorics.
2. Be able to develop a small presentation over the search during the semester.
3. Get in touch with mathematical research questions and test my ideas on the topic.

This work, although somehow focused on chromatic symmetric function of graphs, gave the author the opportunity to get in touch with some other branches of mathematics, such as *Quasi-symmetric function* through consulting Luoto et al. (2013), some basic definitions and properties in combinatorics - for instance, the *centroid of a tree* in 2.5.1 - as well as P- partitions from Stembridge (1997) and some poset theory. The author has also developed a 15 minute presentation, in Graz University, regarding the topic.

There are works in this topic haven't been mentioned, for instance the work in Gebhard and Sagan (2001) where it is introduced a non-commutative version of the chromatic symmetric function of graphs, namely the variables don't commute and some parallels to the theory of ordinary chromatic symmetric function are drawn: from the basis coefficient formulas to a deletion-contraction relation, among other things.

For the research experience, in this project there was a lot of space to work with some examples and obtain some personal results. In that regard, we were able to devise a formula for the expected value of the chromatic symmetric function in the random graph setting, in Proposition 2.1.2, which paves the way for another type of further work regarding the study of the chromatic symmetric function in random models of graphs.

We have as well discussed the definition of Tailings, in 2.3.1, introduced by this work, obtaining a relation between the tailings and the coefficients of the chromatic symmetric function in the p-basis, in Proposition 2.3.11.

4.3 Recommendations for Further Work

As we have mentioned at the end of every section, every introduced theme has its own further work proposals. We use this chapter to gather all the proposals.

- It would be interesting, though not a big advance towards Conjecture 0.1.4, if we could distinguish between forks and non-fork trees with the chromatic symmetric function of graphs, or between caterpillars and non-caterpillars. In that case, for the initial conjecture 0.1.4, we would only need to focus on non-fork trees. Also the methods introduced by such study can be applicable to other families of trees.

- We have here introduced the notion of Tailings of a graph G of a certain type T, and computed some of its values. The connection between these tailings and the coefficients θ_λ was clarified in simple terms through Proposition 2.3.11 and also as a consequence we were able to show that the chromatic symmetric function of any tree lies within an algebraic curve, according to Corollary 2.3.12. As such, an improvement of the proof methods presented in Proposition 2.3.8 is useful and might present a way to tackle Conjecture 0.1.4.

- By evaluating which algebraic invariants of a graph are determined by the chromatic symmetric function of a graph, we were able to find in Martin et al. (2008) some algebraic invariants that encode the degree sequence. So the natural further problem is to find which algebraic invariants besides these are determined by the chromatic symmetric function of graphs, and what sort of properties do these invariants reveal.

- We observed that the caterpillars are indexed by the compositions, and fortunately from the close relation between algebraic invariants in the caterpillars and in the compositions, we were able to extract a description of the caterpillars with the same U_L-polynomials. A further idea is to find families of trees that are indexed with some combinatorial structures like compositions, and which algebraic properties are preserved between these two structures.

- Motivated from the counter-examples shown in the Section 2.5, namely from Figure 2.11, we are interested to ask for similar problems for higher cardinality. For instance, are there examples of non-isomorphic trees T, R with $\Theta_T(A) = \Theta_R(A)$ for all sets of size $\#A = k$ fixed, where $2 < k < n$?

- And, finally, of course, the natural question that arises from the work on Guay-Paquet (2013), is whether any weakened part-listing's underlying chromatic symmetric function is e-positive. Note that a positive answer to this problem would prove Conjecture 0.1.8 in its full generality.

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