Research Proposal - Algebraic combinatorics, patterns and polytopes

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1 Introduction

In algebraic combinatorics we wish to study the interaction of combinatorial objects like graphs and posets, through algebraic operations. At the heart of this effort lie Hopf monoids, objects in the interface of category theory and combinatorics and target of much attention, for instance in [AA17]. Intuitively, these algebraic operations reflect how combinatorial objects can be merged and split. Fundamental examples are symmetric functions and the shuffle Hopf algebra. In this research project we will focus on Hopf monoids and Hopf algebras that describe two topics with deep roots in combinatorics: geometric combinatorics and substructures of combinatorial objects.

In particular, our project will be split into two parts: **pattern functions** that count substructures and form Hopf algebras, and the investigation of the Hopf monoid of **generalised permutahedra of type B**.

1.1 Hopf algebras and Hopf monoids

Let us delve a bit into the details of Hopf algebras and Hopf monoids. A Hopf algebra is an algebra A over a field k endowed with a coproduct Δ that is compatible with the multiplication in A, and other maps s, ι, ϵ that satisfy some additional properties, usually depicted in a commutative diagram like in Fig. 1. The main topics and definitions, as well as applications to combinatorics, can be found in the following survey [GR14].

In this vein, we are interested in endowing combinatorial objects with compatible products and coproducts.

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "id \otimes s"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\iota"] & & H \\
& H \otimes H \arrow[rr, "s \otimes id"] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Figure 1: One of the compatibility axioms on Hopf algebras

In [AM10], an algebraic tool with the same flavour was introduced called a **Hopf monoid**. This is a category theory parallel of Hopf algebras detailed in [AA17], of which we stress the particular case of Hopf species. In fact, **combinatorial objects** are better described as species, so we chose to use this language in algebraic combinatorics. A **set species** a is a functor that maps finite sets X to sets $a[X]$ for each set X, and bijections between $a[X]$ and $a[Y]$, whenever $\#X = \#Y$. To the elements of some $h[X]$ we call a combinatorial object. We will use the monoidal structure under the Cauchy product, but we remark that there are other products of interest, like the substitution and Hadamard, see [AM10].

A **monoid structure** in this setting is a product map that elaborates on how the combinatorial objects are *merged* as follows: for X, Y disjoint sets:

$$\cdot_{X,Y} : a[X] \times a[Y] \ni (a_1, a_2) \mapsto a_1 \cdot a_2 \in a[X \sqcup Y] \,. \tag{1}$$

A **comonoid structure** in this setting is a coproduct rule that elaborates on how the combinatorial objects are *split*: for X, Y disjoint sets:

$$\Delta_{X,Y} : a[X \sqcup Y] \ni b \mapsto (b|_X, b\backslash_X) \in a[X] \otimes a[Y] \,. \tag{2}$$

A species is said to be a Hopf species if both operations are compatible in a precise sense that reminisces the compatibility on Hopf algebras. A simple example is the Hopf species of graphs, that is defined as $\mathtt{G}[A] := \{\text{graphs supported in the vertex set } A\}$.

The main feature relating Hopf algebras and Hopf species in combinatorics is that there are functors, called **Fock functors**, that map a given Hopf species to Hopf algebras. In fact, many of the Hopf algebras in vouge in combinatorics can be lifted in this way to Hopf species. This motivates us to work on the more combinatorially flavoured Hopf monoids instead of the more classical algebraic objects of Hopf algebras, while still enjoying most of the machinery built in the Hopf algebra world.

2 Pattern Hopf algebras in combinatorial objects

The study of substructures is widespread in combinatorics. In graphs, the notion of minors and subgraphs are the main examples, but other examples like permutation patterns are also studied, see for instance [Bón16]. With this in mind, the notion of **combinatorial presheaf** marries this study with the Hopf monoid world. We introduced this in [Pen19b]:

Definition 1 (Combinatorial presheaf). A combinatorial presheaf is a coutervariant functor from finite sets with injective maps to sets. Then h is given by the following information: for each X finite set, $h[X]$ is a set of h-objects in X; for each pair of sets X, Y with $\#X = \#Y$, we are given a bijection $h[X] \to h[Y]$; and restriction maps $h[X] \to h[Y]$ whenever $Y \subseteq X$.

It turns out that from a combinatorial presheaf h we can construct an algebra $\mathcal{A}(h)$, following the steps in [Var14]. The algebra $\mathcal{A}(h)$ is the linear span of the following functions $p_a : \biguplus_X h[X] \to \mathbb{Q}$ that are defined, for each a combinatorial object, as:

$$p_a : b \mapsto p_a(b) = \#\{\text{substructures of } a \text{ inside } b\} \,.$$

We can further enrich the combinatorial presheaf h with a **monoid structure**, which is a product, following the Cauchy rule, compatible with the presheaf structure. These objects are called **monoids in combinatorial presheaves**. In fact, in the vein of finding Hopf algebras supported on combinatorial objects, this turns out to be an interesting object in light of the following result:

Theorem 2. *The algebra $\mathcal{A}(h)$ endowed with a coproduct dual to the monoid structure is a Hopf algebra.*

This theorem was motivated in [Var14], where this was established for the permutation monoid in combinatorial presheaves, and proved in full generality in [Pen19b].

The first general goal of this project is to explore these Hopf algebras, find algebraic and coalgebraic generators, enumerate algebraic elements like the number of primitive elements, describe their coradical filtration, and study their character group.

2.1 Freeness of the algebra structure

A major focus of this project will be investigating the freeness of the pattern Hopf algebras. For some examples, this question has been answered positively. Previous works include the proof of Vargas on the permutation case in [Var14], and the marked permutations and marked graph case in [Pen19b].

Since the concept of monoids in combinatorial presheaves is a very general one, we can easily devote some attention to other examples In particular, we hope to find connections between the Hopf algebras indexed by set partitions and set compositions, like **WQSym** and **WSym**, and the relevant pattern algebras.

> **Problem 1.** *Is the pattern Hopf algebra on set compositions and set partitions free? Describe a family of algebraic generators in these cases. How about other combinatorial presheaves?*

It is a long term goal of the project to establish a generic result, for instance to describe an infinite family of pattern Hopf algebras that are free, or a broad sufficient conditions for freeness.

The methods used in [Var14] and in [Pen19b] have the same basis. We start by finding a **unique factorisation theorem** of our objects under a given product.

For instance, in the permutation case studied by Vargas, the $\oplus$ decomposition is used, in the marked permutation case the inflation product is used, in the graph case the disjoint union is used, and in the marked graph case the joint of marked graphs is used. Here we see that the choice of a product is very delicate, as it highly depends on the object at hand and need not be related with the underlying monoidal structure that gives us a Hopf algebra, as discussed in Theorem 2.

When the product is commutative, the freeness of the algebraic structure follows immediately, as discussed in the case of graphs and marked graphs in [Pen19b]. In the non-commutative case, Vargas came up with a beautiful application of the Lyndon word theory in [Var14]. Finally, some tools developed in the marked permutation case in [Pen19b] allow us some flexibility on the **factorisation theorem**, where uniqueness is no longer a requirement.

With these tools we plan to tackle Problem 1.

2.2 The character group of the underlying Hopf algebra

In a forthcoming work by Supina [Sup19], it is established that the Hopf algebra $\mathcal{O}$ on the so called orbit polytopes is free, and the coproduct of the free generators $\mathcal{F} = \{f_i | i \in I\}$ are well behaved, as they satisfy the following rule in the Hopf monoid world:

$$\Delta_{A,B} f_i = f_j \otimes f_k \text{ for some } f_j, f_k \in \mathcal{F}, \tag{3}$$

which allows for a description of the character group of $\mathcal{O}$ as a subspace of a power series ring.

The phenomena in (3) occurs also in the pattern Hopf algebra of marked graphs: that is, it is a free algebra with a well behaved coproduct, so the character group of the Hopf algebra of marked graphs is also a subspace of a power series ring. This opens a new way of exploring the character group of this particular pattern Hopf algebra.

> **Problem 2.** *Can we describe the character group of other known free pattern Hopf algebras? What can we say about a character group once we have a description as the multiplicative group of a power series ring?*

2.3 Monoidal categories

As it turns out, the product and coproduct rules in (1) and (2) depend on what is called in [AM10] the **monoidal structure** that we endow upon the category of species. Thus, we want to investigate other monoidal structures on presheaves, like the Hadamard, substitution and Heisenberg monoidal structures. The following problems arise from discussions with Vargas, who is also interested in exploring further questions related to these problems:

> **Problem 3.** *To what extent do these results depend on the Cauchy monoidal category on presheaves? What sort of Hopf algebras do we obtain when considering the substitution product? What happens with other monoidal structures like Hadamard product or the Heisenberg product?*

The motivation behind exploring the substitution product lies in the following known fact, which was an observation in [ML14]. We recall that an operad is a monoid with respect to the substitution product.

Proposition 3. *If h is an operad in the category of species, then its derivative h' is also a species, and it is naturally a monoid with respect to the Cauchy product.*

The relationship between the combinatorial presheaves of permutations/graphs, which have an operad structure, and the combinatorial presheaves of marked permutations/marked graphs, which have a monoid structure, are precisely this one. The following problem would generalise this relationship:

> **Problem 4.** *If h is an operad in the category of presheaves, is h' a monoid in combinatorial presheaves? What is the corresponding Hopf algebra in general?*

3 Behaviour on the limiting objects on permutations

This section will focus on a growing connection between **algebraic combinatorics** and **discrete probability theory**. The latter area dedicates some efforts into describing the structure of the limit of combinatorial objects, and the notion of limiting object pertains to the behaviour of a sequence of said objects growing in size. Specifically, limit of combinatorial objects satisfy propositions with the flavour of Theorem 4 We are interested in exploring the notion of patterns in these limiting objects.

Consider the example on permutations, where we obtain the *permuton*. This object was first studied in [HKM+13] and [HKMS11]. A permuton can be described as the limit of a sequence of permutations but also a probability measure in the square $[0,1] \times [0,1]$. As with usual permutations, we can also compute the proportion of occurrences of a given permutation π inside a permuton Ψ, so our favourite functions p_π from Section 2 defined in permutations extend to permutons. Further, this general definition is compatible for limits as follows:

Theorem 4. *If $\pi_n \to P$ is a sequence of permutations converging to the permuton P, and σ is another permutation, then*

$$p_\sigma(\pi_n) \to p_\sigma(P)\,.$$

In [GHK+17] a nice geometric result was established regarding the image of permutons through the pattern functions indexed by indecomposible permutations. Consider the following set:

$$\Phi_{\mathcal{F}} = \{(p_\pi(P))_{\pi \in \mathcal{F}} | P \text{ a permuton}\} \subseteq \mathbb{R}^{\#\mathcal{F}}\,.$$

In Fig. 2 one can see $\Phi_{\mathcal{F}}$ for the family of permutations $\mathcal{F} = \{12, 123\}$, as well as some permutons that are in the preimage of the respective points. With only two indecomposible permutations, we already obtain a complicated set, so we are interested in general geometric properties like non-empty interior.

In fact, in [GHK+17] it is established that, if we consider the family $\mathcal{I}_q$ of indecomposible permutations of size $\leq q$, the image $\Phi_{\mathcal{I}_q}$ of permutions has a non-empty interior. Therefore the image $\Phi_{\mathcal{I}_q}$ does not live inside any codimension one manifold, and it follows that there are no analytic relations in between the pattern functions of indecomposible permutations.

Together with Jacopo Borga, we are interested in finding a generalisation of this property to a wider set of permutations than the indecomposible permutations.

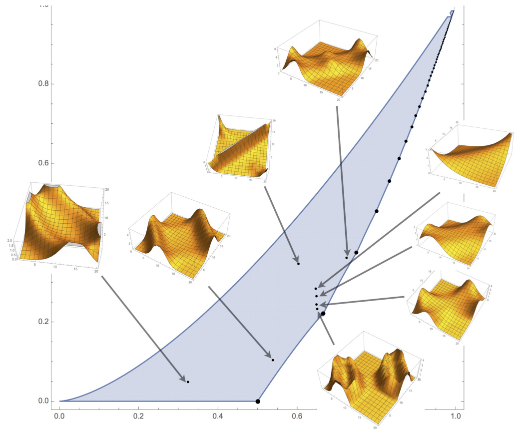

Figure 2: Image of permutons through permutations 123 and 12, from [KKRW15].

The motivation for these questions is the following: In [Var14] it was established that the *Lyndon permutations* form an algebraic basis of the pattern function Hopf algebra $\mathcal{A}(\texttt{Pat})$, so we know that there are no algebraic relations in between pattern functions of Lyndon permutations. It is natural, then, to wonder if no analytic relations exist, and one of the goals of this project is to establish that:

Problem 5. *Show that $\Phi_{\mathcal{L}_q}$ for the Lyndon permutations of size smaller than q, denoted as $\mathcal{L}_q$, has a non-empty interior.*

We are also interested in partial results, i.e. establishing that said image is dense when we consider some big family of Lyndon permutations.

In general, we want to describe the same result for several other pattern Hopf algebras. The corresponding problem in graphs has been already fully solved in [ELS79], where it was shown that the family of graphs' pattern functions map the *graphons*, the corresponding limiting objects of graphs, to a set with non-empty interior.

However, nothing in this direction is known for the marked graph case or the marked permutation case, where the free algebraic elements are presented in [Pen19b]. With this, a further application of the tools developed to tackle Problem 5 is the following:

Problem 6. *Establish the equivalent of Problem 5 for marked graphs and marked permutations.*

In parallel with this problem, we want to further explore what happens with consecutive patterns in the limit of sequences. This follows the same motivation as before, together with the fact that in a recent work in [Bor18], Borga described a new notion of local convergence on permutations, the so called **infinite rooted permutation**. In this case, we are interested in studying the so called consecutive pattern functions $\widetilde{c - occ}_\pi$. We know that for the indecomposible permutations $\mathcal{I}$, the image

$$\Phi^{cc}_{\mathcal{I}_q} := \{(\widetilde{c - occ}_\pi(LP))_{\pi \in \mathcal{I}_q} | LP \text{ is an infinite rooted permutation}\} \subseteq \mathbb{R}^{\#\mathcal{I}_q} ,$$

will be contained in a positive codimention submanifold. We wish to establish a manifold of minimal dimension that contains $\Phi^{cc}_{\mathcal{I}_q}$:

Problem 7. *Describe all the relations that arise when we consider the consecutive patterns of indecomposible permutations in sequences of permutations.*

4 Type B generalised permutahedra

We finally reach the part of the project that regards geometry in combinatorics. In particular, it regards generalised permutahedra, combinatorial objects that were introduced in [Pos09]. This is a family of polytopes that arise from a natural generalisation of associahedra and permutahedra. In fact, the generalised permutahedra embed not only associahedra, but also several other combinatorial objects, like graphs, matroids, set compositions, as is emphasised in [AA17].

Definition 5 (Generalised Permutahedra). The $(n-1)$-**permutahedron**, Per_{n-1} is the polytope given as $\mathrm{conv}\{(\pi(1), \dots, \pi(n)) | \pi \in S_n\} \subseteq \mathbb{R}^n$. A **generalised permutahedron** is a polytope which every face is parallel to some face of the generalised permutahedron.

Incidentally, Per_n results from the orbit of the point $(1, \dots, n)$ through the reflections of a Coxeter group of type A. If we change the role of said Coxeter group, we obtain the new notion of type B permutahedra, or $\mathbf{B}Per_n$, and type B generalised permutahedra, as introduced in [HLT11].

The problem of finding combinatorial objects embedded in the Hopf monoids of generalised permuitahedra of type B was tackled recently in [ACEP19]. There it is established that Coxeter-graphic polytopes, Coxeter matroids and Coxeter generalised associahedra are all generalised permutahedra of some type.

This project follows though a different path. Let us digress a bit about universal Hopf species. On [Pen18], a universal property on `WQSym` is discussed, where for each **combinatorial Hopf monoid** $(\mathtt{h}, \eta)$ it is shown that there is a unique combinatorial Hopf monoid morphism $\mathbf{\Psi} : h \Rightarrow \mathtt{WQSym}$.

This highly relates to chromatic problems in graphs. Incidentally, the Stanley's chromatic symmetric function is a particular case of a unique map from the Hopf algebra of graphs. On several combinatorial objects this map also has a combinatorial flavour, and a new universal property on the type B Hopf species entails a type-B chromatic function on graphs and other combinatorial objects.

With the directions of the faces of $\mathbf{B}Per_n$ in mind, we found a type-B equivalent of the *word quasisymmetric functions*, `BWQSym`, that not only generalises the Hopf species of word quasisymmetric functions `WQSym`, but also generalises the work of Chow on his PhD thesis in [Cho01], where a concept of quasisymmetric functions of type B is explored.

Our first task is to explore a **Hopf-like structure**, distinct from the Hopf species discussed above, on species that endows `BWQSym` with a universal property as it is the case with `BWQSym`. That is, we wish to describe a **type-B combinatorial Hopf species** structure.

> **Problem 8.** *Can we describe the category of type-B combinatorial Hopf species that has* `BWQSym` *as its the terminal object?*

Further, once this universal property has been established, and we have a notion of **type-B combinatorial Hopf species** $\mathtt{h}$, a map $\Psi^B_{\mathtt{BGPer}} : \mathtt{BGPer} \Rightarrow \mathtt{BWQSym}$ arises that gives us chromatic problems to analyse. A natural candidate for $\Psi^B_{\mathtt{BGPer}}$ have been already described in [Pen19a], where map satisfies the following commutative diagram:

$$\begin{tikzcd} \mathtt{GPer} \arrow[r, hook] \arrow[d, "\mathbf{\Psi}_{\mathbf{GPer}}"] & \mathtt{BGPer} \arrow[d, "\mathbf{\Psi}^B_{\mathbf{BGPer}}"] \\ \mathtt{WQSym} & \mathtt{BWQSym} \arrow[l, two heads] \end{tikzcd} \tag{4}$$

Our goal is to explore the map $\Psi^B_{\mathtt{BGPer}}$ as we did for its type A counterpart in [Pen18]. Particularly, generators of the kernel of the chromatic morphism are described for the so called **nestohedra**, which are a subfamily of generalised permutahedra. We note that these methods extend to the type-B nestohedra, as described in [Pen19a]. The following question is then of interest in the type B case:

> **Problem 9.** *What new elements can we find in the kernel of the map* $\Psi^B_{\mathtt{BGPer}}$*?*

A consequence of the **kernel problem** in type B is its applications in the kernel problem in type A, originally discussed in [Pen18]. Specifically, information on the kernel of the map $\mathbf{\Psi}^{B}_{\mathbf{BGPer}}$ gives us information about the kernel of the map $\mathbf{\Psi}_{\mathbf{GPer}}$, according to (4), which is the original unanswered question in [Pen18].

Back to the Hopf algebra world, the Hopf algebra of word quasisymmetric functions in non-commutative variables results from applying a Fock functor to the Hopf species `WQSym`. In the recent work from Yannic Vargas, duality between two bases in this algebra is established. This result reveals some properties of the bases involved, and we expect to be able to extend the methods to the type B case:

Problem 10. *Find dual basis in the algebra* **BWQSym**.

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