Probability 2

Exercise sheet nb. 12

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Due until: 10th December at 5 p.m.

Exercise 1 (3 points). Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{1,2\}$ with transition matrix

$$Q = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

1. Find the eigenvalues of Q and the corresponding left eigenvector. Does Q have a stationary probability distribution?
2. Does the Markov chain have a limit distribution? Compute for any $n \geq 0$,

$$P(X_n = 2 \,|\, X_0 = 1).$$

   What is its limit when n tends to infinity?

3. Compute $\lim_{n\to\infty} P(X_n = 2, X_{n+1} = 1)$ and $\lim_{n\to\infty} P(X_n = 1, X_{n+1} = 2)$

Exercise 2 (2 points). Let S be finite and Q be an irreducible transition matrix on Q. We assume that there exists S_1 and S_2 such that $S_1 \uplus S_2 = S$ and $Q_{x,y} = 0$ if both x and y are in S_1 or both x and y are in S_2 (in other words, the only possible transitions are from S_1 to S_2 and from S_2 to S_1.)

1. Show that the period of the chain is a multiple of 2;
2. Show that -1 is an eigenvalue of Q.

Exercise 3 (5 points). Let $(X_n)_{n\geq 0}$ be an irreducible non-null recurrent Markov chain on S with transition matrix Q and initial distribution δ_x for some x in S. We denote by μ the unique stationary probability distribution on S. Fix a real function f on S, either nonnegative or in $L^1(S,\mu)$. The goal of the exercise is to show that

$$\frac{1}{n}\sum_{i=0}^{n} f(X_i) \to \int f d\mu, \quad a.s.,$$

without aperiodicity assumption (and without using the limit theorem). In questions 1-4, we assume f nonnegative and bounded.

1. We define $T_0 = 0$ and for $k \geq 1$, we set $T_k = \inf\{n > T_{k-1} : X_n = x\}$ (these are the successive passage time in x). Let

$$Z_k = Z_k(f) = \sum_{i=T_k}^{T_{k+1}-1} f(X_i).$$

Show that the Z_k are independent and identically distributed. (Hint: use the strong Markov property.)

2. Show that

$$\mathbb{E}_x[Z_0(f)] = \frac{\int f d\mu}{\mu(x)}.$$

(Hint: recall that μ is proportional to the measure ν_x introduced in the lecture.)

Use the law of large number to conclude that, when k tends to infinity,

$$\frac{1}{k} \sum_{i=0}^{T_k-1} f(X_i) \to \frac{\int f d\mu}{\mu(x)}, \quad a.s. .$$

3. For $n \geq 0$, call N_n the biggest k such that $T_k \leq n$ (this is the number of visits of the chain at x before time n). Show that, when n tends to infinity,

$$\frac{1}{N_n} \sum_{i=0}^{n} f(X_i) \to \frac{\int f d\mu}{\mu(x)}, \quad a.s. .$$

4. Show that

$$\lim \frac{n}{N_n} = \frac{1}{\mu(x)}, \quad a.s. .$$

(Hint: applying the previous question to a suited function f.)
Deduce the convergence

$$\frac{1}{n} \sum_{i=0}^{n} f(X_i) \to \int f d\mu, \quad a.s. .$$

5. Extend this convergence result to all functions f, either nonnegative or in $L^1(S, \mu)$.