Homework Assignment 6

Hopf algebras - Spring Semester 2018

Exercise 1

Let H be a bialgebra.

a) Show that H^{op} is a bialgebra. (Recall that for any algebra A we let A^{op} denote the algebra with $A^{\mathrm{op}} := \{a^{\mathrm{op}} \mid a \in A\}$ and $a^{\mathrm{op}}b^{\mathrm{op}} = (ba)^{\mathrm{op}}$ for all $a^{\mathrm{op}}, b^{\mathrm{op}} \in A^{\mathrm{op}}$.)

b) Show that H^{cop} is a bialgebra. (Recall that for any coalgebra C we let C^{cop} denote the coalgebra with $C^{\mathrm{cop}} := \{x^{\mathrm{cop}} \mid x \in C\}$ and $\Delta_{C^{\mathrm{cop}}}(x^{\mathrm{cop}}) = x_2^{\mathrm{cop}} \otimes x_1^{\mathrm{cop}}$ for all $x^{\mathrm{cop}} \in C^{\mathrm{cop}}$.)

c) Show that if H is a Hopf algebra then so is H^{opcop}.

d) Show that if H is a Hopf algebra with a bijective antipode, then so are H^{op} and H^{cop}.

Exercise 2

Let H be a Hopf algebra and (A, δ) an H right comodule algebra. The elements of the subalgebra

$$A^{co\ H} = \{a \in A \mid a_0 \otimes a_1 = a \otimes 1\}$$

are termed H-coinvariant. If the map

$$\mathrm{can} : A \otimes_{A^{co\ H}} A \to A \otimes_{A^{co\ H}} H, \quad x \otimes y \mapsto xy_0 \otimes y_1$$

is bijective, we say $A^{co\ H} \subset A$ is an H Galois extension and A is H-Galois.

Now, let A be an H left module algebra. Recall that the smash product $A\#H$ is an H right comodule algebra via $\mathrm{id} \otimes \Delta$. Show that $A \subset A\#H$ is the subalgebra of H-coinvariant elements and that $A \subset A\#H$ is an H Galois extension.

Exercise 3

Let $k \subset L$ be a Galois extension with Galois group $G = \mathrm{Aut}_k(L)$. Clearly G operates on L, making L a $k[G] = (k^G)^*$ left module algebra and hence a k^G right comodule algebra.
Show that $k \subset L$ is a k^G Galois extension.

Exercise 4

Suppose that $\mathrm{char} k = p > 0$ and let $m, n \geq 1$, $\alpha, \beta \in k$. Show that

$$H = k < t \mid t^{p^{n+m}} = 0 >$$

is a commutative Hopf algebra with

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \alpha t^{p^n} \otimes t^{p^m} + \beta t^{p^m} \otimes t^{p^n}.$$

Describe the affine algebraic group $\mathrm{Sp}(H)$.