Groups Formulary

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Semigroups and groups

The simplest algebraic structure to recognize is a semigroup, which is defined as a nonempty set S with an associative binary operation.

Definition 1.1. *Let $(S, \cdot)$ be a semigroup. If there is an element e, in S such that*

$$ex = x = xe \qquad for\ \ all\ \ x \in S,$$

then e is called the identity of the semigroup $(S, \cdot)$.

Definition 1.2. *Let $(S, \cdot)$ be a semigroup with identity e. Let $a \in S$. If there exist an element b in S such that*

$$ab = e = ba$$

then b is called the inverse of a, and a is said to be invertible

Definition 1.3. *A nonempty set G with a binary operation $\cdot$ on G is called a group if the following axioms hold:*

$$\begin{aligned} (i)\quad & a(bc) = (ab)c\ for\ all\ a, b, c \in G.\\ (ii)\quad & There\ exist\ e \in G\ such\ that\ ea = a\\ & for\ all\ a \in G.\\ (iii)\quad & For\ every\ a \in G\\ & there\ exist\ a' \in G\ such\ that\ a'a = e \end{aligned}$$

Theorem 1.1. *A semigroup G is a group if and only if for all a, b in G, each of the equations $ax = b$ and $ya = b$ has a solution.*

Theorem 1.2. *A finite semigroup G is a group if and only if the cancelation laws hold for all elements in G; that is,*

$$ab = ac \Rightarrow b = c \quad and \quad ba = ca \Rightarrow b = c$$

for all $a, b, c \in G$

Homomorphism

Definition 1.4. *Let G, H be groups. A mapping*

$$\phi : G \to H$$

is called a homomorphism if for all $x, y \in G$

$$\phi(xy) = \phi(x)\phi(y)$$

Furthermore, if ϕ is bijective, then ϕ is called an isomorphism of G onto H, and we write $G \simeq H$. If ϕ is just injective, that is, $1-1$, then we say that ϕ is an isomorphism (or monomorphism) of G into H. if ϕ is surjective, that is, onto, then ϕ is called an epimorphism, A homomorphism of G into itself is called an endomorphism of G that is both $1-1$ and onto is called an automorphism of G.

If $\phi : G \to H$ is called an intro homomorphism, then H is called a homomorphic image of G; also, G is said to be homomorphic to H. If $\phi : G \to H$ is a $1-1$ homomorphism, then G is said to be embeddable in H, and we write $G \circlearrowleft H$.

Theorem 1.3. *Let G and H be groups with identities e and e', respectively, and let $\phi : G \to H$ be a homomorphism. Then*

$$\begin{aligned} (i)\quad & \phi(e) = e'\\ (ii)\quad & \phi(x^{-1}) = (\phi(x))^{-1}\ for\ each\ x \in G. \end{aligned}$$

Definition 1.5. *Let G and H be groups, and let $\phi : G \to H$ be a homomorphism. The kernel of ϕ is defined to be the set*

$$Ker\phi = \{x \in G | \phi(x) = e'\}$$

where e' is the identity in H

Theorem 1.4. *A homomorphism $\phi : G \to H$ is injective if and only if $Ker\phi = \{e\}$*

Subgroups and cosets