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High-Order and Model Reference Consensus Algorithms in Cooperative Control of MultiVehicle Systems

In this paper we study ℓth-order ($\ell \geqslant 3$) consensus algorithms, which generalize the existing first-order and second-order consensus algorithms in the literature. We show necessary and sufficient conditions under which each information variable and its higher-order derivatives converge to common values. We also present the idea of higher-order consensus with a leader and introduce the concept of an ℓth-order model-reference consensus problem, where each information variable and its high-order derivatives not only reach consensus, but also converge to the solution of a prescribed dynamic model. The effectiveness of these algorithms is demonstrated through simulations and a multi-vehicle cooperative control application, which mimics a flocking behavior in birds.
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1 Introduction

Cooperative control for multivehicle systems has been applied both to formation control problems, with applications to mobile robots, unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs), satellites, aircraft, spacecraft, and automated highway systems [1–7], and to nonformation cooperative control problems such as task assignment, payload transport, role assignment, air traffic control, timing, search, and adaptive scheduling [8–11]. For cooperative control strategies to be successful, numerous issues must be addressed, including the definition and management of shared information among a group of vehicles. The cooperation of a multivehicle team is often facilitated if the team members can form a consistent view of the shared information.

In plain language, when several entities or vehicles agree on a common value of a variable of interest, they are said to have come to *consensus*. For a group of networked mobile vehicles with a common mission or task, information consensus can play a pivotal role, particularly when the communication capability for each vehicle is limited and/or purposely constrained. For example, when the dynamic environment changes, the vehicles in a team must be in agreement as to what changes have taken place, even when every vehicle cannot talk directly to every other vehicle. To achieve consensus, there must be a shared variable of interest as well as appropriate algorithmic methods for negotiating to reach consensus on the value of that variable (called a consensus algorithm or protocol). Consensus algorithms have a historical perspective in Refs. [12–15], to name a few. Recently, consensus algorithms have been studied extensively in the field of cooperative control using algebraic graph theory [16–20] and nonlinear mathematical tools [21–23]. Some results in consensus algorithms can be understood in the context of connective stability [24]. Optimality issues in consensus algorithms are also considered in the literature [25]. In addition, information consensus is studied in the context of random networks [26], flocking [27,28], and asynchronous communication [29].

Notice that in the literature, most consensus algorithms focus on the case where the communicating/cooperating vehicles come to consensus about the value of the consensus variable (e.g., Refs. [17,18,21,20]). Although the consensus variable may be a vector, such algorithms are effectively first order, as the typical consensus algorithm adapts the first derivative of the consensus variable on each node based on the value of the consensus variable of its neighbors. The idea of a second-order consensus algorithm under directed information exchange has been suggested in Ref. [30], where each vehicle adapts the second-order derivative of its local consensus variable based on both zero-order and first-order derivatives of the consensus variables of their nearest neighbors. In the current paper, we generalize the first-order and second-order consensus algorithms in the literature. Expanding on our earlier work reported in Ref. [31], we show *necessary* and *sufficient* conditions under which the consensus variable and its higher-order derivatives converge to common values. We also extend these ideas to include setpoint tracking in higher-order derivatives of the consensus variable (higher-order consensus with a leader) and we consider an ℓth-order model-reference consensus problem, where each information variable and its high-order derivatives not only reach consensus, but also converge to the solution of a prescribed dynamic model.

When considering the problem of consensus among a group of cooperating entities, a natural question is "*consensus on what?*" A similar question "*formation to what form or shape?*" has been asked in a general way [32] within the context of mobile actuator and sensor networks [33]. However, in most work on information consensus, the answer is application dependent. Likewise, in our case, with the question "*consensus on what*?" in mind, our motivation for studying higher-order consensus comes from observing the behavior of flocks of birds. It is often noted that such flocks fly somewhat in formation, maintaining a nominal separation from each other, but each traveling with the same velocity vector. In Ref. [34], it is shown how second-order consensus algorithms can produce the behavior of a separation and common velocity under directed information exchange. However, sometimes a bird flock abruptly changes direction, perhaps when one of them suddenly perceives a source of danger or food. Clearly, the birds in this setting need to build consensus on not only their relative position and their velocity, but also on acceleration. This motivates the idea of higher-order consensus. Higher-order consensus also makes sense in swarm-on-swarm scenarios, where a cooperative team of "friendly" UAVs confront another team of "hostile"

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UAVs, necessitating abrupt collective motions of the team. Finally, again driven by the question "*consensus on what*?" we are also motivated to consider what we call the ℓth-order model-reference consensus problem, where each information variable and its high-order derivatives not only reach consensus, but also converge to the solution of a prescribed dynamic model. Convergence not to something arbitrary as defined by initial conditions, but to the solution of a prescribed problem, as defined by the reference model, is important in applications where the team of vehicles has a global task to achieve, but does not have centralized communication. We introduce this model-reference consensus problem and establish sufficient conditions for consensus convergence for the case where all or some of the team members have complete knowledge of the reference model.

The remainder of the paper is organized as follows. Section 2 presents some background materials and some mathematical preliminaries for our later development. Higher-order consensus algorithms are given in Sec. 3, including the standard, unforced case and the case of setpoint tracking, which leads to the idea of higher-order consensus with a leader, and model-reference consensus. The effectiveness of the proposed algorithms is illustrated throughout the paper by simulations, including an example of flocking behavior in Sec. 4. Section 5 concludes the paper.

2 Background and Preliminaries

It is natural to model information exchange among vehicles by directed/undirected graphs. A digraph (directed graph) consists of a pair $(\mathcal{N},\mathcal{E})$, where $\mathcal{N}$ is a finite nonempty set of nodes and $\mathcal{E} \in \mathcal{N}^2$ is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. If there is a directed edge from node v_i to node v_j, then v_i is defined as the parent node and v_j is defined as the child node. A directed path is a sequence of ordered edges of the form $(v_{i_1},v_{i_2}),(v_{i_2},v_{i_3}),\ldots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined accordingly. In a digraph, a cycle is a path that starts and ends at the same node. A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is called connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node has exactly one parent except for one node, called a root, which has no parent, and the root has a directed path to every other node. Note that in a directed tree, each edge has a natural orientation away from the root, and no cycle exists. In the case of undirected graphs, a tree is a graph in which every pair of nodes is connected by exactly one path. A directed spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. A graph has (or contains) a directed spanning tree if there exists a directed spanning tree being a subset of the graph. Note that the condition that a digraph has a directed spanning tree is equivalent to the case that there exists at least one node having a directed path to all the other nodes. In the case of undirected graphs, having an undirected spanning tree is equivalent to being connected. However, in the case of digraphs, having a directed spanning tree is a weaker condition than being strongly connected.

The adjacency matrix $A=[a_{ij}] \in \mathbb{R}^{n\times n}$ of a weighted digraph is defined as $a_{ii}=0$ and $a_{ij}>0$ if $(j,i) \in \mathcal{E}$ where $i \neq j$. The adjacency matrix of a weighted undirected graph is defined accordingly except that $a_{ij}=a_{ji}$, $\forall i \neq j$, since $(j,i) \in \mathcal{E}$ implies $(i,j) \in \mathcal{E}$. Let matrix $L=[\ell_{ij}]$ be defined as $\ell_{ii}=\Sigma_{j\neq i}a_{ij}$ and $\ell_{ij}=-a_{ij}$, where $i \neq j$. The matrix L satisfies the following conditions:

$$\ell_{ij} \leqslant 0, \quad i \neq j$$

$$\sum_{j=1}^{n} \ell_{ij} = 0, \quad i=1,\ldots,n \tag{1}$$

For an undirected graph, L is called the Laplacian matrix [35], which has the property that it is symmetric positive semidefinite. However, L for a digraph does not have this property. In both cases of undirected graphs and digraphs, 0 is an eigenvalue of L with an associated eigenvector $\mathbf{1}$, where $\mathbf{1} \triangleq [1,\ldots,1]^T$ is an $n \times 1$ column vector of all ones, since all of the row sums of L are 0. In the case of undirected graphs, all of the nonzero eigenvalues of L are positive. In the case of digraphs, all of the nonzero eigenvalues of L have positive real parts following Gershgorin's disk theorem [36]. In the case of undirected graphs, 0 is a simple eigenvalue of L if and only if the undirected graph is connected [37]. In the case of digraphs, 0 is a simple eigenvalue of L if and only if the digraph has a directed spanning tree [38].

Let $\mathbf{0}$ denote the $n \times 1$ column vector of all zeros. Let I_n denote the $n \times n$ identity matrix and 0_n denote the $n \times n$ zero matrix. Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $A=[a_{ij}] \in M_n(\mathbb{R})$, the digraph of A, denoted by $\Gamma(A)$, is the digraph on n nodes v_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from v_j to v_i if and only if $a_{ij} \neq 0$ [36].

3 Higher-Order Consensus Algorithms

We begin by presenting the general ℓth-order extension to the standard consensus algorithm, followed by two extensions: (1) Setpoint tracking and higher-order consensus with a leader, and (2) model-reference consensus.

3.1 ℓth-Order Consensus. Consider information variables with ℓth-order dynamics given by

$$\begin{aligned} \dot{\xi}_i^{(0)} &= \xi_i^{(1)} \\ &\vdots \\ \dot{\xi}_i^{(\ell-2)} &= \xi_i^{(\ell-1)} \\ \dot{\xi}_i^{(\ell-1)} &= u_i \quad i \in \{1,\ldots,n\} \end{aligned} \tag{2}$$

$\dot{x}_1 = x_2$
$\dot{x}_2 = x_3$
$\vdots$
$\dot{x}_1 = u$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{bmatrix}$$

$$u \to \boxed{\tfrac{1}{s}} \to \boxed{\tfrac{1}{s}} \cdots \boxed{\tfrac{1}{s}} \to x_\ell$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

Si $u = -k_1 x_1 - k_2 x_2 - \cdots - k_\ell x_\ell$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -k_1 & -k_2 & -k_3 & \cdots & -k_n \end{bmatrix} \mathbf{x} \quad \circledast$$

where $\xi_i^{(k)} \in \mathbb{R}^m$, $k=0,1,\ldots,\ell-1$, are the states, $u_i \in \mathbb{R}^m$ is the control input, and $\xi_i^{(k)}$ denotes the kth derivative of ξ_i with $\xi_i^{(0)} = \xi_i$, $i=1,\ldots,n$. Consensus is said to be reached among the n vehicles if $\xi_i^{(k)} \to \xi_j^{(k)}$, $k=0,1,\ldots,\ell-1$, $\forall i \neq j$. The goal of a consensus algorithm is to derive a control law u_i such that consensus is reached among the vehicles.

In the case of $\ell=1$, a consensus algorithm is proposed in Refs. [16–19,38] as

$$u_i = -\sum_{j=1}^{n} g_{ij}k_{ij}(\xi_i - \xi_j) \quad i \in \{1,\ldots,n\} \tag{3}$$

where $k_{ij}>0$, $g_{ii} \triangleq 0$, and g_{ij} is 1 if information flows from vehicle j to vehicle i and 0 otherwise. The motivation behind (3) is to drive the information variable of each vehicle toward the information variables of its neighbors.

Motivated by (3), we propose the following higher-order consensus algorithm:

$$u_i = -\sum_{j=1}^{n} g_{ij}k_{ij}\left[\sum_{k=0}^{\ell-1} \gamma_k(\xi_i^{(k)} - \xi_j^{(k)})\right], \quad i \in \{1,\ldots,n\} \tag{4}$$

where $k_{ij}>0$, $\gamma_k>0$, and g_{ij} is defined in Eq. (3). The motivation behind (4) is to drive each vehicle's information variable and its high-order derivatives toward the states of its neighbors. Note that

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the linear consensus strategies reported in the literature can be considered special cases of (4) when $l=1$ or $l=2$.

Let $\xi^{(k)}$ be an $mn\times 1$ column vector with components $\xi_i^{(k)}$, $i=1,\ldots,n$, where $\xi_i^{(k)}$ is defined in Eq. (2). By applying (4), Eq. (2) can be written in matrix form as

$$\begin{bmatrix}\dot{\xi}^{(0)}\\ \dot{\xi}^{(1)}\\ \vdots\\ \dot{\xi}^{(\ell-1)}\end{bmatrix}=(\Gamma\otimes I_m)\begin{bmatrix}\xi^{(0)}\\ \xi^{(1)}\\ \vdots\\ \xi^{(\ell-1)}\end{bmatrix} \tag{5}$$

$$\xi^{(0)} \to \begin{bmatrix}\xi_1^{(0)}\\ \xi_2^{(0)}\\ \vdots\\ \xi_n^{(0)}\end{bmatrix}\in\mathbb{R}^n$$

where $\otimes$ denotes the Kronecker product, and

$$\Gamma=\begin{bmatrix}0_n & I_n & 0_n & \cdots & 0_n\\ 0_n & 0_n & I_n & \cdots & 0_n\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0_n & 0_n & 0_n & \cdots & I_n\\ -\gamma_0 L & -\gamma_1 L & -\gamma_2 L & \cdots & -\gamma_{\ell-1}L\end{bmatrix}_{\ell n\times \ell n}$$

where $L=[\ell_{ij}]\in M_n(\mathbb{R})$ with $\ell_{ii}=\Sigma_{j\neq i}g_{ij}k_{ij}$ and $\ell_{ij}=-g_{ij}k_{ij}$, $\forall i\neq j$. Note that L satisfies the property (1).

In the following, we assume $m=1$ for simplicity. However, all the results hereafter remain valid for $m>1$. In addition, we only consider the case when $\ell=3$. Similar analyses are applicable to the case when $\ell>3$.

Before stating our main results, we need the following lemma.

LEMMA 3.1. *In the case of $\ell=3$, Γ has at least three zero eigenvalues. It has exactly three zero eigenvalues if and only if $-L$ has a simple zero eigenvalue. Moreover, if $-L$ has a simple zero eigenvalue, the zero eigenvalue of Γ has geometrical multiplicity equal to one.*

Proof. Let λ be an eigenvalue of Γ and $q=[p^T,r^T,s^T]^T$ be its associated eigenvector, where p, r, and s are $n\times 1$ column vectors. Note that

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$$\Gamma q=\begin{bmatrix}0_n & I_n & 0_n\\ 0_n & 0_n & I_n\\ -\gamma_0 L & -\gamma_1 L & -\gamma_2 L\end{bmatrix}\begin{bmatrix}p\\ r\\ s\end{bmatrix}=\lambda\begin{bmatrix}p\\ r\\ s\end{bmatrix}$$

that implies that

$$r=\lambda p$$

$$s=\lambda r$$

$$-\gamma_0 Lp-\gamma_1 Lr-\gamma_2 Ls=\lambda s$$

Thus, it follows that $q=[p^T,\lambda p^T,\lambda^2 p^T]^T$. It also follows that $-\gamma_0 Lp-\gamma_1 L\lambda p-\gamma_2 L\lambda^2 p=\lambda^3 p$, which can be written as

$$-Lp=\frac{\lambda^3}{\gamma_0+\gamma_1\lambda+\gamma_2\lambda^2}p$$

Thus, it follows that $\lambda^3/(\gamma_0+\gamma_1\lambda+\gamma_2\lambda^2)$ is an eigenvalue of $-L$ with an associated eigenvector p. Let $\mu=\lambda^3\big/(\gamma_0+\gamma_1\lambda+\gamma_2\lambda^2)$, where μ is an eigenvalue of $-L$ with p being its associated eigenvector. Therefore, it follows that

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$$\lambda^3-\gamma_2\mu\lambda^2-\gamma_1\mu\lambda-\gamma_0\mu=0 \tag{6}$$

that implies that there are three roots for λ corresponding to each μ. That is, each eigenvalue of $-L$ corresponds to three eigenvalues of Γ.

Let μ_i, $i=1,\ldots,n$, be the ith eigenvalue of $-L$. In addition, Let λ_{3i-2}, λ_{3i-1}, and λ_{3i}, $i=1,\ldots,n$ be the eigenvalues of Γ corresponding to μ_i. From Eq. (6), we can see that $\mu_j=0$ implies that $\lambda_{3j-2}=\lambda_{3j-1}=\lambda_{3j}=0$. It is straightforward to see that $-L$ has at least one zero eigenvalue with an associated eigenvector $\mathbf{1}$, since all of its row sums are equal to zero. Therefore, we know that Γ has at least three zero eigenvalues.

From Eq. (6), we can also see that $-L$ has a simple zero eigenvalue if and only if Γ has exactly three zero eigenvalues. In addition, if $-L$ has a simple zero eigenvalue, denoted as $\mu_1=0$ without loss of generality, then there is only one linearly independent eigenvector p for $-L$ associated with the eigenvalue zero. Note that $\mu_1=0$ implies that $\lambda_1=\lambda_2=\lambda_3=0$, which in turn implies that $q=[p^T,\mathbf{0}^T,\mathbf{0}^T]^T$. Therefore, there is only one linearly independent eigenvector q for Γ associated with eigenvalue zero. That is, the zero eigenvalue of Γ has geometric multiplicity equal to one if $-L$ has a simple zero eigenvalue. ■

Using this lemma we can prove the following result.

THEOREM 3.1. *In the case of $\ell=3$, the algorithm* (4) *achieves consensus exponentially if and only if Γ has exactly three zero eigenvalues and all of the other eigenvalues have negative real parts.*

Proof. (Sufficiency.) If Γ has exactly three zero eigenvalues, we know that the eigenvalue zero has geometric multiplicity equal to one from Lemma 3.1. As a result, it follows that Γ can be written in Jordan canonical form as

$$\Gamma=\underbrace{[w_1,\ldots,w_{3n}]}_{P}\underbrace{\begin{bmatrix}0 & 1 & 0 & 0_{1\times(3n-3)}\\ 0 & 0 & 1 & 0_{1\times(3n-3)}\\ 0 & 0 & 0 & 0_{1\times(3n-3)}\\ 0_{(3n-3)\times 1} & 0_{(3n-3)\times 1} & 0_{(3n-3)\times 1} & J'\end{bmatrix}}_{J}\times\underbrace{\begin{bmatrix}\nu_1^T\\ \vdots\\ \nu_{3n}^T\end{bmatrix}}_{P^{-1}} \tag{7}$$

where $w_j\in\mathbb{R}^{3n}$, $j=1,\ldots,3n$, can be chosen to be the right eigenvectors or generalized eigenvectors of Γ, $\nu_j\in\mathbb{R}^{3n}$, $j=1,\ldots,3n$, can be chosen to be the left eigenvectors or generalized eigenvectors of Γ, and J' is the Jordan upper diagonal block matrix corresponding to $3n-3$ nonzero eigenvalues of Γ.

Without loss of generality, we choose $w_1=[\mathbf{1}^T,\mathbf{0}^T,\mathbf{0}^T]^T$, $w_2=[\mathbf{0}^T,\mathbf{1}^T,\mathbf{0}^T]^T$, and $w_3=[\mathbf{0}^T,\mathbf{0}^T,\mathbf{1}^T]^T$, where it can be verified that w_1, w_2, and w_3 are an eigenvector and two generalized eigenvectors of Γ associated with the eigenvalue 0, respectively. Noting that Γ has exactly three zero eigenvalues, denoted as $\lambda_1=\lambda_2=\lambda_3=0$ without loss of generality, we know that $-L$ has a simple zero eigenvalue, which, in turn, implies that there exists a nonnegative $n\times 1$ vector p such that $p^TL=0$ and $p^T\mathbf{1}=1$, as shown in Ref. [38]. It can be verified that $\nu_1=[p^T,\mathbf{0}^T,\mathbf{0}^T]^T$, $\nu_2=[\mathbf{0}^T,p^T,\mathbf{0}^T]^T$, and $\nu_3=[\mathbf{0}^T,\mathbf{0}^T,p^T]^T$ are two generalized left eigenvectors and a left eigenvector of Γ associated with eigenvalue 0, respectively, where $\nu_j^T w_j=1$, $j=1,2,3$. Note that

$$e^{\Gamma t}=\underbrace{[w_1,\ldots,w_{3n}]}_{P}\underbrace{\begin{bmatrix}1 & t & \frac{1}{2}t^2 & 0_{1\times(3n-3)}\\ 0 & 1 & t & 0_{1\times(3n-3)}\\ 0 & 0 & 1 & 0_{1\times(3n-3)}\\ 0_{(3n-3)\times 1} & 0_{(3n-3)\times 1} & 0_{(3n-3)\times 1} & e^{J't}\xrightarrow{} 0\end{bmatrix}}_{e^{Jt}}\times\underbrace{\begin{bmatrix}\nu_1^T\\ \vdots\\ \nu_{3n}^T\end{bmatrix}}_{P^{-1}}$$

Note also that $\lim_{t\to\infty}e^{J't}\to 0_{3n-3}$ exponentially since the eigenvalues λ_{3i-2}, λ_{3i-1}, and λ_{3i}, $i=2,\ldots,n$ have negative real parts. Therefore, it follows by computation that for large t, the dominant terms in $e^{\Gamma t}$ are

$G(s) = \frac{1}{(s+1)^3}$

$$\begin{bmatrix} \mathbf{1}p^T & t\mathbf{1}p^T & \frac{1}{2}t^2\mathbf{1}p^T \\ 0_n & \mathbf{1}p^T & t\mathbf{1}p^T \\ 0_n & 0_n & \mathbf{1}p^T \end{bmatrix}$$

$g(t) = e^{-t} + te^{-t} + \frac{1}{2}t^2 e^{-t}$

where the rows $(kn+1)$ to $(k+1)n$, $k=0,1,2$ are identical. Noting that

$$\begin{bmatrix} \xi^{(0)}(t) \\ \xi^{(1)}(t) \\ \xi^{(2)}(t) \end{bmatrix} = e^{\Gamma t} \begin{bmatrix} \xi^{(0)}(0) \\ \xi^{(1)}(0) \\ \xi^{(2)}(0) \end{bmatrix}$$

where $\xi^{(k)}=[\xi_1^{(k)},\ldots,\xi_n^{(k)}]^T$, we know that $\xi_i^{(k)}(t)\to\xi_j^{(k)}(t)$ exponentially, $\forall i\neq j$, $k=0,1,2$, as $t\to\infty$.

(Necessity.) Suppose that the sufficient condition that Γ has exactly three zero eigenvalues and all the other eigenvalues have negative real parts does not hold. Noting that Γ has at least three zero eigenvalues, the fact that the sufficient condition does not hold implies that Γ has either more than three zero eigenvalues or it has three zero eigenvalues but has at least another eigenvalue having positive real part. In either case, we know that $\lim_{t\to\infty}e^{Jt}$ has a rank larger than three, which implies that $\lim_{t\to\infty}e^{\Gamma t}$ has a rank larger than three. Note that consensus is reached asymptotically if and only if

$$\lim_{t\to\infty}e^{\Gamma t} \to \begin{bmatrix} \mathbf{1}q^T \\ \mathbf{1}r^T \\ \mathbf{1}s^T \end{bmatrix}$$

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where q, r, and s are $3n\times 1$ vectors. As a result, the rank of $\lim_{t\to\infty}e^{\Gamma t}$ cannot exceed three. This results in a contradiction. ■

In the case of $\ell=3$, let λ_k, $k=1,\ldots,3n$, be the eigenvalues of Γ. Note that consensus is reached exponentially if and only if Γ has exactly three zero eigenvalues and all of the other eigenvalues have negative real parts. The convergence speed of the algorithm (4) is related to the nonzero eigenvalues of Γ. Let λ_j be the nonzero eigenvalue such that $|\mathrm{Re}(\lambda_j)|=\min|\mathrm{Re}(\lambda_k)|$, $\forall\lambda_k\neq 0$, where $\mathrm{Re}(\cdot)$ represents the real part of a number. In particular, for $\ell=3$, $e^{\lambda_j t}$ is the dominant term when $e^{\Gamma t}$ converges to a matrix of the form

$$\begin{bmatrix} \mathbf{1}q^T \\ \mathbf{1}r^T \\ \mathbf{1}s^T \end{bmatrix}$$

where q, r, and s are $3n\times 1$ vectors.

Note that L has a simple zero eigenvalue and all of the other eigenvalues have positive real parts if and only if the information-exchange topology has a directed spanning tree [38]. In the case of $\ell=3$, if (4) achieves consensus exponentially, we know that Γ has exactly three zero eigenvalues following Theorem 3.1. Thus, we see that $-L$ has a simple zero eigenvalue, which in turn implies that the information-exchange topology has a directed spanning tree. Therefore, in the case of $\ell=3$, having a directed spanning tree is a necessary condition for consensus seeking. However, similar to the case of $l=2$ [30], having a directed spanning tree is not a sufficient condition for consensus seeking. Both the information-exchange topology and values of γ_k, $k=0,1,2$, will affect the convergence of the ℓth-order consensus algorithm with $l\geqslant 3$. In contrast, in the case of $\ell=1$, having a directed spanning tree is a necessary and sufficient condition for consensus seeking [38].

From Eq. (6) we can see that γ_k, $k=0,1,2$, plays an important role in the eigenvalues of Γ. In the case of $\ell=2$, we know that if $-L$ has a simple zero eigenvalue and all of the other eigenvalues are real and therefore negative (e.g., an undirected connected information-exchange topology or a leader-follower information-exchange topology), then (4) achieves consensus for arbitrary $\gamma_k>0$, $k=0,1$ [30]. In contrast, this argument is no longer valid for the case of $\ell=3$. In the case that the digraph of $-L$ is itself a directed spanning tree (i.e., a leader-follower information-exchange topology), $-L$ has a simple zero eigenvalue and all of the other eigenvalues are real, and therefore negative [38]. As a result, all of the coefficients of the polynomial (6) are positive real numbers when $\mu\neq 0$, where μ is the eigenvalue of $-L$. From the Routh criterion, there always exist γ_k, $k=0,1,2$, such that all of the roots of (6) have negative real parts when $\mu\neq 0$. It is straightforward to see that when a graph has a directed spanning tree, the graph can be constructed by adding information-exchange links to a graph that is itself a directed spanning tree. Noting that μ is continuously dependent on the entries of $-L$ (i.e., the digraph of $-L$) and the roots of (6) are continuously dependent on its coefficients, we know that for each $-L$ whose graph has a directed spanning tree, there always exist γ_k, $k=0,1,2$, such that all of the roots of (6) have negative real parts when $\mu\neq 0$. The parameters γ_k, $k=0,1,2$, can be chosen according to the Routh-Hurwitz theorem. As a result, the conditions of Theorem 3.1 are satisfied.

To illustrate these points, consider the following simulation example. Let L be given by

$$L_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad (8)$$

Note that the digraph of L has a directed spanning tree. In fact, the graph of L is itself a directed spanning tree in this case. Note also that $\mu_j=-1$, where μ_j is the nonzero eigenvalue of $-L$. In Case 1, we choose $\gamma_0=2$, $\gamma_1=1$, and $\gamma_2=2$. From Eq. (6), we then see that $\lambda_{3j-2}=-2$, $\lambda_{3j-1}=i$, and $\lambda_{3j}=-i$, where λ_* is the eigenvalue of Γ corresponds to μ_j. As a result, consensus cannot be achieved. However, if we choose $\gamma_0=1$, $\gamma_1=2$, and $\gamma_2=3$ in Case 2, then $\lambda_{3j-2}=-2.3247$, $\lambda_{3j-1}=-0.3376+0.5623i$, and $\lambda_{3j}=-0.3376-0.5623i$. As a result, consensus can be reached. Figure 1 shows the plots of $\xi_i^{(2)}$, $i=1,\ldots,4$, for Cases 1 and 2 with different γ_j, $j=0,1,2$, values. Note that although the digraph of L has a directed spanning tree in both cases, the consensus system is not stable in Case 1, whereas it is stable in Case 2. Thus, the gains γ_k must be chosen properly to ensure that consensus is achieved.

3.2 Setpoint Tracking and Higher-Order Consensus With a Leader.

In Ref. [39] the idea of a leader-node is introduced, whereby a single node is chosen that ignores all the other nodes, but continues to broadcast, and the controllability properties of the resulting graph are explored. In Ref. [40] it was shown how to modify the first-order consensus algorithm to introduce setpoint tracking, but with a less stringent requirement than full state controllability (and not requiring that the leader ignore all the other nodes, though this is effectively what happens). The algorithm in Ref. [40] caused all the nodes to converge to the leader's setpoint. This is called *consensus with a leader*.

Though Ref. [40] is considered only first-order consensus, in the same way for higher-order consensus, we can modify Eq. (4) as follows:

$$u_i = -\sum_{j=1}^{n} g_{ij}k_{ij}\left[\sum_{k=0}^{\ell-1}\gamma_k(\xi_i^{(k)}-\xi_j^{(k)})\right] - \alpha_i(\xi_i^{(\ell-1)}-\xi_i^{(\ell-1)*}),$$

$$i\in\{1,\ldots,n\} \quad (9)$$

where $\alpha>0$ and $\xi_i^{(\ell-1)*}$ is the local setpoint on node i. Using Eq. (9), we claim that if $\alpha_i=0$ for all but node k, with $\alpha_k=1$ (i.e., higher-order consensus with a leader), and node k has a directed path to all the other nodes, then $\xi_i^{(l-1)}\to\xi_k^{(\ell-1)*}$, $\forall i\in\{1,\ldots,n\}$ by

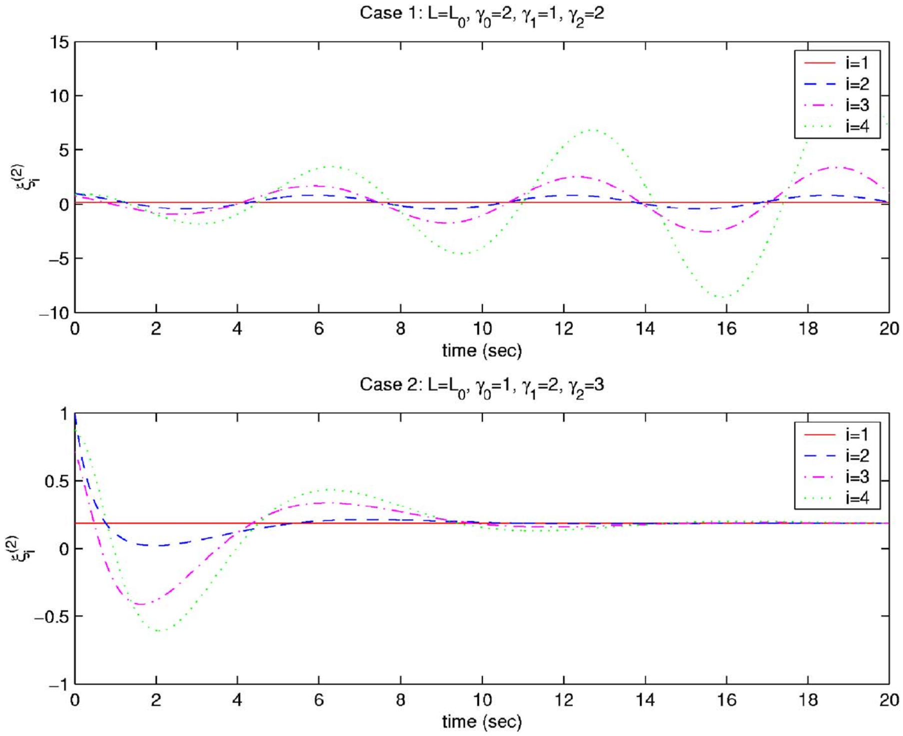

Fig. 1 Plots of $\xi_i^{(k)}$, $k=2$, for Cases 1 and 2 with different γ_j, $j=0,1,2$ values

following a similar argument in Ref. [40]. Note that this assertion requires $\xi_k^{(\ell-1)*}$ to be piecewise constant. However, in the next subsection we generalize these ideas further by extending them to include setpoints for all the derivatives, where the setpoints come from a reference model.

To illustrate, we choose $\xi_1^{(2)*}=0.5$, $\alpha_1=1$, and $\alpha_i=0$, $\forall i\neq 1$. Figure 2 shows the plots of $\xi_i^{(2)}$, where $L=L_0$ is given by Eq. (8). Note that each $\xi_i^{(2)}$ converges to the setpoint.

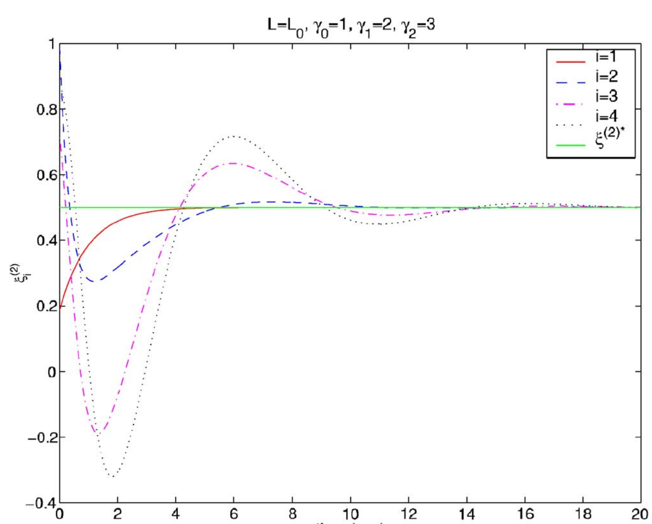

Fig. 2 Plots of $\xi_i^{(k)}$, $k=2$, with $\xi_1^{(2)*}=0.5$

3.3 Model-Reference Consensus. Consider a prescribed reference dynamic model given by

$$\dot{\xi}_r^{(0)} = \xi_r^{(1)}$$

$$\vdots$$

$$\dot{\xi}_r^{(\ell-2)} = \xi_r^{(\ell-1)}$$

$$\dot{\xi}_r^{(\ell-1)} = u_r \tag{10}$$

where $\xi_r^{(k)}\in\mathbb{R}^m$, $k=0,1,\ldots,\ell-1$, are the reference states, and $u_r\in\mathbb{R}^m$ is the reference control input.

A model reference consensus problem is said to be solved if $\xi_i^{(k)}\to\xi_r^{(k)}$, $k=0,\ldots,\ell-1$, asymptotically and $\xi_i^{(k)}\to\xi_j^{(k)}$, $\forall i\neq j$, during the transition.

3.3.1 Full Access to the Reference Model. In the case that the reference model is available to each vehicle in the team, we propose the following model-reference consensus algorithm:

$$u_i = -\sum_{j=1}^{n} g_{ij}k_{ij}\left[\sum_{k=0}^{\ell-1}\gamma_k(\xi_i^{(k)}-\xi_j^{(k)})\right] - \eta\sum_{k=0}^{\ell-1}\gamma_k(\xi_i^{(k)}-\xi_r^{(k)}) + u_r$$

$$i\in\{1,\ldots,n\} \tag{11}$$

where $\eta>0$.

Let $\tilde{\xi}_i^{(k)}=\xi_i^{(k)}-\xi_r^{(k)}$, $k=0,\ldots,\ell$, and $\tilde{\xi}^{(k)}$ be an $mn\times 1$ column vector with components $\tilde{\xi}_i^{(k)}$, $i=1,\ldots,n$. By applying (11), Eq. (2) can be written in matrix form as