EE102 Week 0, Lecture 1 (Fall 2025)

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1 Goals

- Logistics, grading, extensions, expectations
- Motivation to study signal processing
- Pre-requisites to signal processing: vectors and complex numbers

2 Why study signal processing?

Signal processing and linear systems theory is foundational in engineering. It has revolutionized engineering in more ways than we realize — machine learning/AI, RF amplifiers, satellite communications, airplanes, medical devices, automotive vehicles, MRI scans, and pretty much every other engineering and science discipline out there *directly* uses concepts from this course. This is fantastic but there is also a downside! Learning signal processing is dependent on many prerequisites as it builds on various other fundamental courses in engineering. As electrical engineers, you are required to learn signal processing and linear systems theory. Other engineering disciplines typically do not require such a course. This gives electrical engineers an edge because what you learn in this course is not just applicable to EE but also to all other areas. So, despite many pre-requisite requirements, I hope that you will be motivated to cross the technical barriers in learning signal processing.

2.1 Real-world significance

Count the number of questions you answer "yes" to from the list below:

- Do you enjoy music? Did you ever find songs that sounded very similar to each other? Would you like to be able to explain why that's the case (mathematically)?
- Would you like design your own electrical circuits that are able to meet performance specifications of your (future) clients?

- Do you anticipate that you will be working on radio frequency circuits in your career where you need to design circuits and systems that communicate at specific frequencies?
- Are you attracted to the rigor of electrical engineering? Or perhaps, put another way, are you looking forward to setting aside time to learn the mathematical underpinnings of electrical engineering?
- Would you like to gain a better understanding of how various "scanners" scan our body parts to provide useful medical information (like X-rays, MRI, CT scans, etc.)?
- Do you want to be able to explain to others how images and colors on any digital display are created and manipulated?
- Would you like to *mathematically* create new music that takes the best parts of some of the songs that you like? Or perhaps, create entirely new sounds that have never been heard before? By doing it mathematically, you will be making the process general, easily customizable, and reproducible.
- Do you anticipate that your career choice after graduation will involve image processing / machine learning / artificial intelligence?
- Are you interested in understanding the underpinnings of the controllers that are used in automotive vehicles, or robotics, or even in the design of precise drugs that target pathogens in the human body?
- Noise canceling in audio tech is a huge industry! Are you someone who fancies designing / understanding these systems?
- Do you want to understand how Shazam (or other apps that can recognize a song just based on a few beats) work?

If you counted more than a few "yes" answers, you are in the right place! This course will help you understand the mathematical underpinnings of many of these applications. Of course, this course will not go into the technical specifics of any of the applications. There won't be enough time for it. Other courses exist for such details. See below for what this course is not.

2.2 What signal processing is not?

In signal processing, you will **not** learn the fundamentals of circuit analysis, AC analysis, transistors, communication algorithms, design of RF circuits, or controller design. Most of those topics are already assigned to other specific courses. In signal processing and linear systems course, we will focus on the mathematical tools and techniques used to analyze and process signals and systems.

With that motivation, let us jump into a discussion about various pre-requisites that you will need to be familiar with to succeed in this course.

3 Pre-requisite #1: Vectors

When studying problems with many entities/observations, we structure our variables into vectors.

An n-dimensional vector $\mathbf{x}$ can be written as

$$\mathbf{x} = [\, x_1,\ x_2,\ \ldots,\ x_n \,], \qquad \mathbf{x} \in \mathbb{R}^n.$$

3.1 Matrices are transformations

If you transform a vector $\mathbf{x}$ to a new vector $\mathbf{y}$ such that all elements in $\mathbf{y}$ are linear combinations of elements in $\mathbf{x}$, then the transformation is called a matrix.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

such that

$$y_1 = \sum_{i=1}^{n} \alpha_{1i} x_i, \qquad y_2 = \sum_{i=1}^{n} \alpha_{2i} x_i, \quad \ldots, \quad y_m = \sum_{i=1}^{n} \alpha_{mi} x_i.$$

Then $A\mathbf{x} = \mathbf{y}$, where

$$A \in \mathbb{R}^{m \times n}, \qquad A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}.$$

We write

$$A:\ X \to Y,$$

where X is the vector space in $\mathbb{R}^n$ where $\mathbf{x}$ lies and Y is the vector space in $\mathbb{R}^m$ where $\mathbf{y}$ lies.

Recall

- Diagonal matrix
- Identity matrix
- Symmetric matrix
- Zero matrix
- Matrix transpose
- Matrix algebra ($+$, $-$, $\times$, inverse)

3.2 Real-world significance

Note that a transformation is called an "affine" transformation if it is linear

$$\mathbf{y} = A\mathbf{x} + \mathbf{b},$$

where A is a linear transformation (matrix) and $\mathbf{b}$ is a translation vector. Affine transformations are common in many practical applications such as image processing, computer-aided design in engineering, medical imaging, graphic design, and many more. On a lighter note, check this fun meme template out which uses matrix transformations at its core — the content aware scale gif and some related Reddit discussion on it. Creating memes often requires very specific image transforms (such as the Wide Keanu or the general Stretched Resolution meme)! On a more technical note, you can check out the Adobe Photoshop tool called "Transform" (or the equivalent rotate, scale, and skew tools in Microsoft Paint) — these tools allow users to manipulate images using affine transformations. The same concepts are at the core of many research-grade affine transform tools. Some examples are rasterio for geographical applications, flirt for affine transformations of MRI images and the RandomAffine tool for affine transformations in image augmentation used in machine learning applications.

In summary, vector and matrix algebra is the centerpiece in signal processing and you will see the mathematical preliminaries being used throughout the course.

4 Pre-requisite #2: Complex numbers

Although the usual way we learn about the complex unit "j" is as a convenient notation for a solution of

$$x^2 + 1 = 0 \;\;\Rightarrow\;\; x = \sqrt{-1} := \mathrm{j},$$

it is useful to recognize other places where this convenience is beneficial. In signal processing we are often looking for easy ways to analyze physical signals, not only to solve algebraic equations.

4.1 From vectors to a complex scalar.

Given a vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

define the (complex) scalar

$$z_{\mathbf{x}} = x_1 + \mathrm{j}x_2.$$

The entries x_1 and x_2 are not "added" in $\mathbb{R}$; they are bound only because they are components of the same vector. Writing $z_{\mathbf{x}}$ lets us treat the vector like a single scalar living in $\mathbb{C}$.

4.2 Inner products and linear dependence

In simple terms, the inner product between two vectors is a scalar quantity that quantifies a relationship between two vectors: how much they align with each other. In quantifying this, the inner product takes into account the lengths of the two vectors and the angle between them. The inner product can be used to define orthogonality (perpendicularity) — which is one the most fundamental concepts in signal processing.

Why? The key idea in EE 102 is that a linear combination of a set of orthogonal signals can be used to represent *any* signal (no matter how complicated), under some conditions, of course. So, understanding orthogonality, linear independence, and linear combinations is key to this course. Consequently, inner product is an important concept for this course.

If two vectors are orthogonal (that is, their inner product is zero), then these vectors are linearly independent. Indeed, we can prove that a set of non-zero mutually orthogonal vectors (say, $v_1, v_2, \ldots, v_n$) are linearly independent. You can show this by writing the linear combination of the vectors: $S = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ and showing that it is zero only if all constants α_i, $i = \{1, \ldots, n\}$ are equal to zero. To prove this linear independence, you can take the inner product of the above with any vector in the set (or any linear combination, thereof) say v_k:

$$S = v_k \cdot (c_1 v_1 + c_2 v_2 + ... + c_n v_n)$$
$$S = v_k \cdot c_1 v_1 + v_k \cdot c_2 v_2 + ... + v_k \cdot c_n v_n = 0$$

since the pairwise dot products (the inner product between each pair of vectors) are zero, we are only left with

$$c_k(v_k \cdot v_k) = 0$$

which is only possible if $c_k = 0$ since $v_k \cdot v_k$ is non-zero. Since this is true for any k, we have that all coefficients are zero. So, inner products play an important role in proving orthogonality (and thus, linear independence of vectors).

4.3 Inner product via complex numbers.

All the vector algebra can be tedious work! Complex numbers come to our rescue as we can represent a 2D vector as a complex number. For $\mathbf{x} = [x_1, x_2]^\top$ and $\mathbf{y} = [y_1, y_2]^\top$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = x_1 y_1 + x_2 y_2.$$

With the complex representations

$$z_{\mathbf{x}} = x_1 + \mathrm{j}x_2, \qquad z_{\mathbf{y}} = y_1 + \mathrm{j}y_2,$$

their product with conjugation is

$$\begin{aligned} \overline{z}_{\mathbf{x}}\, z_{\mathbf{y}} &= (x_1 - \mathrm{j}x_2)(y_1 + \mathrm{j}y_2) \\ &= (x_1 y_1 + x_2 y_2) \ + \ \mathrm{j}\,(x_1 y_2 - x_2 y_1). \end{aligned}$$

Taking the real part gives the vector inner product:

$$\Re(\overline{z}_{\mathbf{x}}\, z_{\mathbf{y}}) = x_1 y_1 + x_2 y_2 = \langle \mathbf{x}, \mathbf{y} \rangle.$$

Polar form viewpoint. Write $\mathbf{x}$ in polar coordinates with $r_x = \|\mathbf{x}\|$ and angle θ_x:

$$x_1 = r_x \cos\theta_x, \qquad x_2 = r_x \sin\theta_x,$$

so

$$z_{\mathbf{x}} = r_x(\cos\theta_x + \mathrm{j}\sin\theta_x) = r_x e^{\mathrm{j}\theta_x}.$$

Similarly $z_{\mathbf{y}} = r_y e^{\mathrm{j}\theta_y}$ from Euler's identity[1]. Then

$$\begin{aligned} \overline{z}_{\mathbf{x}}\, z_{\mathbf{y}} &= r_x e^{-\mathrm{j}\theta_x}\ r_y e^{\mathrm{j}\theta_y} = r_x r_y e^{\mathrm{j}(-\theta_x + \theta_y)} \\ &= r_x r_y \Big[\cos(-\theta_x + \theta_y) + \mathrm{j}\sin(-\theta_x + \theta_y) \Big], \end{aligned}$$

hence

$$\Re(\overline{z}_{\mathbf{x}}\, z_{\mathbf{y}}) = r_x r_y \cos(-\theta_x + \theta_y) = r_x r_y \cos(\theta_x - \theta_y) = \langle \mathbf{x}, \mathbf{y} \rangle.$$

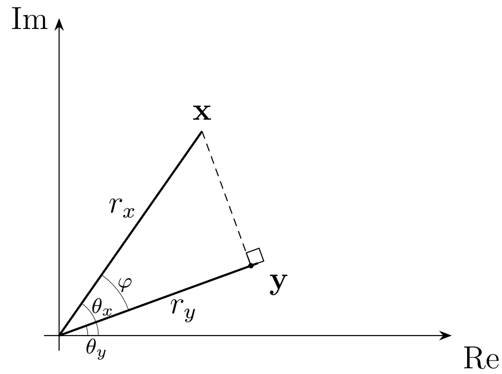


Figure 1: Geometric view of the inner product using polar form

Note that cosine is an even function, which allowed us to write the last equality above.

In Figure 1, observe that vectors $\mathbf{x}$ and $\mathbf{y}$ make angles θ_x and θ_y with the real axis and the angle between them is $\varphi = \theta_x - \theta_y$. You can make intuitive sense of the inner product in polar form by understanding its geometric interpretation (see Figure 1). Specifically, recall how we defined inner products in the previous section — a quantification of the alignment between two vectors. For our example in Figure 1, decompose $\mathbf{x}$ relative to $\mathbf{y}$: drop a perpendicular from the tip of $\mathbf{x}$ to the line span$\{\mathbf{y}\}$ (the direction spanned by $\mathbf{y}$). This splits $\mathbf{x}$ into a part *parallel* to $\mathbf{y}$ and a part *perpendicular* to $\mathbf{y}$:

$$\mathbf{x} = \underbrace{(\mathbf{x}\cdot\hat{\mathbf{y}})\,\hat{\mathbf{y}}}_{\text{projection onto span}\{\mathbf{y}\}\text{, parallel to }\mathbf{y}} + \underbrace{\Big(\mathbf{x} - (\mathbf{x}\cdot\hat{\mathbf{y}})\,\hat{\mathbf{y}}\Big)}_{\text{perpendicular (rejection) to }\mathbf{y}} .$$

The second part that is perpendicular to $\mathbf{y}$ is called the rejection because that's the part that is remaining (you can see that it is quite literally the remaining part as it is obtained by subtracting the projection from $\mathbf{x}$). Note that the inner product of $\mathbf{x}$ with the unit vector $\hat{\mathbf{y}}$ gives us the projection of x onto span$\{\mathbf{y}\}$. By computing the inner product, you can also check that the remaining (perpendicular part) of $\mathbf{x}$ is orthogonal to $\mathbf{y}$:

$$\operatorname{rej}_{\mathbf{y}}(\mathbf{x})\cdot\hat{\mathbf{y}} = \mathbf{x}\cdot\hat{\mathbf{y}} - (\mathbf{x}\cdot\hat{\mathbf{y}})(\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}) = \mathbf{x}\cdot\hat{\mathbf{y}} - \mathbf{x}\cdot\hat{\mathbf{y}} = 0.$$

Thus

$$\mathbf{x} = \operatorname{proj}_{\mathbf{y}}(\mathbf{x}) + \operatorname{rej}_{\mathbf{y}}(\mathbf{x}), \qquad \|\mathbf{x}\|^2 = \|\operatorname{proj}_{\mathbf{y}}(\mathbf{x})\|^2 + \|\operatorname{rej}_{\mathbf{y}}(\mathbf{x})\|^2.$$

In summary, the inner product measures how much of $\mathbf{x}$ points along $\mathbf{y}$, scaled by the length of $\mathbf{y}$. This is given by

$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{y}\|\,(\mathbf{x}\cdot\hat{\mathbf{y}}) = \|\mathbf{x}\|\,\|\mathbf{y}\|\cos\varphi.$$

[1] You can practice proving Euler's identity that $e^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\sin\theta$ by expanding the left-hand side using the exponential series and collecting the real and imaginary parts together (the real part will be the cosine series and the imaginary part will be the sine series).

The sign of $\cos\varphi$ carries the orientation: it is positive when the angle is acute and negative when obtuse. Complex numbers in their polar form express the same geometric intuition:

$$\overline{z_{\mathbf{x}}}\, z_{\mathbf{y}} = r_x r_y e^{\mathrm{j}(\theta_x - \theta_y)}$$

so the real part matches the dot product:

$$\Re\big(\overline{z_{\mathbf{x}}}\, z_{\mathbf{y}}\big) = r_x r_y \cos\varphi = \mathbf{x} \cdot \mathbf{y}.$$

Therefore, the scalar projection of $\mathbf{x}$ onto $\mathbf{y}$ is $r_x \cos(\theta_x - \theta_y)$, which when multiplied by the absolute value of y gives the inner product (that is, $r_x r_y \cos(\theta_x - \theta_y)$ in complex polar form).

4.4 Real-world significance

We discussed three main topics in this section — vectors, complex numbers, and their products. Representing quantities as vectors has many advantages, which mirrors the advantage of using lists and arrays in computer programming. Inner products are useful in quantifying the alignment between vectors. A simple example is in machine learning, where the similarity between data points can be measured using inner products, which has applications in face detection, recommendation systems, and more. Finally, as discussed, complex numbers help us analyze vectors in a more nuanced way by providing a framework for understanding their magnitude and direction. You will see many more real-world application examples of complex numbers in signal processing. In Fourier analysis, complex exponentials are used as the orthogonal basis functions for representing signals in the frequency domain.

5 Pre-requisite #3: Circuits

Without going into the specific details about various electrical circuits, this section will briefly discuss the fundamental concepts of circuits that are relevant to signal processing. In signal processing, we will use circuits only as examples. In fact, equivalent examples can be devised that are relevant for other disciplines. For example, in circuit theory, Kirchoff's voltage and current laws are essential tools that are commonly used to analyze currents and voltages. These laws frame the conservation of energy for an electrical circuit setting. An equivalent mechanical engineering example is the analysis of forces in a static system (such as a spring-mass damper system), where conservation of energy provides a similar framework for understanding the system's behavior.

Historically, signal processing has been a field that has drawn heavily from electrical engineering concepts, particularly in the analysis and manipulation of signals. The design of the feedback amplifier in the 1920s is the prime example. Scientists and engineers were interested in maximizing the signal to noise ratio for amplifier circuits, which led to the use of many of the foundational mathematical theory that existed at the time. The formalization of these mathematical tools for electrical engineering applications led to the development of the field of signal processing. Since then, these tools have found applications in many other disciplines, including mechanical engineering, civil engineering, computer science, biomedical engineering, and more. But for better or for worse, the pedagogy of signal processing has remained set in that historical context. So, without challenging the years of history too much, we will continue to use circuit examples in this course. However, wherever possible, we will try to incorporate broader application examples too.

EE 102 Week 1, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: September 3, 2025

1 Goals

- Introduction to signals: continuous-time and discrete-time
- Basic properties of signals: scaling, offset, linearity, and time invariance, and more
- Quantifying the energy and power of signals

2 What are signals?

A *signal* is a set of data or information. This is intentionally defined in a very broad manner. A simple way to understand signals: all mathematical functions that you have studied in your calculus classes are signals if you can attach a physical meaning to the function. Note that a signal need not be a function of time. It is often intuitive to think about functions of time and physical signals are functions of time (often, but not always!).

A *system* maps (that is, it processes) input signals into output signals. So, systems are characterized by their input-output relationships. See Figure 1 for a visual representation of signals and systems.

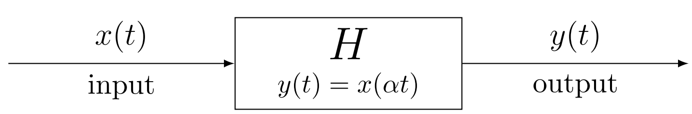


Figure 1: A system H that maps input signal $x(t)$ to output signal $y(t)$.

2.1 Continuous-time and discrete-time signals

Continuous-time domain is $\mathbb{R}$, and we write continuous-time signals as $x(t)$, if they are continuous functions of time (recall: continuous functions from your math classes). On the other hand, discrete-time domain is $\mathbb{Z}$, and a discrete-time signal is written as $x[n]$, where $n \in \mathbb{Z}$. This means that discrete-time signals are defined only at integer time indices.

2.2 Sketching signals

To sketch, draw and label axes, mark key values (peaks, zeros, discontinuities), and indicate any symmetry, periodicity, decay/growth, or piecewise structure (try to identify as many properties as you can before starting to sketch). The best way to start your sketch is to compute the values of the signal at "easy" points like, zero, the max time, etc.

3 Properties of signals

3.1 Scaling

Time scaling changes the horizontal axis by a constant α:

$$x_s(t) = x(\alpha t).$$

If $0 < \alpha < 1$, the signal expands in time whereas if $\alpha > 1$, it compresses the signal in time.

3.2 Offset

Time shifting offsets the horizontal axis by a constant T. A (right) delay of T seconds is defined by

$$x_d(t) = x(t - T).$$

Equivalently, $x_d(t_1 + T) = x(t_1)$ for every t_1.

3.3 Linearity of systems

A system H is *linear* if it satisfies additivity and homogeneity:

$$H\{x_1 + x_2\} = H\{x_1\} + H\{x_2\}, \qquad H\{k\,x\} = k\,H\{x\} \quad (\forall k \in \mathbb{C}).$$

Example: the exponential-weighting system $y(t) = e^{-at}x(t)$ is linear since

$$H\{x_1 + x_2\} = e^{-at}(x_1 + x_2) = e^{-at}x_1 + e^{-at}x_2 = H\{x_1\} + H\{x_2\}.$$

Remark. "Linear system" is a property of the *mapping*, not of the input/output signals. You should not confuse it with a straight-line graph of a scalar function, which you are used to thinking about when thinking about "linearity".

3.4 Time invariance of systems

A system H is *time invariant* if delaying the input by T produces the same delay at the output:

$$\text{If } y(t) = H\{x\}(t), \quad \text{then } H\{x(t-T)\} = y(t-T), \quad \forall T \in \mathbb{R}.$$

Intuition: If your opinion of a friend is dependent on the input about the friend, let's say that input is $x(t)$ (the friend descriptor signal), and seeing that input, you decide your opinion of your friend with an opinion signal called $y(t)$. Then, if your opinion about your friend does not change with time, that is, if you have the same opinion about your friend in the morning, in the evening (and even as the day changes), then your "opinion-defining" system (the one that outputs $y(t)$) is time-invariant! However, if your opinion of your friend keeps changing based on the time that you're meeting your friend, then you have a time-varying system of opinion generation (probably not a good trait!). Note that for time-invariant systems, the output is the "same response" delivered at the new time. It should not become, e.g., $k\,y(t)$ or $k\,y(t-T)$ depending on T.

3.5 A special signal — the unit step function

A unit step function is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

It is a special signal because it models the "start" of something, or an "onset" of an event, or more simply, a "switching on" of a process. You can shift the time to $t - T$ to delay the start by T seconds, so it's a very versatile signal. Therefore, the unit step function finds use in various applications.

Quick check (in-class): Is the *unit step* $u(t)$ time-invariant? (Trick question: time invariance is a *system* property, not a signal property.)

4 Energy and power of signals

We quantify the "size" of signals using energy and (time-averaged) power. For continuous-time signals, we define

$$E_\infty \triangleq \int_{-\infty}^{+\infty} |x(t)|^2 \, dt, \qquad P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt.$$

For discrete-time signals:

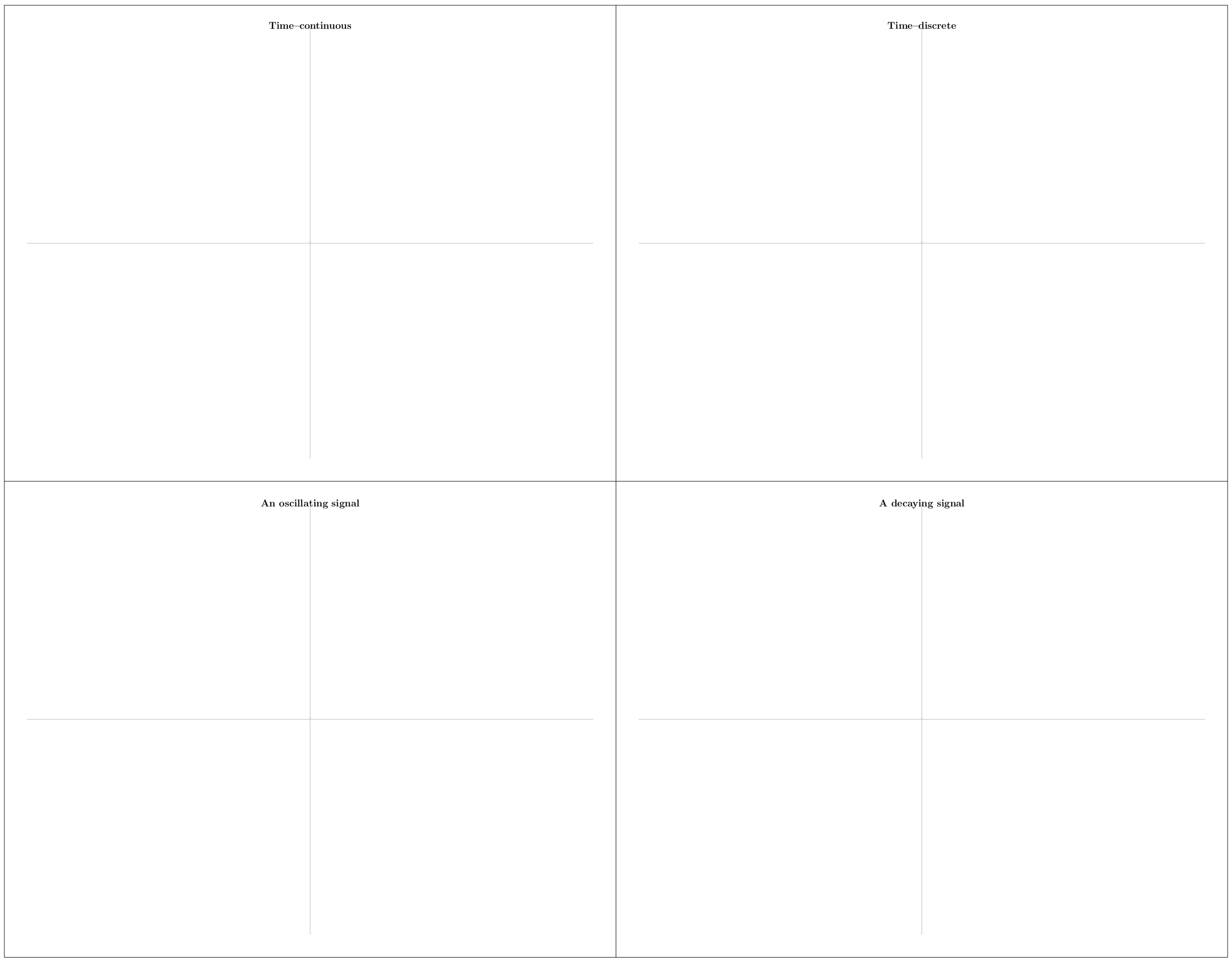
$$E_\infty \triangleq \sum_{n=-\infty}^{+\infty} |x[n]|^2, \qquad P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2.$$

With a desired signal s and noise n, one practical signal-to-noise ratio is

$$\mathrm{SNR} = \frac{E_\infty(s)}{E_\infty(n)} \quad (\text{or } P_\infty \text{ for power signals}).$$

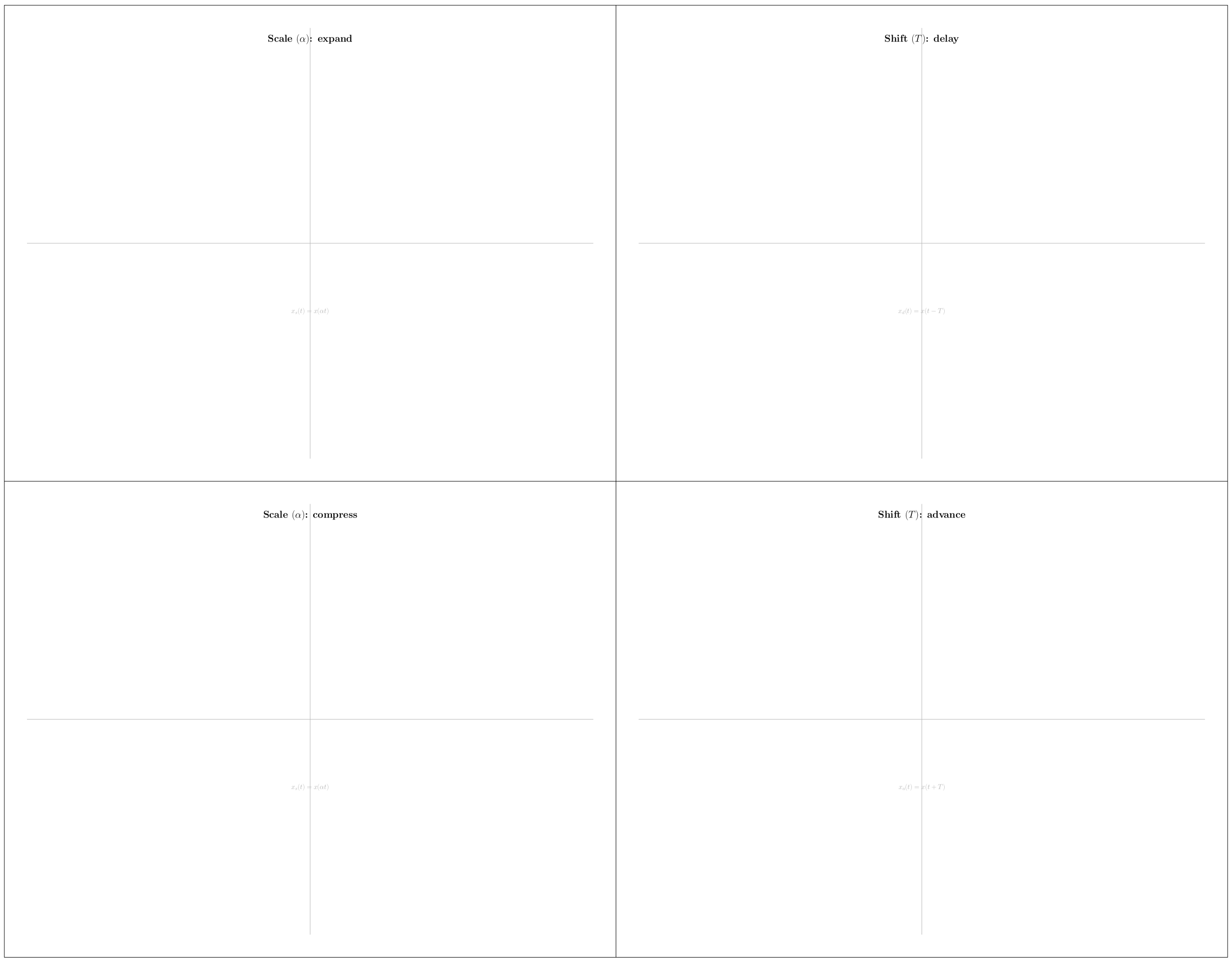
Worksheet #1: Sketching Signals (Part 1) — Groups of 4.

Each student takes one quadrant. Label axes clearly and annotate *what is your signal?*, *where is the signal likely to be observed?*, and key *properties of the signal*.



Worksheet #2: Transforming Signals (Part 2) — Groups of 2.

Each pair of students should scale and shift the two signals drawn by the other pair of students. Agree as a pair what scaling and shifting would mean and then draw it out. Clearly show the parameters and the transformed axes.



EE 102 Week 2, Lecture 1 (Fall 2025)

Instructor: Ayush Pandey

Date: September 8, 2025

1 Goals

- Review: time scaling, shifting, and combined operations on time-domain signals
- Review: energy and power — metrics to quantify signals
- Understand periodic signals using time shifting operations
- Derive the fundamental period of a signal
- Understand even and odd signals and their properties
- Apply signal operations to real-world signals using a guitar audio distortion example
- Next class: Complex exponentials, the unit impulse and step functions

2 Review: transforming signals

For a signal $x(t)$, common time operations include:

1. Reversal: $x(-t)$
2. Compression: $x(2t)$
3. Expansion: $x\left(\frac{t}{2}\right)$
4. Delay: $x(t-6)$
5. Advance: $x(t+6)$

2.1 How to sketch signal transformations?

To sketch signal transformations, first note down the key points on the X-axis (the time axis for time-domain signals). Then evaluate the value at the new domain (the shifted/scaled time) by looking at the values of the original signal at the corresponding time.

A quick summary: keep the vertical axis unchanged; apply horizontal changes only. For $x(at)$, compress if $|a| > 1$ and expand if $0 < |a| < 1$; for $x(t \pm T)$, shift right by T for $x(t - T)$ and left by T for $x(t + T)$; for $x(-t)$, reflect across the vertical axis.

Example: scaling and shifting a sinusoidal signal

Consider a sinusoidal signal $x(t) = \sin(t)$. A time-shifting transformation of $x(t)$ is given by

$$y(t) = x(t + t_0) = \sin(t + t_0)$$

where $t_0 \in \mathbb{R}$. For $t_0 > 0$, the signal is shifted to the left by t_0 while for $t_0 < 0$, the signal is shifted to the right by $|t_0|$. Similarly, a time-scaling transformation of $x(t)$ is given by

$$y(t) = x(\alpha t) = \sin(\alpha t)$$

where $\alpha \in \mathbb{R}$. For $|\alpha| > 1$, the signal is compressed by a factor of α while for $0 < |\alpha| < 1$, the signal is expanded by a factor of $\frac{1}{\alpha}$. If $\alpha < 0$, the signal is also reflected across the vertical axis.

Let us look at some specific values of t_0 and α to see how the signal is transformed.Figure 1 shows the original signal $x(t) = \sin(t)$ along with its time-shifted and time-scaled versions for different values of t_0 and α. Note that for $\alpha = -1$, we get a reflection of the original signal across the vertical axis. Expanding on this example, we can also combine time-shifting and time-scaling operations to get more complex transformations. For instance, consider the transformation

$$y(t) = x(\alpha t + t_0)$$

where both α and t_0 are non-zero. This transformation first scales the time by α and then shifts it by t_0. The order of operations matters here — if we were to shift first and then scale, we would have

$$y(t) = x(\alpha(t + t_0)) = x(\alpha t + \alpha t_0)$$

which is different from the previous transformation unless $\alpha = 1$.

If you intend to reverse the order of operations, you can redefine the shift parameter accordingly. For example, to achieve the same effect as $x(\alpha t + t_0)$ by shifting first and then scaling, you would need to use $x(\alpha(t + \frac{t_0}{\alpha}))$. Similarly, for the second combined transformation, you

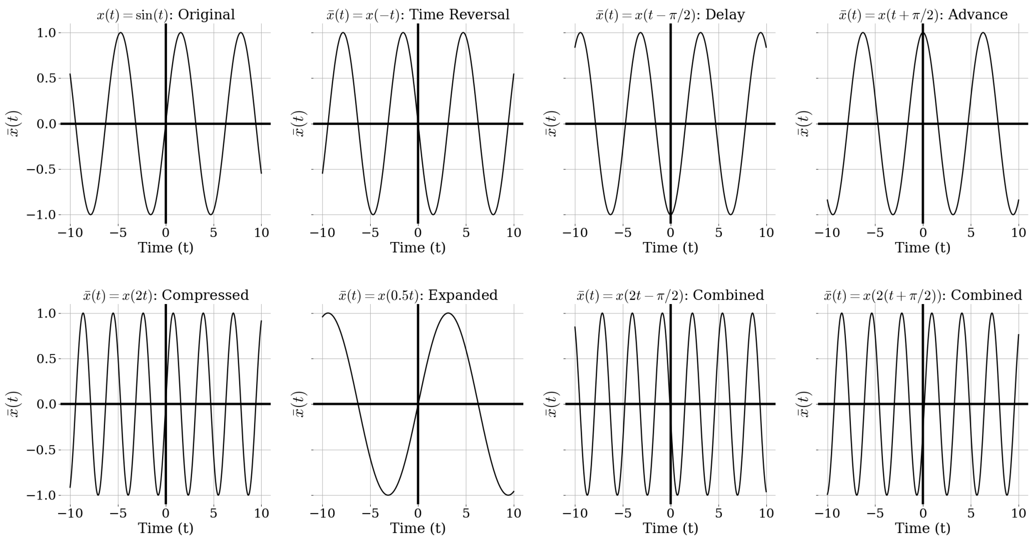


Figure 1: Time-shifting and time-scaling transformations of the signal $x(t) = \sin(t)$.

would need to use $x(\alpha t + \alpha t_0)$ to achieve the same effect as shifting first and then scaling by α next. Python code for generating signal transformations is available on Github[1].

2.2 Measuring the energy and power of signals

Previously, we defined energy as the integral of the squared magnitude of a signal over all time:

$$E_\infty(x) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

and power as the time-averaged measure for a time period $[-T, T]$:

$$P_\infty(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt.$$

It is natural to wonder how we ended up with those specific definitions. You can build the intuition behind these definitions in two ways: (a) by considering an electrical signal $x(t)$ as a voltage across a 1-ohm resistor, and (b) by considering the mathematical convenience that these definitions provide. For (a), it is pretty clear that the energy dissipated in a resistor is given by the integral of the square of the voltage over time divided by the value of the resistor

[1] github.com/ee-ucmerced/ee102-signals-systems

(in this case, 1 ohm). To fully appreciate the mathematical meaning of these definitions, consider the following alternate definition of energy as the integral of the absolute value of the signal over all time:

$$E'_{\infty}(x) = \int_{-\infty}^{\infty} |x(t)| dt.$$

This would be a valid way to quantify the "energy" of a signal as well. Note that our goal is to not physically define "energy", the electrical engineering concept, rather we are interested in coming up with measures of signals that we can use to compare two different signals. Note that taking absolute value is *atleast* required to prevent the integral from being zero for signals that oscillate between positive and negative values. Despite this, the integral of the square of the absolute value (E) is preferred over just the integral of the absolute value (E') because it is the metric that lets us compare signals in the L^2 space, which is a Hilbert space[2]. Simply stated, this means that the space of signals with finite energy (i.e., $E_{\infty}(x) < \infty$) has mathematical properties that enable better analysis and manipulation of signals. For example, we can define an inner product between two signals as a metric that quantifies the similarity between two signals $x(t)$ and $y(t)$ as

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt,$$

where $y^*(t)$ is the complex conjugate of $y(t)$. This inner product allows us to define concepts like orthogonality and projection in the space of signals. Remember that being able to represent complex signals as linear combinations of standard signals is the core concept in signal processing — this is not possible without a clear notion of orthogonality! In fact, the definition of E above is simply the inner product of a signal with itself, i.e., $E_{\infty}(x) = \langle x, x \rangle$. The L^1 space (signals with finite E') does not have these properties, which is why we prefer to use E as our measure of energy. Power can then be defined as the time-averaged energy over a time period.

3 Periodic signals

Definition 1. A signal $x(t)$ is periodic if $\exists\, T_0 > 0$ such that $x(t + T_0) = x(t)$ for all t. The smallest such T_0 is the *fundamental period* of the signal.

In more verbose language, we say that a signal is a periodic signal if we can find a time shift T_0 such that shifting the signal by T_0 does not change the signal. If such a T_0 does not exist, then the signal is aperiodic. For example, $x(t) = \sin(t)$ is periodic. To find the fundamental period, we would have to find the smallest shift T_0 to satisfy $\sin(t + T_0) = \sin(t)$ for all t.

[2]Read more on Hilbert spaces here: https://en.wikipedia.org/wiki/Hilbert_space

Using the periodicity of the sine function, we know that $\sin(t + 2\pi) = \sin(t)$ for all t. Thus, $T_0 = 2\pi$ is the fundamental period of $\sin(t)$. Note that $T_0 = 4\pi$ also satisfies the periodicity condition, but it is not the fundamental period since it is not the smallest such T_0. In fact, any integer multiple of 2π would satisfy the periodicity condition, but we are only interested in the smallest such value for the fundamental period.

3.1 Why periodic signals?

Just like many other concepts in signal processing, periodic signals are also a mathematical convenience! Intuitively, it is clear that if something repeats over and over again, then we can analyze just one cycle of it and extend the results to the entire signal. Therefore, studying periodic signals is often a good starting point. But you may wonder — what if the signal is not periodic? Read on.

3.2 Periodic extensions

When a signal is defined on a finite interval (e.g., a single cycle), it is often useful to *periodically extend* it by repeating that interval end-to-end. This makes the time-averaged power well defined and makes symmetries/harmonics easier to see.

3.3 Example: Periodic or not?

Consider the following three signals. Our goal is to find out whether they are periodic or not. If they are periodic, we will report the fundamental period of these signals.

1. $x(t) = \cos(t)$
2. $x(t) = \cos(t)$ for $t \geq 0$ and $x(t) = -sin(t)$ for $t < 0$
3. $x(t) = e^{j\omega t}$ where $\omega \neq 0$

How to prove periodicity? We can simply *propose a T_0 and check* if the periodicity condition is satisfied. If you cannot find a T_0, that does not mean that the signal is aperiodic — it just means that you have not found the right T_0 yet! To prove aperiodicity, you have to show that no such T_0 exists. This is often done by contradiction. Proofs by contradiction is an important mathematical trick where you assume that the statement you want to prove is false and then show that this assumption leads to a contradiction (something that will be *obviously* incorrect). This implies that the original statement must be true. You will find the signal transformation properties useful in proving (a)periodicity. Finally, if you can exploit the properties of the given signal, then you will find a much easier path to the proof.

For the first example, we know that the cosine function is "oscillatory", which indicates that it is probably periodic. Let's try to find a T_0 such that $\cos(t + T_0) = \cos(t)$ for all t. You might propose a $T_0 = \pi$ and observe that $\cos(t + \pi) = -\cos(t)$, which is not what we were looking for. So, $T_0 = \pi$ is not a good choice for T_0. Let's try one more time. Propose $T_0 = 2\pi$. Then, compute $\cos(t + 2\pi) = \cos(t)$, which is a valid choice. To check if it is the fundamental period, we can see that any integer multiple of 2π would also satisfy the periodicity condition, but 2π is the smallest such value. Therefore, the fundamental period of $x(t) = \cos(t)$ is $T_0 = 2\pi$. Note that you can use trignometric identities to help you prove periodicity in an alternate way too.

For the second signal, sketch the signal to first get an intuition about whether it is periodic or not. You will see that the signal is a cosine wave for $t \geq 0$ and a negative sine wave for $t < 0$. The two parts do not match at $t = 0$, which indicates that the signal is not periodic. To prove this, we can use contradiction. Assume that the signal is periodic with period T_0. Then, we have $\cos(t+T_0) = x(t+T_0) = x(t)$ for all t. Now, consider the case when $t = -\frac{T_0}{2}$. Then, we have $\cos(-\frac{T_0}{2} + T_0) = \cos(\frac{T_0}{2}) = x(-\frac{T_0}{2}) = -\sin(-\frac{T_0}{2}) = \sin(\frac{T_0}{2})$. This is clearly not true! So, our assumption that the signal is periodic must be false. Therefore, the signal is aperiodic.

For the third signal, we can use the properties of the complex exponential function to prove periodicity. Write $x(t + T_0) = e^{j\omega(t+T_0)} = e^{j\omega t}e^{j\omega T_0}$. If we can find a T_0 such that $e^{j\omega T_0} = 1$, then we have periodicity. This is satisfied if $T_0 = \frac{2\pi}{\omega}$. Therefore, the fundamental period of $x(t) = e^{j\omega t}$ is $T_0 = \frac{2\pi}{\omega}$.

3.4 Power and energy of periodic signals

We can revisit the power and energy definitions for periodic signals. If x is periodic with period T_0, then the time-average power is well defined and can be computed over any interval of length T_0. Note that for a finite-duration input, E_∞ is finite and $P_\infty = 0$ (time average over an unbounded window goes to zero) whereas E_∞ is finite because we have finite-duration signal. For periodic signals, we have

$$P_\infty(x) = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt \quad \text{(independent of } t_0\text{)}, \qquad E_\infty(x) = \infty \text{ unless } x \equiv 0.$$

Thus, periodic signals are *power signals* (finite power, infinite energy).

Energy and power for a periodic input

If $x(t)$ is periodic with fundamental period T_0, then $y_d(t)$ is also periodic with the *same* T_0 (memoryless mapping preserves period). Hence

$$E_\infty(y_d) = \int_{-\infty}^{\infty} |y_d(t)|^2 \, dt = \infty, \qquad P_\infty(y_d) = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |y_d(t)|^2 \, dt \text{ (finite)}.$$

4 Even and odd signals

Recall that a mathematical function is called even if $f(-t) = f(t)$ for all t and odd if $f(-t) = -f(t)$ for all t. Examples of even functions include $\cos(t)$, t^2, and $|t|$. Examples of odd functions include $\sin(t)$, t^3, and the sign function $\text{sgn}(t)$.

Definition 2. A signal $x(t)$ is even if $x(-t) = x(t)$ for all t and odd if $x(-t) = -x(t)$ for all t.

Proposition 1. *Any signal $x(t)$ can be uniquely decomposed as the sum of an even signal $x_e(t)$ and an odd signal $x_o(t)$.*

Proof. Let $x_e(t) = \frac{x(t)+x(-t)}{2}$ and $x_o(t) = \frac{x(t)-x(-t)}{2}$. Then, we have

$$x_e(-t) = \frac{x(-t) + x(t)}{2} = x_e(t)$$

so $x_e(t)$ is even. Similarly,

$$x_o(-t) = \frac{x(-t) - x(t)}{2} = -x_o(t).$$

which is is odd. Now, we can see that

$$x_e(t) + x_o(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} = x(t).$$

To prove uniqueness, we again use the proof by contradiction method. Assume, to the contrary, that there exist another pair of even and odd signals $x_e'(t)$ and $x_o'(t)$ such that $x(t) = x_e'(t) + x_o'(t)$. Then, we have

$$x_e(t) - x_e'(t) = x_o'(t) - x_o(t).$$

The left side is even (difference of two even functions), and the right side is odd (difference of two odd functions). The only function that is both even and odd is the zero function. Therefore, we have $x_e(t) - x_e'(t) = 0$ and $x_o'(t) - x_o(t) = 0$, which implies that $x_e(t) = x_e'(t)$ and $x_o(t) = x_o'(t)$. Hence, the decomposition is unique. □

5 Application demonstration: a guitar amplifier

An amplifier system can be modeled as $y(t) = \alpha x(t)$ where $\alpha > 1$ is the amplifier gain. However, real-world amplifiers have limits on the maximum and minimum output levels they can produce. When the input signal is too large, the output signal gets "clipped" at these limits. Although this is an undesirable effect for most audio applications, it is often used intentionally by musicians to create a distorted sound effect. This is common in many music genres such as rock, metal, and punk.

We can model a simple hard-clipping (overdrive) amplifier system as:

$$y_d(t) = \begin{cases} -\beta, & \alpha\, x(t) < -\beta, \\ \alpha\, x(t), & |\alpha\, x(t)| \le \beta, \\ \beta, & \alpha\, x(t) > \beta, \end{cases} \qquad \alpha > 0,\ \beta > 0.$$

This is a *memoryless* nonlinearity: at each t, $y_d(t)$ depends only on $x(t)$. It amplifies small inputs by α and saturates at $\pm\beta$ for large inputs. Here, the parameter α controls the amount of gain (loudness) and β controls the amount of distortion to apply. Vierinen has a YouTube demonstration for this effect[3].

The transfer curve for this system is shown in Figure 2.

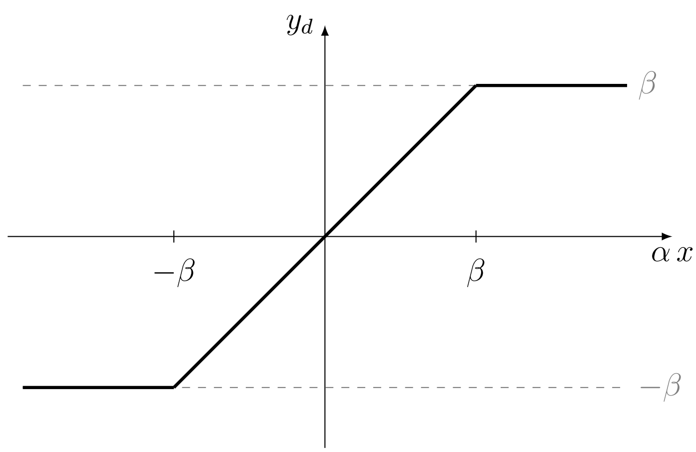


Figure 2: Hard-clipping nonlinearity: linear region $|\alpha x| \le \beta$, saturation outside.

Transforming the system: You can apply all the signal transformations to transform the system by transforming the transfer curve of the system shown above. Using Python, try to draw all time operations discussed above for $y_d(t)$ — the distorting amplifier system. Figure 3 shows the results.

[3] https://youtu.be/I30Mn_-yYF8.

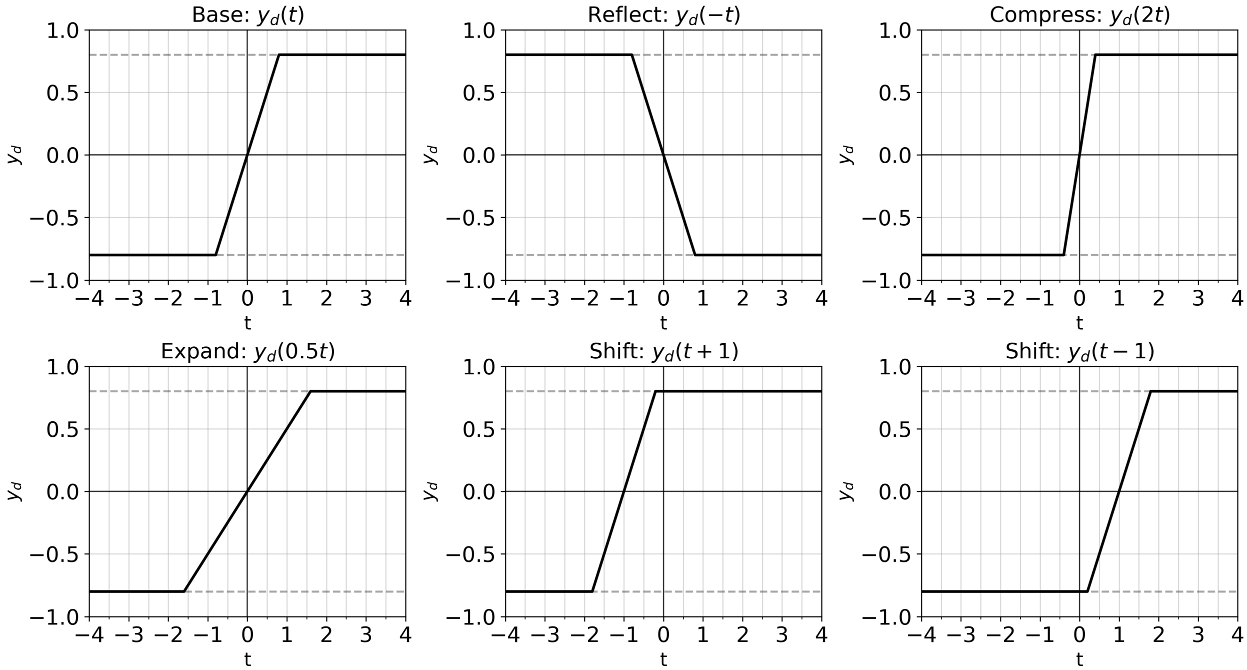


Figure 3: Time operations on the signal $y_d(t)$.

5.1 Distortion effects

By running the provided code, you can test various distortion effects by changing the parameters α and β. The code loads a .wav file for a sample guitar tone. Practice your Python (and music design) skills with this example!

5.2 Optional: Audio tone signal example and time operations

In the supplementary notes, you will find a Python notebook that creates a guitar-like audio tone. You can use computer programming to compute various time-transformed versions yourself.

Next class

The unit impulse $\delta(t)$ and step $u(t)$; convolution preview.

EE 102 Week 2, Lecture 2 (Fall 2025)

Instructor: Ayush Pandey

Date: September 10, 2025

1 Goals

- The timeless trio of signals: the complex exponential, the unit step, and impulse.
- The unit step function as a switch and an accumulator.
- The unit impulse function as an exciter and a sampler.
- The complex exponential signal as a sinusoid and a phasor.
- Applications of the timeless trio in real-world signal processing.

2 The unit step function

When we defined signals informally, we discussed how any mathematical function from your calculus class could be a signal as long as it represents something physical. Then it will not come as a surprise that for physical system applications, we would usually be interested only in positive values of time and we would want our signal to take the value of zero for all $t < 0$. Since we are extending the general concept of mathematical functions (which are defined for all t), it is important to have a mathematical way to write signals that are zero for $t < 0$. The unit step function does exactly that! Formally, we define the unit step function as

Definition 1. The unit step is a function defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

2.1 The unit step as a switch

As you can see in Figure 1, it is a discontinuous function that "steps" from 0 to 1 at $t = 0$. You may find the unit step function with different names: like the Heaviside function, or the ultrasensitive switch function.

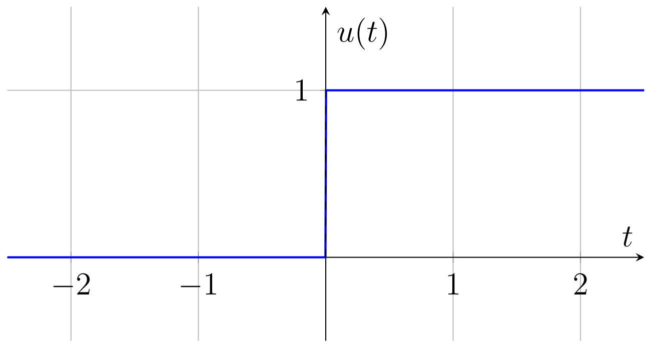


Figure 1: The unit step function $u(t)$.

So, any function that is zero for $t < 0$ can be written as the product of the unit step function and another function that defines the behavior of the signal for $t \geq 0$. For example, if we have a signal that is zero for $t < 0$ and equals $f(t)$ for $t \geq 0$, we can write it as $x(t) = f(t)u(t)$.

2.2 Example: A sinusoidal audio wave that starts at zero time

If $f(t) = Asin(\omega t + \phi)$, then we can write the signal as

$$x(t) = A\sin(\omega t + \phi)u(t).$$

This signal is zero for $t < 0$ and equals a sinusoidal wave for $t \geq 0$. The unit step function effectively "switches on" the sinusoidal wave at $t = 0$. But did we *really* need the step function here? You can argue that we could have defined all signals using two cases,

$$x(t) = \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

but this type of definition will quickly get cumbersome. So, yet again, we are introducing a mathematical object to make our lives easier, at least in the long run (at the moment, it may seem that we are making our life harder by learning another new function). For the specific sinusoidal example, the alternative way to define it is using a piecewise function that would need two different cases:

$$x(t) = \begin{cases} 0, & t < 0 \\ A\sin(\omega t + \phi), & t \geq 0 \end{cases}.$$

We would prefer $x(t) = A\sin(\omega t + \phi)u(t)$ over the piecewise definition so that we can work with just a single expression.

2.3 Unit step as a general switch

Beyond the switching behavior of unit step, we can also use it to define arbitrary "pulse" signals and also other piecewise continuous signals. For example, we can define a rectangular pulse of width τ as

$$p_\tau(t) = u(t) - u(t-\tau).$$

This pulse is 1 for $0 \leq t < \tau$ and zero otherwise (can you prove this without relying on sketching?). It is also possible to write the equation of the pulse signal by using a time reversal:

$$p_\tau(t) = u(t) + u(\tau - t) - 1$$

The sketch in both cases is the same: a pulse that starts at $t = 0$ and ends at $t = \tau$. More generally, we can write a pulse that starts at t_1 and ends at t_2 as

$$p_{t_1,t_2}(t) = u(t-t_1) - u(t-t_2).$$

See Figure 2 for a sketch of the rectangular pulse that starts at t_1 and ends at t_2.

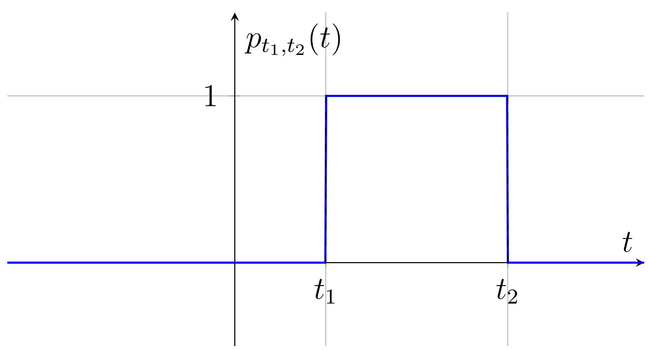


Figure 2: A rectangular pulse $p_{t_1,t_2}(t)$ that is 1 for $t \in [t_1, t_2)$ and 0 otherwise.

The pulse function provides us with more control over when the signal "turns on" and "turns off" — in Figure 2, it turns on at $t = t_1$ and turns off at $t = t_2$. We would expect such knobs of any modern switching system!

We can also define a triangular pulse of width 2τ that is zero for all $t < 0$ and $t > 2\tau$, and peaks at $t = \tau$. This triangular pulse would increase linearly from 0 to 1 in the interval $[0, \tau]$ and then decrease linearly from 1 to 0 in the interval $[\tau, 2\tau]$. By writing the equations of the two linear segments and then combining with unit step to decide when would each linear segment be "active", we can write the equation of the triangular pulse. Let's build it step-by-step. We write the two parts that "activate" the two linear segments using unit step functions:

$$w_1(t) = \underbrace{u(t) - u(t-\tau)}_{\text{active on } [0,\tau)}, \qquad w_2(t) = \underbrace{u(t-\tau) - u(t-2\tau)}_{\text{active on } [\tau,2\tau)}.$$

now, we define the two linear segments:

$$r(t) = \underbrace{\frac{t}{\tau}}_{\text{linear rise}}, \qquad f(t) = \underbrace{2 - \frac{t}{\tau}}_{\text{linear fall}}.$$

Then, the triangular pulse is obtained by gating each linear segment with its corresponding window and then adding the two gated segments:

$$tr_\tau(t) = \underbrace{r(t)}_{\text{rise}} \underbrace{w_1(t)}_{\text{gate }[0,\tau)} + \underbrace{f(t)}_{\text{fall}} \underbrace{w_2(t)}_{\text{gate }[\tau,2\tau)}.$$

$$tr_\tau(t) = \frac{t}{\tau}\big[u(t) - u(t-\tau)\big] + \left(2 - \frac{t}{\tau}\right)\big[u(t-\tau) - u(t-2\tau)\big].$$

See Figure 3 for a sketch of the triangular pulse.

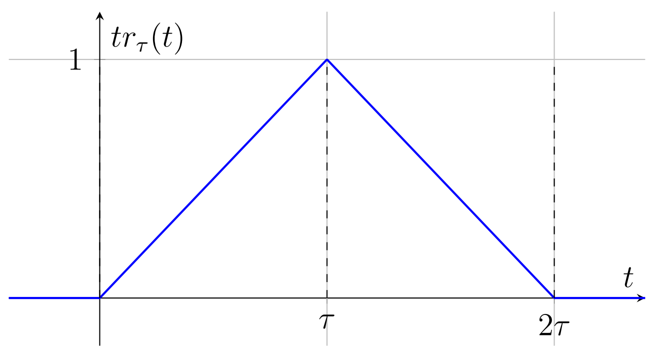


Figure 3: Triangular pulse of width 2τ: zero outside $[0, 2\tau]$, peak 1 at $t = \tau$.

The triangular pulse provides a smoother transition between the off and the on states compared to the rectangular pulse and is constructed using the unit step function as well. So, the unit step function is not just a simple switch it is also a building block for constructing more complex signals. The next "pulse" type signal that you should try is a trapezoidal pulse!

2.4 The unit step as an accumulator

The unit step function can also be viewed as an accumulator. If we integrate the unit step function, we obtain the ramp function (Problem 2.1 in HW #2):

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

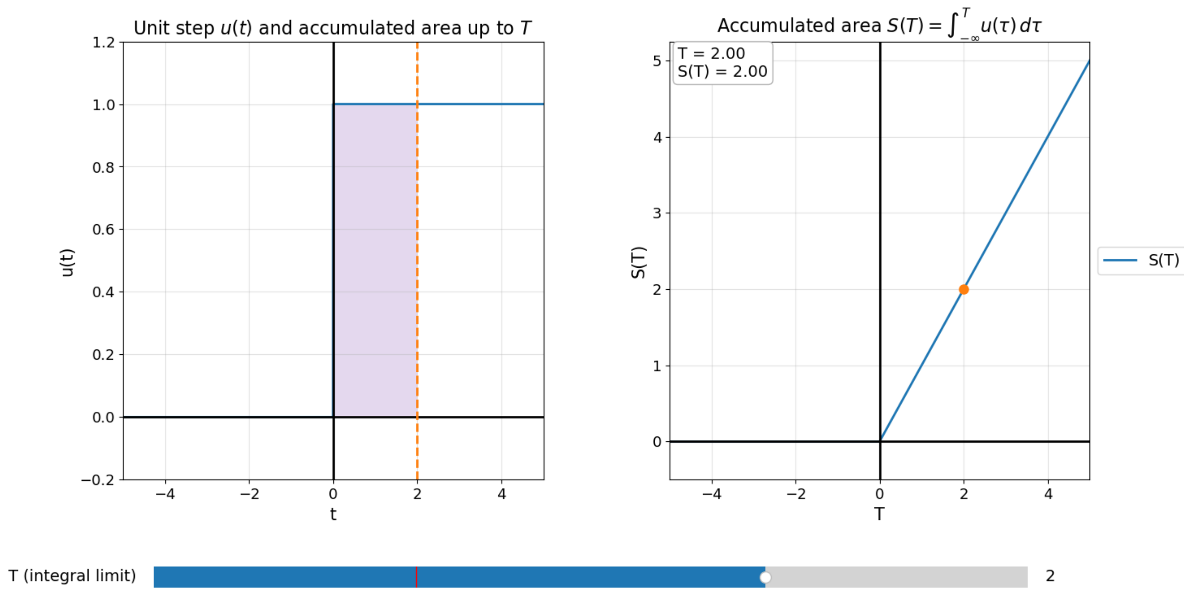


Figure 4: The unit step function $u(t)$ (left) and its accumulated area, the ramp function $r(t)$ (right). The slider below sets the upper limit of integration T.

This ramp function increases linearly for $t \geq 0$ and is zero for $t < 0$. It effectively accumulates the area under the unit step function. See Figure 4 for a sketch of the unit step function and its accumulated area (the ramp function) on the right side. A virtual manipulative is available for you to explore this concept interactively on the course Github page[1].

3 The unit impulse function

The unit impulse function is our second of the "timeless trio" of signals — one of the three most important signals in signal processing. In other fields, such as physics, it is also known as the Dirac delta function. In discrete mathematics, it is known as the Kronecker delta function. So, the same mysterious function has many different names! The reason is clear — it is a very useful mathematical object, while not even being a function in the traditional sense! We will define it formally later, but for now, let's understand it informally. You only need to remember two properties about the impulse function:

- It is zero at all times except at $t = 0$, that is $\delta(t) = 0$ for all $t \neq 0$.

[1]Here is the link for virtual manipulative on Github for unit step as an accumulator

- It has an area of 1 under its curve, that is, $\int_{-\infty}^{\infty} \delta(t)dt = 1$.

How is that possible? If a signal is zero everywhere except at one point, then there must be something unique happening at that point. To build intuition for this function, consider the rectangular pulse we defined earlier, with a slight modification. Let's define a rectangular pulse of width ϵ and height $\frac{1}{\epsilon}$ around the origin:

$$p_\epsilon(t) = \frac{1}{\epsilon}\left[u\left(t+\frac{\epsilon}{2}\right) - u\left(t-\frac{\epsilon}{2}\right)\right].$$

A sketch of this pulse is shown in Figure 5.

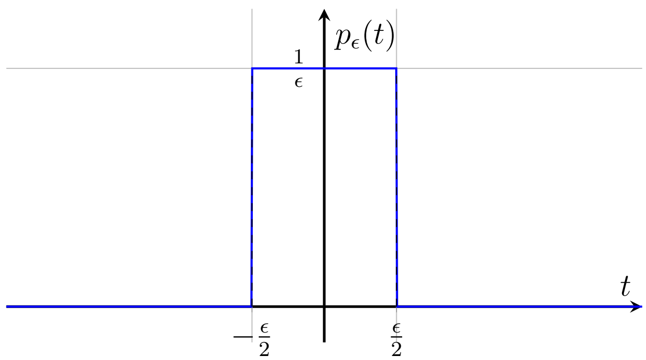


Figure 5: Rectangular pulse $p_\epsilon(t) = \frac{1}{\epsilon}\left[u\left(t+\frac{\epsilon}{2}\right) - u\left(t-\frac{\epsilon}{2}\right)\right]$ of width ϵ centered at the origin.

As you can see in the figure, the area of the rectangle is equal to 1 for all values of ϵ. But this does not satisfy the two properties of the impulse function listed above as there are points $t \neq 0$ where the pulse is non-zero. So, we need to modify this pulse further. As we make ϵ smaller, visualize how the rectangle becomes taller and narrower, while still maintaining an area of 1. You can interactively explore this concept using the virtual manipulative available on the course Github page[2].

3.1 Impulse as the limit of a rectangular pulse

Formally, we can write the unit impulse function as the limit of the rectangular pulse as ϵ approaches zero:

$$\delta(t) = \lim_{\epsilon\to 0} p_\epsilon(t) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[u\left(t+\frac{\epsilon}{2}\right) - u\left(t-\frac{\epsilon}{2}\right)\right].$$

Despite the definition above, it is not possible to write a closed-form expression for the impulse function because it is not defined at $t = 0$ and is zero everywhere else. The only

[2] Here is the link for virtual manipulative on Github for impulse approximation using a pulse. Additionally, you can approximate an impulse using a Gaussian function, see here.

quantifiable property that we know so far is that the area under the curve of a delta function is equal to 1. To visually describe an impulse function, we draw an arrow pointing upwards at the point at which the impuse is located. That is, if we have $\delta(t)$, we draw an arrow at $t = 0$; if we have $\delta(t - t_0)$, we draw an arrow at $t = t_0$. The height of the arrow is not important, but we label it with a number to indicate the area under the impulse. For example, if we have $A\delta(t - t_0)$, we draw an arrow at $t = t_0$ and label it with A to indicate that the area under the impulse is equal to A.

In practice, we can never generate a true impulse function, we can only get infinitesimally close to it as we keep making the width of the rectangular pulse infinitesimally close to zero and the height of the pulse close to infinity. Even though this may sound needlessly confusing, it provides us with a very powerful mathematical tool. One example is discussed next.

3.2 Impulse as a time-sampler

The impulse function can sample any test function at a specific point in time. Note that if we write $f(t)\delta(t)$, the product will be zero for all $t \neq 0$ because $\delta(t)$ is zero for all $t \neq 0$. The only point where the product is non-zero is at $t = 0$. So we can write $f(t)\delta(t) = f(0)\delta(t)$. This is also an impulse function located at $t = 0$ with an area of $f(0)$. Now if we integrate this product over all time, we get

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)\int_{-\infty}^{\infty} \delta(t)dt = f(0) \cdot 1 = f(0).$$

This property is known as the sifting property of the impulse function. By integrating any test function multiplied by an impulse function, we can extract the value of the test function at the location of the impulse $\rightarrow$ we have sampled that function! More generally, if we have an impulse located at $t = t_0$, we can write this impulse as $\delta(t - t_0)$. Then, by the same reasoning as above, we can show that

$$f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt.$$

To prove the above, write the integral as

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt = f(t_0)\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = f(t_0) \cdot 1 = f(t_0).$$

We obtained the latter equality by observing that $\delta(t - t_0)$ is zero at every point other than $t = t_0$. Since $f(t_0)$ is independent of t, we can take it outside the integral. The remaining integral is equal to 1 because the area under the impulse function is equal to 1. This property makes the impulse function a powerful tool in signal processing and system analysis.

In formal mathematical analysis, the above is not seen as a property but is instead used to *define* the impulse function.

Definition 2. The unit impulse function $\delta(t)$ is defined as the function for which the area under curve of its product with any test function $f(t)$ that is continuous at $t = 0$, is equal to the value of the test function at the time instant at which the impulse is located ($t = 0$ for $\delta(t)$). That is,

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0). \tag{1}$$

3.3 Relationship between unit step and unit impulse

If you look back at the unit step function, you will notice that it is discontinuous at $t = 0$. So, can we define the differentiation of unit step function with time, du/dt? Generally, the answer will be no since the derivative of a function is not defined at points where the function is discontinuous. Let's try an alternate approach. Consider the following integral:

$$\int_{-\infty}^{\infty} f(t)\frac{du(t)}{dt}dt.$$

We can evaluate this integral using integration by parts to write

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\frac{du(t)}{dt}dt &= f(t)u(t)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\frac{df(t)}{dt}dt \\
&= f(\infty) - \int_{0}^{\infty} \frac{df(t)}{dt}dt \\
&= f(\infty) - [f(t)]_0^{\infty} \\
&= f(\infty) - f(\infty) + f(0).
\end{aligned}$$

So, we derived that

$$\int_{-\infty}^{\infty} f(t)\frac{du(t)}{dt}dt = f(0), \tag{2}$$

which is the same as the definition of the impulse function discussed earlier — the area under the product of any test function and the impulse function is equal to the value of the test function at the location of the impulse. So, we can conclude by comparing equations (1) and (2) that

$$\frac{du(t)}{dt} = \delta(t).$$

Using a similar approach, you can also show that (HW #2)

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau.$$

4 Complex exponential signals

Continuous time

$$x(t) = A\,e^{j(\omega_0 t+\phi)} = A\cos(\omega_0 t+\phi) + j\,A\sin(\omega_0 t+\phi).$$

Real and imaginary parts are orthogonal sinusoids. Fundamental period $T_0 = \frac{2\pi}{\omega_0}$.

Discrete time

$$x[n] = A\,e^{j(\Omega_0 n+\phi)}.$$

This is periodic iff $\dfrac{\Omega_0}{2\pi} = \dfrac{M}{N}$ with integers M, N coprime. Then the fundamental period is $N_0 = N$. Otherwise, it is *aperiodic* on $\mathbb{Z}$.

Geometric phasor

The complex exponential traces a circle of radius A in the complex plane at angular speed ω_0 (continuous) or advances by a fixed angle Ω_0 per sample (discrete). The real part is the projection on the horizontal axis and the imaginary part is the vertical projection.