EE 102 Week 6, Lecture 2 (Fall 2025)

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Date: October 8, 2025

1 Introduction and Review

At the end of the previous lecture, we established that $e^{j\omega t}$ is an eigenfunction of LTI systems. That is, for an input $x(t) = e^{j\omega t}$, the output $y(t)$ is also a complex exponential at the same frequency ω but scaled by a complex number $H(j\omega)$:

$$y(t) = H(j\omega)e^{j\omega t}.$$

where

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau.$$

As a result of this *very important* result, we proposed that if we are able to write any signal $x(t)$ as a linear combination of complex exponentials, then we can find the output $y(t)$ of an LTI system by simply scaling each complex exponential by $H(j\omega)$ and adding them up!

The previous line is a one-line summary of Fourier analysis and synthesis — something that we will be spending a lot of time on in the next few weeks.

We start this journey by positing the following problem: Given a T-periodic signal $x(t)$, can we write it as a linear combination of complex exponentials? If so, how? More concretely, can we write

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}, \qquad \omega_0 = \tfrac{2\pi}{T},$$

the complex Fourier series coefficients are $\{a_k\}$. Note that $a_k \in \mathbb{C}$ are complex numbers, in general. The big question is; how do we find a_k?

2 Goals

Represent any periodic signal $x(t)$ as a linear combination of complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}. \tag{1}$$

Find the coefficients a_k.

3 Introduction to Fourier Series

Since the first part of the goal is "any periodic signal $x(t)$" and we are claiming that $x(t)$ can be written as a linear combination of complex exponentials. So, it must be true that (and we should make sure of it) that $e^{jk\omega_0 t}$ is periodic for every integer k.

Pop Quiz 3.1: Check your understanding!

Prove that (a) $e^{j\omega_0 t}$, (b) $e^{jk\omega_0 t}$ for $k \in \mathbb{Z}$, and (c) $\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$ are all periodic and find their fundamental periods.

Solution on page 8

Let's start to answer the second part of the goal: how do we find a_k? We will start with a simple example.

3.1 Example: A mix-sinusoidal audio signal

Consider the signal below that represents a combination of three sinusoids (added together). When you play a note of music at one specific frequency, you are playing one sinusoid. When you play a chord, you are playing multiple sinusoids at the same time by combining them together. So, in this example, we are representing an audio chord as a linear combination of complex exponentials to start our journey of representing *any* periodic signal as a linear combination of complex exponentials.

Consider

$$x(t) = \sin(6t) + \cos(2t) + \sin(12t), \qquad \omega_0 = 2.$$

Write $x(t)$ as a linear combination of $e^{jk\omega_0 t}$:

$$\sin(6t) = \frac{1}{2j}\left(e^{j6t} - e^{-j6t}\right) = \frac{1}{2j}\left(e^{j(3)\omega_0 t} - e^{-j(3)\omega_0 t}\right),$$

$$\cos(2t) = \frac{1}{2}\left(e^{j2t} + e^{-j2t}\right) = \frac{1}{2}\left(e^{j(1)\omega_0 t} + e^{-j(1)\omega_0 t}\right),$$

$$\sin(12t) = \frac{1}{2j}\left(e^{j12t} - e^{-j12t}\right) = \frac{1}{2j}\left(e^{j(6)\omega_0 t} - e^{-j(6)\omega_0 t}\right).$$

Hence

$$x(t) = \sum_{k=-6}^{6} a_k \, e^{jk\omega_0 t}, \qquad \omega_0 = 2,$$

with the nonzero Fourier series coefficients as

$$a_{\pm 2} = \frac{1}{2}, \qquad a_{\pm 3} = \pm\frac{1}{2j}, a_{\pm 6} = \pm\frac{1}{2j},$$

where the "$\pm$" pairs obey $a_{-k} = a_k^*$ for this real $x(t)$.

Here, the cos term contributes the even coefficients $a_{\pm 2}$; the sin terms contribute the odd, purely imaginary coefficients at $k = \pm 3, \pm 6$. All other a_k are zero.

4 The Trigonometric Form of Fourier Series

If the linear combination form in equation (1) is confusing and the fact that "we are representing everything as a linear combination of sinusoids" is not obvious to you, you can see how we can rewrite the Fourier series synthesis equation (1) in a more familiar trigonometric form. Although you might not find the formulation below much useful, it will at least convince you that we are indeed representing everything as a linear combination of sinusoids.

4.1 From exponentials to trigonometry

For real $x(t)$, $x^*(t) = x(t)$, and

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right) + a_0.$$

Since $x(t)$ is real, we must have $a_{-k} = a_k^*$ (you can see that this is indeed the case in the example above with real signals). Therefore

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right] = a_0 + 2\sum_{k=1}^{\infty} \mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\},$$

where we used the fact that for any complex number z, $z + z^* = 2\,\mathrm{Re}\{z\}$.

Writing $a_k = B_k + jC_k$ with $B_k, C_k \in \mathbb{R}$, we obtain the trigonometric form

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} \left[\, B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t) \,\right]. \tag{2}$$

Equation (2) shows that $x(t)$ is a linear combination of sinusoids at frequencies $k\omega_0$, $k = 1, 2, \ldots$ with real coefficients. The constant term a_0 is the DC component (average value) of $x(t)$ (as we will see again in the next section). It's also finally an equation without any complex numbers or the imaginary term j in it! So, it's hopefully more intuitive now. Let's continue towards our main goal — finding a_k.

4.2 The Fourier coefficients

To find a_k generally, let's start by multiplying both sides of equation (1) by $e^{-jn\omega_0 t}$ for some integer $n \in \mathbb{Z}$, we get

$$x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{j(k-n)\omega_0 t}.$$

Next, let us integrate this equation over $[0, T)$,

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt$$

Now, we need to evaluate the integral on the right-hand side. We have two cases:

- If $k = n$, then
  $$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T 1\, dt = T$$
  since $e^{j0} = 1$.
- If $k \neq n$, then we have
  $$\int_0^T e^{j(k-n)\omega_0 t}\, dt$$
  Using orthogonality over one period T, this integral is 0 (since integrating a sinusoid over one period will lead to the positive and negative areas canceling out). You can verify this by direct integration too:
  $$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \left[\frac{e^{j(k-n)\omega_0 t}}{j(k-n)\omega_0}\right]_0^T = \frac{e^{j(k-n)\omega_0 T} - 1}{j(k-n)\omega_0} = 0$$
  since $e^{j(k-n)\omega_0 T} = e^{j(k-n)2\pi} = 1$ for every integer $k - n$ (from the pop-quiz above).

Hence, we have the Fourier series analysis equation (the equation that gives us values of a_k):

$$a_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \qquad n \in \mathbb{Z},$$

if you replace n by k, you get the more familiar form:

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt, \qquad k \in \mathbb{Z},$$

which is the formula for the Fourier series coefficients. Note that for $k = 0$, we have

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt,$$

which is the average value (or DC component) of $x(t)$ over one period.

4.3 Properties of Fourier Series coefficients

There are many helpful properties that you should know about Fourier series coefficients. Here are a few of them (you can find the full list in the textbook):

Linearity For two periodic signals $x(t)$ and $y(t)$ with Fourier Series coefficients a_k and b_k, if we construct another signal by linear superposition, $z(t) = Ax(t) + By(t)$, then the Fourier Series coefficients for $z(t)$ satisfy

$$z(t) \iff \{\, Aa_k + Bb_k \,\}_{k\in\mathbb{Z}}.$$

Periodic convolution The Fourier series coefficients for the output $y(t)$ of a system with impulse response $h(t)$ to a periodic input $x(t)$ can be computed using periodic convolution.

Let $y(t) = (x * h)(t)$ denote periodic convolution with period T. We write the Fourier series expansion of $x(t)$ and $h(t)$ as

$$x(t) = \sum_{\ell=-\infty}^{\infty} a_\ell e^{j\ell\omega_0 t}, \qquad h(t) = \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t}, \qquad \omega_0 = \tfrac{2\pi}{T}.$$

Let $y(t) = (x * h)(t)$ denote the periodic convolution (integrate only for one period):,

$$y(t) = \int_0^T x(\tau)\, h(t-\tau)\, d\tau.$$

Let $\{c_k\}$ be the Fourier Series coefficients of $y(t)$, defined as

$$c_k = \frac{1}{T}\int_0^T y(t)\, e^{-jk\omega_0 t}\, dt.$$

We can find c_k in terms of a_k and b_k using convolution as follows. Start by substituting the expression for $y(t)$ into the definition of c_k:

$$c_k = \frac{1}{T}\int_0^T \left[\int_0^T x(\tau)\, h(t-\tau)\, d\tau\right] e^{-jk\omega_0 t}\, dt.$$

Now, using the following equations for the Fourier Series expansions of $x(\tau)$ and $h(t-\tau)$ (this is also an in-place proof for the time-shift property of Fourier Series!):

$$x(\tau) = \sum_{\ell} a_\ell e^{j\ell\omega_0\tau}$$

and

$$h(t-\tau) = \sum_m b_m e^{jm\omega_0(t-\tau)} = \sum_m b_m e^{jm\omega_0 t}\, e^{-jm\omega_0\tau},$$

we get

$$c_k = \frac{1}{T}\int_0^T\int_0^T \left(\sum_\ell a_\ell e^{j\ell\omega_0\tau}\right)\left(\sum_m b_m e^{jm\omega_0 t} e^{-jm\omega_0\tau}\right) e^{-jk\omega_0 t}\, d\tau\, dt.$$

Interchange sums and integrals and collect factors together to write,

$$\begin{aligned} c_k &= \frac{1}{T}\sum_\ell\sum_m a_\ell b_m \int_0^T\int_0^T e^{j\ell\omega_0\tau}\, e^{-jm\omega_0\tau}\, e^{jm\omega_0 t}\, e^{-jk\omega_0 t}\, d\tau\, dt \\ &= \frac{1}{T}\sum_\ell\sum_m a_\ell b_m \left[\int_0^T e^{j(m-k)\omega_0 t}\, dt\right]\left[\int_0^T e^{j(\ell-m)\omega_0\tau}\, d\tau\right]. \end{aligned}$$

Finally, note that periodic integral over one period is 0 unless the integrand is constant. So, for any integers p, $\int_0^T e^{jp\omega_0 t}\, dt = \begin{cases} T, & p=0, \\ 0, & p\neq 0. \end{cases}$ Hence the t-integral is zero unless $m=k$, and the τ-integral is zero unless $\ell = m$:

$$\int_0^T e^{j(m-k)\omega_0 t}\, dt = T\,\delta_{m,k}, \qquad \int_0^T e^{j(\ell-m)\omega_0\tau}\, d\tau = T\,\delta_{\ell,m}.$$

So, only the terms with $\ell = m = k$ survive!

$$c_k = \frac{1}{T}\sum_\ell\sum_m a_\ell b_m\, (T\delta_{m,k})\,(T\delta_{\ell,m}) = \frac{1}{T}\,(T)(T)\, a_k b_k = T\, a_k b_k.$$

$$c_k = T\, a_k\, b_k\,, \qquad k\in\mathbb{Z}.$$

Thus, periodic convolution corresponds to a *line-by-line* product of Fourier Series coefficients: each harmonic k of the output equals T times the product of the input and impulse response harmonics at the same k.

Filtering of output using aperiodic impulse response If $h(t)$ is aperiodic (but LTI) and $x(t)$ is T-periodic with Fourier Series coefficients $\{a_k\}$, then

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \sum_k a_k e^{jk\omega_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-jk\omega_0 \tau}\, d\tau}_{H(jk\omega_0)} .$$

Hence

$$y(t) = \sum_{k=-\infty}^{\infty} a_k\, H(jk\omega_0)\, e^{jk\omega_0 t}$$

i.e., each harmonic is scaled by the continuous-time frequency response $H(j\omega)$ evaluated at $\omega = k\omega_0$. So, the Fourier Series coefficients of $y(t)$ are

$$c_k = a_k\, H(jk\omega_0).$$

This will be very useful for your homework problems!

5 Practice Problems

1. Solved Example 3.6 in Oppenheim and Willsky (2nd Edition) — the square wave
2. Solved Example 3.7 in Oppenheim and Willsky (2nd Edition) — the ramp function
3. Work through the properties in Table 3.1 in Oppenheim and Willsky (2nd Edition)
4. Solved Example 3.5 in Oppenheim and Willsky (2nd Edition) — the square wave **(this is similar to HW 6 problem 1 and 2!)**

Pop Quiz Solutions

Pop Quiz 3.1: Solution(s)

We can just consider the general case: for $x(t) = e^{jk\omega_0 t}$ to be periodic with period T we need $x(t+T) = x(t)$, i.e.,

$$e^{jk\omega_0(t+T)} = e^{jk\omega_0 t} \iff e^{jk\omega_0 T} = 1 \iff k\omega_0 T = 2\pi m,\ m \in \mathbb{Z}.$$

Thus any $T = \dfrac{2\pi m}{k\omega_0}$ is a period and the smallest period (the fundamental period) is

$$T_0 = \frac{2\pi}{|k|\omega_0}.$$

Since each harmonic $e^{jk\omega_0 t}$ has a period that is an integer divisor of $\dfrac{2\pi}{\omega_0}$, the sum of all harmonics is periodic with the common (fundamental) period

$$T_0 = \frac{2\pi}{\omega_0}.$$

For $k = 1$, we get the simpler result for $e^{j\omega_0 t}$.
Moreover, note that $e^{jk2\pi} = \cos(2\pi k) + j\sin(2\pi k) = 1$ for every integer k, confirming periodicity.