EE 102 Week 5, Lecture 2 (Fall 2025)

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1 Goals

The main goal of this lecture is to learn how to visualize the process of convolution using graphs.

2 Review: Convolution definition

Recall that in continuous-time, the output $y(t)$ of an LTI system with input $x(t)$ and impulse response $h(t)$ is given by the convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau.$$

Pop Quiz 2.1: Check your understanding!

Prove that convolution is commutative, i.e., show that $x(t) * h(t) = h(t) * x(t)$.

Solution on page 15

3 Discrete time convolution

Similar to the derivation for continuous-time convolution, we can derive the discrete-time convolution sum. Consider a discrete-time LTI system with input $x[n]$, output $y[n]$, and impulse response $h[n]$. Note that for a discrete-time impulse $\delta[n]$, the output is $h[n]$. Recall the sifting property of the discrete-time impulse:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Using linearity and time-invariance of the system, we can write the output as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

This is the discrete-time convolution sum, denoted by $y[n] = x[n] * h[n]$.

4 Example: A discrete-time echo system

An audio receiver system produces an echo. When excited by a unit impulse, it responds with an echo of magnitude 1 at $n = 0$ that decays exponentially as α^n for $\alpha \in (0, 1)$ until $n = 5$ (that is, for six seconds in total). You may assume that $\alpha = \frac{1}{2}$ for numerical parts. Answer the following:

(A) Sketch the impulse response $h[n]$ and label $h[0], h[1], \ldots, h[5]$.

(B) We want to understand the kind of echo that will be produced when the audio receiver system is excited by a pulse input of unit amplitude lasting three seconds, starting at $n = 0$ and staying at unit amplitude until $n = 3$. Find $y[n]$ for this input using convolution and show your steps.

The impulse response of the system is

$$h[n] = \begin{cases} \alpha^n, & 0 \le n \le 5, \\ 0, & \text{otherwise}, \end{cases}$$

The input is a unit amplitude tone that starts at $n = 0$ and lasts three seconds. So, we can write the pulse signal for the input $x[n]$ as

$$x[n] = u[n] - u[n-3] = \begin{cases} 1, & n = 0, 1, 2, \\ 0, & \text{otherwise}. \end{cases}$$

Now, we can compute the output $y[n]$ using convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k],$$

and give $y[n]$ explicitly for all n where it is nonzero. It is important that we are careful about all values of n for which $y[n]$ is nonzero. Echos can last longer than the original sound!

4.1 Convolution computation (without visualizing)

Let us compute the output for various values of n using the convolution sum directly:

$$\begin{array}{r|l} n & y[n] \\ \hline -2: & 0 \\ -1: & 0 \\ 0: & 1 \\ 1: & 1+\alpha \\ 2: & 1+\alpha+\alpha^2 \\ 3: & \alpha+\alpha^2+\alpha^3 \\ 4: & \alpha^2+\alpha^3+\alpha^4 \\ 5: & \alpha^3+\alpha^4+\alpha^5 \\ 6: & \alpha^4+\alpha^5+\alpha^6 \\ 7: & \alpha^5+\alpha^6 \\ 8: & \alpha^6 \\ 9: & 0 \\ 10: & 0 \end{array} \quad \Rightarrow \quad \text{with } \alpha = \tfrac{1}{2}: \; y[0..8] = \left[1, \tfrac{3}{2}, \tfrac{7}{4}, \tfrac{7}{8}, \tfrac{7}{16}, \tfrac{7}{32}, \tfrac{7}{64}, \tfrac{3}{64}, \tfrac{1}{64}\right].$$

So, we find that the output $y[n]$ is nonzero for $n = 0, 1, \ldots, 8$. In general, the output of convolution in discrete-time is equal to $N + M - 1$ where N and M are the lengths of the two signals being convolved. Here, the length of $x[n]$ is 3 and the length of $h[n]$ is 6, so the length of $y[n]$ is $3 + 6 - 1 = 8$. **This is important!**

4.2 Visualizing convolution (with graphs)

Now, we will solve this by using illustrations of convolution. For each index $n = 0, 1, \ldots,$ draw three plots in a row for each n:

$$x[k], \qquad h[n-k] \text{ (as a function of } k), \qquad \text{and the resulting single sample } y[n],$$

so that the overlap of $x[k]$ and $h[n-k]$ and the accumulation giving $y[n]$ are visually clear.

Let's start by drawing $h[n]$ for $\alpha = \frac{1}{2}$:

4.3 Idea: Flip 'h' and slide through 'x'

Note the x-axis labels carefully! We have $x[k]$ and $h[k]$ because we need these for the convolution sum. We are interested in finding $y[n]$ for each value of n. For each n, we have a $x[k]$ and $h[k]$ that we use for all values of k to solve the convolution sum. Notice that $h[k]$

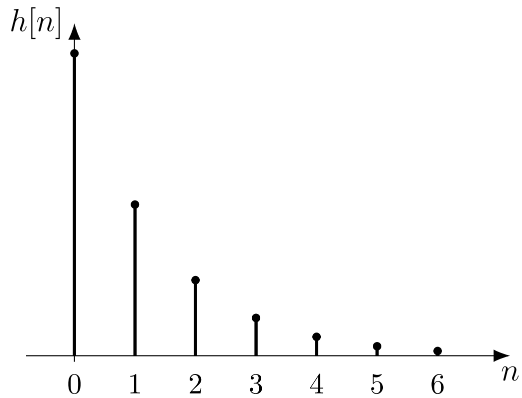

Figure 1: Impulse response $h[n]$ for $\alpha = \frac{1}{2}$.

is not directly used in the convolution sum, instead we have $h[n-k]$. This means that for each value of n, we need to flip $h[k]$ around the vertical axis and then shift it by n units to get $h[n-k]$.

For $n = 0$, the convolution is visualized in Figure 2.

Then, for $n = 1$, the convolution is visualized in Figure 3 and for all other values of n see Figures 4, 5, 6, 7, 8, 9, 10, and 11.

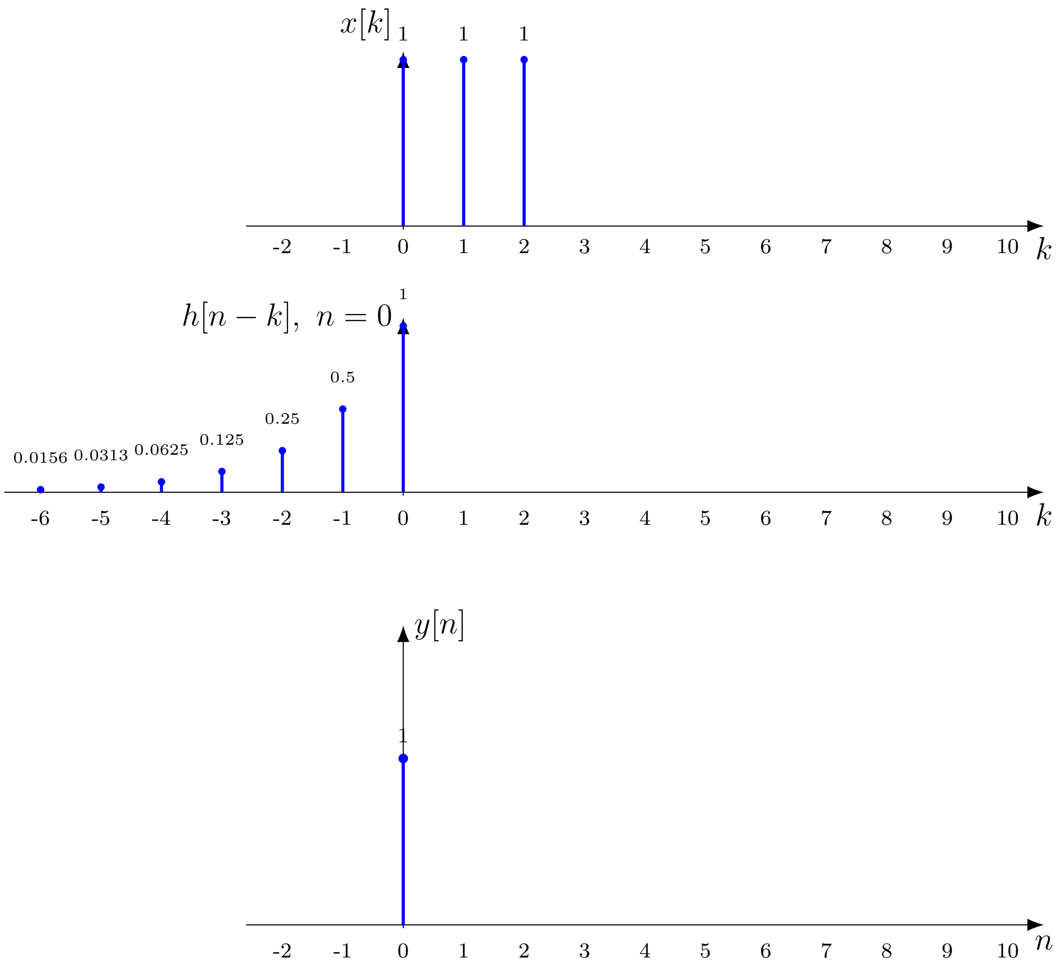

Figure 2: Convolution for $n = 0$.

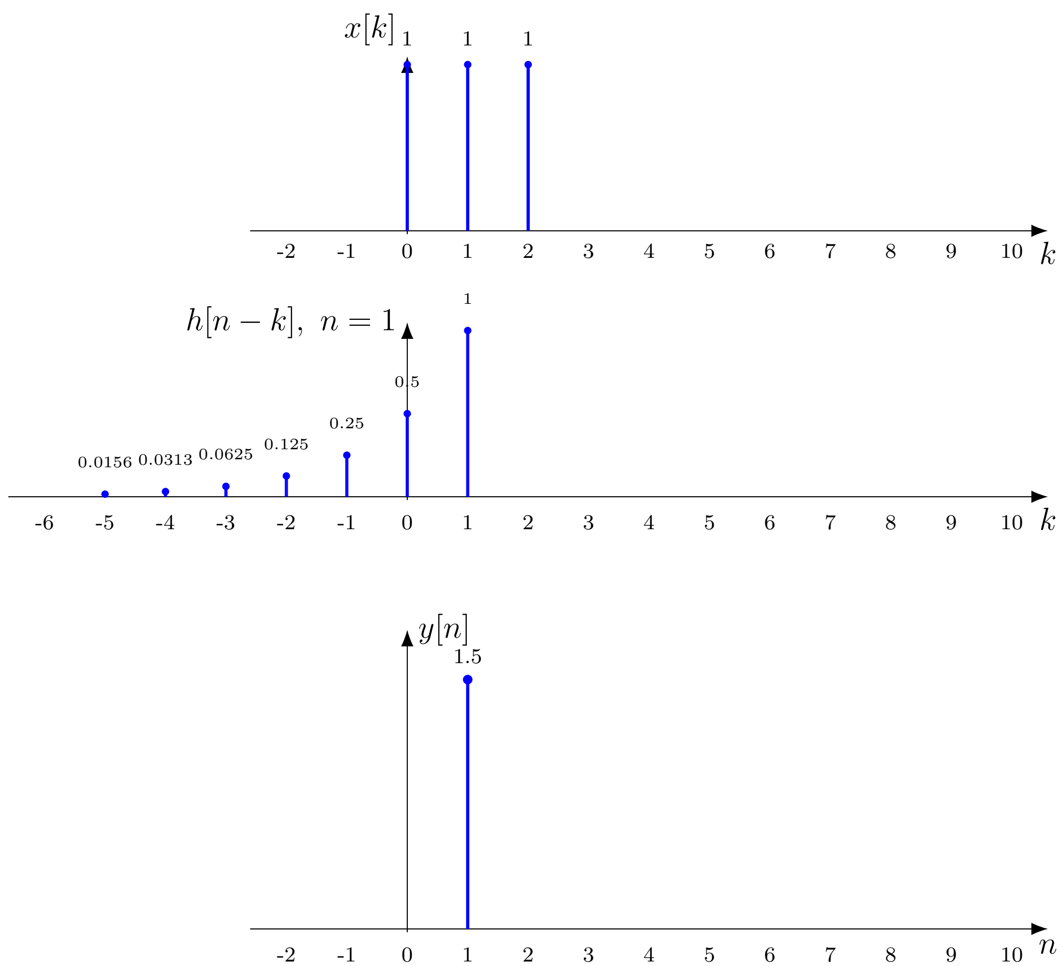

Figure 3: Convolution for $n = 1$.

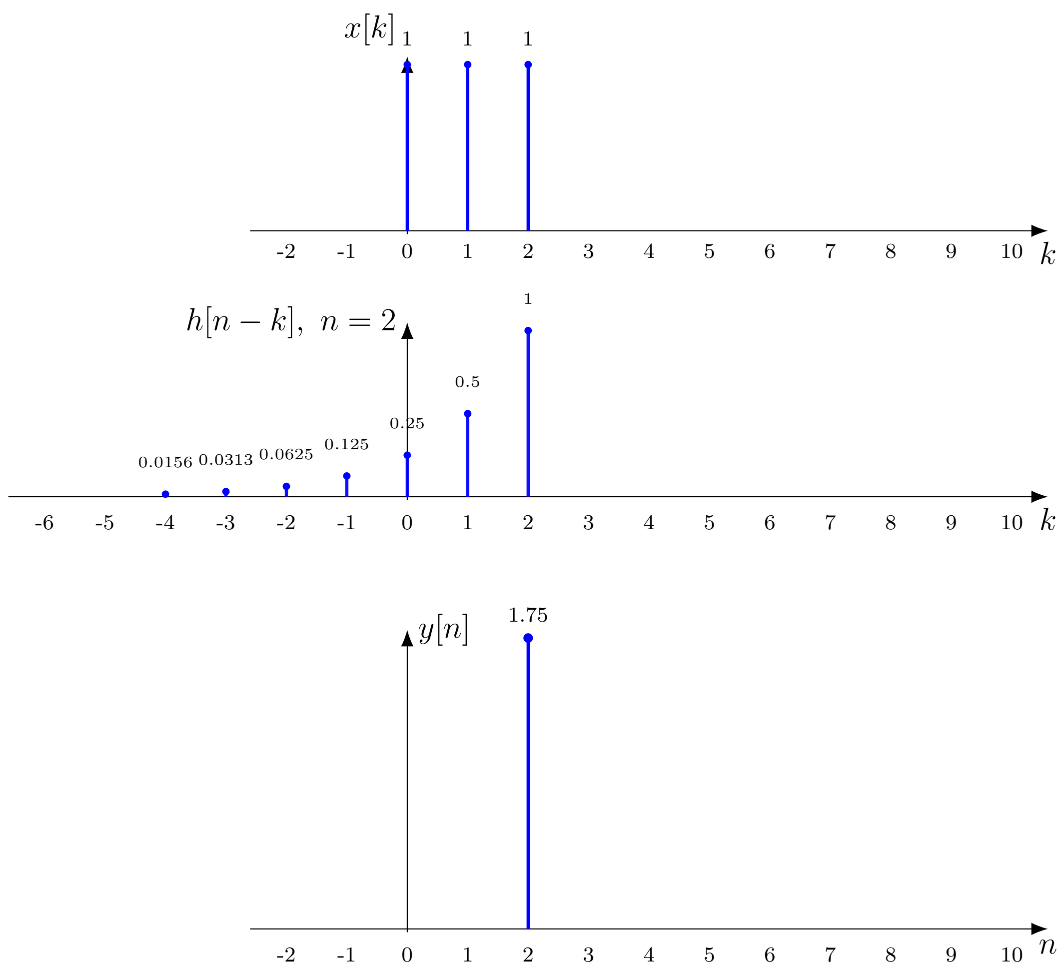

Figure 4: Convolution for $n = 2$.

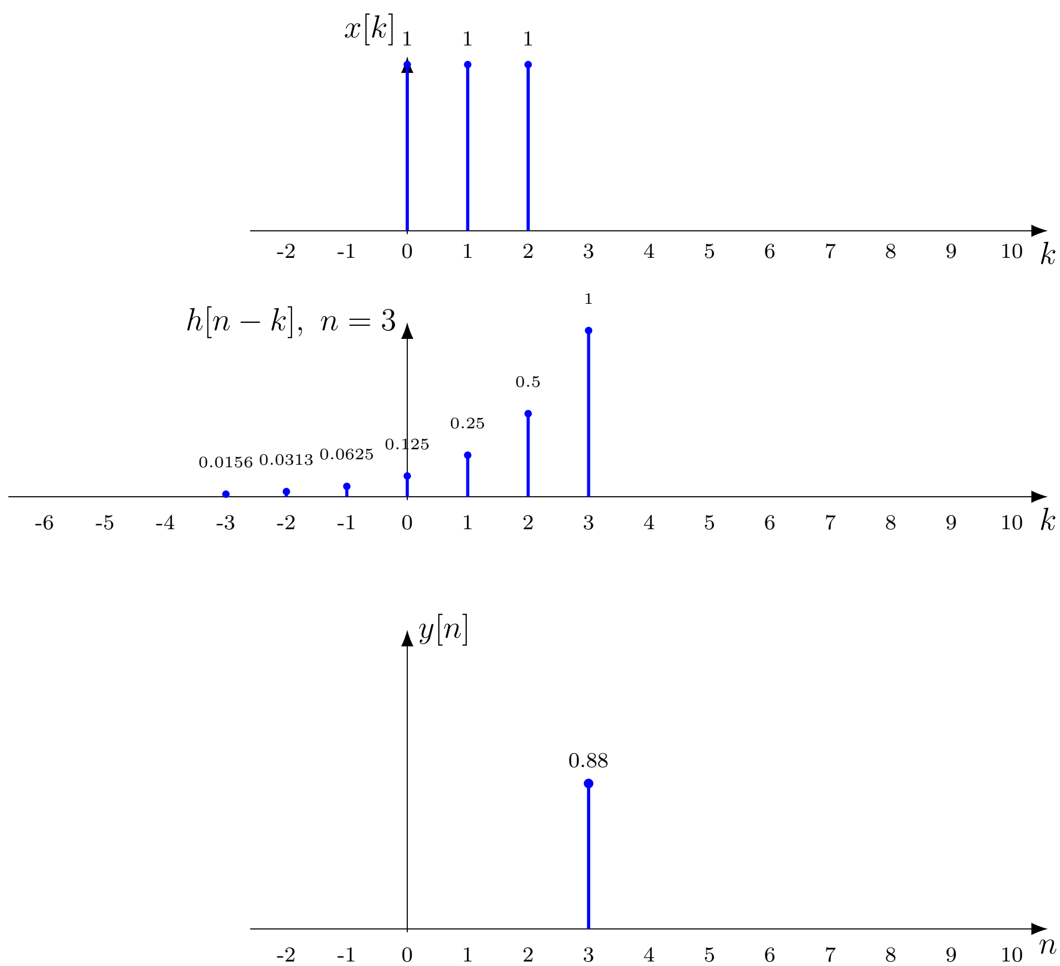

Figure 5: Convolution for $n = 3$.

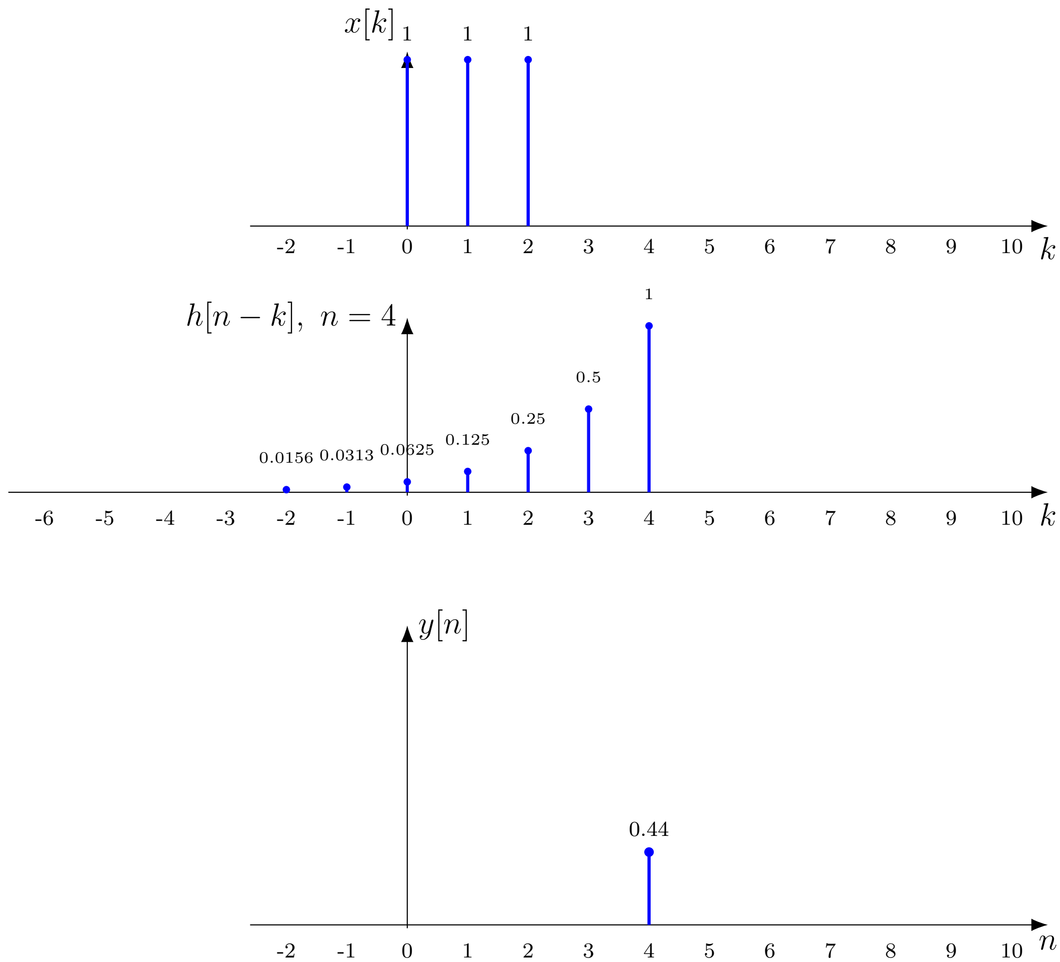

Figure 6: Convolution for $n = 4$.

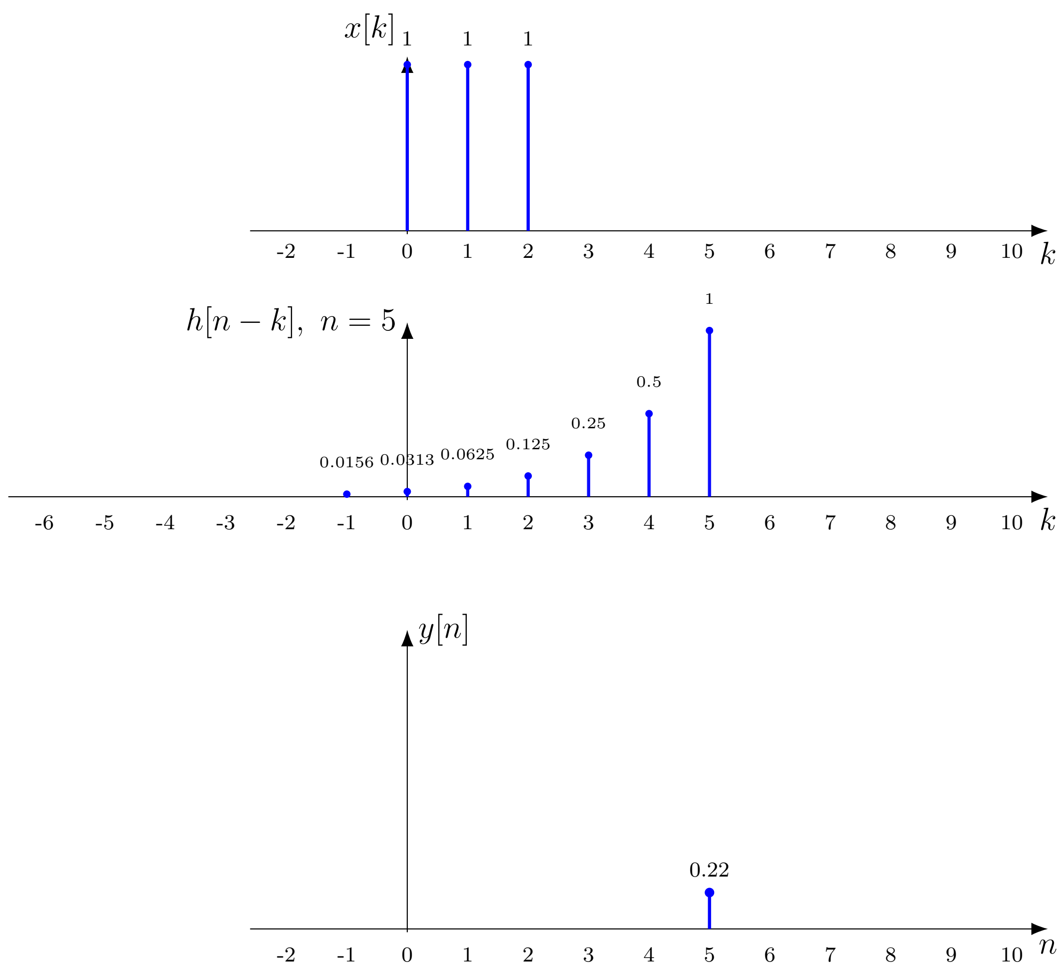

Figure 7: Convolution for $n = 5$.

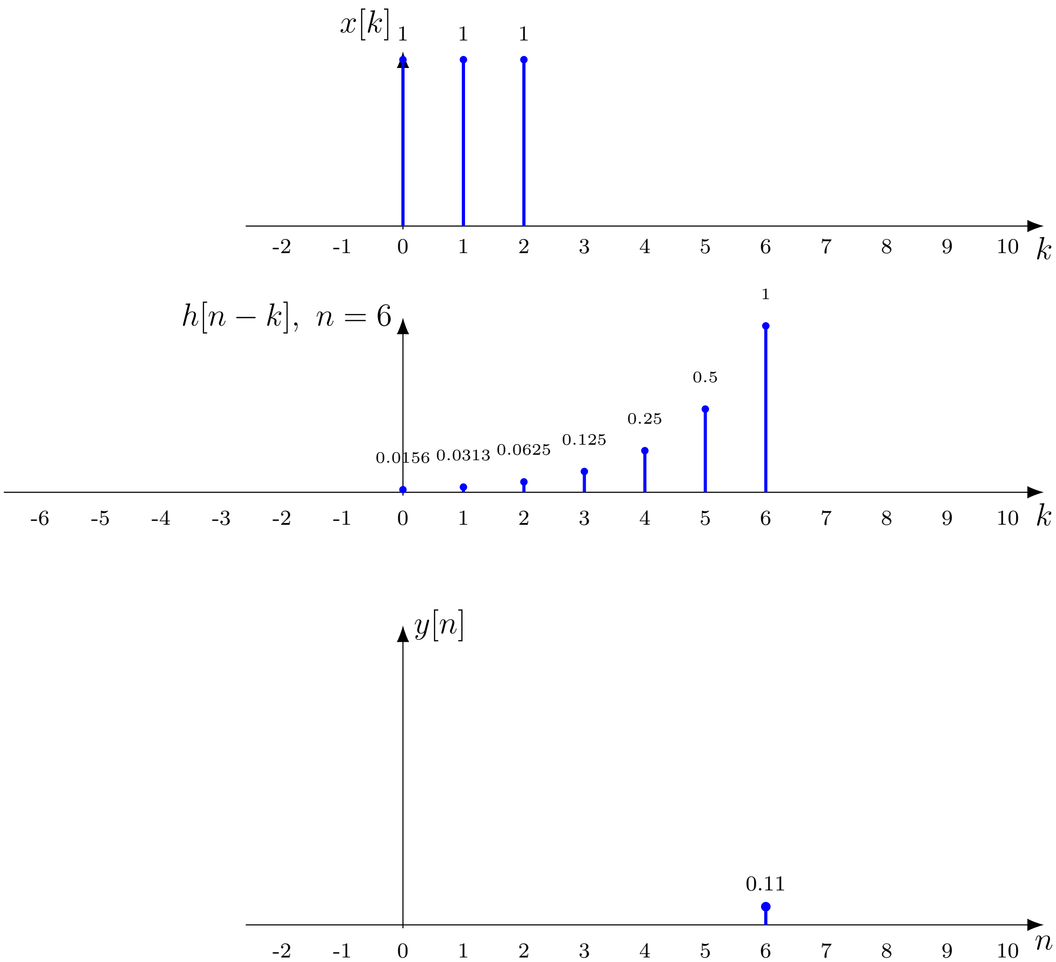

Figure 8: Convolution for $n = 6$.

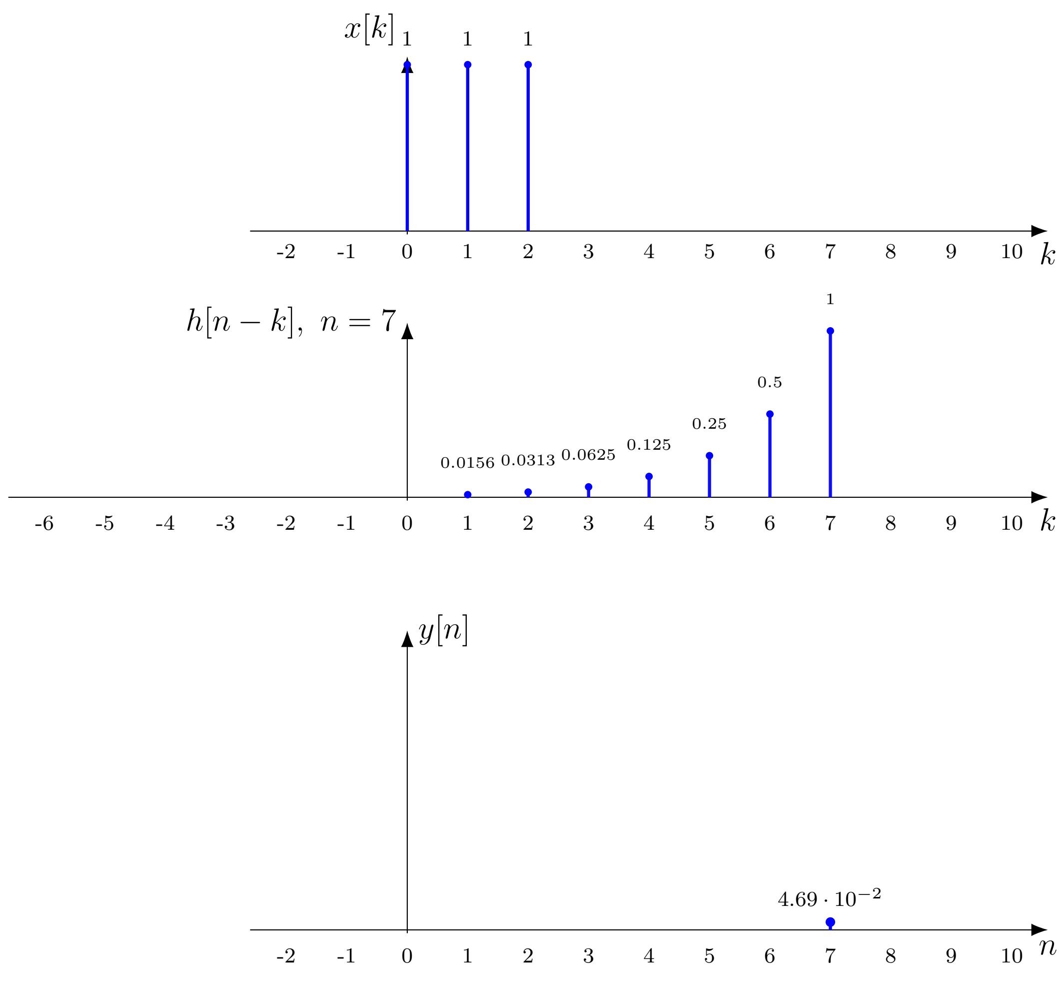

Figure 9: Convolution for $n = 7$.

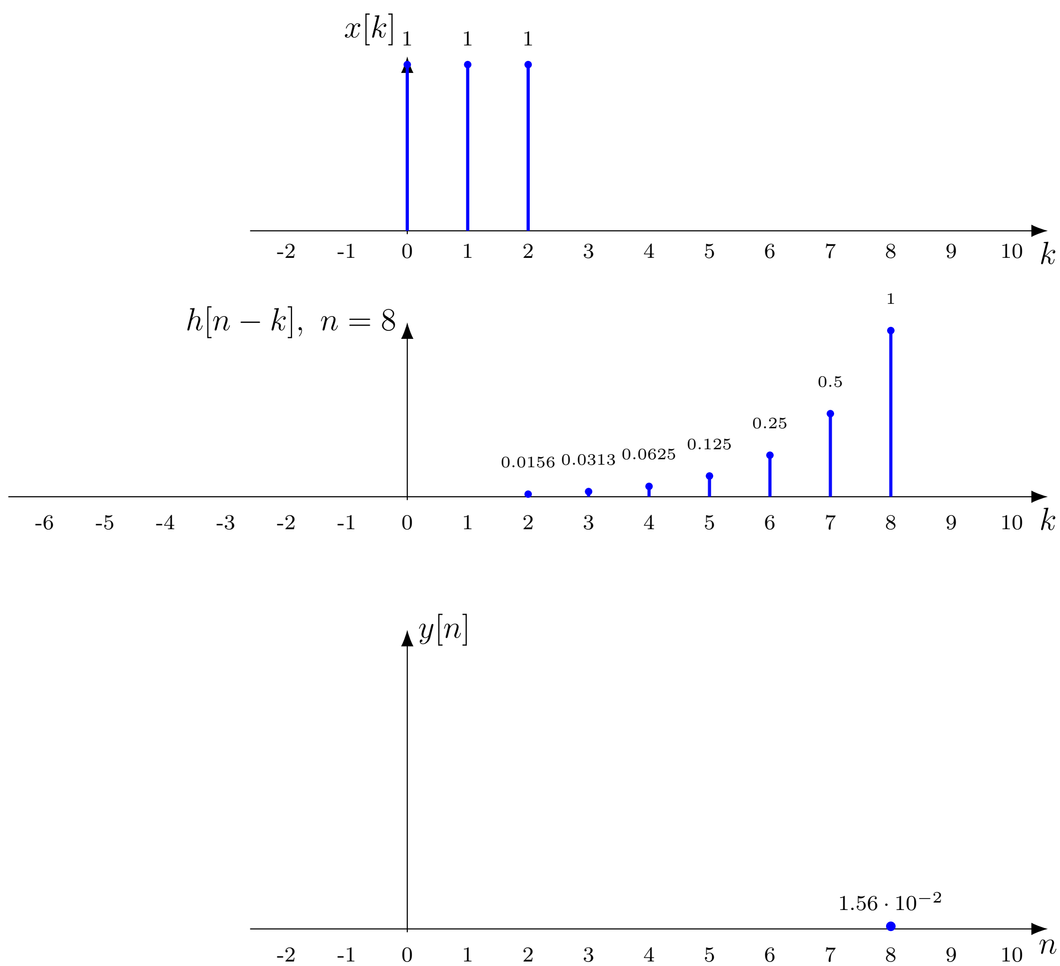

Figure 10: Convolution for $n = 8$.

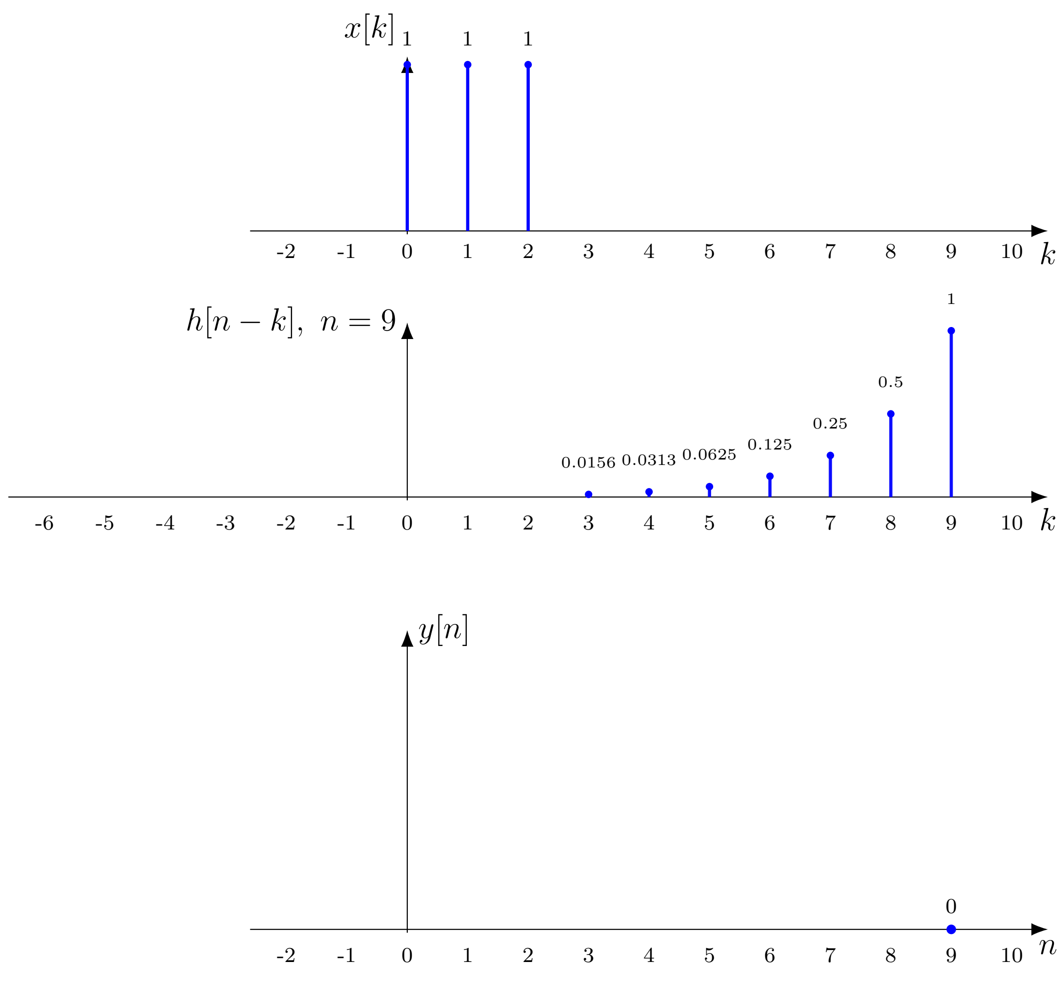

Figure 11: Convolution for $n = 9$.

Pop Quiz Solutions

Pop Quiz 2.1: Solution(s)

Write the convolution integral for left-hand side as

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$$

Now, let $s = t - \tau$. Then, $\tau = t - s$ and $d\tau = -ds$. When τ goes from $-\infty$ to ∞, s goes from ∞ to $-\infty$. Thus, we can rewrite the integral as

$$x(t) * h(t) = \int_{\infty}^{-\infty} x(t-s)h(s)(-ds) = \int_{-\infty}^{\infty} h(s)x(t-s)\,ds.$$

This is exactly the convolution integral for $h(t) * x(t)$. Hence, convolution is commutative.