Tucker's Lemma Applications

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1 A constructive proof of Fan's weak generalization

The main motivation for a constructive proof of Fan's Weak Generalization is to use this theorem to prove Tucker's Lemma constructively and, that way, induce algorithms for some applications of Tucker's Lemma [5]. By constructive proof, we mean one that (i) shows the existence of the solution and (ii) locates it by a method other than an exhaustive search. This is the sense in which Freund and Todd [1] use the word **constructive**.

All known constructive proofs of Tucker's Lemma, apparently, requires some condition on the triangulation. For instance, the first constructive proof, due to Freund and Todd [1], requires the triangulation to be a refinement of the octahedral subdivision, and the constructive proof of Yang [6] depends on the AS-triangulation that is closely related to the octahedral subdivision. Prescott and Su [5] give a constructive proof of Tucker's lemma for triangulations with a weaker condition: that it only contain a *flag of hemispheres.*

Definitions.

Let S^n be the unitary n-sphere. If A is a set in S^n, then $-A$ is the antipodal set related to A.

The definition given by Prescott and Su [5] of a *flag of hemispheres* in S^n is a sequence $H_0 \subset \ldots \subset H_n$ where each H_d is homeomorphic to a d-ball, and for $1 \leq d \leq n$, $\partial H_d = \partial(-H_d) = H_d \cap -H_d = H_{d-1} \cup -H_{d-1} \cong S^{d-1}$, $H_n \cup -H_n = S^n$, and $\{H_0, -H_0\}$ are antipodal points. One can think of a flag of hemispheres in the following way: decompose S^n into two balls that intersect along an equatorial S^{n-1}. Each ball can be thought of as a hemisphere. By successively decomposing equators in this fashion (since they are spheres) and choosing one such ball in each dimension, we obtain a flag of hemispheres.

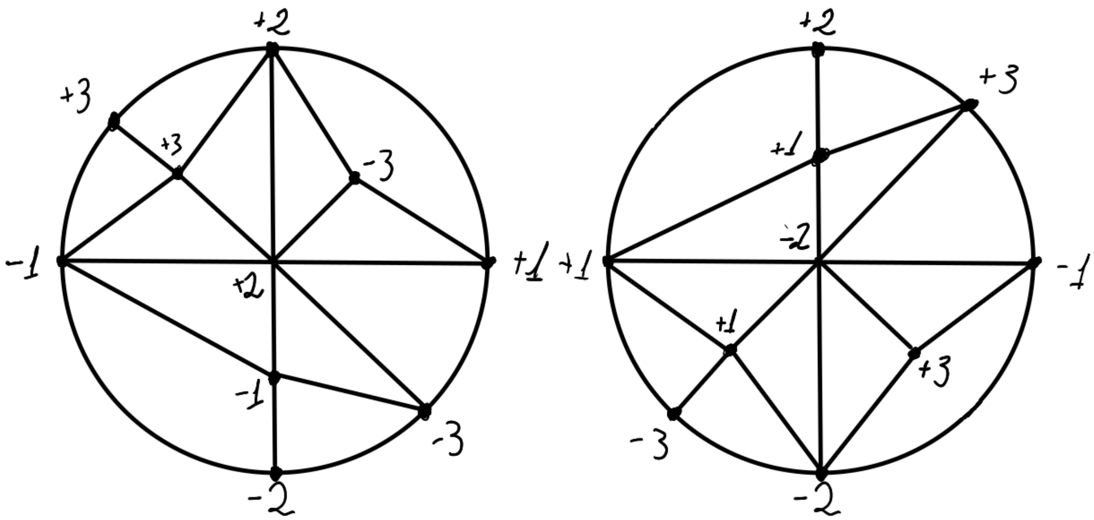

Figure 1: example of triangulation in S^2. Left ball is the front, right ball is the back

A triangulation K of S^n is (centrally) symmetric if when a simplex δ is in K, then $-\delta$ is in K. A symmetric triangulation of S^n is said to be aligned with hemispheres, or that it respect hemispheres, if we can find a flag of hemispheres such that H_d is contained in the d-skeleton of the triangulation. The **carrier hemisphere** of a simplex δ in K is the minimal H_d or $-H_d$ that contains δ.

A labeling of the triangulation assigns a non-zero integer to each vertex of the triangulation. We will say that a symmetric triangulation has an **anti-symmetric labeling** if each pair of antipodal vertices have labels that sum to zero. We say an edge is a **complementary edge** if the labels at its endpoints sum to zero.

We call a simplex in a labelled triangulation **alternating** if its vertex labels are distinct in magnitude and alternate in sign when arranged in order of increasing magnitude, i.e., the labels have the form $\{k_0, -k_1, k_2, ..., (-1)^n k_n\}$ or $\{-k_0, k_1, -k_2, ..., (-1)^{n+1} k_n\}$, where $1 \leq k_0 < k_1 < ... < k_n \leq m$. The sign of the alternating simplex is the sign of k_0.

We define a simplex as *almost-alternating* if it is not an alternating simplex, but has a facet that is alternating.

Note that, to a given simplex, if the simplex has no complementary edge (which is a condition of Fan's Lemma), all of it's alternating facets have the same sign, so we define the sign of an almost-alternating simplex as the sign of one of it's alternating facets.

Another point to note is that if a given simplex δ is not alternating, it implies that there are two vertices, with sorted labels, that contains the same value. Therefore remove one of those vertices would turn δ into a alternating simplex. As the removal of any other vertex of δ would turn it into alternating, we define that δ, an almost-alternating simplex, has exactly two alternating facets.

With that, let's state **Ky Fan's Combinatorial Lemma**.

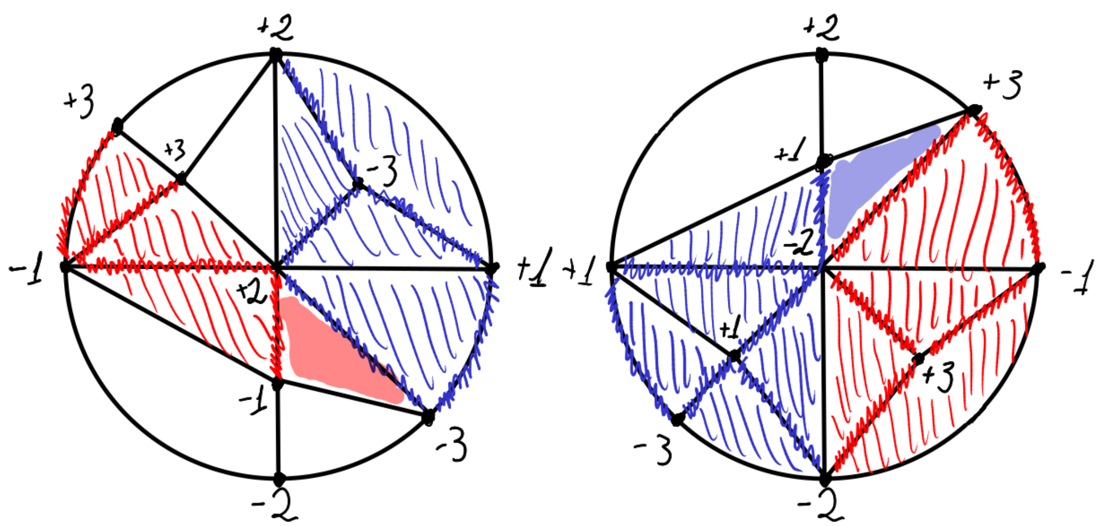

Figure 2: coloration of alternating and almost-alternating simplices of triangulation of Figure 1. Blue indicates a positive alternating simplex and red a negative one. 2-simplices with stripes are almost-alternating.

Theorem 1 (Ky Fan's Combinatorial Lemma) [3]. Let K be a symmetric triangulation of the n-sphere S^n that respects the hemispheres. Suppose that each vertex of K has (i) an anti-symmetric labelling by labels $\{\pm 1, \pm 2, ..., \pm m\}$ and (ii) no complementary edge (an edge whose labels sum to zero). Then there are an *odd* number of positive alternating n-simplices, and an equal number of negative alternating n-simplices. In particular,

$m \geq n+1$. Moreover, there is a constructive procedure to locate an alternating simplex of each sign.

Proof of Theorem 1. Let K be a triangulation of S^n aligned with the flag of hemispheres $H_0 \subset H_1 \subset ... \subset H_n$. We call an alternating, or almost-alternating simplex **agreeable** if it has the same sign of it's **carrier hemisphere**. For instance, the simplex with labels $\{-2, 3, -5, 9\}$ is agreeable if its carrier hemisphere is $-H_d$ for some d.

Let's define a graph G. A simplex δ carried by H_d is a vertex of G if it is one of the following:

1. an agreeable alternating (d-1)-simplex,
2. an agreeable almost-alternating d-simplex, or
3. an alternating d-simplex.

Two vertices δ and τ are adjacent in G if all the following hold:

1. one is a facet of the other,
2. $\delta \cap \tau$ is alternating, and
3. the sign of the carrier hemisphere of $\delta \cup \tau$ matches the sign of $\delta \cap \tau$.

Note that, with this construction, all vertices in the graph G has degree 1 or 2. Furthermore, a vertex has degree 1 if, and only if, the related simplex in K is carried by H_0 or $-H_0$ or the simplex is a n-dimensional alternating simplex.

To see that consider the 3 kind of vertices we have:

(1) An agreeable alternating (d-1)-simplex δ with carrier $\pm H_d$ is the facet of exactly two d-simplices, each of which must be an agreeable alternating or an agreeable almost-alternating simplex in the same carrier. These satisfy the adjacency conditions 1, 2 and 3 with δ, hence δ has degree 2 in G.

(2) An agreeable almost-alternating d-simplex δ with carrier $\pm H_d$ is adjacent in G to its two facets that are agreeable alternating (d-1)-simplices. (Adjacency condition 3 is satisfied because δ is agreeable and an almost-alternating d-simplex must have the same sign as its alternating facets).

(3) An alternating d-simplex δ carried by $\pm H_d$ has one alternating facet τ whose sign agrees with the sign of the carrier hemisphere of δ. That facet is obtained by deleting either the highest or lowest label (by magnitude) of δ so that the remaining simplex satisfies condition 3. Deleting the other label would give a facet with sign opposite that of the carrier hemisphere and thus cannot satisfy 3. Deleting a label that is neither highest nor lowest would give a facet that is necessarily almost-alternating. Thus δ is adjacent to its facet τ in G, and δ is not adjacent to any other of its facets.

Note that δ is the facet of exactly two simplices, one in H_{d+1} and one in $-H_{d+1}$, but it is adjacent in G to exactly one of them; which one is determined by the sign of δ, since the adjacency condition 3 must be satisfied.

So we just demonstrated that an alternating d-simplex has indeed degree 1. And since the vertex related to the 0-simplex δ carried by $\pm H_0$ has no facets, it also has degree 1.

With this construction, we can see that G consists of a collection of disjoint paths, that has endpoints $\pm H_0$ or the vertices related to the top dimension simplices.

Due to the symmetry of the K triangulation, the antipodes of any path in G is also a path in G. No path can have antipodes endpoints (otherwise the central edge of the path would be antipodal to itself, an absurd).

Since exactly two such endpoints are the nodes at H_0 and $-H_0$, there are twice an odd number of alternating n-simplices. And, because every positive alternating n-simplex has a negative alternating n-simplex as its antipode, exactly half of the alternating n-simplices are positive. Thus there are an odd number of positive alternating n-simplices (and an equal number of negative alternating n-simplices).

To locate an alternating simplex, follow the path that begins at H_0; it cannot terminate at $-H_0$ (since a path is never its own antipodal path), so it must terminate in a (negative or positive) alternating simplex. The antipode of this simplex will be an alternating simplex of the opposite sign.

Tucker's Lemma implication. Based in this results, we can construct a proof of Tucker's Lemma. Let $m = n$, as defined in Fan's Lemma. That way, the valid of condition (i) of Fan's Lemma is still valid, however the condition (ii) fails, since, for "lack of labels", we can't construct a alternating n-simplex. In fact, removing condition (ii) of Fan's Lemma allows another group of simplices to have degree 1 in G, that's the agreeable almost-alternating simplices with a complementary edge. By the construction of G in paths, there must

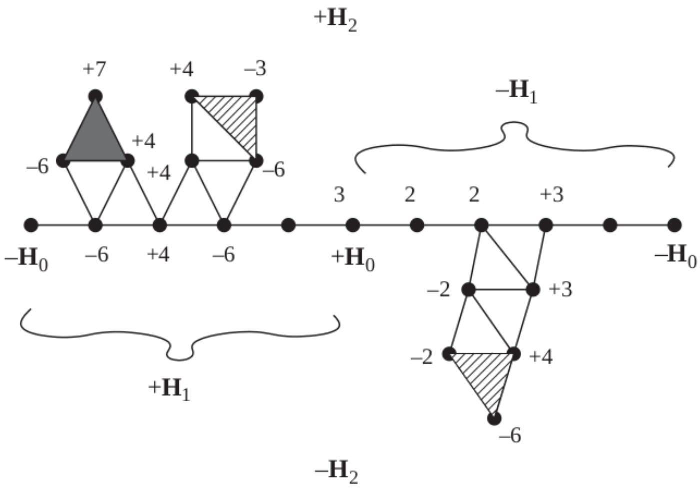

Figure 3: A portion of triangulation of 3-sphere. The two striped triangles are connected by a path in the interior of $-H_3$ (not shown). The black triangle connects to an alternating simplex $\{+4, -6, +7, -9\}$ in the interior of +H 3 (not shown)[3].

be an odd number of agreeable positive almost-alternating simplices with a complementary edge.

Therefore, there is a complementary edge in the K triangulation. And, using a similar process to find an alternating simplex in Fan's Lemma, we can find a complementary edge.

Commentaries on a non-constructive proof of Fan's using unique add and unique remove

It's interesting to notice that we can build a framework to prove Fan's Lemma using the **Unique Add** and **Unique Remove** properties in hipergraphs. That prove can be done in a similar fashion as presented in the prove of Sperner's Lemma using these techniques.

The trick is construct two A, B be n-regular hypergraphs on $V(K)$ (just a reminder that K is a triangulation of a S^n sphere). We can define A as having all n-dimension simplices, therefore having the unique remove property, since by removing any vertex of a triangle, we can add another from the triangle adjacent to the opposite facet of the removed vertex, and that new construction is still in the edge set of A.

Then if we can build a hipergraph B such that the intersection $A \cap B$ only contains alternating n-simplices, we can use **Theorem 16.1** from the course notes, and conclude that there is a even number of alternating n-simplices in K.

By the antipodality of K, we can infer that half of these alternating simplices are positive, and the other half, negative.

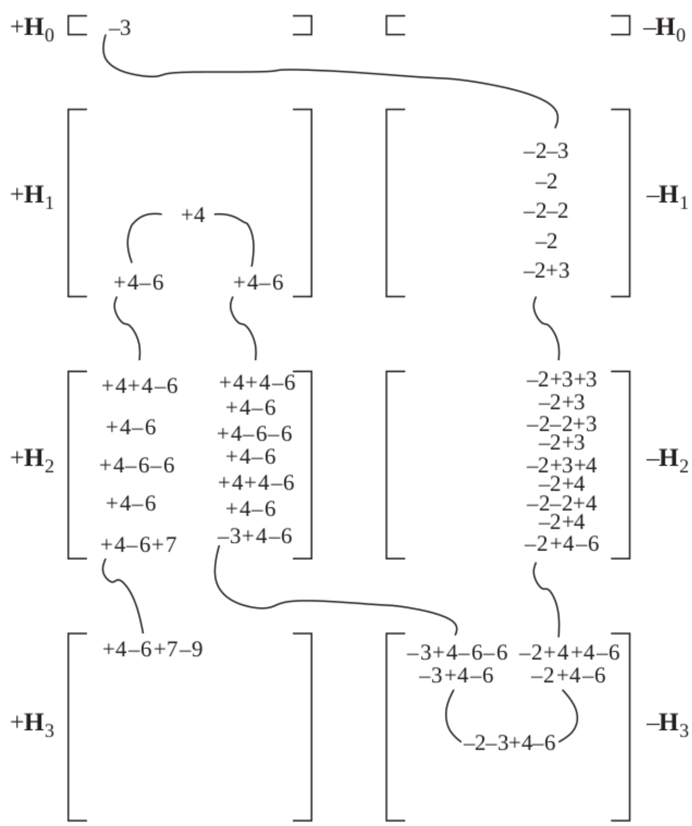

Figure 4: A schematic example of what sets of labels of simplices along a path in G could look like. This path corresponds to the one shown in Figure 3 [3].

2 Ham Sandwich

The informal statement that gave the ham sandwich theorem its name is this: For every sandwich made of ham, cheese, and bread, there is a planar cut that simultaneously halves the ham, the cheese, and the bread. The mathematical ham sandwich theorem says that any d (finite) mass distributions in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane.

Figure 5: Example of a ham sandwich.

We'll present two demonstrations for the Ham Sandwich Theorem. The first one is a continuous version of the Theorem, provided by Longueville [3], and the other one were provided by Matousek [4].

2.1 Consensus 1/2-Division

Suppose two daughters want to divide a piece of land they inherited. Each member of the family has a distinct opinion about the value of each part of the land. We want to prove that there is a division of the land that satisfies every family member.

Let's see the land as an interval $I \in [0, 1]$. Let define each family member's preference as a set $\mu_1, ..., \mu_n$ of probability functions over the interval I. Then, let's define A_{+1} and A_{-1} such that $A_{+1} \cup A_{-1} = I$. That way $A_{\pm 1}$ represents the division of the "interval" of the land each daughter is going to receive.

Theorem 1. Let $\mu_1, ..., \mu_n$ be n continuous probability measures on the unit interval. Then there exists a solution to the consensus 1/2-division problem within the space $X_n \cong \mathbb{Z}_2^{*(n+1)}$. In particular, it suffices to make n cuts.

Proof. We want to find $A_{\pm 1}$ such that $\mu_i(A_{+1}) = \mu_i(A_{-1}) = 1/2$ for every i. Note that we want $A_{\pm 1}$ to be the union of a **finite** number of intervals.

Let's consider a space X_n made of all possible combinations $A_{+1} \cup A_{-1} = I$ made by cutting I n times. (note that n is the number of distinct probabilities functions we have).

$$X_n = \{((\epsilon_0, t_0), ..., (\epsilon_n, t_n)) : \epsilon_i \in \{\pm 1\}, t_i \geq 0, \textstyle\sum_{i=0}^{n} t_i = 1\}$$

Where t_i defines the length of the ith slice of the interval and ϵ_i defines if this slice belongs to A_{+1} or A_{-1}.

With this, we can describe X_n as the space $\mathbb{Z}_2^{*(n+1)}$, or the boundary complex of the (n+1)-dimensional **cross polytope**.

Note that $-1 \cdot ((\epsilon_0, t_0), ..., (\epsilon_n, t_n)) = ((-1 \cdot \epsilon_0, t_0), ..., (-1 \cdot \epsilon_n, t_n))$. In another words, the space is antipodal. Then given a solution $x \in X_n$, we have $-x$, which derives from the x subdivision, but exchange the land each daughter received.

Let's define the map $f : X_n \to \mathbb{R}^n$ by:

$$x \mapsto \begin{pmatrix} \mu_1(A_{+1}(x)) - \mu_1(A_{-1}(x)) \\ \vdots \\ \mu_n(A_{+1}(x)) - \mu_n(A_{-1}(x)) \end{pmatrix} \tag{1}$$

Is a continuous antipodal map, and hence, by Borsuk-Ulam, has a zero that yields the desired partition.

Approximation a Solution.

Besides just define that a solution exists, we want to find (or approximate) a solution $x \in X_n$ such that $\mu_i(A_{+1}(x)) = \mu_i(A_{-1}(x)) = 1/2$.

We can a division $A_{+1} \cup A_{-1} = I$ as ϵ-approximated if $\mu_i(A_{+1}), \mu_i(A_{-1}) \in [1/2 - \epsilon, 1/2 + \epsilon]$ for all i. In order to find such a solution, consider an antipodally symmetric triangulation K subdividing $\mathbb{Z}_2^{*(n+1)}$ that is fine enough, in another words, for any edge xy in K the inequality $|\mu_i(A_{+1}(x)) - \mu_i(A_{+1}(y))| \leq \epsilon$ holds. This exists by continuity of the μ_i.

Let construct an antipodal labelling $\lambda : vert(K) \to \{\pm 1, ..., \pm n\}$ of K.

Let's define:

$$\mu(x) = min\{\mu_i(A_{+1}(x)), \mu_i(A_{-1}(x)) : i \in [n]\}$$
$$i(x) = min\{i \in [n] : \mu_i(A_{+1}(x)) = \mu(x) or \mu_i(A_{-1}(x)) = \mu(x)\}$$

Given a division x, the function $i(x)$ can be seen as which of the family members that considers this division more unjust than any other family member. And $\mu_i(x)$ is the value given by such member to the lower half of the division.

If $\mu(x) = 1/2$, then all family members considers the division x as just. Otherwise we define:

$$x \mapsto \begin{cases} +i(x) & \text{if } \mu_i(x)(A_{+1}(x)) = \mu(x) \\ -i(x) & \text{if } \mu_i(x)(A_{-1}(x)) = \mu(x) \end{cases} \tag{2}$$

This function maps a solution x for $+i(x)$ if the value of the division considered most unjust is smaller to A_{+1} and $-i(x)$, otherwise.

If none of the vertices of K gives an exact solution, then λ is an antipodal labelling of K.

By Tucker's Lemma, there exists an edge xy in K and $i \in [n]$ such that $\lambda(x) = +i$ and $\lambda(y) = -i$. In particular, $\mu_i(A_{+1}(x)) < 1/2$ and $\mu_i(A_{+1}(y)) > 1/2$, since the difference between both is at most ϵ, then both lie in the interval $[1/2 - \epsilon, 1/2 + \epsilon$, in another words, it is a ϵ-approximated solution.

As we have seen in the previous section, a complementary edge can be found by following a path in some graph. In particular, the ϵ-approximation to consensus 1/2-division can be found using this procedure.

2.2 The Ham Sandwich Theorem

This section presents the version of the Ham Sandwich Theorem shown by Matousek [4].

A *finite Borel measure* μ in $\mathbb{R}^d$ is a measure in $\mathbb{R}^d$ such that all open subsets of $\mathbb{R}^d$ are measurable and $0 < \mu(\mathbb{R}^d) < \infty$. In another words, it can be seen as a function that maps a open set to the positive values of $\mathbb{R}^d$.

Theorem 1. (Ham Sandwich theorem for measures) [4]. Let $\mu_1, \mu_2, ..., \mu_d$ be finite Borel measures on $\mathbb{R}^d$ such that every hyperplane has measure 0 for each of the μ_i (in the sequel, we refer to such measures as "mass distributions"). Then there exists a hyperplane h such that $\mu_i(h^+) = 1/2\mu_i(\mathbb{R}^d)$ para $i = 1, 2, ..., d$. And where h^+ denotes one of the half-spaces defined by h.

Proof. Let $u = (u_0, u_1, ..., u_d)$ be a point in the sphere S^d. If, at least, one of the components $u_1, ..., u_d$ is nonzero, then we add u to the half-space defined as:

$$h^+(u) := \{(x_1, ..., x_d) \in \mathbb{R}^d : u_1x_1 + ... + u_dx_d \leq u_0\}$$

Note that, with this construction, antipodal points in S^d corresponds to opposite half-spaces.

For u in the boundary of the half-space or, in another words, having the form $(u_0, 0, 0, ..., 0)$, where $(u_0 = \pm 1)$, we have, by the same formula, $h^+((1, 0, 0, 0, 0, 0)) = \mathbb{R}^d$ and $h^+((-1, 0, 0, 0, 0, 0)) = \emptyset$.

Let's define a function $f : S^d \to \mathbb{R}^d$ by $f_i(u) := \mu_i(h^+(u))$.

We want to find u such that $f_i(u) = 1/2\mu_i(\mathbb{R}^d)$ for every i or, in another words, the hiperplane that cut all mass distributions of the space by half.

It's easy to check that if $f(u_0) = f(-u_0)$ for a u_0 in S^d, then the boundary of the hiperplane is the hiperplane that we are looking for.

To apply Borsuk-Ulam, remains to show that f is continuous. This fact is quite intuitive, given two vectors $v, w \in S^d$, as w gets closer to v, the half-space induced by w gets closer to the half-space induced by v.

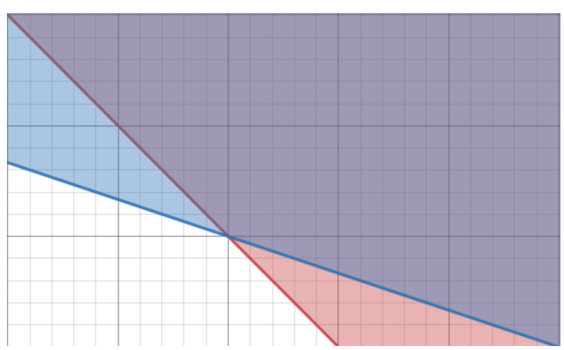

Figure 6: Example of intersection of hiperplanes. (figure generated from *desmos.com*)

So by Borsuk-Ulam, there is u^* such that $f_i(u^*) - f_i(-u^*) = 0$.

3 Necklace Partition

Two thieves have stolen a necklace with various precious stones. As the thieves don't know the value of each stone, they want to split each type of stone evenly. They want to do that division with as few cuts as possible.

We assume the necklace if open (with two ends) and that there are d different kind of stones, with an **even** number of each kind.

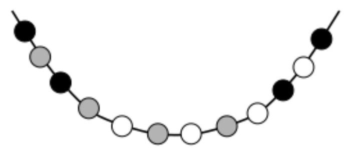

Figure 7: Example of necklace [4].

3.1 Discrete version of necklace division

Theorem 1 (Necklace Theorem). Every (open) necklace with d kinds of stones can be divided between two thieves using no more than d cuts.

Before we prove this Theorem, let's define another one that we'll use.

Theorem 2 (Hobby-Rice Theorem) [2]. Let $\mu_1, \mu_2, ..., \mu_d$ be continuous probability measures on [0, 1]. Then there exists a partition of [0, 1] into $d+1$ intervals $I_0, ..., I_d$ and signs $\epsilon_0, ..., \epsilon_d \in \{-1, +1\}$ with $\sum_{j=0}^{d} \epsilon_j \cdot \mu_i(I_j) = 0$ for $i = 1, 2, ..., d$.

Note that Hobby-Rice Theorem can be derived from the **Continuous Ham Sandwich Theorem**, which was proved in this work, at section 2.1 (Consensus 1/2-Division).

With that in mind, let's prove theorem 1.

Proof Theorem 1. (Necklace Theorem). Suppose we have t_i stones of type i. Therefore $n := \sum_{i=1}^{d} t_i$. Let's imagine that the necklace corresponds to the interval $[0, 1]$ and the kth stone corresponds to the segment $[k - 1/n, k/n)$.

We define the characteristic function $f_i(x) : [0, 1] \to 0, 1$ for $x \in [k - 1/n, k/n)$ as $f_i(x) = 1$, if the kth stone of the necklace is of the ith kind. And $f_i(x) = 0$ otherwise.

Each function f_i defines a measure μ_i on $[0,1]$, by $\mu_i(A) := n/t_i \int_A f_i(x)\,\mathrm{d}x$. Thus $\mu_i(A)$ denotes the fraction of stones of the ith kind that is on the part A of the necklace.

That way, a thief receives the part " + " of the interval, and the other the part " − ".

We then apply **Hobby-Rice Theorem** of f_i to show that there is a division made by this d cuts that is fair, in the sense that divides each kind of stone in half.

Note however that, even though this division is fair, it can be nonintegral (i.e., some stones would have to be cut). We use a induction in the numbers of nonintegral cuts. If a cut subdivides a stone of the ith type, then either the cut is unnecessary, or there is another cut through a stone of type i. In the latter case we can move both cuts away from the stones, without changing the balance.

With this demonstration over, we then define the continuous variant of this theorem.

3.2 Continuous version of necklace division

Proof of the continuous necklace theorem. With every point $x = (x_1, x_2, ..., x_{d+1}) \in S^d$ we associate a division of the interval $[0,1]$ into $d+1$ parts, of lengths $x_1^2, x_2^2, ..., x_{d+1}^2$. That is, with $\mathbf{x}$ we associate the cuts at the points $z_i := x_1^2 + ... + x_i^2$, where $0 = z_0 \leq z_1 \leq ... \leq z_d \leq z_{d+1} = 1$. The sign ϵ_j, of the interval $I_j = [z_{j-1}, z_j]$ is chosen as $sign(x_j)$. This defines a continuous function $g : S^d \to \mathbb{R}^d$.

$$g_i(x) := \textstyle\sum_{j=1}^{d+1} sign(x_j) \cdot \mu_i([z_{z_j-1,z_j}])$$

In another words, $g_i(x)$ is the amount of i-stone given to the first thief minus the quantity of the same stone given to the second thief. This function is clearly antipodal. Thus, an $x \in S^d$ exists with $g(x) = 0$. This $\mathbf{x}$ encodes a just division, and we are done.

References

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