Cinématique

MRU:
$V = cst$
$x = V \cdot t + x_0$

MRUA:
$a = cst$
$V = V_0 + a \cdot t$
$x = x_0 + V_0 t + \frac{1}{2} a t^2$

MC:
$\omega = cst$
$\theta = \omega t + \theta_0$ ($a_{\parallel} = 0$)

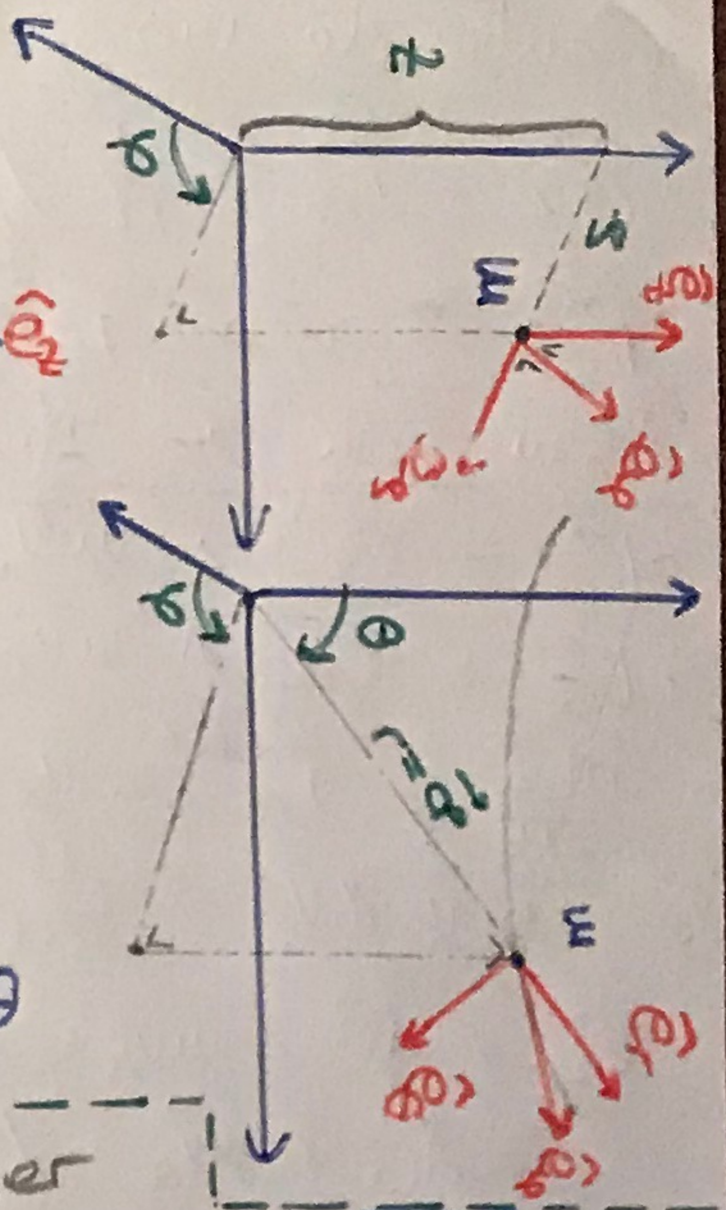
MCUA:
$\alpha = cst$
$\omega = \omega_0 + \alpha t$
$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$

MCU / MCUA

$\vec{a} = \vec{a}_\perp + \vec{a}_\parallel$
$a = (a_\perp^2 + a_\parallel^2)^{1/2}$
$a_\parallel = r\ddot{\theta}$ ⇒ acc. tang. / acc. cent.
$a_\perp = V^2/r = r\omega^2$
$\begin{cases} x(t) = r\cos\theta \\ y(t) = r\sin\theta \end{cases}$
$v = r\omega = \frac{2\pi r}{T}$
$s = r\theta$
$\vec{r} \cdot \vec{v} = 0$
$\vec{a} \cdot \vec{v} = 0$

Repères

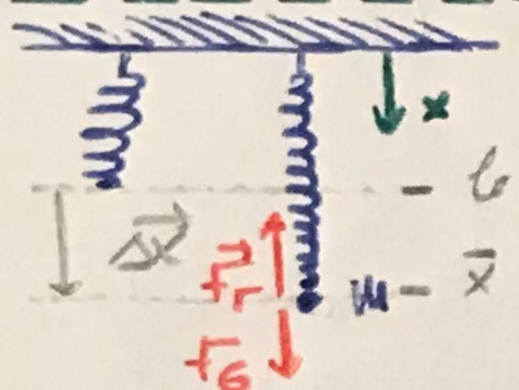
- cylindrique : $\vec{\omega} = \dot{\varphi}\,\hat{e}_z$
  - $\vec{r} = \rho\,\hat{e}_\rho + z\,\hat{e}_z$
  - $\vec{v} = \dot{\rho}\,\hat{e}_\rho + \rho\dot{\varphi}\,\hat{e}_\varphi + \dot{z}\,\hat{e}_z$
  - $\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\,\hat{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\,\hat{e}_\varphi + \ddot{z}\,\hat{e}_z$
- sphérique : $\vec{\omega} = \dot{\varphi}\,\hat{e}_z + \dot{\theta}\,\hat{e}_\varphi$
  - $\vec{r} = \vec{OP} = r\,\hat{e}_r$
  - $\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + r\dot{\varphi}\sin\theta\,\hat{e}_\varphi$
  - $\vec{a} = a_r\,\hat{e}_r + a_\theta\,\hat{e}_\theta + a_\varphi\,\hat{e}_\varphi$
  - $a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 \cdot \sin^2\theta$
  - $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta$
  - $a_\varphi = r\ddot{\varphi}\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta + 2\dot{r}\dot{\varphi}\sin\theta$



Balistique

$t_a = \frac{v_0 \sin\theta}{g}$
$z(t_a) = h_{max} = \frac{v_0^2}{2g}$
$x_{max} = 2 \cdot h_{max} \cdot \sin(2\theta)$
$x(\theta) = x_{max} \cdot \sin(2\theta)$
$z = h - \frac{x^2}{4h}$

OH



peut aussi avoir une cst ⇒ à still OH

k : [N/m]

$\ddot{x} = -k/m \cdot x = -\omega^2 \cdot x$
↳ $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$
↳ $x(t) = C\sin(\omega_0 t + \varphi)$

position eq. ℓ_0 ; $\vec{x} = x - x_{max}$, $x = x - x_{min}$

- à $\bar{x}$, $\sum \vec{F} = \vec{0}$
- à $\bar{x}$, $\frac{dE_{pot}}{dt}(\bar{x}) = \vec{0}$

pour trouver $\bar{x}$

↳ stable si $\frac{d^2E_{pot}}{dt^2}(\bar{x}) > 0$
↳ instable si < 0

$\omega = \sqrt{\frac{k_{eff}}{m}}$
$k_{eff} = \frac{d^2E_{pot}}{dt^2}(\bar{x})$

- Trouver équ. mvt : $\sum \vec{F} = m\ddot{\vec{x}}$

pulsation: $\omega = \sqrt{k/m}$

indép. de l'amplitude: $T = 2\pi\sqrt{m/k}$

⊕ utile : $t_{max} = T/2$

OH : $\vec{F}_r = -k\,\vec{\Delta x} = -k(\vec{x} - \vec{\ell}_0)$
OH amorti : $\vec{F}_r = -k\,\vec{\Delta x} - b\dot{x}$
↳ $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 \cdot x = 0$
OH amorti forcé :
↳ $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 \cdot x = \alpha_0 \sin(\omega t)$

$\gamma = b/2m$
$\omega_0 = \sqrt{k/m}$
$\alpha_0 = f/m$

$A = x_0$
$B = v_0/\omega_0$
$C^2 = A^2 + B^2$
$\tan(D) = A \cdot B^{-1}$

Energy / Work / Momentum

$W_{AB} = \int_A^B dx\,F = \int_A^B \vec{F} \cdot d\vec{r} = K_B - K_A = F \cdot \ell$; $\ell = \delta(A;B)$ ↳ Ein

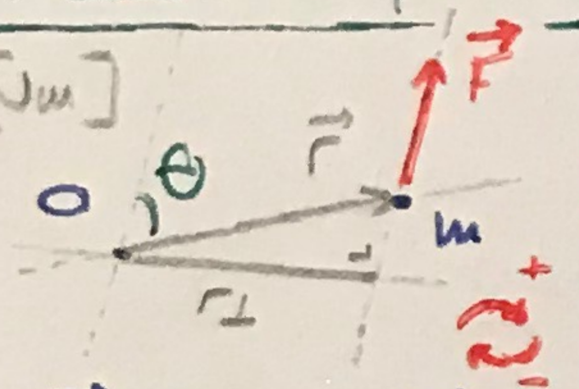
$\vec{I}_{AB} = \int_A^B \vec{F}\,dt$
$P = \vec{F} \cdot \vec{v} = Fv\cos\theta$
$p = mv$
$F = m\dot{v} = \dot{p}$
$\Delta E = W_{AB}^{NC} \Leftrightarrow \frac{dE}{dt} = P^{NC} = \vec{F}^{NC} \cdot \vec{v}$

$P_m = \frac{\Delta W}{\Delta t}$
$\frac{dK}{dt} = P = \vec{F} \cdot \vec{v}$

* $\vec{F} = F_x \cdot \hat{x} + F_y \cdot \hat{y} + F_z \cdot \hat{z}$
$d\vec{r} = dx \cdot \hat{x} + dy \cdot \hat{y} + dz \cdot \hat{z}$
$dW = \vec{F} \cdot d\vec{r}$
$W_{AB} = \int_A^B F_x\,dx + \int_A^B F_y\,dy + \int_A^B F_z\,dz$

Moment de force

[Nm]



$\vec{\tau}_0 = \vec{r} \wedge \vec{F}$
$\tau_0 = r_\perp \cdot F$
$= rF\sin\theta$

Moment cinétique

$\vec{L}_0 = \vec{r} \wedge \vec{p} = \vec{r} \wedge m\vec{v}$
$L_0 = r_\perp \cdot p = r_\perp m v$
$= rp\sin\theta$
avec $\theta = \sphericalangle(\vec{r}, \vec{p})$

th. mom. cin. : $\frac{d\vec{L}_0}{dt} = \vec{\tau}_0$ $[kg\,m^2/s]$

Frottements

$F_{f,max} = \mu \cdot F_N$
- μ_s : get it going
- μ_c : keep it going

$\mu_s = \tan(\alpha)$ ↳ indép. de m ↳ indép. de surface area
$\mu_s > \mu_c$

F conservative

F conservative ⇔ Work ne dépend pas du path mais seulement des points de départ et d'arrivée

Si toutes les forces sont conservatives ⇒ conservation de l'énergie ($E_B = E_A$)

2 intégrales premières du mvt : E et L

Trouver équ. mvt depuis int. premières du mvt :
1) $\frac{dL}{dt} = 0$ 2) $\frac{dE_{pot}}{dt} = 0$

En intégrant l'équ. de mvt, on obtient $E_{tot} = K + V$

E_{pot}	E_{pot} [J]
ressort	$\frac{1}{2} k \cdot x^2$ ($x = \ell - \ell_0$)
pesanteur	mgz
grav.	$-GmM/r$

Kepler

1) traj planètes = ellipse
2) aires égales en temps égaux
3) $\frac{(\text{période})^2}{(\text{grand axe})^3} = cst$

- $G = 6.67 \cdot 10^{-11}$
- $F_{grav} = \frac{mMG}{r^2}$

Système pt matériel

$\vec{\tau}_0' = (\vec{O'O} \wedge \vec{F}) + \vec{\tau}_0$

système isolé: $\begin{cases} p = cst \Rightarrow F^{ext} = 0 \\ L_0 = cst \Rightarrow \tau_0^{ext} = 0 \end{cases}$

système isolé: $\begin{cases} p = 0 \Rightarrow F^{ext} = 0 \\ L_0 = 0 \Rightarrow \tau_0^{ext} = 0 \end{cases}$

⊕ $\sum \vec{F}^{int} = \sum \vec{\tau}_0^{int} = \vec{0}$

$\vec{r}_G = \frac{1}{M} \cdot \sum_\alpha m_\alpha \vec{r}_\alpha$ (masse totale)
M cst ⇒ $\dot{\vec{p}} = \vec{F}^{ext} = M \cdot \vec{a}_G$
$\vec{P} = M \cdot \vec{V}_G$ / th. centre de masse
⇒ centre de masse ≡ pt matériel
équilibre : $\vec{r}_\alpha = cst$; $\vec{V}_\alpha = 0$ $\forall \alpha$
somme des qté de mvt

Problèmes à deux corps

mvt relatif : $\begin{cases} \ddot{\vec{R}} = 0 \\ \vec{F}_{2\to1} = \mu \cdot \ddot{\vec{r}} \end{cases}$

: $\ddot{R}$ = acc. de G
: mvt relatif

$\begin{cases} \vec{R} = \frac{m_1 \cdot \vec{r}_1 + m_2 \cdot \vec{r}_2}{m_1 + m_2} \\ \vec{r} = \vec{r}_1 - \vec{r}_2 \end{cases}$

: $R = \delta(O;G)$
: $r = \delta(m_1, m_2)$
: u = coord. relative

$u = x_2 - x_1$
↳ u cst ⇒ $\delta(m_1, m_2) = cst$
↳ $V_G = 0 \Rightarrow \delta(G, x_1) = \delta(G, x_2)$ $\forall x_1(t), x_2(t)$

$L_{tot,0} = \vec{R} \wedge M\vec{V} + L^*_{tot,0}$
$K_{tot} = \frac{1}{2} M V^2 + K^*_{tot}$

$\mu = \frac{m_1 \cdot m_2}{m_1 + m_2}$
↳ masse réduite

Chocs / dépend de Δt

- choc élastique : K, p conservées
- choc inélastique : p conservée
- choc mou : inélastique + restent collés ⇒ $v_{f,1} = v_{f,2}$

$T = \frac{2\pi}{\omega}$
$f = \nu = T^{-1}$ [Hz]
temps pour faire 1 tour [s]

Centre de masse

$$\vec{OG} = \vec{r}_G = \frac{1}{M}\int_V \vec{r}\, dm(\vec{r}) = \frac{1}{M}\int_V \vec{r}\, f(\vec{r})\, dV$$

pos. de G : $j_G = \frac{1}{M}\int_V j \cdot \rho(x,y,z)\, dV \quad j = x,y,z$

Ciném. solide

$\vec{\omega} \wedge \vec{y}$: vitesse instant. pos. d'un point $P \in S$

$$\vec{v}_P = \vec{v}_A + \vec{\omega} \wedge \vec{AP}$$

$$\vec{a}_P = \vec{a}_A + \dot{\vec{\omega}} \wedge \vec{AP} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{AP})$$

$$\vec{AC} = \frac{\vec{\omega} \wedge \vec{v}_A}{\omega^2}$$

- $P, A \in S$
- C sur l'axe de rot. inst.

Mom. cin. & Inertie

(th. transfert)

$$\vec{L}_Q = \vec{QO} \wedge m\vec{v}_G + \vec{L}_O$$

$$\vec{L}_Q = \vec{QG} \wedge m\vec{v}_G + \vec{L}_G^*$$

$$\frac{d\vec{L}_Q}{dt} = \vec{\tau}_Q^{\,ext} - \vec{v}_Q \wedge m\vec{v}_G$$

si $\vec{v}_Q \parallel \vec{v}_G \Rightarrow \frac{d\vec{L}_Q}{dt} = \vec{\tau}_Q^{\,ext}$

$$\vec{L}_A = \vec{AG} \wedge m\vec{v}_A + \sum_\alpha m_\alpha \left[AP_\alpha^2 \cdot \vec{\omega} - (\vec{AP}_\alpha \cdot \vec{\omega})\vec{AP}_\alpha\right]$$

$$= \vec{AG} \wedge m\vec{v}_A + \int_V \left[(\vec{r} - \vec{OA})^2 \cdot \vec{\omega} - [(\vec{r} - \vec{OA})\vec{\omega}](\vec{r} - \vec{OA})\right] dm$$

⊕ $\vec{L}_G = \vec{L}_G^*$

Mom. inertie + axe rot

$$I_\Delta = \sum_\alpha m_\alpha d_\alpha^2 = \int_V d^2\, dm$$

$$L_\Delta = I_\Delta \cdot \omega$$

$$\vec{L}_C = I_\Delta \cdot \vec{\omega}$$

Tenseurs Inertie

$$K_{rot} = \frac{1}{2}m\vec{v}_A^{\,2} + m\vec{v}_A(\vec{\omega} \wedge \vec{AP}_\alpha) + \frac{1}{2}\vec{\omega}(\tilde{I}_A \cdot \vec{\omega})$$

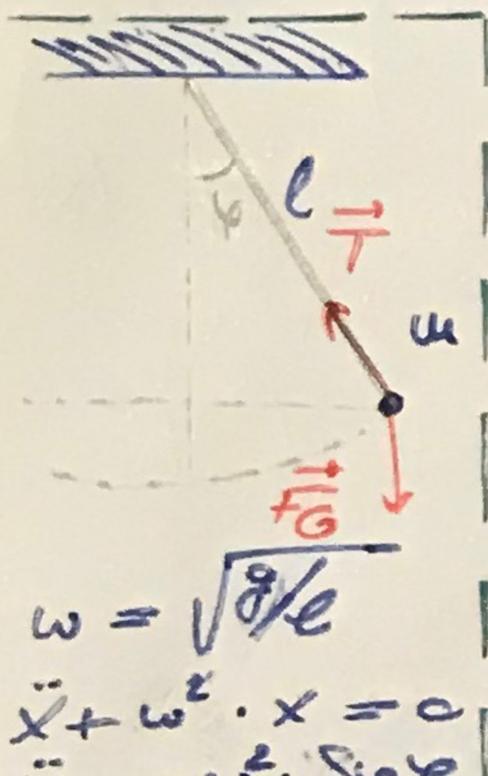
$$(\tilde{I}_G)_{ij} = \sum_\alpha m_\alpha \left[GP_\alpha^2\, \delta_{ij} - (GP_\alpha)_j (GP_\alpha)_i\right]$$

$$(\tilde{I}_A)_{ij} = (I_G)_{ij} + m\left[AG^2 \cdot \delta_{ij} - (AG)_j (AG)_i\right]$$

$$\vec{L}_G = \tilde{I}_G \cdot \vec{\omega} \qquad \tilde{I}_G = \begin{pmatrix} \cdot & \square & \triangle \\ \square & \cdot & \circ \\ \triangle & \circ & \cdot \end{pmatrix}$$

si rot. autour de Δ fixe : $K_{rot} = \frac{1}{2} I_\Delta \cdot \omega^2$

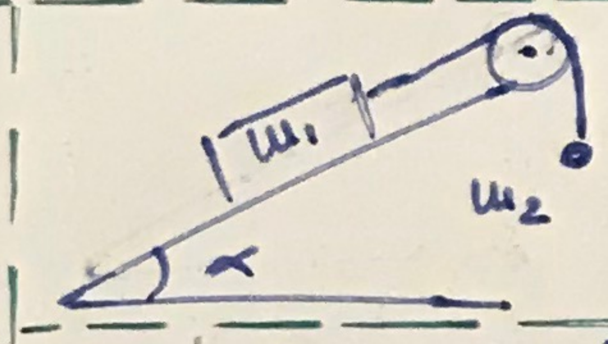
$$I_{\Delta_C} = I_{\Delta_G} + m\, \delta^2(G,C)$$



$\omega = \sqrt{g/l}$

$\ddot{x} + \omega^2 \cdot x = 0$

$\ddot{\varphi} = -\omega^2 \cdot \sin\varphi$



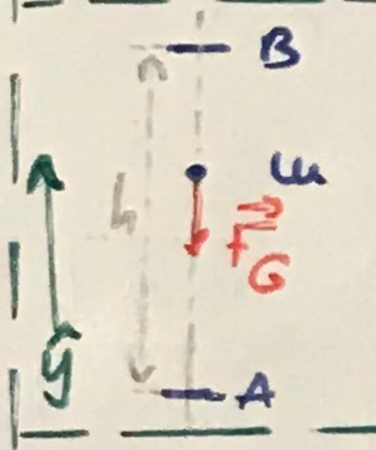
1) acc. uphill : $m_2 g > m_1 g \sin\alpha + f_{max}$

2) acc. downhill : $m_2 g < m_1 g \sin\alpha - f_{max}$

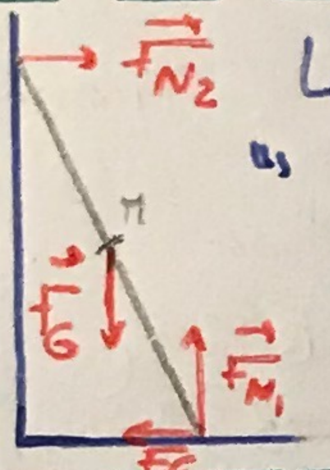
3) $a = 0$: won't move

looping :

condition pour ne pas tomber : $a_c = \frac{v^2}{r} > g$ ou $N_s \neq 0$



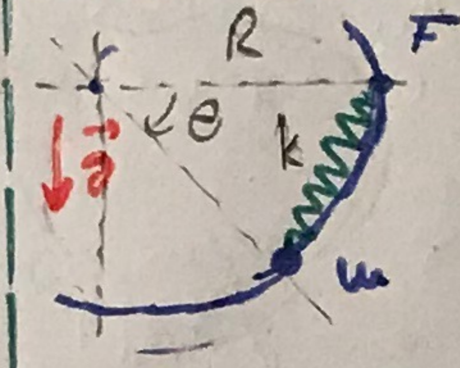
$$W_{AB} = \int_A^B F_G\, dy = -mg(B-A) = -mgh$$



$L_M = cst$

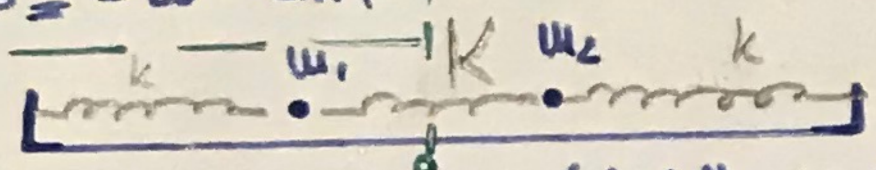
↳ $\frac{dL_M}{dt} = 0$

$\sum \tau_M = 0$



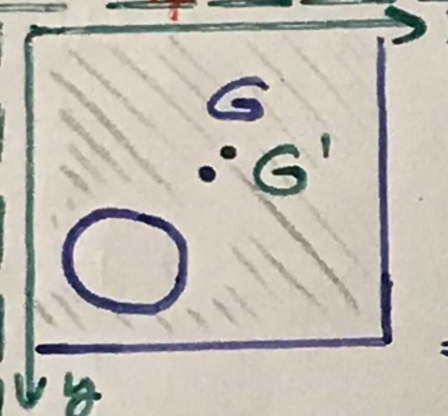
A $\bar{\theta}$ stable,

$$k_{ef} = \left.\frac{d^2E_{pot}}{ds^2}\right|_{\theta=\bar{\theta}} = \frac{1}{R^2}\left.\frac{d^2E_{pot}}{d\theta^2}\right|_{\theta=\bar{\theta}}$$



- $m\ddot{x}_1 = -k(x_1 - (-b/2)) - K(x_1 - (x_2 - l_0))$
- $m\ddot{x}_2 = -k(x_2 - b/2) - K(x_2 - (x_1 + l_0))$

$$\vec{r}_G = \frac{1}{M_{tot}} \sum_\alpha m_\alpha \vec{r}_\alpha \Rightarrow \ddot{x}_G = \frac{1}{M}(m\ddot{x}_1 + m\ddot{x}_2)$$



$$\vec{r}_G = \frac{1}{M_{tot}} \sum_\alpha m_\alpha \vec{r}_\alpha$$

↳ α_1 = plaque percée

↳ α_2 = disque plein

⇒ isoler $\vec{r}_{G'} = \begin{pmatrix} x_{G'} \\ y_{G'} \end{pmatrix}$

Math. utile

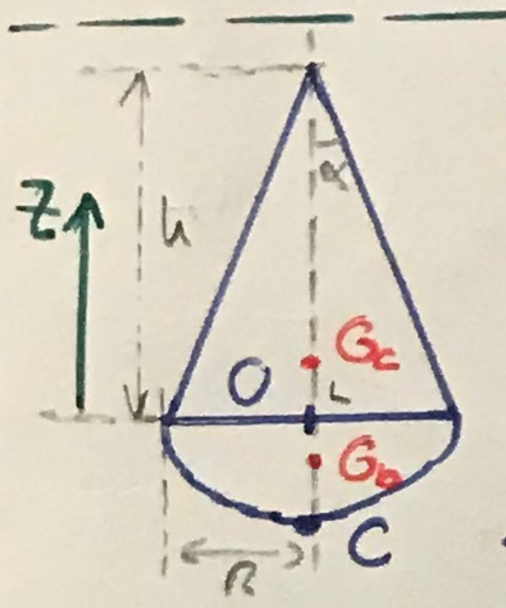
$$\vec{a} \cdot \vec{b} = ab\cos\varphi$$

$$\vec{a} \wedge \vec{b} = ab\sin\varphi$$

$$\vec{OP} \cdot \hat{u} = \vec{OP}'$$

$$\dot{\hat{e}}_i = \frac{d\hat{e}_i}{dt} = \vec{\omega} \wedge \hat{e}_i$$

$$\vec{a} \wedge (\vec{b} \wedge \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$



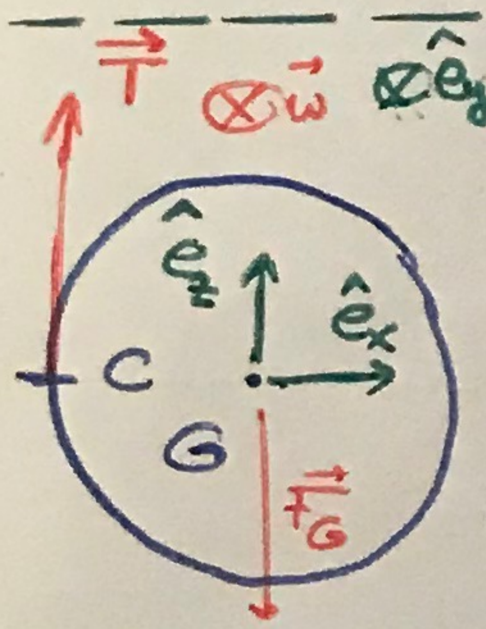
$$z_G = \frac{1}{M}\int_V z\, dm = \frac{M_c \cdot z_{G_c} + M_b \cdot z_{G_b}}{M_c + M_b}$$

équilibre ⇒ $z \leq 0$

- $z_{G_c} = \frac{1}{M_c}\int_{cône} z \cdot \rho\, dV = \ldots \quad \left(r_c = R\left(\frac{h-z}{h}\right)\right)$
- $z_{G_b} = \frac{1}{M_b}\int_{boule} z \cdot \rho\, dV = \ldots \quad \left(r_b = \sqrt{R^2 - z^2}\right)$

Équ différentielles

- $m\ddot{x} = -kx$ ⇒ voir ex recto
- $m\ddot{x} = -kx - mg$
  ↳ $x(t) = A\cos(\omega t) + B\sin(\omega t) - mg/k$ $\quad \begin{cases} A = x_0 + mg/k \\ B = v_0/\omega \end{cases}$
- $m\ddot{x} = -k(x - L) - mg$
  ↳ $x(t) = A\cos(\omega t) + B\sin(\omega t) + L - mg/k$ $\quad \begin{cases} A = x_0 - L + \frac{mg}{k} \\ B = v_0/\omega \end{cases}$



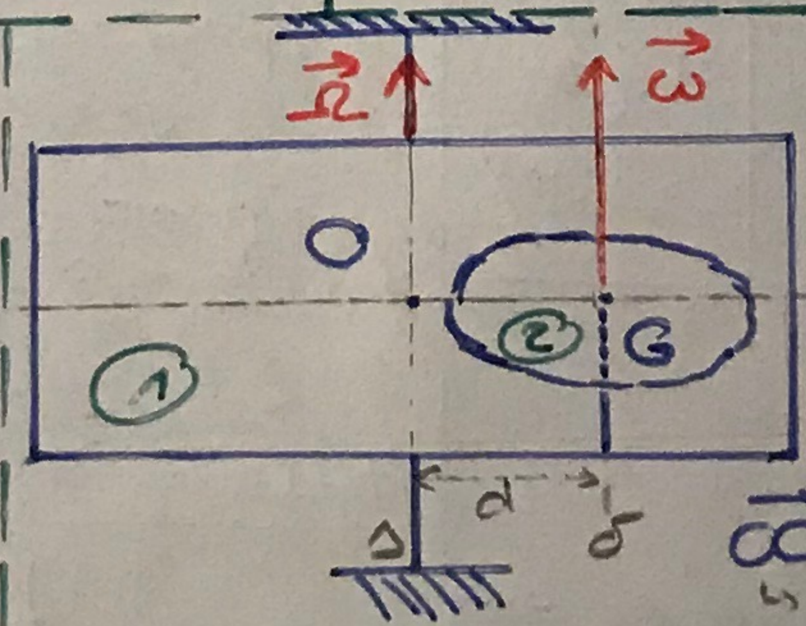
1) $\sum \vec{F} = m\vec{a}$ (pour G)

2) $\frac{d\vec{L}_G}{dt} = I_\omega \cdot \dot{\omega} \cdot \hat{e}_y = \sum \vec{\tau}_G$

↳ proj. résult sur $\hat{e}_y$

3) $\vec{v}_G = \vec{v}_C + \vec{\omega} \wedge \vec{CG}$

↳ $\omega = a_s/R$



1) Calculer : $\vec{L}_{G,2}$, $\vec{L}_{O,1}$, $\vec{L}_{O,2}$ (th. transfert), $\vec{V}_G$

Conservation L_O (si frein ② (interne))

Conservation L_G (si frein ① (externe))

$$\vec{OC} = \vec{OG} + \vec{GC} = \vec{OG} + \frac{(\vec{\Omega} + \vec{\omega}) \wedge \vec{V}_G}{(\Omega + \omega)^2}$$

↳ axe de rotation instant.

- dV cône : $\pi r^2(z)\, dz = \pi \cdot R^2\left(1 - \frac{z}{h}\right)^2 dz$
- dV cône : $r\, dr\, d\theta\, dz$
- dV sphère : $\pi r^2(z)\, dz = \pi(R^2 - z^2)\, dz$
- dV sphère : $r^2 \sin\theta\, dr\, d\theta\, d\varphi$
- dV disque : $r\, d\varphi\, dr$
- cylindre ⇄ plein : $I = \frac{1}{2}mr^2$; vide : $I = mr^2$ $\quad I_{disc} = \frac{1}{2}mr^2$
- sphère ⇄ pleine : $I = \frac{2}{5}mr^2$; vide : $I = \frac{2}{3}mr^2$
- parall. rect. plein : $I = \frac{1}{12}M(b^2 + c^2) = \frac{1}{12}M(a^2 + b^2) = \frac{1}{2}M(a^2 + c^2)$

Tips pour commencer

- $\sum \vec{F} = m\vec{a}$

  Conservation de E → car N ne trav. pas
- Kepler : conserv. E et L
- Solides indéf. :

  $\vec{v}_P = \vec{v}_A + \vec{\omega} \wedge \vec{AP}$

  $\frac{d\vec{L}_G}{dt} = \sum \vec{\tau}_G$ calcul ...

  2 parties

  considérer syst. séparément

  ↳ proj. sur les axes

$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgh$

Seule ∥ travaille

F_N ne travaille jamais

$F \propto \delta(O, P)$ ⇒ $F = k \cdot r$