

explain these topics one by one try to cover hidden aspects of them

Calculus (Single Variable)

Limits & continuity

Differentiation: rules, product/quotient rule, chain rule

Applications: slopes, optimization problems

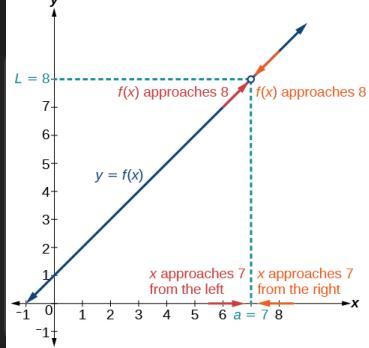
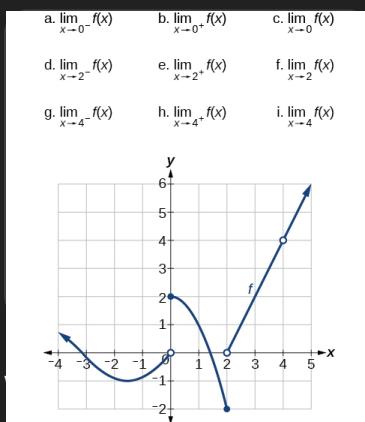
Integration: definite, indefinite, substitution

Area under curves, basic probability integrals

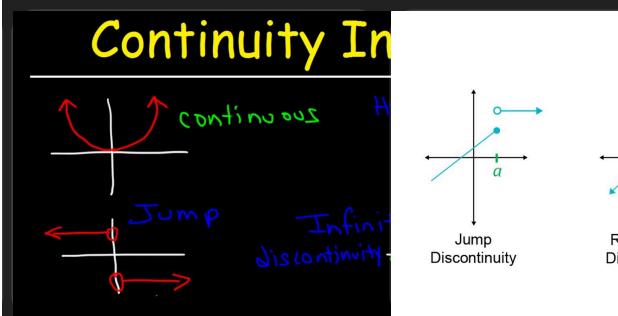
Great — let's make **Single-Variable Calculus** *crystal clear*, covering all hidden aspects, geometric intuition, pitfalls, and ML-relevant ideas.

We'll go **topic by topic**, with visuals (conceptual), deep explanations, and "why this matters".

1 Limits & Continuity — The Engine of Calculus



(Visual Meaning)



A limit answers one question:

As x gets close to a number a , what value does $f(x)$ get close to?

Important:

You do NOT care what $f(a)$ actually is.

Limits ignore the actual value at that point — only the *approach* matters.

Example hidden trap:

- A function can have a limit at $x = a$ even if it's not defined at a .
- A function can have a value at a that is completely different from the limit.

🔍 Hidden Aspect: *Left-hand vs Right-hand limits*

- Limit exists only when $\lim(x \rightarrow a^-) f(x) = \lim(x \rightarrow a^+) f(x)$
- If even slightly different \rightarrow limit does NOT exist.

This is the reason why discontinuous functions break models in ML (e.g., step functions).

Continuity

A function is continuous at $x = a$ if:

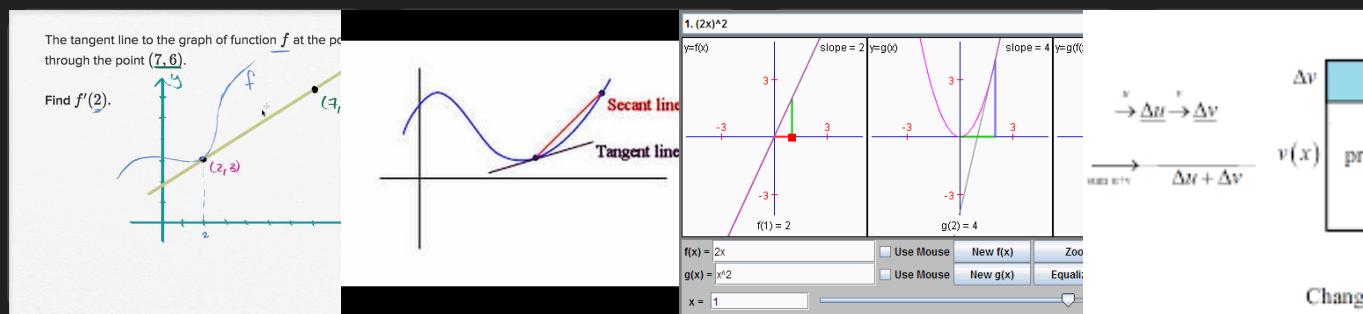
1. $f(a)$ is defined
2. $\lim(x \rightarrow a) f(x)$ exists
3. $\lim(x \rightarrow a) f(x) = f(a)$

Hidden Aspect: Three Types of Discontinuities

1. **Removable** \rightarrow hole in graph
2. **Jump** \rightarrow left and right differ
3. **Infinite** \rightarrow vertical asymptote

In machine learning, differentiability/continuity matter because backpropagation assumes smoothness.

2 Differentiation: Rules, Chain Rule, Product, Quotient



Differentiation measures:

How fast something changes at an instant.

Slope of tangent = derivative = instantaneous rate.

Basic Rules

1. Power Rule

$$\frac{d}{dx} (x^n) = n \cdot x^{n-1}$$

2. Constant Rule

Derivative of constant = 0

3. Sum Rule

Derivative distributes over addition.

Product Rule

$$(fg)' = f'g + fg'$$

💡 Hidden Insight:

Product rule is necessary because change in a product has TWO sources:

- f changes
 - g changes
 - both change simultaneously
-

Quotient Rule

$$(f/g)' = (f'g - fg') / g^2$$

Hidden trick:

You can avoid quotient rule by rewriting:

bash

$$f/g = f \cdot g^{-1}$$

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Then apply:

- product rule
 - chain rule
-

Chain Rule — THE MOST IMPORTANT RULE

If

$$y = f(g(x))$$

then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

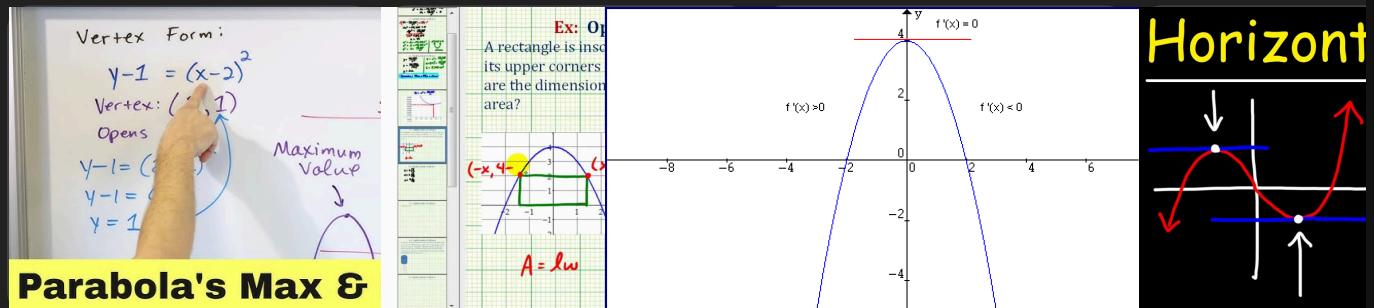
Hidden Insight:

Chain rule is simply the rule of **propagating change through layers**.

This is the mathematical basis of:

- Backpropagation
- Neural networks
- Composition of transformations

3 Applications of Derivatives — Slopes, Optimization



Slopes

Derivative sign means:

- $f'(x) > 0 \rightarrow$ increasing
- $f'(x) < 0 \rightarrow$ decreasing
- $f'(x) = 0 \rightarrow$ stationary (could be max, min, or inflection)

Optimization Problems

Steps:

1. Set up function (profit, area, speed, cost, etc.)
2. Differentiate
3. Set derivative = 0
4. Check:

- second derivative
- endpoints
- physical constraints

Hidden Aspect:

Critical points include more than points where derivative is zero

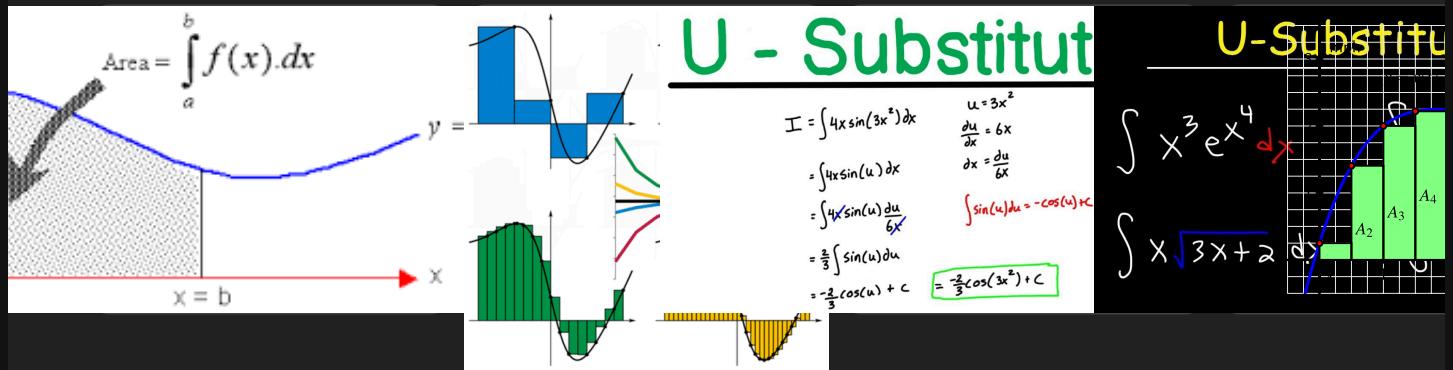
Also points where derivative is undefined.

Examples:

- sharp corners (like $|x|$ at 0)
- vertical tangents

In ML, optimization is solving for minima of loss functions.

4 Integration — Definite, Indefinite, Substitution



Integration is reverse differentiation, but also area accumulation.

Indefinite Integrals

General antiderivative:

$$\int f(x) dx = F(x) + C$$

Why $+C$?

Because many different functions share the same derivative.

Definite Integrals

$$\int [a \rightarrow b] f(x) dx = \text{area under the curve (signed)}$$

Hidden aspect:

- Area below x-axis counts negative
- If curves cross, you must split integral

Fundamental Theorem of Calculus (FTC)

The MOST powerful link:

$$\frac{d}{dx} \int [a \rightarrow x] f(t) dt = f(x)$$

and

$$\int [a \rightarrow b] f(x) dx = F(b) - F(a) \text{ where } F' = f.$$

This connects:

- slopes (derivatives)
- areas (integrals)
- accumulation (physics, ML)

Substitution (u-substitution)

Reverse chain rule.

If you see a composite like $f(g(x))g'(x)$ → try:

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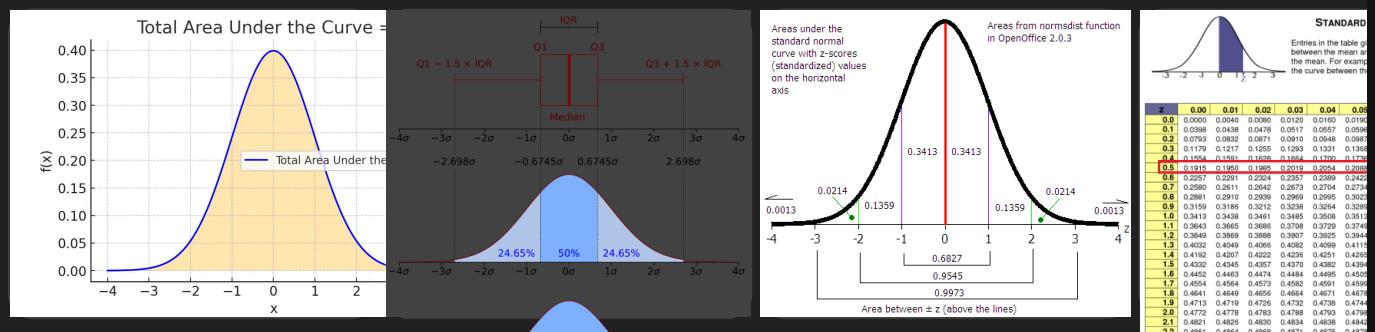
$$\begin{aligned} u &= g(x) \\ du &= g'(x) dx \end{aligned}$$

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Hidden trick:

Choose u as the inner function whose derivative also appears.

5 Area Under Curves & Probability Integrals



Area Under Curve



2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946
2.6	0.4952	0.4955	0.4956	0.4957	0.4959	0.4959
2.7	0.4967	0.4970	0.4971	0.4972	0.4973	0.4973
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984
3.0	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989
3.1	0.4993	0.4991	0.4991	0.4992	0.4992	0.4992
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994
3.3	0.4995	0.4995	0.4995	0.4995	0.4995	0.4995
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997

Area = accumulation of infinitesimal rectangles

width = dx , height = $f(x)$.

Probability Integrals

If $p(x)$ is a probability density function (PDF):

1. Total area = 1

$$\int p(x) dx = 1$$

2. Probability between a and b

$$P(a \leq X \leq b) = \int_{[a \rightarrow b]} p(x) dx$$

Example:

- Part of ML uses Gaussian integrals.
- Many loss functions (cross-entropy, KL divergence) come from integrals.

⚡ Want to go deeper?

I can now expand each topic with:

- Hidden tricks and shortcuts
- Visual intuition using animations
- ML connections
- Hard problem sets + solutions
- Derivation of all formulas

Which topic should we fully expand next?

1. Limits & continuity
2. Differentiation rules
3. Chain rule (very deep)
4. Optimization problems
5. Integration (including FTC)
6. Probability integrals

Just tell me the number.

□ ⌂ ⌂ ⌂ ⌂ ...

Connecting from limits show how formulas are derived and connect everything

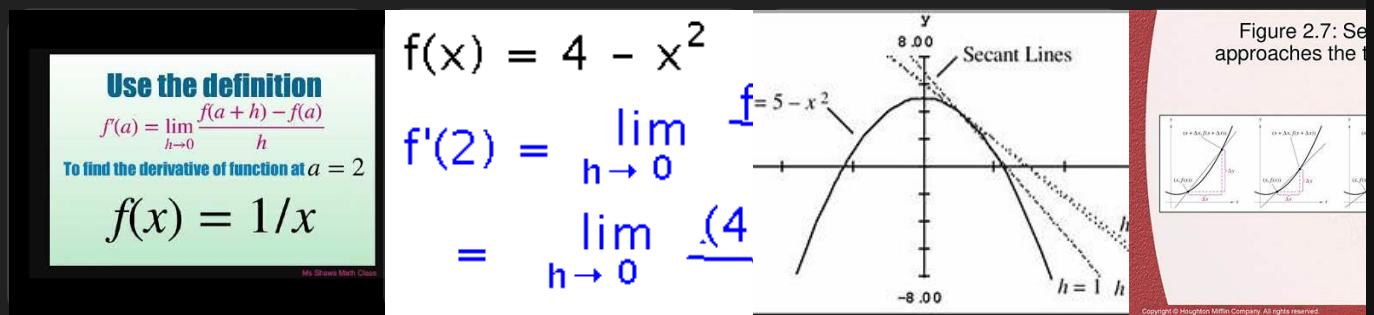
Perfect — this is the **deepest way** to learn calculus:

Start from limits → derive every formula → connect all rules → show how differentiation and integration emerge from the same limit idea.

I will build the entire chain of calculus from scratch using **only limits**, showing how every rule you know (power rule, product rule, chain rule, integrals) actually *comes from* the structure of limits.

I'll include visuals where needed.

0 The Starting Point: The Limit Definition of the Derivative



Everything in differentiation comes from ONE definition:

Definition

For a function $f(x)$,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Interpretation:

- $f(x+h) - f(x) =$ change in output
- $h =$ change in input

- Limit → slope as you zoom infinitely close

This limit is the **source** of:

- Power rule
 - Product rule
 - Quotient rule
 - Chain rule
 - Trig derivatives
 - Exponential/log derivatives
-

1 Deriving the Power Rule Using Only Limits

Let

$$f(x) = x^n$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x + h)^n - x^n}{h}$$

Expand using binomial theorem:

$$(x + h)^n = x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \cdots + h^n$$

Subtract x^n :

$$(x + h)^n - x^n = nx^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \cdots + h^n$$

Divide by h :

$$\frac{(x + h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2} x^{n-2} h + \cdots + h^{n-1}$$

Now take limit as $h \rightarrow 0$:

All terms containing h vanish.

$$f'(x) = nx^{n-1}$$

This gives the **power rule**.

📌 Hidden connection:

Power rule is NOTHING but the binomial expansion + limit.

Nothing mysterious.

2 Deriving the Sum Rule

Using limit definition:

Let

$$F(x) = f(x) + g(x)$$

Then:

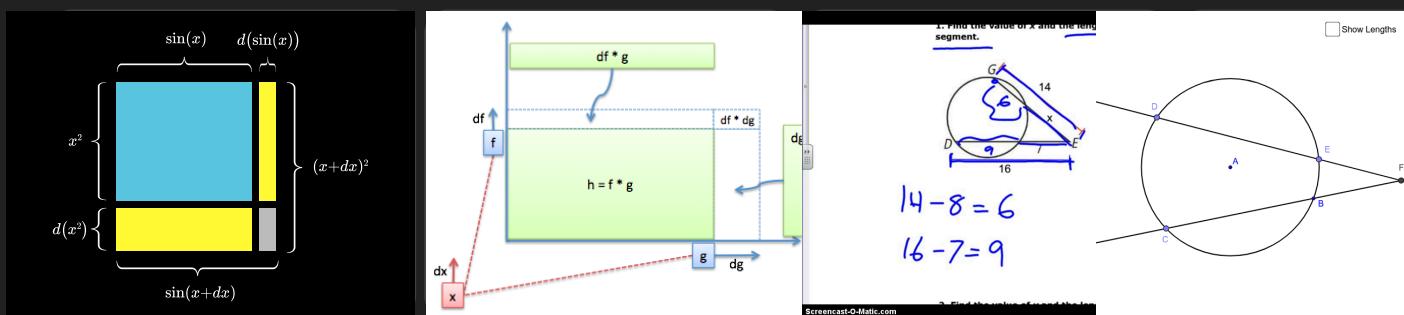
$$F'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

Split terms:

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

📌 Hidden idea: limits preserve addition → derivative preserves addition.

3 Deriving the Product Rule from Limits



Let $F(x) = f(x)g(x)$.

$$F'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Add and subtract a middle term:

$$= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

Group:

$$= \lim_{h \rightarrow 0} \left[\frac{f(x+h)(g(x+h) - g(x))}{h} + \frac{g(x)(f(x+h) - f(x))}{h} \right]$$

Take limits:

- $f(x+h) \rightarrow f(x)$
- $\frac{g(x+h)-g(x)}{h \rightarrow g'(x)}$
- $\frac{f(x+h)-f(x)}{h \rightarrow f'(x)}$

So:

$$F'(x) = f(x)g'(x) + g(x)f'(x)$$

📌 Hidden idea:

This rule arises because both functions change when you move from x to $x+h$.

4 Deriving the Quotient Rule from Limits

Let

$$F(x) = \frac{f(x)}{g(x)}$$

Using product rule trick:

$$\frac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}$$

Differentiate:

$$F'(x) = f'(x)g^{-1}(x) + f(x) \cdot (-1)g^{-2}(x)g'(x)$$

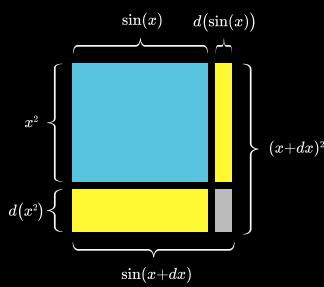
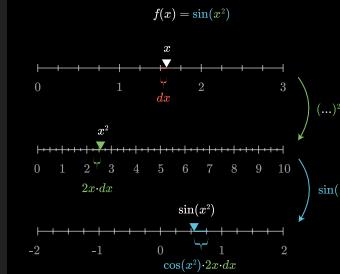
Simplify:

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

📌 Hidden idea:

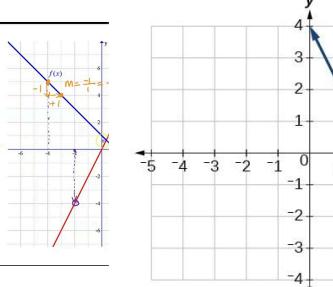
Quotient rule is just product rule + chain rule.

5 Deriving the Chain Rule Using Limits



Suppose $f(x)$ is graphed in blue, and $g(x)$ is graphed in red. Find the following derivatives.

- a. Let $u(x) = f(g(x))$. Find $u'(-2)$.
- $$u'(x) = f'(g(x)) \cdot g'(x)$$
- $$u'(-2) = f'(g(-2)) \cdot g'(-2)$$
- $$u'(-2) = (-1) \cdot ()$$



Let

$$y = f(g(x))$$

We want:

$$\frac{dy}{dx} = ?$$

Start with:

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

Add intermediate variable:

Let

$$u = g(x)$$

As $h \rightarrow 0$,

$$g(x+h) = u + \Delta u$$

So:

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(u + \Delta u) - f(u)}{\Delta u} \cdot \frac{\Delta u}{h}$$

Break into two limits:

$$\begin{aligned} &= \left(\lim_{\Delta u \rightarrow 0} \frac{f(u + \Delta u) - f(u)}{\Delta u} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \right) \\ &= f'(u) \cdot g'(x) \end{aligned}$$

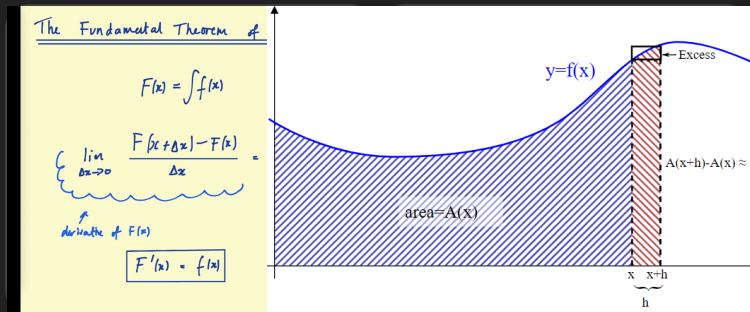
Substitute back:

$$= f'(g(x)) \cdot g'(x)$$

★ Hidden connection:

Chain rule is the limit version of backpropagation.

6 Connecting Derivatives to Integrals (FTC from Limits)



Ex 1: Rewrite the limit of the Riemann sum as a integral.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \cos^2 \left(3 - \frac{2i}{n} \right) \frac{2}{n}$$

$$\int_0^2 \cos^2(3-x) dx$$

5.e – The Definite Integral of a Riemann Sum Techniques for Evaluating Integrals

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n$$

Integration is defined by **Riemann sums**, also a limit:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

The **Fundamental Theorem of Calculus (FTC)** comes from this same structure.

FTC Part 1 — Derivative of Area Function

Define:

$$F(x) = \int_a^x f(t) dt$$

Compute derivative:

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$$

But:

$$F(x+h) - F(x) = \int_x^{x+h} f(t) dt$$

For very small h :

$$\int_x^{x+h} f(t) dt \approx f(x) \cdot h$$

So:

$$F'(x) = \lim_{h \rightarrow 0} \frac{f(x)h}{h} = f(x)$$

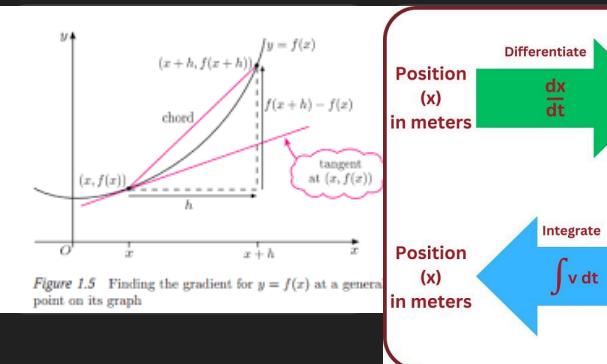
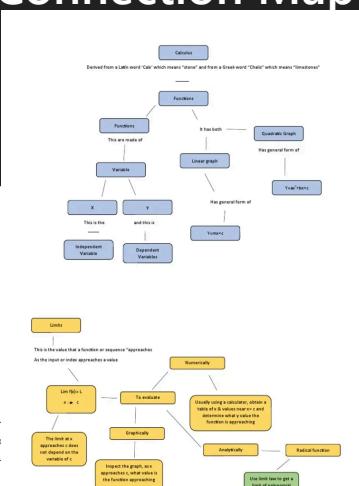
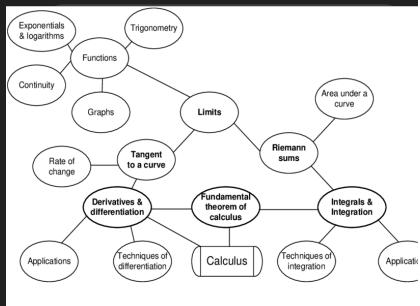
Thus:

$$\boxed{\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x)}$$

Derivatives and integrals become inverse operations.

7 The Ultimate Connection Map (Everything from Limits)

Everything we have done so far



1. Limit → Derivative definition

From

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

we derived:

- power rule
- product rule
- quotient rule
- chain rule
- trig derivatives
- exponential/log derivatives

2. Limit → Integral definition

From

$$\lim_{n \rightarrow \infty} \sum f(x_i) \Delta x$$

integration emerges.

3. Limits connect both through FTC

$$\frac{d}{dx} \int f = f$$

$$\int f' = f$$

This closes the loop.

★ If you'd like, we can now go even deeper:

Which direction should we expand?

- 1 Derive trig derivatives using only limits
- 2 Derive exp/log derivatives from limits
- 3 Show chain rule as composition of infinitesimal changes (ML version)
- 4 Derive integration techniques from limit definitions
- 5 Full calculus formula sheet derived step-by-step

Just tell me the number.

