

Show case span and basis by differentiating with graphs and mathematical example

Great question — **Span** and **Basis** look similar on the surface, but their **roles are fundamentally different**. I'll differentiate them using:

- ✓ Geometric graphs (2D & 3D intuition)
- ✓ Exact mathematical equations
- ✓ Hidden structural meaning
- ✓ What they *guarantee* vs what they *describe*

## 1 Span — “What space do these vectors fill?”

### ◆ Definition (Mathematical Core)

Given vectors

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$$

their **span** is:

$$\text{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k \mid a_i \in \mathbb{R}\}$$

👉 This means:

All possible linear combinations of the given vectors

### ◆ Graphical Meaning of Span

Case 1: One non-zero vector in 2D

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

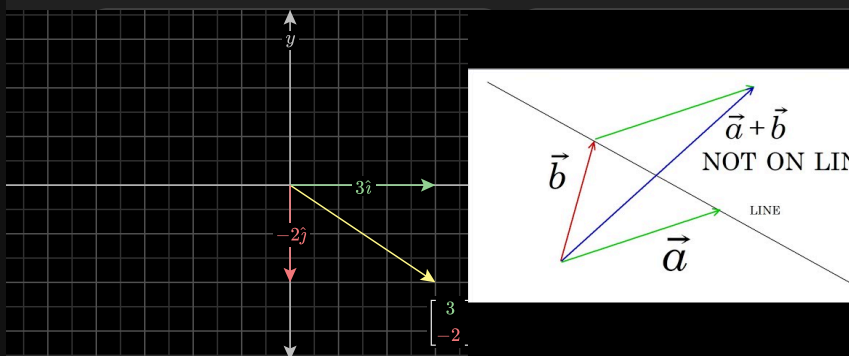
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  ...

This creates:

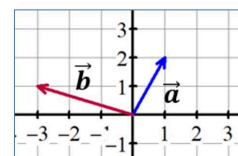
- ✓ A line through the origin

✗ NOT the entire plane



## Vectors

### Sketching a Scalar Multiple of a Vector

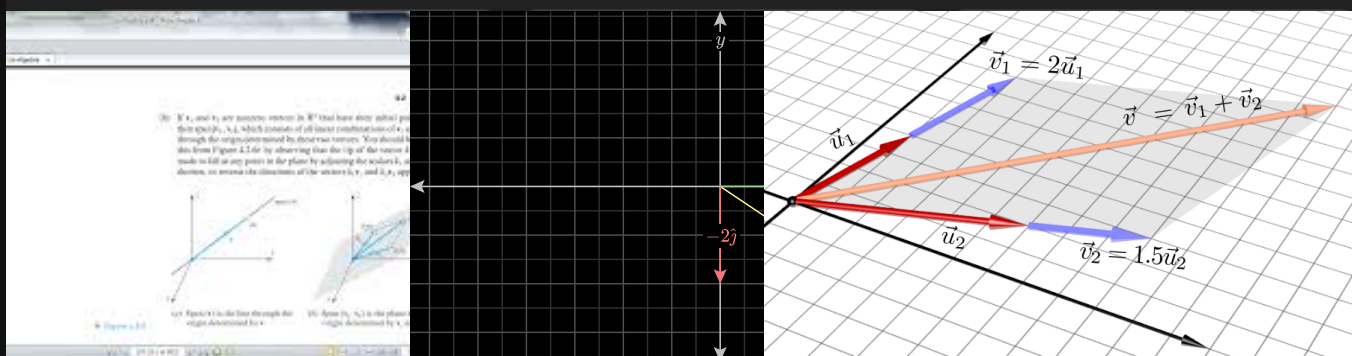


## Case 2: Two non-collinear vectors in 2D

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{Span}(\vec{v}_1, \vec{v}_2) = \mathbb{R}^2$$

✓ Entire 2D plane is filled



## Case 3: Two collinear vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

They lie on the same line, so:

$$\text{Span}(\vec{v}_1, \vec{v}_2) = \text{a line, NOT the plane}$$

✓ Span depends on directional independence, not number of vectors.

## ◆ Hidden Truth About Span

Fact	Meaning
Span is a set	It's a geometric object
Span always includes zero vector	Set all coefficients = 0
Span can be a line, plane, or full space	Depends on independence
You can have many different sets with same span	Basis solves this ambiguity

## 2 Basis — “Minimal perfect generators of the space”

A basis is a special set of vectors that:

1. ☒ Span the entire space
2. ☒ Are linearly independent
3. ☒ Are the minimum possible number

### ◆ Formal Definition

A set

$$\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$$

is a basis of a vector space  $V$  if:

$$\text{Span}(\vec{b}_1, \dots, \vec{b}_n) = V$$

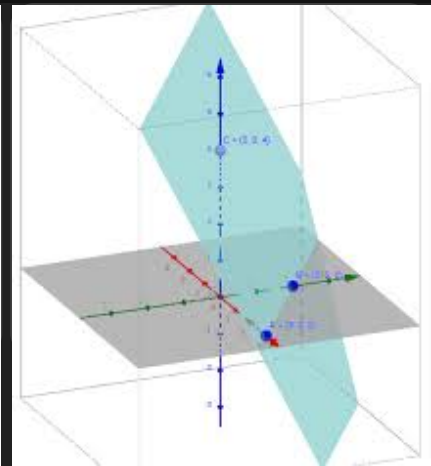
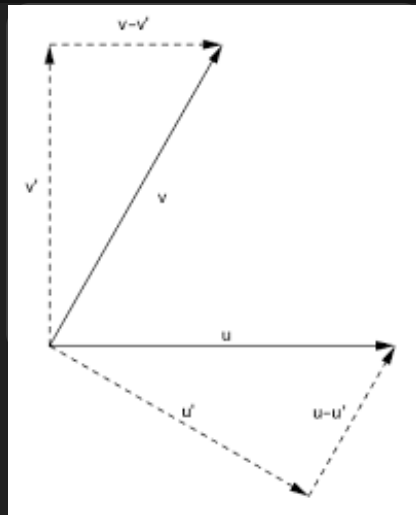
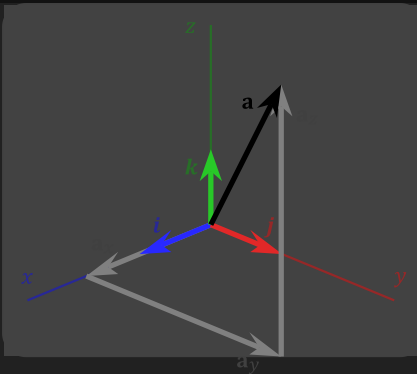
AND

$$c_1\vec{b}_1 + \dots + c_n\vec{b}_n = \vec{0} \Rightarrow c_1 = \dots = c_n = 0$$

### ◆ Graphical Meaning of a Basis (2D)

Standard basis of  $\mathbb{R}^2$ :

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



✓ These two:

- Are **independent**
- Span the **entire plane**
- Are the **smallest possible number**

### ◆ Non-standard Basis Example

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

These are:

- ✓ Independent
- ✓ Span  $\mathbb{R}^2$
- ✓ A valid basis

Even though NOT perpendicular in axis sense.

### ◆ Hidden Power of Basis

Property	Deep Meaning
Every vector has <b>unique coordinates</b>	No ambiguity
Dimension = <b>number of basis vectors</b>	Measures degrees of freedom
Change of basis = <b>coordinate rotation</b>	Used in PCA, physics
Basis compresses information	Minimal representation

## 3 Span vs Basis — The Deep Difference

Feature	Span	Basis
What is it?	A set of all combinations	A minimal generator set
Output	A space	A set of vectors
Can be redundant?	✓ Yes	✗ Never
Linearly independent required?	✗ Not necessary	✓ Required
Uniqueness of coordinates	✗ Not guaranteed	✓ Guaranteed
Dimension connection	✗ Indirect	✓ Direct

## 4 One Clean Mathematical Example Showing Both

Let:

$$V = \mathbb{R}^2$$

Take:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

✓ Span:

$$\text{Span}(S) = \mathbb{R}^2$$

✗ But this is NOT a basis because:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So one vector is dependent.

✓ Valid basis extracted:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

## 5 Final One-Line Intuition

- ✓ Span tells you *what space is formed*
  - ✓ Basis tells you *the smallest perfect toolkit to build that space*
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If you want, next I can visually derive:

- Why dimension equals number of basis vectors
- How PCA is actually just “rotating the basis”
- How SVD secretly builds optimal bases

Just say “Show PCA as a basis rotation” 🚀

