

$$A^T A = V \Lambda V^T$$

this is eigen decomposition use a example metrics and so case eigen decomposition

You're very close with the idea — the **correct eigen-decomposition form** is:

$$A^T A = V \Lambda V^T$$

(not  $V A V^T$ ; it must be the **diagonal eigenvalue matrix  $\Lambda$** )

Let's derive this completely with a concrete matrix example and verify it step-by-step.

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- ◆ **Step 1: Choose a Real Matrix  $A$**

Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This is a **3×2 matrix** (tall matrix).

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- ◆ **Step 2: Compute  $A^T A$**

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

This matrix is:

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- Positive semi-definite

So eigen decomposition is guaranteed to exist.

◆ Step 3: Find Eigenvalues of  $A^T A$

Solve:

$$\det(A^T A - \lambda I) = 0$$

$$\begin{aligned} \det \begin{bmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} &= (2-\lambda)(5-\lambda) - 4 \\ &= \lambda^2 - 7\lambda + 6 = 0 \\ (\lambda-1)(\lambda-6) &= 0 \end{aligned}$$

Eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = 6$$


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◆ Step 4: Find Eigenvectors

► For  $\lambda = 1$ :

$$(A^T A - I)v = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} v = 0$$

Solving gives:

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Normalize:

$$\|v_1\| = \sqrt{5} \Rightarrow \hat{v}_1 = \boxed{\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}}$$


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► For  $\lambda = 6$ :

$$(A^T A - 6I)v = 0 \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} v = 0$$

Eigenvector:

$$v_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Normalize:

$$\|v_2\| = \sqrt{5}/2 \Rightarrow \hat{v}_2 = \boxed{\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}}$$


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- ◆ Step 5: Build  $V$  and  $\Lambda$

$$V = \boxed{\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}$$

$$\Lambda = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}}$$

- Columns of  $V$  are orthonormal eigenvectors
  - $\Lambda$  contains eigenvalues
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- ◆ Step 6: Final Eigen-Decomposition

$$\boxed{A^T A = V \Lambda V^T}$$

If you multiply:

$$V \Lambda V^T = \boxed{\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}} = A^T A \quad \checkmark$$

- Perfect reconstruction
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- ◆ Deep Meaning (VERY Important)

Object	Meaning
$A^T A$	Measures correlation & energy in feature space
$V$	Principal directions (right singular vectors)
$\Lambda$	Squared singular values
$V^T$	Rotates data into eigen-basis

This is exactly why:

- PCA uses eigen-decomposition of  $A^T A$
  - SVD is a generalization of this
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## Final Correct Identity

$$A^T A = V \Lambda V^T$$

✗ Not  $V A V^T$

Always  $V \Lambda V^T$

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If you want, I can now:

- Derive **SVD from this exact example**
- Show **geometric meaning with projections**
- Connect this directly to **PCA from raw data**
- Or derive why  $V$  is orthonormal mathematically

Just tell me what to unfold next 😊

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$$A^T A v_i = \lambda_i v_i$$

how this could be written as form in image

You're starting from the **scalar eigenvector equation** shown in the image:

$$A^T A v_i = \lambda_i v_i$$

Now I'll show exactly how this turns into the matrix form:

$$A^T A = V \Lambda V^T$$

Step by step, with zero jumps.

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## 1. Single Eigenvector Equation (What Your Image Shows)

For each eigenvector  $v_i$ :

$$A^T A v_i = \lambda_i v_i$$

This means:

- $v_i$  = direction that **does NOT** rotate
  - $\lambda_i$  = how much it **stretches**
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## 2. Stack All Eigenvectors into a Matrix

If there are  $n$  eigenvectors:

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

And put all eigenvalues into a diagonal matrix:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$


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## 3. Write All Eigen-Equations at Once (Matrix Form)

Each equation:

$$A^T A v_1 = \lambda_1 v_1$$

$$A^T A v_2 = \lambda_2 v_2$$

$$\vdots$$

$$A^T A v_n = \lambda_n v_n$$

Put together:

$$A^T A V = V \Lambda$$

This is the **matrix version of your image equation**

## 4. Isolate $A^T A$

Since  $A^T A$  is **symmetric**, its eigenvectors are **orthonormal**:

$$V^T V = I$$

Multiply both sides by  $V^T$ :

$$A^T A V V^T = V \Lambda V^T$$

But:

$$V V^T = I$$

So:

$$A^T A = V \Lambda V^T$$

This is the **exact matrix form of the image equation**

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## Final Conceptual Mapping

Your Image

$$A^T A v_i = \lambda_i v_i$$

Single direction

Single stretch

One equation

Matrix Form

$$A^T A V = V \Lambda$$

All directions

All stretches

Full decomposition

## Deep Intuition (One Line)

Eigen-decomposition is nothing but stacking all individual eigenvector stretching rules into one matrix equation.

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If you want, I can now:

- Derive SVD directly from this step
- Show how this becomes PCA

- Or give a **geometric visualization with projections**

Just say the word.

