

$$A^T A = V \Lambda V^T$$

this is eigen decomposition use a example metrics and so case eigen decomposition

You're very close with the idea — the **correct eigen-decomposition form** is:

$$A^T A = V \Lambda V^T$$

✅ (not $V A V^T$; it must be the **diagonal eigenvalue matrix Λ**)

Let's derive this completely with a concrete matrix example and verify it step-by-step.

◆ Step 1: Choose a Real Matrix A

Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This is a 3×2 matrix (tall matrix).

◆ Step 2: Compute $A^T A$

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

✅ This matrix is:

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- Positive semi-definite

✅ So eigen decomposition is guaranteed to exist.

◆ Step 3: Find Eigenvalues of $A^T A$

Solve:

$$\det(A^T A - \lambda I) = 0$$

$$\det \begin{bmatrix} 2 - \lambda & 2 \\ 2 & 5 - \lambda \end{bmatrix} = (2 - \lambda)(5 - \lambda) - 4$$

$$= \lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

✓ Eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = 6$$

◆ Step 4: Find Eigenvectors

► For $\lambda = 1$:

$$(A^T A - I)v = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} v = 0$$

Solving gives:

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Normalize:

$$\|v_1\| = \sqrt{5} \Rightarrow \hat{v}_1 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

► For $\lambda = 6$:

$$(A^T A - 6I)v = 0 \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} v = 0$$

Eigenvector:

$$v_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Normalize:

$$\|v_2\| = \sqrt{5}/2 \Rightarrow \hat{v}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

◆ Step 5: Build V and Λ

$$V = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

✓ Columns of V are **orthonormal eigenvectors**

✓ Λ contains eigenvalues

◆ Step 6: Final Eigen-Decomposition

$$A^T A = V \Lambda V^T$$

If you multiply:

$$V \Lambda V^T = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} = A^T A \quad \checkmark$$

✓ Perfect reconstruction

◆ Deep Meaning (VERY Important)

Object	Meaning
$A^T A$	Measures correlation & energy in feature space
V	Principal directions (right singular vectors)
Λ	Squared singular values
V^T	Rotates data into eigen-basis

This is exactly why:

- ☒ PCA uses eigen-decomposition of $A^T A$
- ☒ SVD is a generalization of this

☒ Final Correct Identity

$$A^T A = V \Lambda V^T$$






☒ Not $V A V^T$

☒ Always $V \Lambda V^T$

If you want, I can now:

- Derive **SVD** from this exact example
- Show **geometric meaning** with projections
- Connect this directly to **PCA** from raw data
- Or derive why V is orthonormal mathematically

Just tell me what to unfold next 😊

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$$A^T A v_i = \lambda_i v_i$$

how this could be written as form in image

You're starting from the **scalar eigenvector equation** shown in the image:

$$A^T A v_i = \lambda_i v_i$$

Now I'll show **exactly how this turns into the matrix form**:

$$A^T A = V \Lambda V^T$$

Step by step, with zero jumps.

✓ 1. Single Eigenvector Equation (What Your Image Shows)

For each eigenvector v_i :

$$A^T A v_i = \lambda_i v_i$$

This means:

- v_i = direction that **does NOT** rotate
- λ_i = how much it **stretches**

✓ 2. Stack All Eigenvectors into a Matrix

If there are n eigenvectors:

$$V = \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{bmatrix}$$

And put all eigenvalues into a diagonal matrix:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

✓ 3. Write All Eigen-Equations at Once (Matrix Form)

Each equation:

$$A^T A v_1 = \lambda_1 v_1$$

$$A^T A v_2 = \lambda_2 v_2$$

$$\vdots$$

$$A^T A v_n = \lambda_n v_n$$

Put together:

$$\boxed{A^T A V = V \Lambda}$$

This is the matrix version of your image equation ✓

✓ 4. Isolate $A^T A$

Since $A^T A$ is **symmetric**, its eigenvectors are **orthonormal**:

$$V^T V = I$$

Multiply both sides by V^T :

$$A^T A V V^T = V \Lambda V^T$$

But:

$$V V^T = I$$

So:

$$\boxed{A^T A = V \Lambda V^T}$$

✓ This is the **exact matrix form of the image equation**

✓ Final Conceptual Mapping

Your Image	Matrix Form
$A^T A v_i = \lambda_i v_i$	$A^T A V = V \Lambda$
Single direction	All directions
Single stretch	All stretches
One equation	Full decomposition

💡 Deep Intuition (One Line)

Eigen-decomposition is nothing but stacking all individual eigenvector stretching rules into one matrix equation.

If you want, I can now:

- Derive **SVD** directly from this step
- Show how this becomes **PCA**

- Or give a **geometric visualization with projections**

Just say the word.

