Test Code PSB (Short answer type) 2014

Syllabus for Mathematics

Combinatorics; Elements of set theory. Permutations and combinations. Binomial and multinomial theorem. Theory of equations. Inequalities.

Linear Algebra: Vectors and vector spaces. Matrices. Determinants. Solution of linear equations. Trigonometry. Co-ordinate geometry.

Complex Numbers: Geometry of complex numbers and De Moivres theorem.

Calculus: Convergence of sequences and series. Functions. Limits and continuity of functions of one or more variables. Power series. Differentiation. Leibnitz formula. Applications of differential calculus, maxima and minima. Taylor's theorem. Differentiation of functions of several variables. Indefinite integral. Fundamental theorem of calculus. Riemann integration and properties. Improper integrals. Double and multiple integrals and applications.

Syllabus for Statistics and Probability

Probability and Sampling Distributions: Notions of sample space and probability. Combinatorial probability. Conditional probability and independence. Random variables and expectations. Moments and moment generating functions. Standard univariate discrete and continuous distributions. Joint probability distributions. Multinomial distribution. Bivariate normal and multivariate normal distributions. Sampling distributions of statistics. Weak law of large numbers. Central limit theorem.

Descriptive Statistics: Descriptive statistical measures. Contingency tables and measures of association. Product moment and other types of correlation. Partial and multiple correlation. Simple and multiple linear regression.

Statistical Inference: Elementary theory of estimation (unbiasedness, minimum variance, sufficiency). Methods of estimation (maximum likelihood method, method of moments). Tests of hypotheses (basic concepts and simple applications of Neyman-Pearson Lemma). Confidence intervals. Inference related to regression. ANOVA. Elements of nonparametric inference.

Design of Experiments and Sample Surveys: Basic designs such as CRD, RBD, LSD and their analyses. Elements of factorial designs. Conventional sampling techniques (SRSWR/SRSWOR) including stratification. Ratio and regression methods of estimation.

Sample Questions

1. Let $E = \{1, 2, \dots, n\}$, where n is an odd positive integer. Let V be the vector space of all functions from E to $\mathbb{R}^3$, where the vector space operations are given by

$$
\begin{aligned}
(f+g)(k) &= f(k) + g(k), \qquad \text{for } f, g \in V, k \in E, \\
(\lambda f)(k) &= \lambda f(k), \qquad \text{for } f \in V, \lambda \in \mathbb{R}, k \in E.
\end{aligned}
$$

   (a) Find the dimension of V.

   (b) Let $T\colon V \to V$ be the map given by

$$Tf(k) = \frac{1}{2}\left(f(k) + f(n+1-k)\right), \quad k \in E.$$

   Show that T is linear.

   (c) Find the dimension of the null space of T.

2. Let $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$ be real numbers such that

$$\sum_{i=1}^{m} |a_i - x| \;=\; \sum_{j=1}^{n} |b_j - x| \;\text{ for all } x \in \mathbb{R}.$$

   Show that $m = n$ and $a_j = b_j$ for $1 \le j \le n$.

3. Let $S = \{1, 2, \dots, n\}$.

   (a) In how many ways can we choose two subsets A and B of S so that $B \neq \emptyset$ and $B \subseteq A \subseteq S$?

   (b) In how many of these cases is B a proper subset of A?

4. Consider a machine with three components whose times to failure are independently distributed as exponential random variables with mean λ. The machine continues to work as long as at least two components work. Find the expected time to failure of the machine.

5. A pin whose centre is fixed on a flat table is randomly and independently spun twice. Each time, the final position is noted by drawing a line segment.

   (a) What is the probability that the smallest angle between the two segments is more than half of the largest angle ?

   (b) What is the probability that at least one of the two segments makes an angle which is less than 45° with the x-axis (when measured in the anti-clockwise direction) ?

6. There are twenty individuals numbered $1, 2, \ldots, 20$. Each individual chooses 10 others from this group in a random fashion, independently of the choices of the others, and makes one phone call to each of the 10.

   (a) Let X be the number of calls handled (incoming as well as outgoing) by Individual 1. Find $E(X)$.

   (b) Let Y be the number of calls between Individual 1 and Individual 2. Find $E(Y)$.

   (c) Find $E(X|Y = 1)$.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $Uniform(\theta, 1)$ population, where $\theta < 1$.

   (a) Find the MLE $\hat{\theta}$ of θ.

(b) Find constants c and d (possibly depending on n) such that, $c+d\,\hat{\theta}$ is unbiased for θ.

8. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from some distribution with mean μ and variance σ^2. Let

$$s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

where $\bar{X}$ is the sample mean. Show that s underestimates σ (that is, it has negative bias).

9. Let $X_1 \sim Geo(p_1)$ and $X_2 \sim Geo(p_2)$ be independent random variables, where $Geo(p)$ refers to the Geometric distribution whose p.m.f. f is given by

$$f(k) = p(1-p)^k, \quad k = 0, 1, \ldots .$$

We are interested in testing the null hypothesis $H_0 : p_1 = p_2$ against the alternative $H_1 : p_1 < p_2$. Intuitively, it is clear that we should reject if X_1 is large, but unfortunately we cannot compute a cutoff because the distribution of X_1 under H_0 depends on the unknown (common) value of p_1 and p_2.

(a) Let $Y = X_1 + X_2$. Find the conditional distribution of $X_1|Y = y$ when $p_1 = p_2$.

(b) Based on the result obtained in (a), derive a level 0.05 test for H_0 against H_1 that rejects H_0 when X_1 is large.

10. Let y_1, y_2, y_3, y_4 be uncorrelated observations with common variance σ^2 and expectations given by

$$E(y_1) = E(y_2) = \beta_1 + \beta_2 + \beta_3$$
$$E(y_3) = E(y_4) = \beta_1 - \beta_2$$

An observational function $\sum_{i=1}^{4} a_i y_i$ is said to be an *error function* if its expectation is zero.

(a) Obtain a maximal set of linearly independent error functions for the above model (that is, the set should be such that adding any other error function to the set would make the set linearly dependent). Justify your answer.

(b) Obtain an unbiased estimator of $3\beta_1 - \beta_2 + \beta_3$ such that it is uncorrelated with each of the error functions obtained in (a).

Note: For more sample questions you can visit `http://www.isical.ac.in/~deanweb/MSTATSQ.html`.