2024

Booklet Number: 00848 TEST CODE: **PSA**

Forenoon **Time: 2 hours**

- This test contains thirty (30) multiple-choice questions (MCQs).
- The questions are to be answered on a separate *Optical Mark Recognition* (OMR) Answer Sheet.
- Please write your *Name, Registration Number, Test Centre, Test Code* and the *Number of this Question Booklet* in the appropriate places on the OMR Answer Sheet. Please do not forget to put your signature in the designated place.
- For each of the questions there are four suggested answers, of which only one is correct. For each question, indicate your choice of the correct answer by darkening the appropriate circle (●) completely on the OMR Answer Sheet, using ball-point pen with BLACK ink only.
- You will score
  - 4 marks for each correctly answered question,
  - 0 mark for each incorrectly answered question, and
  - 1 mark for each unattempted question.
- ALL ROUGH WORK MUST BE DONE ONLY IN THE SPACE AVAILABLE ON THIS QUESTION BOOKLET.
- USE OF CALCULATORS, MOBILE PHONES AND ALL TYPES OF ELECTRONIC DEVICES IS STRICTLY PROHIBITED.

STOP! WAIT FOR THE SIGNAL TO START.

Notations and Abbreviations

The following are used throughout the question paper.

$\mathbb{R}$	Set of real numbers
$\mathbb{E}$	Expectation
$\mathbb{V}$	Variance
Cov	Covariance
$\lfloor x \rfloor$	Largest integer less than or equal to x
iid	independent and identically distributed
pmf	probability mass function
cdf	cumulative distribution function
MLE	Maximum likelihood estimator

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1. Let n be an odd positive integer. For any permutation σ of $\{1, 2, 3, \ldots, n\}$, consider the product

$$P(\sigma) = \prod_{j=1}^{n} (j - \sigma(j)) = (1 - \sigma(1))(2 - \sigma(2)) \cdots (n - \sigma(n)).$$

Then

(A) $P(\sigma) = 0$ for all σ.

(B) $P(\sigma)$ is odd for all σ.

(C) $P(\sigma)$ is even for all σ.

(D) None of the above.

2. Let a, b, c be three distinct non-zero real numbers. The expression

$$a\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\frac{(x-c)(x-a)}{(b-c)(b-a)} + c\frac{(x-a)(x-b)}{(c-a)(c-b)} - x$$

is equal to zero for

(A) exactly 2 values of x. (B) exactly 3 values of x.

(C) infinitely many values of x. (D) no value of x.

3. Equation of the straight line passing through the centres of the two circles $x^2 + y^2 - 4x = 0$ and $2x^2 + 2y^2 + 4x - 5y = 0$ is

(A) $(x-2)y + (x-1)(y+5/4) = 0$.

(B) $4(x+1)y - (x-2)(4y-5) = 0$.

(C) $(x+1)y - (x-2)(y-5) = 0$.

(D) $2x + y = 3$.

4. Let p be a positive real and T be the set of all triangles that have perimeter p. A triangle in T has maximum area if its angles are

(A) 60°, 60° and 60° (B) 90°, 45° and 45°

(C) 120°, 30° and 30° (D) 50°, 60° and 70°

5. For $x \geq 0$, let $f(x)$ denote the non-negative square root of x. The expression

$$f\left(\sin^4 x + 4\cos^2 x\right) - f\left(\cos^4 x + 4\sin^2 x\right)$$

equals

(A) $\sin 2x$ (B) $\cos x$ (C) $\sin x$ (D) $\cos 2x$

6. Let S be the set of all 3×3 matrices A such that $A^2 = 0$ and the entries of A belong to $\{0, 1\}$. Then the number of elements in the set S is

(A) 9 (B) 12 (C) 15 (D) 6

7. Let A be an $m \times n$ matrix and let P be an $n \times n$ idempotent matrix. If $P \neq I$, then which of the following statements is true?

(A) $\text{rank}\,(AP) < m$

(B) $\text{rank}\,(AP) < \min\{m, n\}$

(C) $\text{rank}\,(AP) = \min\{m, n\}$

(D) $\text{rank}\,(AP) < n$

8. Let a_1, a_2, a_3, a_4 be four distinct positive numbers. Consider the 4×4 matrix $B = ((b_{ij}))$ where $b_{ij} = a_i + a_j$ for $i, j = 1, 2, 3, 4$. Then $\text{rank}\,(B)$ is

(A) 1 (B) 4 (C) 2 (D) 3

9. Let S be the set of vectors $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ such that $x_i \in \{1, 2, 3, 4, 5\}$ for all i and not all x_i's are equal. For how many ordered pairs $(\mathbf{x}, \mathbf{y}) \in S \times S$ are $\mathbf{x}$ and $\mathbf{y}$ linearly dependent vectors in $\mathbb{R}^3$?

(A) 137 (B) 125 (C) 126 (D) 132

10. Let $f : [0, 1] \to [0, 1]$ be a function such that

$$|f(x) - f(y)| < |x - y|$$

for all $x \neq y \in [0, 1]$. Consider the following two statements:

(i) There exists at least one point $a \in [0, 1]$ such that $f(a) = a$.

(ii) There exists at most one point $a \in [0, 1]$ such that $f(a) = a$.

Choose the correct option.

(A) Both (i) and (ii) are true.

(B) (ii) is true but (i) is false.

(C) (i) is true but (ii) is false.

(D) Both (i) and (ii) are false.

11. The number of subsets $\{a, b, c\}$ of $\{1, 2, \ldots, 24\}$ such that a, b and c are in arithmetic progression is

(A) 66 (B) 132 (C) 276 (D) 138

12. Let f and g be two real-valued differentiable functions on $\mathbb{R}$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h(x) = \max\{f(x), g(x)\}$. Consider the following two statements:

(i) If h is differentiable then $f \geq g$ or $f \leq g$.

(ii) If $f \geq g$ or $f \leq g$ then h is differentiable.

Choose the correct option.

(A) (ii) is true but (i) is false.

(B) h is always differentiable.

(C) Both (i) and (ii) are true.

(D) (i) is true but (ii) is false.

13. Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics from a random sample of size n, drawn from an exponential distribution with mean equal to 1. For $z > y > 0$, the conditional density of $X_{(n)}$ given $X_{(1)} = y$ is

(A) $n \exp\{-(z-y)\} \left(1 - \exp\{-(z-y)\}\right)^{n-1}$

(B) $(n-1) \exp\{-(z-y)\} \left(\exp\{-(z-y)\}\right)^{n-2}$

(C) $(n-1) \exp\{-(z-y)\} \left(1 - \exp\{-(z-y)\}\right)^{n-2}$

(D) $\exp\{-(z-y)\}$

14. Let F and G be two cdfs and let $\alpha, \beta > 0$. Which of the following is not a cdf?

(A) $\left(\alpha F(\alpha x) + \beta G(\beta x)\right)/(\alpha + \beta)$ (B) $F(\alpha x)G(\beta x)$

(C) $\left(F(\alpha x) + G(\beta x)\right)/(\alpha + \beta)$ (D) $(F(\alpha x))^{\alpha}(G(\beta x))^{\beta}$

15. Suppose X_1, X_2 and X_3 are three random variables such that X_1, $(X_1 + X_2)/2$ and $(X_1 + X_2 + X_3)/3$ are iid standard normal random variables. Based on the above information choose the correct statement from the options given below.

(A) This is not possible.

(B) X_1 and X_3 are independent normal random variables.

(C) X_1, X_2 and X_3 are identically distributed normal random variables.

(D) X_1, X_2 and X_3 are independent normal random variables.

16. Suppose $E_1, \ldots, E_m$ are disjoint events satisfying $0 < \mathbb{P}(E_j) < 1$, for each $j = 1, \ldots, m$. Let $E = \bigcup_{j=1}^{m} E_j$. Assume $0 < \mathbb{P}(E) < 1$. Let A be any other event and define $p = \mathbb{P}(A \mid E)$. Which of the following statements is true?

(A) $p = \sum_{j=1}^{m} \mathbb{P}(A \mid E_j)\, \mathbb{P}(E)$

(B) $p = \sum_{j=1}^{m} \mathbb{P}(A \mid E_j)\, \mathbb{P}(E_j \mid E)$

(C) $p = \sum_{j=1}^{m} \mathbb{P}(A \mid E_j)$

(D) $p = \sum_{j=1}^{m} \mathbb{P}(A \mid E_j)\, \mathbb{P}(E_j)$

17. Suppose A is the set of all 6-digit numbers formed using each of the digits $1, 2, 3, \ldots, 6$ exactly once. If a number X is chosen uniformly at random from A, then what is the probability that X is divisible by 6 but not by 9 or 11?

(A) 1/2　　(B) 1/6　　(C) 1/11　　(D) 2/33

18. Suppose X is a random variable taking values a^n with probability $(1-p)\,p^n$ for $n \geq 0$, where $a \neq 0$ is a real number and $0 < p < 1$. Consider the following statements.

(i) $\mathbb{E}[X]$ is finite for all values of a and p.

(ii) $\mathbb{V}(X)$ is finite whenever $|a| < 1/\sqrt{p}$.

Which of the following statements is correct?

(A) Only (ii) is true.

(B) Both (i) and (ii) are true.

(C) Only (i) is true.

(D) Neither (i) nor (ii) is true.

19. A coin with success probability 1/4 is independently tossed N times and the outcomes are recorded. Given that the total number of successes is m, what is the probability that the first k tosses are successes, where $k \le m$?

(A) $(1/4)^k(3/4)^{N-k}$

(B) $\dfrac{k}{m}$

(C) $\dfrac{m(m-1)\cdots(m-k+1)}{N(N-1)\cdots(N-k+1)}$

(D) $\dbinom{m}{k}(1/4)^k(3/4)^{m-k}$

20. Suppose X and Y are two independent random variables both following Poisson distribution with parameter 10. What is the value of $\mathbb{E}(X-Y)^2$?

(A) 30 (B) 10 (C) 40 (D) 20

21. Let M denote the region bounded by the parallelogram with vertices at $(0,1), (2,1), (0,-1)$ and $(-2,-1)$. Let (X, Y) denote the x and y co-ordinates of a randomly chosen point from M. Consider the folowing statements.

(i) The marginal distribution of X is uniform on $(-2, 2)$.

(ii) $\mathbb{E}(XY) = 0$.

(iii) X and Y are independent.

Which of the following statements is correct?

(A) Only (ii) and (iii) are true.

(B) Only (ii) is true.

(C) Only (i) is true.

(D) None of the statements is true.

22. Suppose $\{X_i : i \geq 1\}$ are iid Poisson random variables with standard deviation 2. Then as $n \to \infty$

$$\frac{1}{n}\sum_{i=1}^{n} X_i(X_i - 2)$$

converges in probability to

(A) 16 (B) 8 (C) 12 (D) 2

23. For $x_1, x_2, \ldots, x_{2n} \in [-1, 1]$, where the x_i's are not necessarily distinct, define

$$m = \frac{1}{2n}\sum_{i=1}^{2n} x_i \qquad \text{and} \qquad s^2 = \frac{1}{2n}\sum_{i=1}^{2n}(x_i - m)^2.$$

Let s^2_{max} denote the maximum possible value that s^2 can take over all such $x_1, x_2, \ldots, x_{2n}$.

Based on the above information, choose the FALSE statement from the options given below.

(A) $s^2 \in [0, 1]$ for all such $x_1, x_2, \ldots, x_{2n}$.

(B) $m \in [-1, 1]$ for all such $x_1, x_2, \ldots, x_{2n}$.

(C) If $m = 0$ then the corresponding $s^2 = s^2_{\text{max}}$.

(D) If $s^2 = s^2_{\text{max}}$ then the corresponding $m = 0$.

24. Consider a random variable with pmf given by

$$P(X = x) = \begin{cases} p & \text{if } x = 1, \\ 3p & \text{if } x = 2, \\ 2p & \text{if } x = 3, \\ 1 - 6p & \text{if } x = 4, \end{cases}$$

where the unknown parameter $p \in [1/20, 1/10]$. A random sample of size 4 from this distribution yielded the observations $x_1 = 2, x_2 = 3, x_3 = 2, x_4 = 1$. What is the maximum likelihood estimate of p based on these observations?

(A) 1/10 (B) $1/\sqrt[4]{18}$ (C) 1/18 (D) 1/20

25. Suppose that we have observations $(x_i, y_i, z_i), i = 1, \ldots, n$. Let $\widehat{\beta}_x$ be the estimated coefficient obtained by least-squares linear regression of z on x, without intercept. Let $\widehat{\gamma}_x, \widehat{\gamma}_y$ be the estimated coefficients obtained by similarly regressing z on x and y jointly, also without intercept. Which of the following statements is true?

(A) $\sum_{i=1}^{n}(z_i - \widehat{\beta}_x x_i)^2 > \sum_{i=1}^{n}(z_i - \widehat{\gamma}_x x_i)^2$.

(B) $\sum_{i=1}^{n}(z_i - \widehat{\beta}_x x_i)^2 < \sum_{i=1}^{n}(z_i - \widehat{\gamma}_x x_i - \widehat{\gamma}_y y_i)^2$.

(C) $\sum_{i=1}^{n}(z_i - \widehat{\beta}_x x_i)^2 \leq \sum_{i=1}^{n}(z_i - \widehat{\gamma}_y y_i)^2$.

(D) $\sum_{i=1}^{n}(z_i - \widehat{\beta}_x x_i)^2 \geq \sum_{i=1}^{n}(z_i - \widehat{\gamma}_x x_i - \widehat{\gamma}_y y_i)^2$.

26. Suppose that we have observations $(x_1, y_1), \ldots, (x_n, y_n)$. We fit a simple linear regression of y on x, with intercept, using these observations via the least-squares method. Let e_i, $i = 1, \ldots, n$, be the residuals of the fitted regression model. Consider the following statements.

(i) $\frac{1}{n}\sum_{i=1}^{n} e_i = 0$.

(ii) $\frac{1}{n}\sum_{i=1}^{n}(e_i - \bar{e})(x_i - \bar{x}) = 0$ where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{e} = \frac{1}{n}\sum_{i=1}^{n} e_i$.

(iii) $\frac{1}{n}\sum_{i=1}^{n} e_i x_i = 0$.

Which of the following statements is correct?

(A) (i), (ii) and (iii) are all true.

(B) Only (ii) and (iii) are true.

(C) Only (i) and (ii) are true.

(D) Only (i) and (iii) are true.

27. Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ random variables, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown parameters. Let $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ denote the cdf of $N(0,1)$ and its inverse respectively. Define

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

For $\alpha \in (0,1)$, let $\xi_\alpha(\mu, \sigma)$ be a parametric function satisfying

$$\mathbb{P}(X_1 \le \xi_\alpha(\mu, \sigma)) = \alpha.$$

What is the MLE of $\xi_\alpha(\mu, \sigma)$?

(A) $\Phi\left(\bar{X}_n + \alpha S_n\right)$.

(B) $\bar{X}_n + \alpha S_n$.

(C) $X_{(\lfloor n\alpha \rfloor)}$, the $\lfloor n\alpha \rfloor$-th order statistic of $X_1, \ldots, X_n$.

(D) $\bar{X}_n + \Phi^{-1}(\alpha)\, S_n$.

28. A random sample is obtained from a $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is an unknown parameter. Using this sample, a most powerful (MP) test ϕ of size $\alpha = 1/2$ is constructed for testing $H_0 : \mu = 0$ against $H_1 : \mu = 1$. Let β denote the power of this test. Which of the following statements is correct?

(A) ϕ is an MP test of size $< 1/2$ with power β for testing $H_0' : \mu = -1$ against $H_1' : \mu = 1$.

(B) ϕ is an MP test of size $1/2$ with power $> \beta$ for testing $H_0' : \mu = 0$ against $H_1' : \mu = 1/2$.

(C) $\psi = (1 - \phi)$ is not an MP test of size $(1 - \beta)$ for testing $H_0' : \mu = 1$ against $H_1' : \mu = 0$.

(D) ϕ is an MP test of size $1/2$ for testing $H_0' : \mu = 0$ against $H_1' : \mu = a$ for some $a \in (-\infty, 0)$.

29. Let $\lambda > 0$. Let X_1, X_2, X_3 be independent Poisson random variables with parameters $\lambda, 2\lambda$ and $\lambda/2$ respectively. Which of the following statements is sufficient for λ?

(A) $T = 2X_1 + 4X_2 + X_3$

(B) $T = 2X_1 + X_2 + 4X_3$

(C) $T = \max\{2X_1, 4X_2, X_3\}$

(D) $T = X_1 + X_2 + X_3$

30. Let $N \geq 3$ be an integer. A box contains N balls numbered $\{1, 2, \ldots, N\}$. A simple random sample s of size n $(1 < n < N)$ is selected **with replacement** from the box. For $i = 1, \ldots, N$, define

$$W_i = \begin{cases} 1 & \text{if the } i\text{-th ball is selected in } s, \\ 0 & \text{otherwise.} \end{cases}$$

Which of the following statements is true?

(A) $\mathbb{C}\text{ov}(W_1, W_2) < 0$.

(B) $\sum_{i=1}^{N} W_i = n$.

(C) $\mathbb{E}(W_i) = n/N$ for each $i = 1, \ldots, N$.

(D) $\mathbb{C}\text{ov}(W_1, W_2) = 0$.