1. Let $f : R \to R$ be a function which is continuous at 0 and $f(0) = 1$. Also assume that f satisfies the following relation for all x:

$$f(x) - f(x/2) = \frac{3x^2}{4} + x.$$

Find $f(3)$.

2. For any $n \times n$ matrix $A = ((a_{ij}))$, consider the following three properties:

   1. a_{ij} is real valued for all i, j and A is upper triangular.
   2. $\sum_{j=1}^{n} a_{ij} = 0$, for all $1 \leq i \leq n$.
   3. $\sum_{i=1}^{n} a_{ij} = 0$, for all $1 \leq j \leq n$.

   Define the following set of matrices:

$$\mathcal{C}_n = \{A : \ A \text{ is } n \times n \text{ and satisfies (1), (2) and (3) above}\}.$$

   (a) Show that $\mathcal{C}_n$ is a vector space for any $n \geq 1$.

   (b) Find the dimension of $\mathcal{C}_n$, when $n = 2$ and $n = 3$.

3. Let A be a real valued and symmetric $n \times n$ matrix with entries such that $A \neq \pm I$ and $A^2 = I$.

   (a) Prove that there exist non-zero column vectors v and w such that $Av = v$ and $Aw = -w$.

   (b) Prove that every vector z has a unique decomposition $z = x + y$ where $Ax = x$ and $Ay = -y$.

4. Suppose 15 identical balls are placed in 3 boxes labeled A, B and C. What is the number of ways in which Box A can have more balls than Box C?

5. Suppose that X and Y are random variables such that

$$\begin{aligned} E(X+Y) = E(X-Y) &= 0\,, \\ \mathrm{Var}(X+Y) &= 3\,, \\ \mathrm{Var}(X-Y) &= 1\,. \end{aligned}$$

   (a) Evaluate $\mathrm{Cov}(X,Y)$.

   (b) Show that $E|X+Y| \leq \sqrt{3}$.

   (c) If in addition, it is given that (X,Y) is bivariate normal, calculate $E\big(|X+Y|^3\big)$.

6. Suppose $X_1, \ldots, X_n$ are i.i.d. random variables with mean μ and variance 1. Also assume that $Y_1, \ldots, Y_n$ are i.i.d. with probability mass function $\mathbf{P}(Y_i = \pm 1) = 1/2$ for all $1 \leq i \leq n$ and independent of $X_1, \ldots, X_n$. Define T_n as follows:

$$T_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \cdot |X_i|, \quad n \geq 1.$$

   (a) For any fixed $z \in R$, find

$$\lim_{n\to\infty} \mathbf{P}\left(\sqrt{n}T_n \leq z\right).$$

   (b) Using the result in part (a) above, find random quantities L_n and U_n, based on T_n, such that

$$\lim_{n\to\infty} \mathbf{P}\left(L_n \leq \mu \leq U_n\right) = 0.95\ .$$

7. Suppose that $X_1, \ldots, X_n$ are i.i.d. with probability density function

$$f_\theta(x) = \begin{cases} \frac{1}{2\sqrt{\theta x}} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $X_{(n)}$ is sufficient for θ.

(b) Consider a test of size α ($0 < \alpha < 1$) for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ ($> \theta_0$), that rejects H_0 if and only if $X_{(n)} > k$.

i. Determine the value of k.

ii. Find the minimum sample size required such that the power of the test is at least β ($\alpha < \beta < 1$).

8. Consider the regression model:

$$y_i = bx_i + e_i, \quad 1 \leq i \leq n,$$

where x_i's are fixed non-zero real numbers and e_i's are independent random variables with mean zero and equal variance.

(a) Consider estimators of the form $\sum_{i=1}^n a_i y_i$ (where a_i's are non random real numbers) that are unbiased for b. Show that the least squares estimator of b has the minimum variance in this class of estimators.

(b) Suppose that x_i's take values -1 or $+1$ and e_i's have density

$$f(t) = \frac{1}{2}\, e^{-|t|}, \quad t \in R.$$

Find the maximum likelihood estimator of b.

9. Consider a collection of N cards, numbered $1, 2, \ldots, N$, where $N \geq 2$. A card is drawn at random and set aside. Suppose n cards are selected from the remaining $(N - 1)$ cards using SRSWR and their numbers noted as $Y_1, \ldots, Y_n$. If $S = \sum_{i=1}^{n} Y_i$, find $E(S)$ and $Var(S)$.