ISI Mock 3 PSA

April 25, 2024

1) Let F be the set of all bijections f from the set $S = \{1, 2, 3, 4, 5, 6\}$ onto itself such that $f(i) \neq i$,for any $i \in S$. The number of elements in F are

$a)$ 160
$b)$ 265
$c)$ 120
$d)$ 200

2) Find $max\{xyz\}$ subject to $x^2 + 2y^2 + 9z^2 = 6$ where $x > 0, y > 0, z > 0$.

$a)$ $\frac{1}{3}$
$b)$ $\frac{2}{3}$
$c)$ 1
$d)$ NONE

3) The value of sum $cos\left(\frac{\pi}{1000}\right) + cos\left(\frac{2\pi}{1000}\right) + ... + cos\left(\frac{999\pi}{1000}\right)$ is

$a)$ 0
$b)$ 1
$c)$ $\frac{1}{1000}$
$d)$ an irrational number

4) If $z_1, z_2, .., z_7$ are the roots of $(z+1)^7 + z^7 = 0$, the value of $\sum\limits_{k=1}^{7} Re\,(z_k)$ is (where Re denotes the real part)

$a)$ $-\frac{7}{2}$
$b)$ $-\frac{5}{2}$
$c)$ $\frac{7}{2}$
$d)$ NONE

5) The last digit of $(2137)^{754}$ is

$a)$ 1
$b)$ 3
$c)$ 7
$d)$ 9

6) The remainder when $3^{12} + 5^{12}$ is divisible by 13 is

$a)$ 1
$b)$ 2
$c)$ 3
$d)$ 4

7) The number of $0's$ at the end of the integer $100! - 101! + ... + 108! - 109! + 110!$ is

$a)$ 24
$b)$ 25
$c)$ 26
$d)$ 27

8) The general solution of $sin(x) - 3sin(2x) + sin(3x) = cos(x) - 3cos(2x) + cos(3x)$ is

$a)$ $\frac{n\pi}{2} + \frac{\pi}{8}$
$b)$ $n\pi + \frac{\pi}{4}$
$c)$ $n\pi + \frac{\pi}{2}$
$d)$ $n\pi + \frac{3\pi}{4}$

9) In a triangle ABC, AD is the median. If length of AB is 7, length of AC is 15 and length of BC is 10, then, length of AD equals

$a)$ $\sqrt{125}$
$b)$ $\frac{69}{5}$
$c)$ $\sqrt{112}$
$d)$ $\frac{\sqrt{864}}{5}$

10) Let f is continous in $[0,1]$,then,

$$\lim_{n\to\infty} \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{n} f\left(\frac{j}{n}\right)$$

(where $[y]$ is the largest integer less than or equal to y)

$a)$ does not exist

$b)$ exists and equal to $\frac{1}{2}\int_0^1 f(x)dx$

$c)$ exists and equal to $\int_0^1 f(x)dx$

$d)$exists and equal to $\int_0^{\frac{1}{2}} f(x)dx$

11) Let f be a twice differentiable function defined on R such that $f(0) = 1$, $f'(0) = 2$ and $f'(x) \neq 0$ for all $x \in R$. If $\begin{vmatrix} f(x) & f'(x) \\ f'(x) & f"(x) \end{vmatrix} = 0$, for all $x \in R$, then the value of $f(1)$ lies in the interval

$a)$ $(9, 12)$

$b)$ $(6, 9)$

$c)$ $(3, 6)$

$d)$ $(0, 3)$

12) Let $f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 1 & x = 0 \end{cases}$, $g(x) = \begin{cases} \frac{sin(x+1)}{(x+1)} & x \neq -1 \\ 1 & x = -1 \end{cases}$, $h(x) = 2[x] - f(x)$, where $[x]$ is the greatest integer function $\leq x$. Then, the value of $\lim_{x\to 1} g(h(x-1))$ is ?

$a)$ Does not exist

$b)$ 1

$c)$ -1

$d)$ $sin(1)$

13) Let M be the collection of all 3×3 real symmetric positive definite matrices. Consider the set

$$S = \left\{ A \in M : A^{50} - \frac{1}{4} A^{48} = 0 \right\},$$

where 0 denotes the 3×3 zero matrix. Then, the number of elements in S equals

$a)$ 0
$b)$ 1
$c)$ 8
$d)$ ∞

14) Let $A = [a, u_1, u_2, u_3]\,, B = [b, u_1, u_2, u_3]$and $C = [u_2, u_3, u_1, a+b]$be 4×4 real matrices, where a, b, u_1, u_2, u_3 are 4×1 real column vectors. Let $det(A)$,$det(B)$ and $det(C)$ denotes the determinants of matrices A, B and C respectively. If $det(A) = 6$ and $det(B) = 2$, then, $det(A+B) - det(C)$ equals

$a)$ 8
$b)$ 16
$c)$ 0
$d)$ 72

15)$\lim\limits_{n \to 0} \left(2^n + n2^n sin^2\left(\frac{n}{2}\right)\right)^{\overline{\frac{1}{2n - ncos\left(\frac{1}{n}\right)}}}$ is

$a)$ 0
$b)$ 1
$c)$ 2
$d)$ $NONE$

16) Let $T : R^3 \to R^3$ be a linear transformation whose matrix with respect to the standard basis $\{e_1, e_2, e_3\}$ of $R^3$is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Then, T

a) maps the subspace spanned by e_1 and e_2 into itself.
b) has distinct eigen-values
c) has eigen-vectors that span R^3
d) has non-zero null space

17) The total number of distinct $x^3 \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

a) 0
b) 1
c) 2
d) 3

18) Let X and Y be two random variables such that the moment generating function of X is $M_X(t)$ and the moment generating function of Y is $H(t) = \left(\frac{3}{4}e^{2t} + \frac{1}{4}\right) M(t)$ where $t \in (-h, h)$, $h > 0$. If the mean and variance of X are $\frac{1}{2}$ and $\frac{1}{4}$, respectively, then, the variance of Y (**in integer**) is equal to

a) 1
b) 2
c) 3
d) 4

19) 3 points are marked on a circle. The probability that the triangle formed is obtuse is

$a)$ $\frac{1}{3}$
$b)$ $\frac{1}{4}$
$c)$ $\frac{1}{6}$
$d)$ $NONE$

20) Let the random vector $\underset{\sim}{X} = (X_1, X_2, X_3)$ have the joint pdf

$$f_{\underset{\sim}{X}}(x_1, x_2, x_3) = \begin{cases} \frac{1 - sinx_1 sinx_2 sinx_3}{8\pi^3} & 0 \le x_1, x_2, x_3 \le 2\pi \\ 0 & otherwise \end{cases}$$

Which of the following statements is TRUE?

$a)$ $X_1, X_2$and $X_3$are mutually independent
$b)$ $X_1, X_2$and $X_3$are pairwise independent
$c)$ $(X_1, X_2)$and $X_3$are independently distributed
$d)$ Variance of $X_1 + X_2$ is π^2

21) Let $\{X_n\}_{n\ge1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on $[0, 2]$. For $n \ge 1$, let

$$Z_n = -log_e\left(\prod_{i=1}^{n}(2 - X_i)\right)^{\frac{1}{n}}.$$

Then, as $n \to \infty$, the sequence $\{Z_n\}_{n\ge1}$ converges almost surely to

$a)$ 1
$b)$ $log_e(\frac{4}{e})$
$c)$ $log_e(\frac{2}{2})$
$d)$ NONE

22) Let $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ follow $N_2\left(\mu, \Sigma\right)$ with $\mu = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ and variance-covariance matrix $\Sigma = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 2 & a \\ 1 & a & 2 \end{bmatrix}$ where $a \in R$. Suppose that the partial correlation coefficient between X_2 and X_3 keeping X_1 fixed is $\frac{5}{7}$. Then, a is equal to

$a)$ 1
$b)$ -1
$c)$ 0
$d)$ $NONE$

23) Let (X, Y) have the joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{4}{(x+y)^3} & x > 1, y > 1 \\ 0 & otherwise \end{cases}.$$

Then which one of the following statements is NOT true?

$a)$ The probability density function of $X + Y$ is

$$f_{X+Y}(z) = \begin{cases} \frac{4}{z^3}(z-2) & z > 2 \\ 0 & otherwise \end{cases}$$

$b)$ $P(X + Y > 4) = \frac{3}{4}$
$c)$ $E(X + Y) = 4log_e(2)$
$d)$ $E(Y|X = 2) = 4$

24) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having probability density function

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & otherwise \end{cases}.$$

Let $X_{(n)} = max\left\{X_1, X_2, .., X_n\right\}$for $n \geq 1$. If Z is the random variable to which $\left\{X_{(n)} - log_e(n)\right\}_{n\geq 1}$ converges in distribution, as $n \to \infty$, then, the median Z equals

$a)$ 0.125
$b)$ 0.366
$c)$ 0.245
$d)$ 0.087

25) Let $X_1, X_2, .., X_n$ be a random sample of size $n(\geq 2)$ from $N\left(\theta, \theta^2\right)$ distribution, where $\theta \in (0, \infty)$. Which of the following statements is TRUE?

a) $\frac{1}{n(n+2)}\left(\sum\limits_{i=1}^{n} X_i\right)^2$ is the unique unbiased estimator of θ^2that is a function of a minimal sufficient statistic.
b)$\frac{1}{(3n+1)}\sum\limits_{i=1}^{n} X_i^2$ is an unbiased estimator of θ^2.
c) There exist infinite number of unbiased estimators of θ^2which are functions of minimal sufficient statistic.
d) There does NOT exist any unbiased estimator of $\theta(\theta+1)$ that is a function of minimal sufficient statistic.

26) Suppose that (X_i, Y_i) ,$i = 1, .., 2n$, are *iid* $N_2(0, 0, 1, 1, \rho)$ with $-1 < \rho < 1$. Let us denote

$$U_i = \begin{cases} 0 & if\ X_{2i-1}Y_{2i-1} + X_{2i}Y_{2i} \leq 0 \\ 1 & if\ X_{2i-1}Y_{2i-1} + X_{2i}Y_{2i} \geq 0 \end{cases} ; i = 1, 2, .., n$$

a) $U_i's$ are *iid* Bernoulli with $p = \frac{(1-\rho)}{2}$
b)$U_i's$ are *iid* Bernoulli with $p = \frac{(1+\rho)}{2}$
c) $U_i's$ are *iid* Bernoulli with $p = \frac{1-\rho}{(1+\rho)^2}$
d) *NONE*

27) Suppose that the bivariate data $(x_1, y_1), .., (x_n, y_n)$ lie on the straight line $y = a + bx$, for some $a, b \in R$. Assume further that neither all the $x_i's$ are same, nor all the $y_i's$. Which of the following values is not a possibility for the correlation coefficient calculated for the above data?

a) 1
b) -1
c) 0
d) *NONE*

28) A simple random sample of size n is drawn with replacement from a population of N units. The expected number of distinct units in the sample is

a) $n\left[1-\left(\frac{N-1}{N}\right)^n\right]$
b) $n\left[1-\left(\frac{N-2}{N}\right)^n\right]$
c) $N\left[1-\left(\frac{N-1}{N}\right)^n\right]$
d) $N\left[1-n\left(\frac{N-1}{N}\right)\right]^n$

29) Let $X_1, X_2, .., X_5$ be a random sample from a distribution with the probability density function

$$f(x;\theta)=\frac{1}{2}e^{-|x-\theta|}, x\in(-\infty,\infty),$$

where $\theta \in(-\infty,\infty)$. For testing $H_0:\theta=0$ against $H_1:\theta<0$. Let $\sum\limits_{i=1}^{5} Y_i$ be the sign test statistic, where

$$Y_i=\begin{cases}1 & X_i>0\\ 0 & otherwise\end{cases}.$$

Then, the size of the test, which rejects H_0 if and only if $\sum\limits_{i=1}^{5} Y_i \le 2$ equals

a) $\frac{1}{4}$
b) $\frac{1}{2}$
c) $\frac{3}{4}$
d) $NONE$

30) Let $(X_1, X_2, .., X_n)$ be iid RVs from a continous distribution where density is symmetric around 0. Suppose $E|X_1|=2$. Define, $Y=\sum\limits_{i=1}^{n} X_i$ and $Z=\sum\limits_{i=1}^{n} 1(X_i>0)$. Calculate the covariance between Y and Z.

a) n
b) $2n$
c) $3n$
d) NONE