ISI Mock 1 PSB

April 5, 2024

Group A

1. Let A and B be $n \times n$ real matrices. Let I_n denote the identity matrix of order n. Show that the matrix $\begin{bmatrix} A & I_n \\ I_n & B \end{bmatrix}$ has rank n if and only if A is nonsingular and $B = A^{-1}$.

2. Let $f : (-1, 1) \to R$ be a continuous function with the property that $f(x) = f(x^4)$ for all x and $f(\frac{1}{2}) = a$. Show that the only function f satisfying this property is the constant function $f(x) = a$

3. Find the total number of isosceles triangles such that the length of each side is a positive integer less than or equal to 40. (Here equilateral triangles are also counted as isosceles triangles.)

Group B

1. Suppose that $X_1, X_2, ...$ are independent and identically distributed $N(0, 1)$ random variables. Let

$$Y_i = \begin{cases} X_i - 1 & X_i \leq 0 \\ X_i & X_i > 0 \end{cases} ; i = 1, 2, ..$$

   (a) Find the mean and variance of Y_1. (b) Find constants α_n and β_n, depending on n, such that $\alpha_n \sum\limits_{i=1}^{n} Y_i - \beta_n$ converges in distribution to Z as $n \to \infty$, where Z has a standard normal distribution.

2. A fair coin is tossed repeatedly and let T be the number of tosses till two consecutive tails are observed for the first time. (a) Show that E(T | tail is observed in the first toss) $= 2 + \frac{1}{2}$ E(T). (b) Find a similar formula for E(T | head is observed in the first toss). (c) Compute E(T).

3. Let U and V be two dependent discrete random variables, each being uniformly distributed on $1, 2, ..., k$. Let W be another random variable having the same uniform distribution but independent of U and V . Define a random variable $X = (V + W) mod(k)$. Show that (a) X is uniformly distributed on $0, 1, 2, ..., k - 1$, (b) U and X are independent.

4. Let X be a random variable having a density $\frac{1}{\theta} e^{-\frac{x}{\theta}}, x > 0, \theta > 0$. Consider $H_0 : \theta = 1$ $vs.$ $H_1 : \theta = 2$. Let ω_1 and ω_2 be two critical regions given by ω_1: $\sum\limits_{i=1}^{n} X_i \geq C_1$ and ω_2 : (number of X_i's $\geq 2) \geq$ C_2. (a) Determine approximately the values of C_1 and C_2 for large n so that both tests are of size α. (b) Show that the powers of both tests tend to 1 as $n \to \infty$.

5. Let $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$ be the ordered random variables of a sample of size n from the rectangular $(0, \theta)$ distribution with θ unknown, $0 < \theta < \infty$. By a careless mistake the observations $Y_{(k+1)}, \cdots, Y_{(n)}$ were recorded incorrectly and so they were discarded subsequently (Here $1 \leq k < n$). (a) Show that the conditional distribution of $Y_{(1)}, \cdots, Y_{(k-1)}$ given $Y_{(k)}$ is independent of θ. (b) Hence, or otherwise, obtain the maximum likelihood estimator of θ and show that it is a function of $Y_{(k)}$.

6. A population contains 10 units, labelled $U_1, U_2, ..., U_{10}$. The value, of a character Y under study, for U_i is Y_i $(1 \leq i \leq 10)$. In order to estimate the population mean, $\overline{Y}$, a sample of size 4 is drawn in the following manner: (i) a simple random sample of size 2 is drawn without replacement from the units $U_2, U_3, ..., U_9$; (ii) the sample drawn in step (i) is augmented by the units U_1 and U_{10}. Based on the above sample in (ii), suggest an unbiased estimator of $\overline{Y}$ and obtain its variance.