2020

BOOKLET No. TEST CODE: PSB

Afternoon

Questions: 9	**Time: 2 hours**

- Answer two questions from GROUP A and four questions from GROUP B.
- Best six answers subject to the above conditions will be considered.
- Each question carries 20 marks.
- The maximum possible marks is 120.
- Answer to each question should start on a fresh page.
- Please write your Registration Number, Test Centre, Test Code and the Number of this Question Booklet in the appropriate places on the cover page of the Answer Booklet.
- Please use pens with black/blue ink to answer the questions.
- ALL ROUGH WORK MUST BE DONE ONLY ON THE ANSWER BOOKLET.
- THE USE OF CALCULATORS, MOBILE PHONES AND ALL TYPES OF ELECTRONIC DEVICES IS STRICTLY PROHIBITED.

STOP! WAIT FOR THE SIGNAL TO START.

GROUP A

1. Let $f(x) = x^2 - 2x + 2$. Let L_1 and L_2 be the tangents to its graph at $x = 0$ and $x = 2$ respectively. Find the area of the region enclosed by the graph of f and the two lines L_1 and L_2.

2. Find the number of 3×3 matrices A such that the entries of A belong to the set $\mathbb{Z}$ of all integers, and such that the trace of $A^t A$ is 6.
   (A^t denotes the transpose of the matrix A).

3. Consider n independent and identically distributed positive random variables $X_1, X_2, \ldots, X_n$. Suppose S is a fixed subset of $\{1, 2, \ldots, n\}$ consisting of k distinct elements where $1 \leq k < n$.

   (a) Compute
   $$\mathbb{E}\left[\frac{\sum_{i \in S} X_i}{\sum_{i=1}^{n} X_i}\right].$$

   (b) Assume that X_i's have mean μ and variance σ^2, $0 < \sigma^2 < \infty$. If $j \notin S$, show that the correlation between $(\sum_{i \in S} X_i)X_j$ and $\sum_{i \in S} X_i$ lies between $-\frac{1}{\sqrt{k+1}}$ and $\frac{1}{\sqrt{k+1}}$.

GROUP B

4. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables. Let $S_n = X_1 + \cdots + X_n$. For each of the following statements, determine whether they are true or false. Give reasons in each case.

   (a) If $S_n \sim Exp$ with mean n, then each $X_i \sim Exp$ with mean 1.

   (b) If $S_n \sim Bin(nk, p)$, then each $X_i \sim Bin(k, p)$.

5. Let $U_1, U_2, \ldots, U_n$ be independent and identically distributed random variables each having a uniform distribution on $(0, 1)$. Let

$$X = \min\{U_1, U_2, \ldots, U_n\}, \qquad Y = \max\{U_1, U_2, \ldots, U_n\}.$$

Evaluate $\mathbb{E}[X|Y = y]$ and $\mathbb{E}[Y|X = x]$.

6. Suppose individuals are classified into three categories C_1, C_2 and C_3. Let $p^2, (1-p)^2$ and $2p(1-p)$ be the respective population proportions, where $p \in (0, 1)$. A random sample of N individuals is selected from the population and the category of each selected individual recorded. For $i = 1, 2, 3$, let X_i denote the number of individuals in the sample belonging to category C_i. Define $U = X_1 + \frac{X_3}{2}$.

   (a) Is U sufficient for p? Justify your answer.

   (b) Show that the mean squared error of $\frac{U}{N}$ is $\frac{p(1-p)}{2N}$.

7. Consider the following model:

$$y_i = \beta x_i + \varepsilon_i x_i, \;\; i = 1, 2, \ldots, n,$$

where $y_i, i = 1, 2, \ldots, n$ are observed; $x_i, i = 1, 2, \ldots, n$ are known positive constants and β is an unknown parameter. The errors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent and identically distributed random variables having the probability density function

$$f(u) = \frac{1}{2\lambda} \exp\left(-\frac{|u|}{\lambda}\right), \;\; -\infty < u < \infty,$$

and λ is an unknown parameter.

   (a) Find the least squares estimator of β.

   (b) Find the maximum likelihood estimator of β.

8. Assume that $X_1, \ldots, X_n$ is a random sample from $N(\mu, 1)$, with $\mu \in \mathbb{R}$. We want to test $H_0 : \mu = 0$ against $H_1 : \mu = 1$. For a fixed integer $m \in \{1, \ldots, n\}$, the following statistics are defined:

$$\begin{aligned} T_1 &= (X_1 + \ldots + X_m)/m, \\ T_2 &= (X_2 + \ldots + X_{m+1})/m, \\ \vdots &= \qquad \vdots \\ T_{n-m+1} &= (X_{n-m+1} + \ldots + X_n)/m. \end{aligned}$$

Fix $\alpha \in (0, 1)$. Consider the test

$$\text{reject } H_0 \quad \text{if} \qquad \max\{T_i : 1 \le i \le n - m + 1\} > c_{m,\alpha}.$$

Find a choice of $c_{m,\alpha} \in \mathbb{R}$ in terms of the standard normal distribution function Φ that ensures that the size of the test is at most α.

9. A finite population has N units, with x_i being the value associated with the i^{th} unit, $i = 1, 2, \ldots, N$. Let $\bar{x}_N$ be the population mean. A statistician carries out the following experiment.

- Step 1: Draw a SRSWOR of size n ($< N$) from the population. Call this sample S_1 and denote the sample mean by $\bar{X}_n$.
- Step 2: Draw a SRSWR of size m from S_1. The x-values of the sampled units are denoted by $\{Y_1, \ldots, Y_m\}$.

An estimator of the population mean is defined as,

$$\widehat{T}_m = \frac{1}{m} \sum_{i=1}^{m} Y_i.$$

(a) Show that $\widehat{T}_m$ is an unbiased estimator of the population mean.

(b) Which of the following has lower variance: $\widehat{T}_m$ or $\bar{X}_n$?

PSA
2019

1. Let $A = ((a_{ij}))$ be an $m \times n$ matrix with all non-zero real entries. Let B be obtained from A by replacing a_{11} by 0 and keeping all other entries unchanged. If r is the rank of A, then what is the set of possible values for the rank of B?

   (A) $\{r\}$ (B) $\{r-1, r, r+1\}$ (C) $\{r, r+1\}$ (D) $\{r-1, r\}$

2. What is the period of the function $g(x) = |\cos x| + |\sin x|$?

   (A) π (B) $\pi/2$ (C) 2π (D) $\pi/4$

3. If $3(\cos 100^\circ + i \sin 100^\circ)(\cos 110^\circ + i \sin 110^\circ) = x + iy$, where x and y are real numbers, then

   (A) $x = -\frac{3\sqrt{3}}{2}, \quad y = -\frac{3}{2}$.
   (B) $x = \frac{3\sqrt{3}}{2}, \quad y = \frac{3}{2}$.
   (C) $x = \frac{3\sqrt{3}}{2}, \quad y = -\frac{3}{2}$.
   (D) $x = -\frac{3\sqrt{3}}{2}, \quad y = \frac{3}{2}$.

ROUGH WORK

4. What is the number of 6 digit positive integers in which the sum of the digits is at least 52?

(A) 66 (B) 24 (C) 28 (D) 120

5. Let the sum

$$3 + 33 + 333 + \cdots + \underbrace{33\ldots3}_{200 \text{ times}}$$

be $\ldots zyx$ in the decimal system, i.e., x is the unit's digit, y the ten's digit, and so on. What is z?

(A) 0 (B) 9 (C) 7 (D) 3

6. How many times does the digit '2' appear in the set of integers $\{1, 2, ..., 1000\}$?

(A) 590 (B) 600 (C) 300 (D) 299

ROUGH WORK

7. Let t be a real number. Then the rank of $\begin{bmatrix} 0 & 1 & t \\ 2 & t & -1 \\ 2 & 2 & 0 \end{bmatrix}$ equals

(A) 2 if $t = -1$, and 3 if $t \neq -1$.

(B) 2 if $t = 1$, and 3 if $t \neq 1$.

(C) 2 if $t = \pm 1$, and 3 if $|t| \neq 1$.

(D) 3 for all t.

8. The number $\binom{200}{100}/4^{100}$ lies in

(A) $[\frac{3}{4}, 1)$ (B) $(0, \frac{1}{2})$ (C) $[1, \infty)$ (D) $[\frac{1}{2}, \frac{3}{4})$

9. Let $P(x) = x^4 + 4x^3 - 8x^2 - 1$. Which of the following is **false**?

(A) $P(x)$ has a real root in $(-4, 1)$

(B) $P(x)$ has a real root < -4

(C) $P(x)$ has a real root > 1

(D) $P(x)$ has at least two real roots.

ROUGH WORK

10. Two friends, one from Kolkata and one from Delhi, start driving towards each other at the same time. It is given that the distance between Kolkata and Delhi is 1455 km. One of them drives at a constant speed of 80 kmph (km per hour), while the other drives at a speed of 50 kmph during the first hour, 55 kmph during the second hour, 60 kmph during the third hour, and so on (i.e., his speeds over successive hours are in an arithmetic progression). How long will it take for them to meet each other?

    (A) 8 hrs 56 mins

    (B) 8 hrs 46 mins

    (C) 10 hrs 26 mins

    (D) 9 hrs 36 mins

11. How many positive divisors of $2^5\,5^3\,11^4$ are perfect squares?

    (A) 60 (B) 18 (C) 120 (D) 4

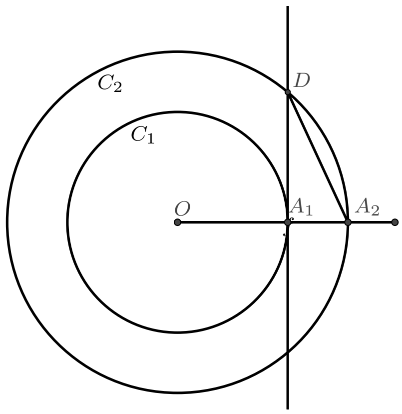
12. What is the set of numbers x in $(0, 2\pi)$ such that $\log\log(\sin x + \cos x)$ is well-defined?

    (A) $[\frac{\pi}{8}, \frac{3\pi}{8}]$ (B) $(0, \frac{\pi}{2})$ (C) $(0, \frac{\pi}{4}]$ (D) $(0, \pi) \cup (\frac{3\pi}{2}, 2\pi)$

ROUGH WORK

13. Let C_1 and C_2 be concentric circles with centre at O and radii r_1 and r_2 respectively. The line OA_2 intersects C_1 at A_1. The line A_1D is tangent to C_1 at A_1. What is the length of the line segment A_2D?

(A) $\sqrt{2r_1(r_2 - r_1)}$ (B) $\sqrt{r_2^2 - r_1^2}$ (C) r_2 (D) $\sqrt{2r_2(r_2 - r_1)}$



14. The reflection of the point $(1, 2)$ with respect to the line $x + 2y = 15$ is

(A) $(3, 6)$. (B) $(6, 3)$. (C) $(10, 5)$. (D) $(5, 10)$.

15. How many solutions does the equation $\cos^2 x + 3 \sin x \cos x + 1 = 0$ have for $x \in [0, 2\pi)$?

(A) 1 (B) 3 (C) 4 (D) 2

ROUGH WORK

16. The functions $f, g : [0,1] \to [0,1]$ are given by $f(x) = \frac{1}{2}x(x+1)$ and $g(x) = \frac{1}{2}x^2(x+1)$. What is the area enclosed between the graphs of f^{-1} and g^{-1}?

(A) 1/8 (B) 1/4 (C) 5/12 (D) 7/24

17. If $f(a) = 2, f'(a) = 1, g(a) = -1$ and $g'(a) = 2$, then what is

$$\lim_{x \to a} \frac{g(x)f(a) - f(x)g(a)}{x - a} \quad ?$$

(A) 5 (B) 3 (C) -3 (D) -5

18. Draw one observation N at random from the set $\{1, 2, \ldots, 100\}$. What is the probability that the last digit of N^2 is 1?

(A) 1/20 (B) 1/50 (C) 1/10 (D) 1/5

ROUGH WORK

19. Let X be the number of tosses of a fair coin required to get the first head. If $Y \mid X = n$ is distributed as Binomial$(n, \frac{1}{2})$, then what is $\mathrm{P}(Y = 1)$?

(A) 4/9 (B) 1/4 (C) 1/3 (D) 5/9

20. Suppose X is distributed uniformly on $(-1, 1)$. For $i = 0, 1, 2, 3$, let $p_i = \mathrm{P}\left(X^2 \in (\frac{i}{4}, \frac{i+1}{4})\right)$. For which value of i is p_i the largest?

(A) 3 (B) 1 (C) 0 (D) 2

21. In a simulation experiment, two independent observations X_1 and X_2 are generated from the Poisson distribution with mean 1. The experiment is said to be successful if $X_1 + X_2$ is odd. What is the expected value of the number of experiments required to obtain the first success?

(A) $2(1 + e^{-2})$
(B) $2/(1 - e^{-2})$
(C) $2/(1 - e^{-4})$
(D) $2(1 + e^{-4})$

ROUGH WORK

22. A shopkeeper has 12 bulbs of which 3 are defective. She sells the bulbs by selecting them at random one at a time. What is the probability that the seventh bulb sold is the last defective one?

(A) 3/44 (B) 9/44 (C) 13/44 (D) 7/44

23. The chances of Smitha getting admitted to colleges A and B are 60% and 40% respectively. Assume that colleges admit students independently of each other. If Smitha is told that she has been admitted to at least one college, what is the probability that she got admitted to college A?

(A) 3/5 (B) 15/19 (C) 10/13 (D) 5/7

24. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from an exponential distribution with mean λ. If $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are, respectively, the maximum likelihood estimators of the mean and the median of the underlying distribution, then

(A) $\hat{\lambda}_1 < \hat{\lambda}_2$.
(B) $\hat{\lambda}_1 = \hat{\lambda}_2$.
(C) $\hat{\lambda}_1 < \hat{\lambda}_2$ and $\hat{\lambda}_1 > \hat{\lambda}_2$ are both possible.
(D) $\hat{\lambda}_1 > \hat{\lambda}_2$.

ROUGH WORK

25. Suppose $Y, X_1, X_2, \ldots, X_n$ are i.i.d. $N(\mu, 1)$ random variables, and $I_n = (\bar{X}_n - a_n, \bar{X}_n + a_n)$ is a 95% confidence interval for μ. Then $\mathrm{P}(Y \in I_n)$

(A) converges to 1 as $n \to \infty$.

(B) is greater than 0.95 for all $n \geq 1$.

(C) is less than 0.95 for all $n \geq 1$.

(D) equals 0.95 for all $n \geq 1$.

26. Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ is an i.i.d. sample from $N_2(0, 0, 1, 1, \rho)$ where $|\rho| \leq 1$. Let $(i_1, \ldots, i_n)$ be a random permutation of $\{1, 2, \ldots, n\}$. Define $T_n = \frac{1}{n} \sum_{j=1}^{n} X_j Y_{i_j}$. What is $\mathrm{E}(T_n)$?

(A) $1/n$ (B) 0 (C) ρ/n (D) ρ

27. Suppose X is a $N(\mu, \sigma^2)$ random variable, and $Y = \Phi(X)$, where Φ is the cumulative distribution function of a standard normal random variable. What is $\mathrm{E}(Y)$?

(A) $\Phi(\mu/\sqrt{2 + \sigma^2})$

(B) $\Phi(\mu/\sqrt{1 + \sigma^2})$

(C) $\Phi(\mu/\sigma)$

(D) $\Phi(\mu/\sqrt{4 + \sigma^2})$

ROUGH WORK

28. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f_\lambda(x) = \begin{cases} (\lambda+1)\, x^\lambda & 0 \le x \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > -1$. What is the maximum likelihood estimator of λ?

(A) $1 + n/(\sum_{i=1}^{n} \log X_i)$

(B) $-1 - n/(\sum_{i=1}^{n} \log X_i)$

(C) $-1 - \frac{1}{n} \sum_{i=1}^{n} \log X_i$

(D) $1 - n/(\sum_{i=1}^{n} \log X_i)$

29. Suppose the joint distribution of (X_1, X_2) is $N_2(0, 0, 2, 2, -0.5)$. What is the value of $\mathrm{P}(2X_1 + X_2 \le 2\sqrt{2})$? Here Φ denotes the cumulative distribution function of a standard normal random variable.

(A) $\Phi(2/\sqrt{7})$ (B) $\Phi(2/\sqrt{3})$ (C) $\Phi(2/\sqrt{5})$ (D) $\Phi(1)$

30. Suppose the joint probability density function of (X, Y) is

$$f(x) = \begin{cases} e^{-x} & 0 \le y \le x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is $\mathrm{E}(X)$?

(A) 2 (B) 1 (C) 6 (D) 1/2

ROUGH WORK

PSB
2019

GROUP A

1. Let $f(x) = x^3 - 3x + k$, where k is a real number. For what values of k will $f(x)$ have three distinct real roots?

2. Let A and B be 4×4 matrices. Suppose that A has eigenvalues x_1, x_2, x_3, x_4 and B has eigenvalues $1/x_1, 1/x_2, 1/x_3, 1/x_4$, where each $x_i > 1$.

   (a) Prove that $A + B$ has at least one eigenvalue greater than 2.

   (b) Prove that $A - B$ has at least one eigenvalue greater than 0.

   (c) Give an example of A and B so that 1 is not an eigenvalue of AB.

3. Elections are to be scheduled on any seven days in April and May. In how many ways can the seven days be chosen such that elections are not scheduled on two consecutive days?

GROUP B

4. Let X and Y be independent and identically distributed random variables with mean $\mu > 0$ and taking values in $\{0, 1, 2, \dots\}$. Suppose, for all $m \geq 0$,

$$\mathrm{P}(X = k \mid X + Y = m) = \frac{1}{m+1}, \quad k = 0, 1, \dots, m.$$

   Find the distribution of X in terms of μ.

5. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables such that

$$\mathrm{P}(X_i = 1) = p_i = 1 - \mathrm{P}(X_i = 0),$$

where $p_1, p_2, \ldots, p_n \in (0, 1)$ are all distinct and unknown. Consider $X = \sum_{i=1}^{n} X_i$ and another random variable Y which is distributed as Binomial$(n, \bar{p})$, where $\bar{p} = \frac{1}{n} \sum_{i=1}^{n} p_i$. Between X and Y, which is a better estimator of $\sum_{i=1}^{n} p_i$ in terms of their respective mean squared errors?

6. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from Uniform$(0, \theta)$ for some unknown $\theta > 0$. Let Y_n be the minimum of $X_1, X_2, \ldots, X_n$.

   (a) Suppose F_n is the cumulative distribution function (c.d.f.) of nY_n. Show that for any real x, $F_n(x)$ converges to $F(x)$, where F is the c.d.f. of an exponential distribution with mean θ.

   (b) Find $\lim_{n \to \infty} \mathrm{P}(n[Y_n] = k)$ for $k = 0, 1, 2, \ldots$, where $[x]$ denotes the largest integer less than or equal to x.

7. Suppose an SRSWOR of size n has been drawn from a population labelled $1, 2, \ldots, N$, where the population size N is unknown.

   (a) Find the maximum likelihood estimator $\widehat{N}$ of N.

   (b) Find the probability mass function of $\widehat{N}$.

   (c) Show that $\frac{n+1}{n}\widehat{N} - 1$ is an unbiased estimator of N.

8. Suppose $\{(x_i, y_i, z_i) : i = 1, 2, \ldots, n\}$ is a set of trivariate observations on three variables: X, Y, and Z, where $z_i = 0$ for $i = 1, 2, \ldots, n-1$ and $z_n = 1$. Suppose the least squares linear regression equation of Y on X based on the first $n-1$ observations is

$$y = \widehat{\alpha}_0 + \widehat{\alpha}_1 x$$

and the least squares linear regression equation of Y on X and Z based on all n observations is

$$y = \widehat{\beta}_0 + \widehat{\beta}_1 x + \widehat{\beta}_2 z.$$

Show that $\widehat{\alpha}_1 = \widehat{\beta}_1$.

9. Let Z be a random variable with probability density function

$$f(z) = \frac{1}{2} e^{-|z-\mu|}, z \in \mathbb{R}$$

with parameter $\mu \in \mathbb{R}$. Suppose we observe $X = \max(0, Z)$.

(a) Find the constant c such that the test that "rejects when $X > c$" has size 0.05 for the null hypothesis $H_0 : \mu = 0$.

(b) Find the power of this test against the alternative hypothesis $H_1 : \mu = 2$.

1. Let A be a 2×2 nonzero real matrix. Which of the following is true?

   (A) A has a nonzero eigenvalue.

   (B) A^2 has at least one positive entry.

   (C) trace (A^2) is positive.

   (D) All entries of A^2 cannot be negative.

2. Let A be a 3×3 real matrix with zero diagonal entries. If $1 + i$ is an eigenvalue of A, the determinant of A equals

   (A) 4. (B) -4. (C) 2. (D) -2.

3. Let A be an $n \times n$ matrix and let b be an $n \times 1$ vector such that $Ax = b$ has a unique solution. Let A' denote the transpose of A. Then which of the following statements is **false**?

   (A) $A'x = 0$ has a unique solution.

   (B) $A'x = c$ has a unique solution for any non-zero c.

   (C) $Ax = c$ has a solution for any c.

   (D) $A^2x = c$ is inconsistent for some vector c.

ROUGH WORK

4. Let A and B be $n \times n$ matrices. Assuming all the inverses exist,

$$(A^{-1} - B^{-1})^{-1}$$

equals

(A) $(I - AB^{-1})^{-1}B$.

(B) $A(B - A)^{-1}B$.

(C) $B(B - A)^{-1}A$.

(D) $B(A - B)^{-1}A$.

5. Let f be a function defined on $(-\pi, \pi)$ as

$$f(x) = (|\sin x| + |\cos x|) \cdot \sin x.$$

Then f is differentiable at

(A) all points.

(B) all points except at $x = -\pi/2, \pi/2$.

(C) all points except at $x = 0$.

(D) all points except at $x = 0, -\pi/2, \pi/2$.

6. The equation of the tangent to the curve $y = \sin^2(\pi x^3/6)$ at $x = 1$ is

(A) $y = \frac{1}{4} + \frac{\sqrt{3}\pi}{4}(x - 1)$.

(B) $y = \frac{\sqrt{3}\pi}{4}x + \frac{1-\sqrt{3}\pi}{4}$.

(C) $y = \frac{\sqrt{3}\pi}{4}x - \frac{1-\sqrt{3}\pi}{4}$.

(D) $y = \frac{1}{4} - \frac{\sqrt{3}\pi}{4}(x - 1)$.

ROUGH WORK

7. Let f be a function defined from $(0, \infty)$ to $\mathbb{R}$ such that

$$\lim_{x \to \infty} f(x) = 1 \text{ and } f(x+1) = f(x) \quad \text{for all } x.$$

Then f is

(A) continuous and bounded.

(B) continuous but not necessarily bounded.

(C) bounded but not necessarily continuous.

(D) neither necessarily continuous nor necessarily bounded.

8. The value of $\lim_{x \to \infty} \big(\log x\big)^{1/x}$

(A) is e. (B) is 0. (C) is 1. (D) does not exist.

9. The number of real solutions of the equation,

$$x^7 + 5x^5 + x^3 - 3x^2 + 3x - 7 = 0$$

is

(A) 5. (B) 7. (C) 3. (D) 1.

ROUGH WORK

10. Let x be a real number. Then

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\cos^{2n}(m!\pi x)\right)$$

(A) does not exist for any x.

(B) exists for all x.

(C) exists if and only if x is irrational.

(D) exists if and only if x is rational.

11. Let $\{a_n\}_{n\geq 1}$ be a sequence such that $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$. Suppose the subsequence $\{a_{2n}\}_{n\geq 1}$ is bounded. Then

(A) $\{a_{2n}\}_{n\geq 1}$ is always convergent but $\{a_{2n+1}\}_{n\geq 1}$ need not be convergent.

(B) both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are always convergent and have the same limit.

(C) $\{a_{3n}\}_{n\geq 1}$ is not necessarily convergent.

(D) both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are always convergent but may have different limits.

12. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers such that $a_{n+1} \leq a_n$ for all n, and $\lim\limits_{n\to\infty} a_n = a$. Let $p_n(x)$ be the polynomial

$$p_n(x) = x^2 + a_n x + 1,$$

and suppose $p_n(x)$ has no real roots for every n. Let α and β be the roots of the polynomial $p(x) = x^2 + ax + 1$. Then

(A) $\alpha = \beta$, α and β are not real.

(B) $\alpha = \beta$, α and β are real.

(C) $\alpha \neq \beta$, α and β are real.

(D) $\alpha \neq \beta$, α and β are not real.

ROUGH WORK

13. Consider the set of all functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$, where $n > m$. If a function is chosen from this set at random, what is the probability that it will be strictly increasing?

(A) $\binom{n}{m}/m^n$. (B) $\binom{n}{m}/n^m$. (C) $\binom{m+n-1}{m}/m^n$. (D) $\binom{m+n-1}{m-1}/n^m$.

14. A flag is to be designed with 5 vertical stripes using some or all of the four colours: green, maroon, red and yellow. In how many ways can this be done so that no two adjacent stripes have the same colour?

(A) 576. (B) 120. (C) 324. (D) 432.

15. Suppose $x_1, \ldots, x_6$ are real numbers which satisfy

$$x_i = \prod_{j \neq i} x_j, \quad \text{for all } i = 1, \ldots, 6.$$

How many choices of $(x_1, \ldots, x_6)$ are possible?

(A) Infinitely many. (B) 2. (C) 3. (D) 1.

ROUGH WORK

16. Suppose X is a random variable with $P(X > x) = 1/x^2$, for all $x > 1$. The variance of $Y = 1/X^2$ is

(A) 1/4. (B) 1/12. (C) 1. (D) 1/2.

17. Let $X \sim N(0, \sigma^2)$, where $\sigma > 0$, and

$$Y = \begin{cases} -1 & \text{if } X \leq -1, \\ X & \text{if } X \in (-1, 1), \\ 1 & \text{if } X \geq 1. \end{cases}$$

Which of the following statements is correct?

(A) $\mathrm{Var}(Y) = \mathrm{Var}(X)$.

(B) $\mathrm{Var}(Y) < \mathrm{Var}(X)$.

(C) $\mathrm{Var}(Y) > \mathrm{Var}(X)$.

(D) $\mathrm{Var}(Y) \geq \mathrm{Var}(X)$ if $\sigma \geq 1$, and $\mathrm{Var}(Y) < \mathrm{Var}(X)$ if $\sigma < 1$.

18. If a fair coin is tossed 5 times, what is the probability of obtaining at least 3 consecutive heads?

(A) 1/8. (B) 5/16. (C) 1/4. (D) 3/16.

ROUGH WORK

19. Let X and Y be random variables with mean λ. Define

$$Z = \begin{cases} \min(X,Y) & \text{with probability } \frac{1}{2}\,, \\ \max(X,Y) & \text{with probability } \frac{1}{2}\,. \end{cases}$$

What is $\mathrm{E}(Z)$?

(A) λ. (B) $4\lambda/3$. (C) λ^2. (D) $\sqrt{3}\lambda/2$.

20. A finite population has $N(\geq 10)$ units marked $\{U_1, \ldots, U_N\}$. The following sampling scheme was used to obtain a sample s. One unit is selected at random: if this is the i-th unit, then the sample is $s = \{U_{i-1}, U_i, U_{i+1}\}$, provided $i \notin \{1, N\}$. If $i = 1$ then $s = \{U_1, U_2\}$ and if $i = N$ then $s = \{U_{N-1}, U_N\}$. The probability of selecting U_2 in s is

(A) $\frac{2}{N}$. (B) $\frac{3}{N}$. (C) $\frac{1}{(N-2)} + \frac{2}{N}$. (D) $\frac{3}{(N-2)}$.

21. Suppose $X_1, \ldots, X_n$ are i.i.d. observations from a distribution assuming values $-1, 1$ and 0 with probabilities p, p and $1-2p$, respectively, where $0 < p < \frac{1}{2}$. Define $Z_n = \prod_{i=1}^{n} X_i$ and $a_n = P(Z_n = 1)$, $b_n = P(Z_n = -1)$, $c_n = P(Z_n = 0)$. Then as $n \to \infty$,

(A) $a_n \to \frac{1}{4}$, $b_n \to \frac{1}{2}$, $c_n \to \frac{1}{4}$.

(B) $a_n \to \frac{1}{3}$, $b_n \to \frac{1}{3}$, $c_n \to \frac{1}{3}$.

(C) $a_n \to 0$, $b_n \to 0$, $c_n \to 1$.

(D) $a_n \to p$, $b_n \to p$, $c_n \to 1-2p$.

ROUGH WORK

22. Suppose X_1, X_2 and X_3 are i.i.d. positive valued random variables. Define $Y_i = \frac{X_i}{X_1+X_2+X_3}$, $i = 1, 2, 3$. The correlation between Y_1 and Y_3 is

(A) 0. (B) $-1/6$. (C) $-1/3$. (D) $-1/2$.

23. Assume (y_i, x_i) satisfies the linear regression model,

$$y_i = \beta x_i + \epsilon_i, \quad \text{for } i = 1, \ldots, n,$$

where, $\beta \in \mathbb{R}$ is unknown, $\{x_i : 1 \leq i \leq n\}$ are fixed constants and $\{\epsilon_i : 1 \leq i \leq n\}$ are i.i.d. errors with mean zero and variance $\sigma^2 \in (0, \infty)$. Let $\widehat{\beta}$ be the least squares estimate of β and $\widehat{y}_i = \widehat{\beta} x_i$ be the predicted value of y_i. For each $n \geq 1$, define

$$a_n = \frac{1}{\sigma^2} \sum_{i=1}^{n} \mathrm{Cov}(y_i, \widehat{y}_i).$$

Then,

(A) $a_n = 1$. (B) $a_n \in (0, 1)$. (C) $a_n = n$. (D) $a_n = 0$.

24. Let X and Y be two random variables with $\mathrm{E}(X|Y = y) = y^2$, where Y follows $N(\theta, \theta^2)$, with $\theta \in \mathbb{R}$. Then $\mathrm{E}(X)$ equals

(A) θ. (B) θ^2. (C) $2\theta^2$. (D) $\theta + \theta^2$.

ROUGH WORK

25. Suppose X is a random variable with finite variance. Define $X_1 = X$, $X_2 = \alpha X_1,\ X_3 = \alpha X_2, \ldots, X_n = \alpha X_{n-1}$, for $0 < \alpha < 1$. Then $\text{Corr}(X_1, X_n)$ is

(A) α^n. (B) 1. (C) 0. (D) α^{n-1}.

26. Let X be a random variable with $P(X = 2) = P(X = -2) = 1/6$ and $P(X = 1) = P(X = -1) = 1/3$. Define $Y = 6X^2 + 3$. Then

(A) $\text{Var}(X - Y) < \text{Var}(X)$.

(B) $\text{Var}(X - Y) < \text{Var}(X + Y)$.

(C) $\text{Var}(X + Y) < \text{Var}(X)$.

(D) $\text{Var}(X - Y) = \text{Var}(X + Y)$.

27. Suppose X is a random variable on $\{0, 1, 2, \ldots\}$ with unknown p.m.f. $p(x)$. To test the hypothesis $H_0 : X \sim \text{Poisson}(1/2)$ against $H_1 : p(x) = 2^{-(x+1)}$ for all $x \in \{0, 1, 2, \ldots\}$, we reject H_0 if $x > 2$. The probability of type-II error for this test is

(A) $\frac{1}{4}$. (B) $1 - \frac{13}{8}e^{-1/2}$. (C) $1 - \frac{3}{2}e^{-1/2}$. (D) $\frac{7}{8}$.

ROUGH WORK

28. Let X be a random variable with

$$P_\theta(X=-1)=\frac{(1-\theta)}{2},\ P_\theta(X=0)=\frac{1}{2},\ \text{and } P_\theta(X=1)=\frac{\theta}{2}$$

for $0<\theta<1$. In a random sample of size 20, the observed frequencies of $-1, 0$ and 1 are $6, 4$ and 10, respectively. The maximum likelihood estimate of θ is

(A) 1/5. (B) 4/5. (C) 5/8. (D) 1/4.

29. Two judges evaluate n individuals, with (R_i, S_i) the ranks assigned to the i-th individual by the two judges. Suppose there are no ties and $S_i = R_i + 1$, for $i = 1, \ldots, (n-1)$, and $S_i = 1$ if $R_i = n$. If the Spearman's rank correlation between the two evaluations is 0, what is the value of n?

(A) 7. (B) 11. (C) 4. (D) 5.

30. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with variance 2. Then for all x,

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(-1)^i X_i \le x\right)$$

equals

(A) $\Phi(x\sqrt{2})$. (B) $\Phi\left(x/\sqrt{2}\right)$. (C) $\Phi(x)$. (D) $\Phi(2x)$.

ROUGH WORK

GROUP A

1. Find all real solutions (x_1, x_2, x_3, λ) for the system of equations

$$x_2 - 3x_3 - x_1\lambda = 0,$$
$$x_1 - 3x_3 - x_2\lambda = 0,$$
$$x_1 + x_2 + x_3\lambda = 0.$$

2. Let $\{x_n\}_{n\geq 1}$ be a sequence defined by $x_1 = 1$ and

$$x_{n+1} = \left(x_n^3 + \frac{1}{n(n+1)(n+2)}\right)^{1/3}, \quad n \geq 1.$$

   Show that $\{x_n\}_{n\geq 1}$ converges and find its limit.

3. Consider all permutations of the integers $1, 2, \ldots, 100$. In how many of these permutations will the 25th number be the minimum of the first 25 numbers and the 50th number be the minimum of the first 50 numbers?

GROUP B

4. An urn contains $r > 0$ red balls and $b > 0$ black balls. A ball is drawn at random from the urn, its colour noted, and returned to the urn. Further, $c > 0$ additional balls of the same colour are added to the urn. This process of drawing a ball and adding c balls of the same colour is continued. Define $X_i = 1$ if at the i-th draw the colour of the ball drawn is red, and 0 otherwise. Compute $\mathrm{E}(\sum_{i=1}^{n} X_i)$.

5. Suppose X_1 and X_2 are identically distributed random variables, not necessarily independent, taking values in $\{1, 2\}$. If $\mathrm{E}(X_1X_2) = 7/3$ and $\mathrm{E}(X_1) = 3/2$, obtain the joint distribution of (X_1, X_2).

6. A fair 6-sided die is rolled repeatedly until a 6 is obtained. Find the expected number of rolls conditioned on the event that none of the rolls yielded an odd number.

7. Suppose $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ is a random sample from a bivariate normal distribution with $\mathrm{E}(X_i) = \mathrm{E}(Y_i) = 0$, $\mathrm{Var}(X_i) = \mathrm{Var}(Y_i) = 1$ and unknown $\mathrm{Corr}(X_i, Y_i) = \rho \in (-1, 1)$, for all $i = 1, \ldots, n$. Define $W_n = \frac{1}{n} \sum_{i=1}^{n} X_i Y_i$.

   (a) Is W_n an unbiased estimator of ρ? Justify your answer.

   (b) For large n, obtain an approximate level $(1 - \alpha)$ two-sided confidence interval for ρ, where $0 < \alpha < 1$.

8. Let $\{X_1, \ldots, X_n\}$ be an i.i.d. sample from $f(x : \theta)$, $\theta \in \{0, 1\}$, with

$$f(x:0) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x:1) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Based on the above sample, obtain the most powerful test for testing $H_0 : \theta = 0$ against $H_1 : \theta = 1$, at level α, with $0 < \alpha < 1$. Find the critical region in terms of the quantiles of a standard distribution.

9. Suppose (y_i, x_i) satisfies the regression model,

$$y_i = \alpha + \beta x_i + \epsilon_i, \quad \text{for } i = 1, \ldots, n,$$

   where $\{x_i : 1 \le i \le n\}$ are fixed constants and $\{\epsilon_i : 1 \le i \le n\}$ are i.i.d. $N(0, \sigma^2)$ errors, where α, β and $\sigma^2 (> 0)$ are unknown parameters.

   (a) Let $\widetilde{\alpha}$ denote the least squares estimate of α obtained assuming $\beta = 5$. Find the mean squared error (MSE) of $\widetilde{\alpha}$ in terms of the model parameters.

   (b) Obtain the maximum likelihood estimator of this MSE.

2017

Booklet No. Test Code : **PSA**

Forenoon

Questions : 30	**Time : 2 hours**

Write your Registration Number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answer sheet.

For each question, there are four suggested answers of which exactly one is correct. For each question indicate your choice of the correct answer by darkening the appropriate oval (●) completely on the answer sheet.

4 marks are allotted for each correct answer,
0 mark for each incorrect answer, and
1 mark for each unattempted question.

All rough work must be done on this booklet only.
You are not allowed to use calculators in any form.

STOP! WAIT FOR THE SIGNAL TO START.

ROUGH WORK

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant function satisfying $f(x+2) = f(x)$ for every real x. Then $\lim_{x\to\infty} f(x)$

   (A) does not exist.

   (B) exists and equals $+\infty$ or $-\infty$.

   (C) exists and is finite.

   (D) may or may not exist depending on f.

2. Let A be a square matrix with real entries such that $A^2 = A$. Then

   (A) $(A+I)^{10} = I + 1024A$.

   (B) $(A+I)^{10} = I + 511A$.

   (C) $(A+I)^{10} = I + 1023A$.

   (D) $(A+I)^{10} = I + 512A$.

3. Let A be a 2×2 matrix such that $\mathrm{trace}(A) = \det(A) = 3$. What is $\mathrm{trace}(A^{-1})$?

   (A) 1/2 (B) 3 (C) 1 (D) 1/3

ROUGH WORK

4. Let A be a nonzero 2×2 matrix such that $A^3 = 0$. Then which of the following statements is always false?

   (A) $\mathrm{trace}(A - A^2) = 0$

   (B) $\mathrm{trace}(A + A^2) = 0$

   (C) $\det(I + A) = 0$

   (D) $A^2 = 0$

5. The vertices A, B, C, D of a square are to be coloured with one of three colours red, blue, or green such that adjacent vertices get different colours. What is the number of such colourings?

   (A) 18 (B) 12 (C) 20 (D) 24

6. Let $S \subset \mathbb{R}$. Consider the statement: "If f is a continuous function from S to S, then $f(x) = x$ for some x." This statement is true if S equals

   (A) $[0, 1]$. (B) $(0, 1]$. (C) $\mathbb{R}$. (D) $[-3, -2] \cup [2, 3]$.

ROUGH WORK

7. The coordinates of the point at which the tangent of the curve $y = x^2 - 1$ makes an angle of $45°$ with the positive X-axis are

(A) $(\frac{\sqrt{5}-1}{2}, \frac{1-\sqrt{5}}{2})$. (B) $(\frac{2}{3}, -\frac{5}{9})$. (C) $(\frac{3}{4}, -\frac{7}{16})$. (D) $(\frac{1}{2}, -\frac{3}{4})$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then f is

(A) continuous everywhere but not differentiable at 0.

(B) differentiable everywhere and its derivative is continuous.

(C) discontinuous at 0.

(D) differentiable everywhere and its derivative is discontinuous at 0.

9. If α, β, and γ are the roots of $x^3 - px + q = 0$, then what is the determinant of

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{pmatrix}?$$

(A) $p^2 + 6q$ (B) 1 (C) p (D) 0

ROUGH WORK

10. Suppose the rank of

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \\ a & b & b & 1 \end{pmatrix}$$

is 2 for some real numbers a and b. What is the value of b ?

(A) $\frac{1}{3}$ (B) 3 (C) 1 (D) $\frac{1}{2}$

11. What is the infinite sum $x + 2x^2 + 3x^3 + \cdots$ for $|x| < 1$?

(A) $\frac{x}{1-x^2}$ (B) $\frac{x}{(1+x)^2}$ (C) $\frac{1}{(1-x)^2}$ (D) $\frac{x}{(1-x)^2}$

12. A function $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = f(-x)$ for all x, has left derivative 5 at $x = 0$. Then, the right derivative of f at $x = 0$

(A) exists and equals -5.

(B) may or may not exist.

(C) does not exist.

(D) exists and equals 5.

ROUGH WORK

13. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{k=0}^{\lfloor |x| \rfloor} \frac{1}{2^k},$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. Then,

(A) f is a left continuous function which is not right continuous.

(B) f is a continuous function.

(C) f is neither left continuous nor right continuous.

(D) f is a right continuous function which is not left continuous.

14. A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 0. Suppose that

$$f(x) < f(0) < f(y) \text{ for all } x < 0 < y.$$

Then, what is the set of all possible values that $f'(0)$ can take?

(A) $\{0\}$ (B) $[0, \infty)$ (C) $(0, \infty)$ (D) $\mathbb{R}$

15. Suppose that the events A, B, and C are pairwise independent such that each of them occurs with probability p. Assume that all three of them cannot occur simultaneously. What is $P(A \cup B \cup C)$?

(A) $1 - (1-p)^3$ (B) p^3 (C) $3p(1-p)$ (D) $3p$

ROUGH WORK

16. Let $X_1, X_2, \ldots$ be a sequence of independent random variables distributed exponentially with mean 1. Suppose that N is a random variable, independent of the X_i-s, that has a Poisson distribution with mean $\lambda > 0$. What is the expected value of $X_1 + X_2 + \cdots + X_{N^2}$?

(A) N^2 (B) $\lambda + \lambda^2$ (C) λ^2 (D) $1/\lambda^2$

17. For which value of k does the inequality

$$\mathrm{Var}(X - Y) \geq |\sigma_X - \sigma_Y|^k$$

hold for all choices of random variables X and Y with finite variances σ_X^2 and σ_Y^2, respectively?

(A) 3 (B) 4 (C) 1 (D) 2

18. Consider a diagnostic test for a disease that gives the correct result with probability 0.9, independently of whether the subject being diagnosed has the disease or not. Suppose that 20% of the population has the disease. For a particular subject, given that the test indicates presence of the disease, what is the conditional probability that the subject has the disease?

(A) $\frac{9}{10}$ (B) $\frac{9}{13}$ (C) $\frac{1}{2}$ (D) $\frac{1}{5}$

ROUGH WORK

19. Let X_1, X_2, X_3, X_4 be independent standard normal random variables. Then what is the distribution of

$$\frac{(X_1 + X_2 + X_3 + X_4)^2}{(X_1 - X_2 + X_3 - X_4)^2}\,?$$

(A) $F_{4,4}$ (B) χ^2_4 (C) $F_{1,1}$ (D) $F_{2,2}$

20. Suppose that $X \sim$ Uniform $(0, 1)$ and $Y \sim$ Bernoulli $(1/4)$, independently of each other. Let $Z = X + Y$. Then,

(A) the distribution of Z is symmetric about 1.

(B) Z has a probability density function.

(C) $E(Z) = 5/4$.

(D) $P(Z \leq 1) = 1/4$.

21. Assume that the events A and B are such that $A \subset B$ and $P(B) > 0$. For every event E, define $\mathbf{1}_E$ to be the random variable which takes value 1 or 0 depending on whether E occurs or not, respectively. Then, $\mathrm{Cov}(\mathbf{1}_A, \mathbf{1}_B)$ equals

(A) $P(A|B)P(B^c)$.

(B) $P(A)P(B^c)$.

(C) $P(A)P(B)$.

(D) $P(B)P(A^c)$.

ROUGH WORK

22. Suppose that X is chosen uniformly from $\{1, 2, \ldots, 100\}$, and given $X = x$, Y is chosen uniformly from $\{1, 2, \ldots, x\}$. What is $P(Y = 30)$?

(A) $\frac{1}{100}$

(B) $\frac{70}{100} \times \frac{1}{30}$

(C) $\frac{1}{100} \times \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30}\right)$

(D) $\frac{1}{100} \times \left(\frac{1}{30} + \cdots + \frac{1}{100}\right)$

23. Three numbers are chosen at random from $\{1, 2, ..., 10\}$ *without* replacement. What is the probability that the minimum of the chosen numbers is 3 or their maximum is 7 ?

(A) $\frac{13}{40}$ (B) $\frac{19}{60}$ (C) $\frac{3}{10}$ (D) $\frac{11}{40}$

24. Let X be a random variable taking values 1, 2, and 3 with respective probabilities $\frac{3}{7}$, $\frac{2}{7}$, and $\frac{2}{7}$. Find the set of values of α for which $E|X - \alpha|$ is minimized.

(A) $[1, 2]$ (B) $\{\frac{13}{7}\}$ (C) $\{2\}$ (D) $\{1\}$

ROUGH WORK

25. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed normal random variables with mean θ and variance θ, where $\theta > 0$. Then,

(A) $\sum_{i=1}^{n} X_i$ is sufficient for θ.

(B) $\sum_{i=1}^{n} X_i^2$ is sufficient for θ.

(C) $\sum_{i=1}^{n} \left(X_i + X_i^2\right)$ is sufficient for θ.

(D) neither $\sum_{i=1}^{n} X_i$ nor $\sum_{i=1}^{n} X_i^2$ is sufficient for θ.

26. Let X be normally distributed with mean μ and variance $\sigma^2 > 0$. What is the variance of e^X ?

(A) $e^{\mu+\sigma^2}(e^{\sigma^2} - 1)$

(B) $e^{2\mu+2\sigma^2}(e^{2\sigma^2} - 1)$

(C) $e^{2\mu+2\sigma^2}(e^{\sigma^2} - 1)$

(D) $e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

27. Consider data from a single replication of a 2^4 factorial experiment. Assume that all interactions of orders higher than two are negligible. What is the error degrees of freedom in the ANOVA table?

(A) 16 (B) 2 (C) 12 (D) 5

ROUGH WORK

28. For $n \geq 2$, let $X_1, \ldots, X_n$ be independent and identically distributed normal random variables with mean 0 and variance σ^2. Consider the most powerful test for the null hypothesis $H_0 : \sigma = 2$ against the alternative $H_1 : \sigma = 1$, at significance level α for some $0 < \alpha < 1$. What is the power of this test?

In what follows, F_m denotes the cumulative distribution function of a χ^2 random variable with m degrees of freedom, and $\chi^2_{m,\beta}$ satisfies the equation $F_m(\chi^2_{m,\beta}) = 1 - \beta$, for every $m \geq 1$ and $\beta \in (0,1)$.

(A) $1 - F_{n-1}(\frac{1}{4}\chi^2_{n-1,\alpha})$

(B) $F_n(4\chi^2_{n,1-\alpha})$

(C) $F_{n-1}(4\chi^2_{n-1,1-\alpha})$

(D) $1 - F_n(\frac{1}{4}\chi^2_{n,\alpha})$

29. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed random variables from a distribution which depends on a parameter θ. For which of the following distributions is the maximum likelihood estimator of θ not unbiased?

(A) Normal with mean θ and variance 1 where $\theta \in \mathbb{R}$

(B) Bernoulli with parameter $\theta \in [0, 1]$

(C) Poisson with mean $\theta \geq 0$

(D) Normal with mean 0 and variance θ^2 where $\theta > 0$

30. Suppose that $X_1, \ldots, X_n$ are independent observations from a uniform distribution on $[0, \theta]$, where $\theta \in \mathbb{N}$. Then, the maximum likelihood estimator of θ

(A) may not exist in some cases.

(B) is $\lfloor X_{(n)} \rfloor$, where $\lfloor a \rfloor$ = greatest integer $\leq a$.

(C) is $\lceil X_{(n)} \rceil$, where $\lceil a \rceil$ = smallest integer $\geq a$.

(D) is $\lfloor X_{(n)} \rfloor$, if $\big(X_{(n)} - \lfloor X_{(n)} \rfloor\big) < 1/2$, otherwise it is $\lceil X_{(n)} \rceil$.

2017

BOOKLET No. TEST CODE: PSB

Afternoon

Questions: 8	Time: 2 hours

Write your Registration number, Test Centre, Test Code, and the Number of this booklet in the appropriate places on the answer-book.

- Answer two questions from GROUP A and four questions from GROUP B.
- Best six answers subject to the above conditions will be considered.
- Each question carries 20 marks.
- The maximum possible marks is 120.

Answer to each question should start on a fresh page.

All rough work must be done on this booklet and/or the answer-book. You are not allowed to use calculators in any form.

STOP! WAIT FOR THE SIGNAL TO START.

GROUP A

1. Let a and b be real numbers. Show that there exists a unique 2×2 real symmetric matrix A with $\text{trace}(A) = a$ and $\det(A) = b$ if and only if $a^2 = 4b$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function and suppose that for some $n \geq 1$,

$$f(1) = f(0) = f^{(1)}(0) = f^{(2)}(0) = \cdots = f^{(n)}(0) = 0,$$

where $f^{(k)}$ denotes the k-th derivative of f for $k \geq 1$. Prove that there exists $x \in (0,1)$ such that $f^{(n+1)}(x) = 0$.

3. Consider an urn containing 5 red, 5 black, and 10 white balls. If balls are drawn *without* replacement from the urn, calculate the probability that in the first 7 draws, at least one ball of each colour is drawn.

GROUP B

4. Let $X_1, X_2, \ldots, X_n$ be independent random variables, with X_i having probability mass function

$$P(X_i = k) = \left(\frac{i}{i+1}\right)^k \frac{1}{i+1}, \text{ for } k = 0, 1, 2, \ldots$$

and for all $i = 1, \ldots, n$. Let $M = \min\{X_i : 1 \leq i \leq n\}$. Derive the probability mass function of M.

5. The lifetime in hours of each bulb manufactured by a particular company follows an independent exponential distribution with mean λ. To test the null hypothesis $H_0 : \lambda = 1000$ against the alternative $H_1 : \lambda = 500$, a statistician sets up an experiment with 50 bulbs, with 5 bulbs in each of 10 different locations, to examine their lifetimes.

   To get quick preliminary results, the statistician decides to stop the experiment as soon as one bulb fails at each location. Let Y_i denote the lifetime of the first bulb to fail at location i. Obtain the most powerful test of H_0 against H_1 based on $Y_1, Y_2, \ldots, Y_{10}$, and compute its power.

6. Suppose you have a 4-digit combination lock, but you have forgotten the correct combination. Consider the following three strategies to find the correct one:

   (i) Try the combinations consecutively from 0000 to 9999.

   (ii) Try combinations using simple random sampling *with* replacement from the set of all possible combinations.

   (iii) Try combinations using simple random sampling *without* replacement from the set of all possible combinations.

   Assume that the true combination was chosen uniformly at random from all possible combinations. Determine the expected number of attempts needed to find the correct combination in all three cases.

7. Consider independent observations $\{(y_i, x_{1i}, x_{2i}) : 1 \leq i \leq n\}$ from the regression model

$$y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \epsilon_i,\ i = 1, \ldots, n\,,$$

   where x_{1i} and x_{2i} are scalar covariates, β_1 and β_2 are unknown scalar coefficients, and ϵ_i are uncorrelated errors with mean 0 and variance $\sigma^2 > 0$. Instead of using the correct model, we obtain an estimate $\hat{\beta}_1$ of β_1 by minimizing

$$\sum_{i=1}^{n} (y_i - \beta_1 x_{1i})^2.$$

   Find the bias and mean squared error of $\hat{\beta}_1$.

8. Let $\theta > 0$ be an unknown parameter, and $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with density

$$f(x) = \begin{cases} 2x/\theta^2 & , 0 \leq x \leq \theta\,, \\ 0 & , \text{otherwise.} \end{cases}$$

   Find the maximum likelihood estimator of θ and its mean squared error.

2016

Booklet No. | Test Code : **PSA**

Forenoon

Questions : 30	**Time : 2 hours**

Write your Registration Number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answersheet.

For each question, there are four suggested answers of which only one is correct. For each question indicate your choice of the correct answer by darkening the appropriate oval (⬤) completely on the answer sheet.

4 marks are allotted for each correct answer,
0 mark for each incorrect answer and
1 mark for each unattempted question.

All rough work must be done on this booklet only.
You are not allowed to use calculators in any form.

STOP! WAIT FOR THE SIGNAL TO START.

ROUGH WORK

1. The number of terms independent of x in the binomial expansion of $\left(3x^2 + \frac{1}{x}\right)^5$ is

(A) 0 (B) 2 (C) 5 (D) 1

2. Let X and Y be independent and identically distributed random variables with moment generating function

$$M(t) = \mathrm{E}\left(e^{tX}\right), \quad -\infty < t < \infty.$$

Then $\mathrm{E}\left(\dfrac{e^{tX}}{e^{tY}}\right)$ equals

(A) $M(t)M(-t)$ (B) 1 (C) $(M(t))^2$ (D) $\frac{M(t)}{M(-t)}$

3. A 6-digit number is to be formed by rearranging the digits of 654321. How many such numbers will be divisible by 12?

(A) 168 (B) 192 (C) 360 (D) 144

4. The value of

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{y} e^{-\frac{1}{2}(x^2+y^2)} \,\mathrm{d}x \,\mathrm{d}y$$

equals

(A) $\frac{1}{4}$ (B) $\frac{\pi}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{2\pi}$

5. Let A be a 2×2 matrix with real entries. If $5 + 3\sqrt{-1}$ is an eigenvalue of A, then the determinant of A equals

(A) 16 (B) 8 (C) 4 (D) 34

ROUGH WORK

6. If $A_1, \ldots, A_n$ are independent events with respective probabilities $p_1, \ldots, p_n$, then $\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right)$ equals

(A) $1 - \prod_{i=1}^{n}(1-p_i)$ (B) $\prod_{i=1}^{n}(1-p_i)$ (C) $\sum_{i=1}^{n} p_i$ (D) $\prod_{i=1}^{n} p_i$

7. Let A and B be matrices such that $B^2 + AB + 2I = \mathbf{0}$, where I denotes the identity matrix. Which of the following matrices must be nonsingular?

(A) B (B) A (C) $A + 2I$ (D) $B + 2I$

8. Let X_1, X_2, and X_3 be independent Poisson random variables with mean 1. Then $\mathrm{P}(\max\{X_1, X_2, X_3\} = 1)$ equals

(A) $1 - e^{-3}$ (B) e^{-3} (C) $1 - 8e^{-3}$ (D) $7e^{-3}$

9. The number of solutions of the equation $2x^3 - 6x + 1 = 0$ in the interval $[-1, 1]$ is

(A) 1 (B) 3 (C) 2 (D) 0

10. Consider a confounded 2^5 factorial design with factors A, B, C, D, E arranged in four blocks each of size eight. If the principal block of this design consists of the treatment combinations $(1), ab, de, ace$, and four others, then the confounded factorial effects would be

(A) $AB, DE, ABDE$
(B) $ABC, CDE, ABDE$
(C) $AB, CDE, ABCDE$
(D) $ABC, DE, ABCDE$

ROUGH WORK

11. The number of ordered pairs (a, b) such that $a + b \leq 60$, where a and b are positive integers, is

(A) 1830 (B) 1770 (C) 885 (D) 3540

12. A box contains 2016 balls labeled $1, 2, 3, \ldots, 2016$. Two balls are selected at random by sampling without replacement. Let X_1 and X_2 be the labels on the first ball and the second ball, respectively. Then

(A) $\mathrm{P}(X_1 < X_2) > \frac{1}{2}$.

(B) $\mathrm{E}(X_2|X_1 = 1008) > 1008$.

(C) $\mathrm{E}(X_1) \neq \mathrm{E}(X_2)$.

(D) X_1 and X_2 are independent.

13. An 8-digit number is to be formed with digits from the set $\{1, 2, 3\}$ such that the sum of the digits in the number is equal to 10. How many such numbers are there?

(A) $\binom{8}{1} + \binom{8}{2}$ (B) $\binom{8}{1}$ (C) $\binom{8}{2}$ (D) $\binom{8}{1} \times \binom{8}{2}$

14. Consider a finite population of size $N > 1$ with units $U_1, U_2, \ldots, U_N$. The following sampling method is used to select a sample: either the sample consists of only one unit U_j with probability $\frac{1}{N+1}$ for any $j = 1, 2, \ldots, N$, or it consists of the whole population with probability $\frac{1}{N+1}$. Then the expected sample size is

(A) $\frac{N+2}{N+1}$ (B) $\frac{2N+1}{N+1}$ (C) $\frac{2N}{N+1}$ (D) 2

15. For any θ, the expression $4 \sin\theta \sin(\frac{\pi}{3} + \theta) \sin(\frac{\pi}{3} - \theta)$ equals

(A) $\sin 2\theta$ (B) $\cos 3\theta$ (C) $\sin 3\theta$ (D) $\cos 2\theta$

ROUGH WORK

16. Let X be a Bernoulli$\left(\frac{1}{3}\right)$ random variable and Y be a Bernoulli$\left(\frac{2}{3}\right)$ random variable independent of X. Let

$$Z = \begin{cases} X & \text{if } Y = 1, \\ 1 - X & \text{if } Y = 0. \end{cases}$$

Given that $Z = 1$, the conditional probability that $X = 1$ is

(A) $\frac{2}{3}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{9}$

17. Let f be a polynomial such that $f''(x) \to 2$ as $x \to \infty$, the minimum of f is attained at 3, and $f(0) = 3$. Then $f(1)$ equals

(A) 1 (B) 2 (C) -1 (D) -2

18. Let X be a random variable with probability density function

$$f(x;\theta) = \begin{cases} \frac{\theta}{x}\left(\frac{4}{x}\right)^{\theta} & \text{if } x > 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$. If a test of size $\alpha = 0.1$ for testing $H_0 : \theta = 1$ vs $H_1 : \theta = 2$ rejects H_0 when $X < m$, then the value of m is

(A) $\frac{80}{9}$ (B) 5 (C) $\frac{40}{9}$ (D) $\frac{40}{3}$

19. Let g be a differentiable function from the set of real numbers to itself. If $g(1) = 1$ and $g'(x^2) = x^3$ for all $x > 0$, then $g(4)$ equals

(A) $\frac{67}{5}$ (B) $\frac{64}{5}$ (C) $\frac{37}{5}$ (D) $\frac{32}{5}$

ROUGH WORK

20. Suppose X and Z are two independent random variables such that $\mathrm{E}(X) = \mu > 0, \mathrm{Var}(X) = \sigma^2 > 0$, and $\mathrm{P}(Z = 1) = \mathrm{P}(Z = -1) = \frac{1}{2}$. Let $Y = ZX$. Then

(A) $\mathrm{Var}(Y) = \mu^2 + \sigma^2$.

(B) X and Y have the same distribution.

(C) $\mathrm{E}(Y) > \mu$.

(D) X and Y are always independent.

21. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many functions $f : A \to A$ are there such that $f(1) < f(2) < f(3)$?

(A) $\binom{8}{3}8^5$ (B) $\binom{8}{3}$ (C) $\binom{8}{3}5^8$ (D) $8!$

22. Let X and Y be independent Bernoulli random variables with

$$\mathrm{P}(X = 1) = p, \quad \mathrm{P}(Y = 1) = 1 - p.$$

Then the distribution of $X + Y - XY$ is

(A) $\mathrm{Binomial}\left(2, \frac{1}{2}(1 - p + p^2)\right)$

(B) $\mathrm{Binomial}\left(2, p^2 + (1 - p)^2\right)$

(C) $\mathrm{Bernoulli}\left(2p - 2p^2\right)$

(D) $\mathrm{Bernoulli}\left(1 - p + p^2\right)$

23. Let $(a_i, \frac{1}{a_i}), i = 1, 2, 3, 4$, be four distinct points on the circle of radius 2 centred at the origin. Then the value of $a_1 a_2 a_3 a_4$ is

(A) -1 (B) $\frac{1}{16}$ (C) 4 (D) 1

ROUGH WORK

24. Let X have a normal distribution with mean 0 and variance 2, and Y have a Poisson distribution with mean 1. Let

$$U = X + Y\,, \quad V = X - Y\,.$$

Then U and V are

(A) negatively correlated.

(B) independent.

(C) positively correlated.

(D) uncorrelated but not independent.

25. Let $f : (0, \infty) \to (0, \infty)$ be a strictly decreasing function. Consider

$$h(x) = \frac{f\left(\frac{x}{1+x}\right)}{1 + f\left(\frac{x}{1+x}\right)}\,, \quad x > 0.$$

Then

(A) h is strictly increasing.

(B) h is strictly decreasing.

(C) h has a maximum.

(D) h has a minimum.

26. Two integers m and n are chosen at random with replacement from $\{1, 2, \ldots, 9\}$. The probability that $m^2 - n^2$ is even is

(A) $\frac{2}{3}$ (B) $\frac{41}{81}$ (C) $\frac{37}{81}$ (D) $\frac{4}{9}$

27. Let f be a function from the set of real numbers to itself. If f is strictly increasing, then

(A) f is not bounded above.

(B) $f'(a) > 0$ if f is differentiable at a.

(C) left and right limits, i.e., $\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a+} f(x)$, exist for all a.

(D) f is not bounded below.

ROUGH WORK

28. Suppose X_1 and X_2 are independent exponential random variables with means λ_1 and λ_2, respectively. To test

$$H_0 : \lambda_1 = \lambda_2 \text{ against } H_1 : \lambda_1 > \lambda_2,$$

consider a test which rejects H_0 if $X_1 > X_2$. What is the power of this test when $\lambda_1 = 2\lambda_2$?

(A) $\frac{1}{3}$ (B) $\frac{4}{9}$ (C) $\frac{8}{27}$ (D) $\frac{2}{3}$

29. Let A be a 3×3 matrix with all diagonal elements equal to a real number x and all remaining entries equal to 1. The set of all possible values of the rank of A is

(A) $\{1, 3\}$ (B) $\{3\}$ (C) $\{1, 2, 3\}$ (D) $\{2, 3\}$

30. Given observed values $(y_1, x_1), (y_2, x_2), \ldots, (y_n, x_n)$, the sum of squares

$$\sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$$

is minimised with respect to α and β to obtain estimators $\hat{\alpha}$ and $\hat{\beta}$. Suppose that the actual model is

$$\mathrm{E}(Y_i | X_i = x_i) = \alpha_0 + \frac{\beta_0}{c} x_i \quad \text{for all } i,$$

where $c \in (0, 1)$ is a fixed number and $\alpha_0 > 0$ and $\beta_0 > 0$ are unknown parameters. Then

(A) $\mathrm{E}(\hat{\alpha}) = \alpha_0$ and $\mathrm{E}(\hat{\beta}) = \beta_0$.
(B) $\mathrm{E}(\hat{\alpha}) \neq \alpha_0$ and $\mathrm{E}(\hat{\beta}) > \beta_0$.
(C) $\mathrm{E}(\hat{\alpha}) = \alpha_0$ and $\mathrm{E}(\hat{\beta}) > \beta_0$.
(D) $\mathrm{E}(\hat{\alpha}) \neq \alpha_0$ and $\mathrm{E}(\hat{\beta}) = \beta_0$.

2016

BOOKLET No. TEST CODE: PSB

Afternoon

Questions: 10	Time: 2 hours

Write your Registration number, Test Centre, Test Code, and the Number of this booklet in the appropriate places on the answer-book.

- All questions carry equal weight.
- Answer two questions from GROUP A and four questions from GROUP B.
- Best six answers subject to the above conditions will be considered.

Answer to each question should start on a fresh page.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR THE ANSWER-BOOK. YOU ARE NOT ALLOWED TO USE CALCULATORS IN ANY FORM.

STOP! WAIT FOR THE SIGNAL TO START.

GROUP A

1. Let x, y be real numbers such that $xy = 10$. Find the minimum value of $|x + y|$ and all also find all the points (x, y) where this minimum value is achieved.

   Justify your answer.

2. Determine the average value of

$$i_1 i_2 + i_2 i_3 + \cdots + i_9 i_{10} + i_{10} i_1$$

   taken over all permutations $i_1, i_2, \ldots, i_{10}$ of $1, 2, \ldots, 10$.

3. For any two events A and B, show that

$$(\mathrm{P}(A \cap B))^2 + (\mathrm{P}(A \cap B^c))^2 + (\mathrm{P}(A^c \cap B))^2 + (\mathrm{P}(A^c \cap B^c))^2 \geq \frac{1}{4}.$$

4. Let X, Y, and Z be three Bernoulli $(\frac{1}{2})$ random variables such that X and Y are independent, Y and Z are independent, and Z and X are independent.

   (a) Show that $\mathrm{P}(XYZ = 0) \geq \frac{3}{4}$.

   (b) Show that if equality holds in (a), then $Z = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{if } X \neq Y. \end{cases}$

GROUP B

5. Let $n \geq 2$, and $X_1, X_2, \ldots, X_n$ be independent and identically distributed Poisson (λ) random variables for some $\lambda > 0$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the corresponding order statistics.

   (a) Show that $\mathrm{P}(X_{(2)} = 0) \geq 1 - n(1 - e^{-\lambda})^{n-1}$.

   (b) Evaluate the limit of $\mathrm{P}(X_{(2)} > 0)$ as the sample size $n \to \infty$.

6. Suppose that random variables X and Y jointly have a bivariate normal distribution with $\mathrm{E}(X) = \mathrm{E}(Y) = 0, \mathrm{Var}(X) = \mathrm{Var}(Y) = 1$, and correlation ρ. Compute the correlation between e^X and e^Y.

7. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with probability mass function

$$f(x;\theta) = \frac{x\theta^x}{h(\theta)} \qquad \text{for } x = 1, 2, 3, \ldots$$

where $0 < \theta < 1$ is an unknown parameter and $h(\theta)$ is a function of θ. Show that the maximum likelihood estimator of θ is also a method of moments estimator.

8. Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent and identically distributed pairs of random variables with $\mathrm{E}(X_1) = \mathrm{E}(Y_1), \mathrm{Var}(X_1) = \mathrm{Var}(Y_1) = 1$, and $\mathrm{Cov}(X_1, Y_1) = \rho \in (-1, 1)$.

   (a) Show that there exists a function $c(\rho)$ such that

   $$\lim_{n\to\infty} \mathrm{P}\left(\sqrt{n}(\overline{X} - \overline{Y}) \leq c(\rho)\right) = \Phi(1)$$

   where Φ denotes the standard normal cumulative distribution function.

   (b) Given $\alpha \in (0, 1)$, obtain a statistic L_n which is a function of $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ such that

   $$\lim_{n\to\infty} \mathrm{P}(L_n < \rho < 1) = \alpha.$$

9. Suppose $X_1, X_2, \ldots, X_N$ are independent exponentially distributed random variables with mean 1, where N is unknown. We only observe the largest X_i value and denote it by T. We want to test $H_0 : N = 5$ against $H_1 : N = 10$. Show that the most powerful test of size 0.05 rejects H_0 when $T > c$ for some c, and determine c.

10. Consider a population with $N > 1$ units having values $y_1, y_2, \ldots, y_N$. A sample of size n_1 is drawn from the population using SRSWOR. From the remaining part of the population, a sample of size n_2 is drawn using SRSWOR. Show that the covariance between the two sample means is

$$\frac{-\sum_{i=1}^{N}(y_i - \bar{y})^2}{N(N-1)},$$

where $\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$.

TEST CODE: MMA (Objective type) 2015

SYLLABUS

Analytical Reasoning

Algebra — Arithmetic, geometric and harmonic progression. Continued fractions. Elementary combinatorics: Permutations and combinations, Binomial theorem. Theory of equations. Inequalities. Complex numbers and De Moivre's theorem. Elementary set theory. Functions and relations. Elementary number theory: Divisibility, Congruences, Primality. Algebra of matrices. Determinant, rank and inverse of a matrix. Solutions of linear equations. Eigenvalues and eigenvectors of matrices. Simple properties of a group.

Coordinate geometry — Straight lines, circles, parabolas, ellipses and hyperbolas.

Calculus — Sequences and series: Power series, Taylor and Maclaurin series. Limits and continuity of functions of one variable. Differentiation and integration of functions of one variable with applications. Definite integrals. Maxima and minima. Functions of several variables - limits, continuity, differentiability. Double integrals and their applications. Ordinary linear differential equations.

Elementary discrete probability theory — Combinatorial probability, Conditional probability, Bayes theorem. Binomial and Poisson distributions.

SAMPLE QUESTIONS

<u>Note:</u> *For each question there are four suggested answers of which only one is correct.*

1. Let $\{f_n(x)\}$ be a sequence of polynomials defined inductively as

$$\begin{aligned} f_1(x) &= (x-2)^2 \\ f_{n+1}(x) &= (f_n(x)-2)^2, \quad n \geq 1. \end{aligned}$$

   Let a_n and b_n respectively denote the constant term and the coefficient of x in $f_n(x)$. Then

   (A) $a_n = 4$, $b_n = -4^n$ (B) $a_n = 4$, $b_n = -4n^2$
   (C) $a_n = 4^{(n-1)!}$, $b_n = -4^n$ (D) $a_n = 4^{(n-1)!}$, $b_n = -4n^2$.

2. If a, b are positive real variables whose sum is a constant λ, then the minimum value of $\sqrt{(1+1/a)(1+1/b)}$ is

   (A) $\lambda - 1/\lambda$ (B) $\lambda + 2/\lambda$ (C) $\lambda + 1/\lambda$ (D) none of the above.

3. Let x be a positive real number. Then

(A) $x^2 + \pi^2 + x^{2\pi} > x\pi + (\pi + x)x^\pi$

(B) $x^\pi + \pi^x > x^{2\pi} + \pi^{2x}$

(C) $\pi x + (\pi + x)x^\pi > x^2 + \pi^2 + x^{2\pi}$

(D) none of the above.

4. Suppose in a competition 11 matches are to be played, each having one of 3 distinct outcomes as possibilities. The number of ways one can predict the outcomes of all 11 matches such that exactly 6 of the predictions turn out to be correct is

(A) $\binom{11}{6} \times 2^5$ (B) $\binom{11}{6}$ (C) 3^6 (D) none of the above.

5. A set contains $2n+1$ elements. The number of subsets of the set which contain at most n elements is

(A) 2^n (B) 2^{n+1} (C) 2^{n-1} (D) 2^{2n}.

6. A club with x members is organized into four committees such that

(a) each member is in exactly two committees,

(b) any two committees have exactly one member in common.

Then x has

(A) exactly two values both between 4 and 8

(B) exactly one value and this lies between 4 and 8

(C) exactly two values both between 8 and 16

(D) exactly one value and this lies between 8 and 16.

7. Let X be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define the set $\mathcal{R}$ by

$\mathcal{R} = \{(x, y) \in X \times X : x$ and y have the same remainder when divided by $3\}$.

Then the number of elements in $\mathcal{R}$ is

(A) 40 (B) 36 (C) 34 (D) 33.

8. Let A be a set of n elements. The number of ways, we can choose an ordered pair (B, C), where B, C are disjoint subsets of A, equals

(A) n^2 (B) n^3 (C) 2^n (D) 3^n.

9. Let $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, n being a positive integer. The value of

$$\left(1+\frac{C_0}{C_1}\right)\left(1+\frac{C_1}{C_2}\right)\cdots\left(1+\frac{C_{n-1}}{C_n}\right)$$

is

(A) $\left(\frac{n+1}{n+2}\right)^n$ (B) $\frac{n^n}{n!}$ (C) $\left(\frac{n}{n+1}\right)^n$ (D) $\frac{(n+1)^n}{n!}$.

10. The value of the infinite product

$$P = \frac{7}{9} \times \frac{26}{28} \times \frac{63}{65} \times \cdots \times \frac{n^3-1}{n^3+1} \times \cdots$$

is

(A) 1 (B) 2/3 (C) 7/3 (D) none of the above.

11. The number of positive integers which are less than or equal to 1000 and are divisible by none of 17, 19 and 23 equals

(A) 854 (B) 153 (C) 160 (D) none of the above.

12. Consider the polynomial $x^5 + ax^4 + bx^3 + cx^2 + dx + 4$ where a, b, c, d are real numbers. If $(1+2i)$ and $(3-2i)$ are two roots of this polynomial then the value of a is

(A) $-524/65$ (B) $524/65$ (C) $-1/65$ (D) $1/65$.

13. The number of real roots of the equation

$$2\cos\left(\frac{x^2+x}{6}\right) = 2^x + 2^{-x}$$

is

(A) 0 (B) 1 (C) 2 (D) infinitely many.

14. Consider the following system of equivalences of integers.

$$\begin{aligned} x &\equiv 2 \bmod 15 \\ x &\equiv 4 \bmod 21. \end{aligned}$$

The number of solutions in x, where $1 \le x \le 315$, to the above system of equivalences is

(A) 0 (B) 1 (C) 2 (D) 3.

15. The number of real solutions of the equation $(9/10)^x = -3 + x - x^2$ is

(A) 2 (B) 0 (C) 1 (D) none of the above.

16. If two real polynomials $f(x)$ and $g(x)$ of degrees m (≥ 2) and n (≥ 1) respectively, satisfy

$$f(x^2 + 1) = f(x)g(x),$$

for every $x \in \mathbb{R}$, then

(A) f has exactly one real root x_0 such that $f'(x_0) \neq 0$
(B) f has exactly one real root x_0 such that $f'(x_0) = 0$
(C) f has m distinct real roots
(D) f has no real root.

17. Let $X = \dfrac{1}{1001} + \dfrac{1}{1002} + \dfrac{1}{1003} + \cdots + \dfrac{1}{3001}$. Then,

(A) $X < 1$ (B) $X > 3/2$
(C) $1 < X < 3/2$ (D) none of the above holds.

18. The set of complex numbers z satisfying the equation

$$(3 + 7i)z + (10 - 2i)\overline{z} + 100 = 0$$

represents, in the complex plane,

(A) a straight line
(B) a pair of intersecting straight lines
(C) a point
(D) a pair of distinct parallel straight lines.

19. The limit $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left| e^{\frac{2\pi ik}{n}} - e^{\frac{2\pi i(k-1)}{n}} \right|$ is

(A) 2 (B) $2e$ (C) 2π (D) $2i$.

20. The limit $\displaystyle\lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right)^n$ equals

(A) e^{-1} (B) $e^{-1/2}$ (C) e^{-2} (D) 1.

21. Let ω denote a complex fifth root of unity. Define

$$b_k = \sum_{j=0}^{4} j\omega^{-kj},$$

for $0 \leq k \leq 4$. Then $\sum_{k=0}^{4} b_k \omega^k$ is equal to

(A) 5 (B) 5ω (C) $5(1+\omega)$ (D) 0.

22. Let $a_n = \left(1 - \frac{1}{\sqrt{2}}\right) \cdots \left(1 - \frac{1}{\sqrt{n+1}}\right), \quad n \geq 1$. Then $\lim_{n\to\infty} a_n$

(A) equals 1 (B) does not exist (C) equals $\frac{1}{\sqrt{\pi}}$ (D) equals 0.

23. Let X be a nonempty set and let $\mathcal{P}(X)$ denote the collection of all subsets of X. Define $f : X \times \mathcal{P}(X) \to \mathbb{R}$ by

$$f(x, A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Then $f(x, A \cup B)$ equals

(A) $f(x, A) + f(x, B)$
(B) $f(x, A) + f(x, B) - 1$
(C) $f(x, A) + f(x, B) - f(x, A) \cdot f(x, B)$
(D) $f(x, A) + |f(x, A) - f(x, B)|$

24. The series $\sum_{k=2}^{\infty} \frac{1}{k(k-1)}$ converges to

(A) -1 (B) 1 (C) 0 (D) does not converge.

25. The limit $\lim_{x\to\infty} \left(\frac{3x-1}{3x+1}\right)^{4x}$ equals

(A) 1 (B) 0 (C) $e^{-8/3}$ (D) $e^{4/9}$

26. $\lim_{n\to\infty} \frac{1}{n} \left(\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n}\right)$ is equal to

(A) ∞ (B) 0 (C) $\log_e 2$ (D) 1

27. Let $\cos^6\theta = a_6\cos 6\theta + a_5\cos 5\theta + a_4\cos 4\theta + a_3\cos 3\theta + a_2\cos 2\theta + a_1\cos\theta + a_0$. Then a_0 is

(A) 0 (B) 1/32. (C) 15/32. (D) 10/32.

28. In a triangle ABC, AD is the median. If length of AB is 7, length of AC is 15 and length of BC is 10 then length of AD equals

(A) $\sqrt{125}$ (B) 69/5 (C) $\sqrt{112}$ (D) $\sqrt{864}/5$.

29. The set $\{x : \left|x + \frac{1}{x}\right| > 6\}$ equals the set

(A) $(0, 3-2\sqrt{2}) \cup (3+2\sqrt{2}, \infty)$
(B) $(-\infty, -3-2\sqrt{2}) \cup (-3+2\sqrt{2}, \infty)$
(C) $(-\infty, 3-2\sqrt{2}) \cup (3+2\sqrt{2}, \infty)$
(D) $(-\infty, -3-2\sqrt{2}) \cup (-3+2\sqrt{2}, 3-2\sqrt{2}) \cup (3+2\sqrt{2}, \infty)$

30. Suppose that a function f defined on $\mathbb{R}^2$ satisfies the following conditions:

$$\begin{aligned} f(x+t, y) &= f(x,y) + ty, \\ f(x, t+y) &= f(x,y) + tx \text{ and} \\ f(0,0) &= K, \text{ a constant.} \end{aligned}$$

Then for all $x, y \in \mathbb{R}$, $f(x,y)$ is equal to

(A) $K(x+y)$. (B) $K - xy$. (C) $K + xy$. (D) none of the above.

31. Consider the sets defined by the real solutions of the inequalities

$$A = \{(x,y) : x^2 + y^4 \leq 1\} \qquad B = \{(x,y) : x^4 + y^6 \leq 1\}.$$

Then

(A) $B \subseteq A$
(B) $A \subseteq B$
(C) Each of the sets $A - B$, $B - A$ and $A \cap B$ is non-empty
(D) none of the above.

32. If a square of side a and an equilateral triangle of side b are inscribed in a circle then a/b equals

(A) $\sqrt{2/3}$ (B) $\sqrt{3/2}$ (C) $3/\sqrt{2}$ (D) $\sqrt{2}/3$.

33. If $f(x)$ is a real valued function such that

$$2f(x) + 3f(-x) = 15 - 4x,$$

for every $x \in \mathbb{R}$, then $f(2)$ is

(A) -15 (B) 22 (C) 11 (D) 0.

34. If $f(x) = \dfrac{\sqrt{3}\sin x}{2 + \cos x}$, then the range of $f(x)$ is

(A) the interval $[-1, \sqrt{3}/2]$
(B) the interval $[-\sqrt{3}/2, 1]$
(C) the interval $[-1, 1]$
(D) none of the above.

35. If $f(x) = x^2$ and $g(x) = x\sin x + \cos x$ then

(A) f and g agree at no points
(B) f and g agree at exactly one point
(C) f and g agree at exactly two points
(D) f and g agree at more than two points.

36. For non-negative integers m, n define a function as follows

$$f(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ f(m-1, 1) & \text{if } m \neq 0, n = 0 \\ f(m-1, f(m, n-1)) & \text{if } m \neq 0, n \neq 0 \end{cases}$$

Then the value of $f(1,1)$ is

(A) 4 (B) 3 (C) 2 (D) 1.

37. Let a be a nonzero real number. Define

$$f(x) = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

for $x \in \mathbb{R}$. Then, the number of distinct real roots of $f(x) = 0$ is

(A) 1 (B) 2 (C) 3 (D) 4.

38. A real 2×2 matrix M such that

$$M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1-\varepsilon \end{pmatrix}$$

(A) exists for all $\varepsilon > 0$

(B) does not exist for any $\varepsilon > 0$

(C) exists for some $\varepsilon > 0$

(D) none of the above is true

39. The eigenvalues of the matrix $X = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ are

(A) 1, 1, 4 (B) 1, 4, 4 (C) 0, 1, 4 (D) 0, 4, 4.

40. Let $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ and y_4 be fixed real numbers, not all of them equal to zero. Define a 4×4 matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} x_1^2 + y_1^2 & x_1x_2 + y_1y_2 & x_1x_3 + y_1y_3 & x_1x_4 + y_1y_4 \\ x_2x_1 + y_2y_1 & x_2^2 + y_2^2 & x_2x_3 + y_2y_3 & x_2x_4 + y_2y_4 \\ x_3x_1 + y_3y_1 & x_3x_2 + y_3y_2 & x_3^2 + y_3^2 & x_3x_4 + y_3y_4 \\ x_4x_1 + y_4y_1 & x_4x_2 + y_4y_2 & x_4x_3 + y_4y_3 & x_4^2 + y_4^2 \end{pmatrix}.$$

Then rank($\mathbf{A}$) equals

(A) 1 or 2. (B) 0. (C) 4. (D) 2 or 3.

41. Let k and n be integers greater than 1. Then $(kn)!$ is not necessarily divisible by

(A) $(n!)^k$. (B) $(k!)^n$. (C) $n!.k!$. (D) 2^{kn}.

42. Let $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}.$$

If $\lambda_1 + \lambda_2 + \lambda_3 = \sqrt{2} + 1$, then the set of possible values of t, $-\pi \leq t < \pi$, is

(A) Empty set (B) $\left\{\frac{\pi}{4}\right\}$ (C) $\left\{-\frac{\pi}{4}, \frac{\pi}{4}\right\}$ (D) $\left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\}$.

43. The values of η for which the following system of equations

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ x & + & 2y & + & 4z & = & \eta \\ x & + & 4y & + & 10z & = & \eta^2 \end{array}$$

has a solution are

(A) $\eta = 1, -2$ (B) $\eta = -1, -2$ (C) $\eta = 3, -3$ (D) $\eta = 1, 2$.

44. Let P_1, P_2 and P_3 denote, respectively, the planes defined by

$$\begin{aligned} a_1x + b_1y + c_1z &= \alpha_1 \\ a_2x + b_2y + c_2z &= \alpha_2 \\ a_3x + b_3y + c_3z &= \alpha_3. \end{aligned}$$

It is given that P_1, P_2 and P_3 intersect exactly at one point when $\alpha_1 = \alpha_2 = \alpha_3 = 1$. If now $\alpha_1 = 2, \alpha_2 = 3$ and $\alpha_3 = 4$ then the planes

(A) do not have any common point of intersection
(B) intersect at a unique point
(C) intersect along a straight line
(D) intersect along a plane.

45. Angles between any pair of 4 main diagonals of a cube are

(A) $\cos^{-1} 1/\sqrt{3}, \pi - \cos^{-1} 1/\sqrt{3}$ (B) $\cos^{-1} 1/3, \pi - \cos^{-1} 1/3$
(C) $\pi/2$ (D) none of the above.

46. If the tangent at the point P with co-ordinates (h, k) on the curve $y^2 = 2x^3$ is perpendicular to the straight line $4x = 3y$, then

(A) $(h, k) = (0, 0)$

(B) $(h, k) = (1/8, -1/16)$

(C) $(h, k) = (0, 0)$ or $(h, k) = (1/8, -1/16)$

(D) no such point (h, k) exists.

47. Consider the family $\mathcal{F}$ of curves in the plane given by $x = cy^2$, where c is a real parameter. Let $\mathcal{G}$ be the family of curves having the following property: every member of $\mathcal{G}$ intersects each member of $\mathcal{F}$ orthogonally. Then $\mathcal{G}$ is given by

(A) $xy = k$

(B) $x^2 + y^2 = k^2$

(C) $y^2 + 2x^2 = k^2$

(D) $x^2 - y^2 + 2yk = k^2$

48. Suppose the circle with equation $x^2 + y^2 + 2fx + 2gy + c = 0$ cuts the parabola $y^2 = 4ax$, $(a > 0)$ at four distinct points. If d denotes the sum of ordinates of these four points, then the set of possible values of d is

(A) $\{0\}$ (B) $(-4a, 4a)$ (C) $(-a, a)$ (D) $(-\infty, \infty)$.

49. The polar equation $r = a \cos\theta$ represents

(A) a spiral (B) a parabola (C) a circle (D) none of the above.

50. Let

$$\begin{aligned} V_1 &= \frac{7^2 + 8^2 + 15^2 + 23^2}{4} - \left(\frac{7 + 8 + 15 + 23}{4}\right)^2, \\ V_2 &= \frac{6^2 + 8^2 + 15^2 + 24^2}{4} - \left(\frac{6 + 8 + 15 + 24}{4}\right)^2, \\ V_3 &= \frac{5^2 + 8^2 + 15^2 + 25^2}{4} - \left(\frac{5 + 8 + 15 + 25}{4}\right)^2. \end{aligned}$$

Then

(A) $V_3 < V_2 < V_1$

(B) $V_3 < V_1 < V_2$

(C) $V_1 < V_2 < V_3$

(D) $V_2 < V_3 < V_1$.

51. A permutation of $1, 2, ..., n$ is chosen at random. Then the probability that the numbers 1 and 2 appear as neighbour equals

(A) $\frac{1}{n}$ (B) $\frac{2}{n}$ (C) $\frac{1}{n-1}$ (D) $\frac{1}{n-2}$.

52. Two coins are tossed independently where P(head occurs when coin i is tossed) $= p_i$, $i = 1, 2$. Given that at least one head has occurred, the probability that coins produced different outcomes is

(A) $\frac{2p_1p_2}{p_1 + p_2 - 2p_1p_2}$ (B) $\frac{p_1 + p_2 - 2p_1p_2}{p_1 + p_2 - p_1p_2}$ (C) $\frac{2}{3}$ (D) none of the above.

53. The number of cars (X) arriving at a service station per day follows a Poisson distribution with mean 4. The service station can provide service to a maximum of 4 cars per day. Then the expected number of cars that do not get service per day equals

(A) 4 (B) 0 (C) $\sum_{i=0}^{\infty} iP(X = i + 4)$ (D) $\sum_{i=4}^{\infty} iP(X = i - 4)$.

54. If $0 < x < 1$, then the sum of the infinite series $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \cdots$ is

(A) $\log \frac{1+x}{1-x}$ (B) $\frac{x}{1-x} + \log(1 + x)$
(C) $\frac{1}{1-x} + \log(1 - x)$ (D) $\frac{x}{1-x} + \log(1 - x)$.

55. Let $\{a_n\}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if

(A) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+2}$ exists

(B) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ exist

(C) $\lim_{n\to\infty} a_{2n}$, $\lim_{n\to\infty} a_{2n+1}$ and $\lim_{n\to\infty} a_{3n}$ exist

(D) none of the above.

56. Let $\{a_n\}$ be a sequence of non-negative real numbers such that the series $\sum_{n=1}^{\infty} a_n$ is convergent. If p is a real number such that the series $\sum \frac{\sqrt{a_n}}{n^p}$ diverges, then

(A) p must be strictly less than $\frac{1}{2}$

(B) p must be strictly less than or equal to $\frac{1}{2}$

(C) p must be strictly less than or equal to 1 but can be greater than $\frac{1}{2}$

(D) p must be strictly less than 1 but can be greater than or equal to $\frac{1}{2}$.

57. Suppose $a > 0$. Consider the sequence

$$a_n = n\{\sqrt[n]{ea} - \sqrt[n]{a}\}, \quad n \geq 1.$$

Then

(A) $\lim_{n\to\infty} a_n$ does not exist

(B) $\lim_{n\to\infty} a_n = e$

(C) $\lim_{n\to\infty} a_n = 0$

(D) none of the above.

58. Let $\{a_n\}$, $n \geq 1$, be a sequence of real numbers satisfying $|a_n| \leq 1$ for all n. Define

$$A_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n),$$

for $n \geq 1$. Then $\lim_{n\to\infty} \sqrt{n}(A_{n+1} - A_n)$ is equal to

(A) 0 (B) -1 (C) 1 (D) none of these.

59. In the Taylor expansion of the function $f(x) = e^{x/2}$ about $x = 3$, the coefficient of $(x-3)^5$ is

(A) $e^{3/2}\frac{1}{5!}$ (B) $e^{3/2}\frac{1}{2^5 5!}$ (C) $e^{-3/2}\frac{1}{2^5 5!}$ (D) none of the above.

60. Let σ be the permutation:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 6 & 2 & 4 & 9 & 8 & 7 & 1, \end{matrix}$$

I be the identity permutation and m be the order of σ i.e.

$$m = \min\{\text{positive integers } n : \sigma^n = I\}.$$

Then m is

(A) 8 (B) 12 (C) 360 (D) 2520.

61. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then

(A) there exists a matrix C such that $A = BC = CB$

(B) there is no matrix C such that $A = BC$

(C) there exists a matrix C such that $A = BC$, but $A \neq CB$

(D) there is no matrix C such that $A = CB$.

62. If the matrix

$$A = \begin{bmatrix} a & 1 \\ 2 & 3 \end{bmatrix}$$

has 1 as an eigenvalue, then $trace(A)$ is

(A) 4 (B) 5 (C) 6 (D) 7.

63. Let $\theta = 2\pi/67$. Now consider the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then the matrix A^{2010} is

(A) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(C) $\begin{pmatrix} \cos^{30}\theta & \sin^{30}\theta \\ -\sin^{30}\theta & \cos^{30}\theta \end{pmatrix}$ (D) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

64. Let the position of a particle in three dimensional space at time t be $(t, \cos t, \sin t)$. Then the length of the path traversed by the particle between the times $t = 0$ and $t = 2\pi$ is

(A) 2π. (B) $2\sqrt{2}\pi$. (C) $\sqrt{2}\pi$ (D) none of the above.

65. Let n be a positive real number and p be a positive integer. Which of the following inequalities is true?

(A) $n^p > \frac{(n+1)^{p+1} - n^{p+1}}{p+1}$ (B) $n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1}$

(C) $(n+1)^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1}$ (D) none of the above.

66. The smallest positive number K for which the inequality

$$|\sin^2 x - \sin^2 y| \leq K|x - y|$$

holds for all x and y is

(A) 2 (B) 1 (C) $\frac{\pi}{2}$ (D) there is no smallest positive value of K; any $K > 0$ will make the inequality hold.

67. Given two real numbers $a < b$, let

$$d(x, [a, b]) = \min\{|x - y| : a \leq y \leq b\} \quad \text{for } -\infty < x < \infty.$$

Then the function

$$f(x) = \frac{d(x, [0, 1])}{d(x, [0, 1]) + d(x, [2, 3])}$$

satisfies

(A) $0 \leq f(x) < \frac{1}{2}$ for every x

(B) $0 < f(x) < 1$ for every x

(C) $f(x) = 0$ if $2 \leq x \leq 3$ and $f(x) = 1$ if $0 \leq x \leq 1$

(D) $f(x) = 0$ if $0 \leq x \leq 1$ and $f(x) = 1$ if $2 \leq x \leq 3$.

68. Let

$$f(x, y) = \begin{cases} e^{-1/(x^2+y^2)} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then $f(x, y)$ is

(A) not continuous at $(0, 0)$

(B) continuous at $(0, 0)$ but does not have first order partial derivatives

(C) continuous at $(0,0)$ and has first order partial derivatives, but not differentiable at $(0,0)$

(D) differentiable at $(0,0)$

69. Consider the function

$$f(x) = \begin{cases} \int_0^x \{5 + |1-y|\}dy & \text{if } x > 2 \\ \\ 5x+2 & \text{if } x \leq 2 \end{cases}$$

Then

(A) f is not continuous at $x = 2$

(B) f is continuous and differentiable everywhere

(C) f is continuous everywhere but not differentiable at $x = 1$

(D) f is continuous everywhere but not differentiable at $x = 2$.

70. Let $w = \log(u^2 + v^2)$ where $u = \mathrm{e}^{(x^2+y)}$ and $v = \mathrm{e}^{(x+y^2)}$. Then

$$\left.\frac{\partial w}{\partial x}\right|_{x=0,y=0}$$

is

(A) 0 (B) 1 (C) 2 (D) 4

71. Let

$$f(x,y) = \begin{cases} 1, & \text{if } xy = 0, \\ xy, & \text{if } xy \neq 0. \end{cases}$$

Then

(A) f is continuous at $(0,0)$ and $\frac{\partial f}{\partial x}(0,0)$ exists

(B) f is not continuous at $(0,0)$ and $\frac{\partial f}{\partial x}(0,0)$ exists

(C) f is continuous at $(0,0)$ and $\frac{\partial f}{\partial x}(0,0)$ does not exist

(D) f is not continuous at $(0,0)$ and $\frac{\partial f}{\partial x}(0,0)$ does not exist.

72. The map $f(x) = a_0 \cos|x| + a_1 \sin|x| + a_2|x|^3$ is differentiable at $x = 0$ if and only if

(A) $a_1 = 0$ and $a_2 = 0$ (B) $a_0 = 0$ and $a_1 = 0$

(C) $a_1 = 0$ (D) a_0, a_1, a_2 can take any real value.

73. $f(x)$ is a differentiable function on the real line such that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} f'(x) = \alpha$. Then

(A) α must be 0 (B) α need not be 0, but $|\alpha| < 1$

(C) $\alpha > 1$ (D) $\alpha < -1$.

74. Let f and g be two differentiable functions such that $f'(x) \leq g'(x)$ for all $x < 1$ and $f'(x) \geq g'(x)$ for all $x > 1$. Then

(A) if $f(1) \geq g(1)$, then $f(x) \geq g(x)$ for all x

(B) if $f(1) \leq g(1)$, then $f(x) \leq g(x)$ for all x

(C) $f(1) \leq g(1)$

(D) $f(1) \geq g(1)$.

75. The length of the curve $x = t^3$, $y = 3t^2$ from $t = 0$ to $t = 4$ is

(A) $5\sqrt{5} + 1$ (B) $8(5\sqrt{5} + 1)$

(C) $5\sqrt{5} - 1$ (D) $8(5\sqrt{5} - 1)$.

76. Given that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$, the value of

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)}\, dxdy$$

is

(A) $\sqrt{\pi/3}$ (B) $\pi/\sqrt{3}$ (C) $\sqrt{2\pi/3}$ (D) $2\pi/\sqrt{3}$.

77. Let R be the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$. The value of the double integral

$$\int\int_R \frac{\sin x}{x}\, dxdy$$

is

(A) $1 - \cos 1$ (B) $\cos 1$ (C) $\frac{\pi}{2}$ (D) π.

78. The value of

$$\lim_{n\to\infty} \left[(n+1) \int_0^1 x^n \ln(1+x)\, dx\right]$$

is

(A) 0 (B) $\ln 2$ (C) $\ln 3$ (D) ∞.

79. Let $g(x, y) = \max\{12 - x, 8 - y\}$. Then the minimum value of $g(x, y)$ as (x, y) varies over the line $x + y = 10$ is

(A) 5 (B) 7 (C) 1 (D) 3.

80. Let $0 < \alpha < \beta < 1$. Then

$$\sum_{k=1}^{\infty} \int_{1/(k+\beta)}^{1/(k+\alpha)} \frac{1}{1+x}\,dx$$

is equal to

(A) $\log_e \dfrac{\beta}{\alpha}$ (B) $\log_e \dfrac{1+\beta}{1+\alpha}$ (C) $\log_e \dfrac{1+\alpha}{1+\beta}$ (D) ∞.

81. If f is continuous in $[0, 1]$ then

$$\lim_{n\to\infty} \sum_{j=0}^{[n/2]} \frac{1}{n} f\left(\frac{j}{n}\right)$$

(where $[y]$ is the largest integer less than or equal to y)

(A) does not exist

(B) exists and is equal to $\dfrac{1}{2}\displaystyle\int_0^1 f(x)\,dx$

(C) exists and is equal to $\displaystyle\int_0^1 f(x)\,dx$

(D) exists and is equal to $\displaystyle\int_0^{1/2} f(x)\,dx$.

82. The volume of the solid, generated by revolving about the horizontal line $y = 2$ the region bounded by $y^2 \leq 2x$, $x \leq 8$ and $y \geq 2$, is

(A) $2\sqrt{2\pi}$ (B) $28\pi/3$ (C) 84π (D) none of the above.

83. If α, β are complex numbers then the maximum value of $\dfrac{\alpha\bar{\beta} + \bar{\alpha}\beta}{|\alpha\beta|}$ is

(A) 2

(B) 1

(C) the expression may not always be a real number and hence maximum does not make sense

(D) none of the above.

84. For positive real numbers $a_1, a_2, \ldots, a_{100}$, let

$$p = \sum_{i=1}^{100} a_i \quad \text{and} \quad q = \sum_{1 \le i < j \le 100} a_i a_j.$$

Then

(A) $q = \frac{p^2}{2}$ (B) $q^2 \ge \frac{p^2}{2}$ (C) $q < \frac{p^2}{2}$ (D) none of the above.

85. The differential equation of all the ellipses centred at the origin is

(A) $y^2 + x(y')^2 - yy' = 0$ (B) $xyy'' + x(y')^2 - yy' = 0$

(C) $yy'' + x(y')^2 - xy' = 0$ (D) none of these.

86. The coordinates of a moving point P satisfy the equations

$$\frac{dx}{dt} = \tan x, \quad \frac{dy}{dt} = -\sin^2 x, \quad t \ge 0.$$

If the curve passes through the point $(\pi/2, 0)$ when $t = 0$, then the equation of the curve in rectangular co-ordinates is

(A) $y = 1/2 \cos^2 x$ (B) $y = \sin 2x$

(C) $y = \cos 2x + 1$ (D) $y = \sin^2 x - 1$.

87. If $x(t)$ is a solution of

$$(1 - t^2)\ dx - tx\ dt = dt$$

and $x(0) = 1$, then $x(\frac{1}{2})$ is equal to

(A) $\frac{2}{\sqrt{3}}(\frac{\pi}{6} + 1)$ (B) $\frac{2}{\sqrt{3}}(\frac{\pi}{6} - 1)$ (C) $\frac{\pi}{3\sqrt{3}}$ (D) $\frac{\pi}{\sqrt{3}}$.

88. Let $f(x)$ be a given differentiable function. Consider the following differential equation in y

$$f(x)\frac{dy}{dx} = yf'(x) - y^2.$$

The general solution of this equation is given by

(A) $y = -\frac{x+c}{f(x)}$ (B) $y^2 = \frac{f(x)}{x+c}$

(C) $y = \frac{f(x)}{x+c}$ (D) $y = \frac{[f(x)]^2}{x+c}$.

89. Let $y(x)$ be a non-trivial solution of the second order linear differential equation

$$\frac{d^2y}{dx^2} + 2c\frac{dy}{dx} + ky = 0,$$

where $c < 0$, $k > 0$ and $c^2 > k$. Then

(A) $|y(x)| \to \infty$ as $x \to \infty$

(B) $|y(x)| \to 0$ as $x \to \infty$

(C) $\lim_{x \to \pm\infty} |y(x)|$ exists and is finite

(D) none of the above is true.

90. The differential equation of the system of circles touching the y-axis at the origin is

(A) $x^2 + y^2 - 2xy\dfrac{dy}{dx} = 0$ (B) $x^2 + y^2 + 2xy\dfrac{dy}{dx} = 0$

(C) $x^2 - y^2 - 2xy\dfrac{dy}{dx} = 0$ (D) $x^2 - y^2 + 2xy\dfrac{dy}{dx} = 0.$

91. Suppose a solution of the differential equation

$$(xy^3 + x^2y^7)\frac{dy}{dx} = 1,$$

satisfies the initial condition $y(1/4) = 1$. Then the value of $\dfrac{dy}{dx}$ when $y = -1$ is

(A) $\dfrac{4}{3}$ (B) $-\dfrac{4}{3}$ (C) $\dfrac{16}{5}$ (D) $-\dfrac{16}{5}$.

92. Consider the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$

with usual matrix multiplication. Let

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.$$

Then,

(A) N is not a subgroup of G

(B) N is a subgroup of G but not a normal subgroup

(C) N is a normal subgroup and the quotient group G/N is of finite order

(D) N is a normal subgroup and the quotient group is isomorphic to $\mathbb{R}^+$ (the group of positive reals with multiplication).

93. Let G be a group with identity element e. If x and y are elements in G satisfying $x^5y^3 = x^8y^5 = e$, then which of the following conditions is true?

(A) $x = e, y = e$

(B) $x \neq e, y = e$

(C) $x = e, y \neq e$

(D) $x \neq e, y \neq e$

94. Let G be the group $\{\pm 1, \pm i\}$ with multiplication of complex numbers as composition. Let H be the quotient group $\mathbb{Z}/4\mathbb{Z}$. Then the number of nontrivial group homomorphisms from H to G is

(A) 4 (B) 1 (C) 2 (D) 3.

1. Let $f : R \to R$ be a function which is continuous at 0 and $f(0) = 1$. Also assume that f satisfies the following relation for all x:

$$f(x) - f(x/2) = \frac{3x^2}{4} + x.$$

Find $f(3)$.

2. For any $n \times n$ matrix $A = ((a_{ij}))$, consider the following three properties:

    1. a_{ij} is real valued for all i, j and A is upper triangular.
    2. $\sum_{j=1}^{n} a_{ij} = 0$, for all $1 \leq i \leq n$.
    3. $\sum_{i=1}^{n} a_{ij} = 0$, for all $1 \leq j \leq n$.

    Define the following set of matrices:

    $$\mathcal{C}_n = \{A : \ A \text{ is } n \times n \text{ and satisfies (1), (2) and (3) above}\}.$$

    (a) Show that $\mathcal{C}_n$ is a vector space for any $n \geq 1$.

    (b) Find the dimension of $\mathcal{C}_n$, when $n = 2$ and $n = 3$.

3. Let A be a real valued and symmetric $n \times n$ matrix with entries such that $A \neq \pm I$ and $A^2 = I$.

    (a) Prove that there exist non-zero column vectors v and w such that $Av = v$ and $Aw = -w$.

    (b) Prove that every vector z has a unique decomposition $z = x + y$ where $Ax = x$ and $Ay = -y$.

4. Suppose 15 identical balls are placed in 3 boxes labeled A, B and C. What is the number of ways in which Box A can have more balls than Box C?

5. Suppose that X and Y are random variables such that

$$\begin{aligned} E(X+Y) = E(X-Y) &= 0\,, \\ \mathrm{Var}(X+Y) &= 3\,, \\ \mathrm{Var}(X-Y) &= 1\,. \end{aligned}$$

   (a) Evaluate $\mathrm{Cov}(X, Y)$.

   (b) Show that $E|X+Y| \leq \sqrt{3}$.

   (c) If in addition, it is given that (X, Y) is bivariate normal, calculate $E\big(|X+Y|^3\big)$.

6. Suppose $X_1, \ldots, X_n$ are i.i.d. random variables with mean μ and variance 1. Also assume that $Y_1, \ldots, Y_n$ are i.i.d. with probability mass function $\mathbf{P}(Y_i = \pm 1) = 1/2$ for all $1 \leq i \leq n$ and independent of $X_1, \ldots, X_n$. Define T_n as follows:

$$T_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \cdot |X_i|, \quad n \geq 1.$$

   (a) For any fixed $z \in R$, find

$$\lim_{n\to\infty} \mathbf{P}\left(\sqrt{n}T_n \leq z\right).$$

   (b) Using the result in part (a) above, find random quantities L_n and U_n, based on T_n, such that

$$\lim_{n\to\infty} \mathbf{P}\left(L_n \leq \mu \leq U_n\right) = 0.95\ .$$

7. Suppose that $X_1, \ldots, X_n$ are i.i.d. with probability density function

$$f_\theta(x) = \begin{cases} \frac{1}{2\sqrt{\theta x}} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $X_{(n)}$ is sufficient for θ.

(b) Consider a test of size α $(0 < \alpha < 1)$ for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ $(> \theta_0)$, that rejects H_0 if and only if $X_{(n)} > k$.

i. Determine the value of k.

ii. Find the minimum sample size required such that the power of the test is at least β $(\alpha < \beta < 1)$.

8. Consider the regression model:

$$y_i = bx_i + e_i, \quad 1 \leq i \leq n,$$

where x_i's are fixed non-zero real numbers and e_i's are independent random variables with mean zero and equal variance.

(a) Consider estimators of the form $\sum_{i=1}^{n} a_i y_i$ (where a_i's are non random real numbers) that are unbiased for b. Show that the least squares estimator of b has the minimum variance in this class of estimators.

(b) Suppose that x_i's take values -1 or $+1$ and e_i's have density

$$f(t) = \frac{1}{2}\, e^{-|t|}, \quad t \in R.$$

Find the maximum likelihood estimator of b.

9. Consider a collection of N cards, numbered $1, 2, \ldots, N$, where $N \geq 2$. A card is drawn at random and set aside. Suppose n cards are selected from the remaining $(N - 1)$ cards using SRSWR and their numbers noted as $Y_1, \ldots, Y_n$. If $S = \sum_{i=1}^{n} Y_i$, find $E(S)$ and $Var(S)$.

Test Code MS (Short answer type) 2012

Syllabus for Mathematics

Combinatorics; Elements of set theory. Permutations and combinations. Binomial and multinomial theorem. Theory of equations. Inequalities.

Linear Algebra: Vectors and vector spacs. Matrices. Determinants. Solution of linear equations. Trigonometry. Co-ordinate geometry.

Complex Numbers: Geometry of complex numbers and De Moivres theorem.

Calculus: Convergence of sequences and series. Functions. Limits and continuity of functions of one or more variables. Power series. Differentiation. Leibnitz formula. Applications of differential calculus, maxima and minima. Taylor's theorem. Differentiation of functions of several variables. Indefinite integral. Fundamental theorem of calculus. Riemann integration and properties. Improper integrals. Double and multiple integrals and applications.

Syllabus for Statistics and Probability

Probability and Sampling Distributions: Notions of sample space and probability. Combinatorial probability. Conditional probability and independence. Random variables and expectations. Moments and moment generating functions. Standard univariate discrete and continuous distributions. Joint probability distributions. Multinomial distribution. Bivariate normal and multivariate normal distributions. Sampling distributions of statistics. Weak law of large numbers. Central limit theorem.

Descriptive Statistics: Descriptive statistical measures. Contingency tables and measures of association. Product moment and other types of correlation. Partial and multiple correlation. Simple and multiple linear regression.

Statistical Inference: Elementary theory of estimation (unbiasedness, minimum variance, sufficiency). Methods of estimation (maximum likelihood method, method of moments). Tests of hypotheses (basic concepts and simple applications of Neyman-Pearson Lemma). Confidence intervals. Inference related to regression. ANOVA. Elements of nonparametric inference.

Design of Experiments and Sample Surveys: Basic designs such as CRD, RBD, LSD and their analyses. Elements of factorial designs. Conventional sampling techniques (SRSWR/SRSWOR) including stratification. Ratio and regression methods of estimation.

Sample Questions

1. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the four vectors

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix}.$$

2. Let f be a polynomial. Assume that $f(0) = 1$, $\lim_{x\to\infty} f''(x) = 4$ and $f(x) \geq f(1)$ for all $x \in \mathbb{R}$. Find $f(2)$.

3. Let X_1 and X_2 be i.i.d. exponential random variables with mean $\lambda > 0$. Let $Y_1 = X_1 - X_2$ and $Y_2 = RX_1 - (1-R)X_2$, where R is a Bernoulli random variable with parameter $1/2$ and is independent of X_1 and X_2.

   (a) Show that Y_1 and Y_2 have the same distribution.

   (b) Obtain the common density function.

4. Let $\begin{pmatrix} X \\ Y \end{pmatrix}$ be a bivariate normal vector such that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$. Let $S \subset \mathbb{R}^2$ be defined by

$$S = \{(a,b) : aX + bY \text{ is independent of } Y\}.$$

   (a) Show that S is a subspace.

   (b) Find its dimension.

5. Let $X_1, X_2, \ldots, X_j \ldots$ be i.i.d. $N(0,1)$ random variables. Show that for any $a > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sum_{i=1}^{n} X_i^2 \leq a\right) = 0.$$

6. There are two biased coins – one which has probability $1/4$ of showing heads and $3/4$ of showing tails, while the other has probability $3/4$ of showing heads and $1/4$ of showing tails when tossed. One of the two coins is chosen at random and is then tossed 8 times.

(a) Given that the first toss shows heads, what is the probability that in the next 7 tosses there will be exactly 6 heads and 1 tail?

(b) Given that the first toss shows heads and the second toss shows tail, what is the probability that the next 6 tosses all show heads?

7. Consider a randomized (complete) block design with 4 treatments and 5 replications and, let t_i be the effect of the i-th treatment ($1 \leq i \leq 4$). Consider the following three treatment contrasts.

$$\frac{1}{\sqrt{2}}(t_1 - t_2), \quad \frac{1}{\sqrt{6}}(t_1 + t_2 - 2t_3) \quad \text{and} \quad \frac{1}{\sqrt{12}}(t_1 + t_2 + t_3 - 3t_4).$$

(a) Find the variances of the best linear unbiased estimators of the above treatment contrasts.

(b) Find all the covariances between them.

8. Let V_1 be the variance of the estimated mean from a stratified random sample of size n with proportional allocation. Assume that the strata sizes are such that the allocations are all integers.

Let V_2 be the variance of the estimated mean from a simple random sample of size n.

Show that the ratio V_1/V_2 is independent of n.

9. Suppose X_1 and X_2 are i.i.d. Bernoulli random variables with parameter p where it is known that $\frac{1}{3} \leq p \leq \frac{2}{3}$. Find the maximum likelihood estimator $\widehat{p}$ of p based on X_1 and X_2.

10. Let $X_1, X_2, \ldots, X_{10}$ be i.i.d. Poisson random variables with unknown parameter $\lambda > 0$. Find the minimum variance unbiased estimator of $\exp\{-2\lambda\}$.

Note: For more sample questions you can visit `http://www.isical.ac.in/~deanweb/MSTATSQ.html`.