1. A sequence of real numbers $\{a_n\}_{n\geq 1}$ has a *peak* at n if $a_n \geq a_k$ for all $k \geq n$. Consider the following statements.

   (I) No sequence of real numbers can have only finitely many peaks.

   (II) No sequence of real numbers can have infinitely many peaks.

   (III) Any sequence of real numbers having finitely many peaks must have the property that $a_n \geq 0$ for all n greater than some k.

   Then

   (A) only (I) is true

   (B) none of (I), (II) and (III) are true

   (C) both (II) and (III) are true

   (D) only (III) is true

2. Let $\mathbb{C}$ denote the set of complex numbers and let $\mathrm{Im}(z)$ denote the imaginary part of $z \in \mathbb{C}$. Consider the set

   $$S = \{s \in \mathbb{R} : \text{ there exists } z \in \mathbb{C} \text{ such that } \mathrm{Im}(z) \neq 0 \text{ and } s = z^2+2z-1\}.$$

   Then,

   (A) $S \neq \mathbb{R}$, but contains infinitely many elements

   (B) S is a non-empty finite set

   (C) $S = \mathbb{R}$

   (D) S is the empty set

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3. For a set S, let S^c denote the complement of S. Also, for two sets P and Q, let $P \setminus Q = P \cap Q^c$. Let A, B_1, B_2 and B_3 be four sets. Which of the following statements is NOT true?

(A) $(A \cup B_1 \cup B_2 \cup B_3)^c = A^c \cap B_1^c \cap B_2^c \cap B_3^c$

(B) $(A \setminus B_1) \setminus (B_2 \cup B_3) = A \setminus (B_1 \cup B_2 \cup B_3)$

(C) $A \setminus (B_1 \cup B_2 \cup B_3) = (A \setminus B_1) \cup (A \setminus B_2) \cup (A \setminus B_3)$

(D) $A \cap (B_1 \cup B_2 \cup B_3) = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$

4. For a complex number z, let $\bar{z}$ be its complex conjugate. Then the equation

$$z\bar{z}^2 + z^2\bar{z} = 0$$

has

(A) exactly three roots

(B) exactly two roots

(C) infinitely many roots

(D) only real roots

5. Let $f(x) = x^2 + (2a+1)x + (a^2+2)$. The number of values of a for which one of the roots of the equation $f(x) = 0$ is twice the other root is

(A) more than 2 (B) 2 (C) 1 (D) 0

6. Let A be a finite set of real numbers having m (≥ 2) elements. Define a function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \min\{|a - x| : a \in A\}.$$

Then,

(A) f is continuous everywhere

(B) f is continuous only at finitely many points

(C) f is discontinuous everywhere

(D) f has m discontinuities

7. Let A be the set of functions $f : \mathbb{R} \to \mathbb{R}$ for which $|f(x) - f(y)| \leq 2|x - y|^2$ for all $x, y \in \mathbb{R}$ and $f(0) = 0$. Then, for any $f \in A$,

(A) the functions $g(x) = P(f(x)) \in A$ for every polynomial P

(B) the function $g(x) = x + f(x) \in A$

(C) the function $g(x) = xf(x) \in A$

(D) the function $g(x) = e^{f(x)} \in A$

8. The number of values of a for which the three lines

$$2x + y - 1 = 0, \quad ax + 3y - 3 = 0, \quad 3x + 2y - 2 = 0$$

are concurrent is

(A) more than 2 (B) 1 (C) 0 (D) 2

9. For a non-constant geometric progression for which the second term is 2 and the common ratio is an integer, the 10th, 20th and 30th terms are in arithmetic progression. Then, the fourth term is

(A) -2 (B) -4 (C) 4 (D) 2

10. $\lim_{x \to 1} \dfrac{\sqrt{x+8} - \sqrt{8x+1}}{\sqrt{5-x} - \sqrt{7x-3}}$ equals

(A) does not exist (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\frac{7}{12}$

11. The rank of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & a & 0 \\ 1 & 0 & 0 & 0 & b \end{pmatrix}.$$

(A) depends on the values of both a and b

(B) is independent of the values of both a and b

(C) depends on the value of a but not on the value of b

(D) depends on the value of b but not on the value of a

12. If the matrix $A = \begin{pmatrix} a & 1 \\ 2 & 3 \end{pmatrix}$ has 1 as an eigenvalue, then the determinant of A is

(A) 5 (B) 2 (C) 4 (D) 3

13. Let $a < 500$ be a positive integer. Consider a box containing balls numbered $a, a+1, \ldots, 500$. Suppose that the ball numbered x is picked with probability

$$\frac{2xa}{(500+a)(500-a+1)} \quad \text{for } x = a, a+1, \ldots, 500.$$

Then the value of a is

(A) 251 (B) 1 (C) 499 (D) 2

14. Let $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. Define $f_n(x)$ inductively by setting

$$f_1(x) = f(x)$$

and

$$f_{n+1}(x) = f(f_n(x)) \quad \text{for} \quad n > 1.$$

If $f_7(x) = 128x + 381$, then a^b equals

(A) $\frac{1}{8}$ (B) 32 (C) $\frac{1}{32}$

15. Let $n = aaaaaaaaabcd$ be a 12-digited number divisible by 45 where the digits a, b, c, d are not necessarily distinct and $a \neq 0$. How many such numbers are there?

(A) 216 (B) 207 (C) 189 (D) 198

16. Let a, b and c be the sides of a triangle such that $c^2 = a^2+b^2-ab$. Then which of the following is always true?

(A) $a \leq c$ and $b \leq c$

(B) $a \leq c \leq b$ or $b \leq c \leq a$

(C) $c \leq a$ and $c \leq b$

(D) None of the above

17. Let X be a discrete random variable and Y be a continuous random variable which is independent of X. Let $U = X + Y$ and $V = XY$. Choose the correct statement from the options given below.

(A) Both U and V are continuous random variables

(B) U is a continuous random variable but V need not be continuous

(C) U is a discrete random variable but V need not be a continuous random variable

(D) U is a discrete random variable and V is a continuous random variable

18. Suppose that the sample mean and sample standard deviation for a set of n observations $x_1, x_2, \ldots, x_n$ are m and s (> 0), respectively. These values are updated to m_1 and s_1 after one more observation x_{n+1} is added to the data set.

Based on the above information, choose the correct statement(s), from the options given below.

(A) If $m_1 = m$ then $s_1 < s$

(B) If $m_1 = m$ then $s_1 = s$

(C) If $m_1 < m$ then $s_1 = s$

(D) If $m_1 < m$ then $s_1 < s$

19. Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with probability density function

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$. Let $Y = X_1 + X_2 + \cdots + X_n$. Then the conditional distribution of X_n given $Y = 1$ is

(A) uniform on $(0, 1)$

(B) exponential with mean 1

(C) beta with parameters $n-1$ and 1

(D) beta with parameters 1 and $n-1$

20. Let $X_1, X_2, \ldots$ be a sequence of random variables such that $E(X_i) = 1$, $Var(X_i) = 1$ for all i and $Cov(X_i, X_j) = \frac{1}{2}$ for all $i \neq j$. Let $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, $\lim_{n\to\infty} Var(Z_n)$ equals

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) 0 (D) 1

21. Suppose that $P(A|B) = 0.4$ and $P(A^c|B^c) = 0.6$. Then, the two equations are sufficient to find

(A) neither $P(A)$ nor $P(B)$

(B) both $P(A)$ and $P(B)$

(C) $P(B)$ but not $P(A)$

(D) $P(A)$ but not $P(B)$

22. Let $X_1, \ldots, X_n$ be independent and identically distributed normal random variables with mean 0 and variance $\sigma^2 > 0$. Define $T_n = \frac{\sqrt{n}\,\overline{X}_n}{S_n}$ where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Then, the distribution of T_n is

(A) Student's t with n degrees of freedom

(B) Student's t with $(n-1)$ degrees of freedom

(C) normal with mean 0 and variance 1

(D) None of the above

23. Consider a matrix $\mathbf{M} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$ where X and Y are independent standard normal random variables. Then the probability that $\mathbf{M}$ is a non-singular matrix is

(A) 0 (B) 1 (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{1}{2}$

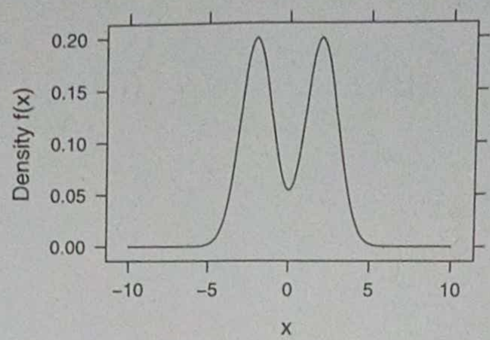
24. Let (U, V) be a point chosen uniformly at random from the unit circle $\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = 1\}$. Then $\mathrm{Var}(U)$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{1}{4}$

25. Suppose that we choose 2 cards simultaneously at random from a deck of 20 cards numbered $1, 2, \ldots, 20$. What is the probability that the smaller of the two numbers divides the larger?

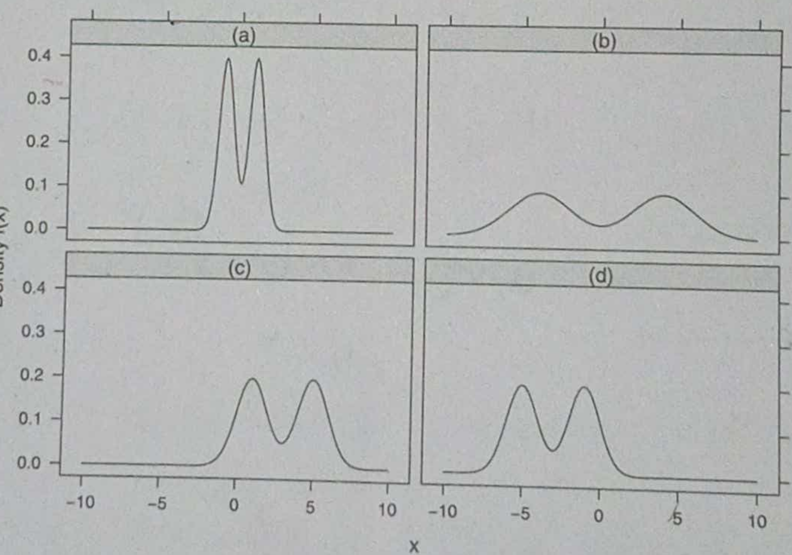
(A) $\frac{36}{190}$ (B) $\frac{46}{190}$ (C) $\frac{56}{190}$ (D) $\frac{66}{190}$

26. Let X be a random variable with probability density function displayed in the following graph.



Match the random variables (i)-(iv) below with the probability density functions depicted in figures (a)-(d).

$$(i)\, X+3 \qquad (ii)\, X-3 \qquad (iii)\, 2X \qquad (iv)\, \frac{X}{2}.$$



(A) (i)–(d), (ii)–(c), (iii)–(b), (iv)–(a)

(B) (i)–(d), (ii)–(c), (iii)–(a), (iv)–(b)

(C) (i)–(c), (ii)–(d), (iii)–(a), (iv)–(b)

(D) (i)–(c), (ii)–(d), (iii)–(b), (iv)–(a)

27. Suppose that we want to fit the regression model

$$y = \beta_1 x + \beta_2 x^2 + \epsilon$$

to 10 pairs of observations $(x_1, y_1), \ldots, (x_{10}, y_{10})$ where each x_i takes one of the two values, 0 and 1, and not all the x_i's are same. Which of the following can be estimated using the method of least squares?

(A) Both β_1 and β_2

(B) β_1 but not β_2

(C) β_2 but not β_1

(D) $\beta_1 + \beta_2$

28. Consider a bivariate sample $(X_1, Y_1), \ldots, (X_9, Y_9)$ where $X_i = i$ for $i = 1, 2, \ldots, 9$. The least squares regression line for this dataset is obtained as $y = 3 + 2x$. Later it turns out that Y_5 was recorded wrongly. When the revised regression line is obtained which of the following are possible?

(A) Intercept can change but slope cannot

(B) Slope can change but intercept cannot

(C) Both intercept and slope can change

(D) Neither intercept nor slope can change

29. Assume $X_1, \ldots, X_n$ are independent and identically distributed $N(\mu, 1)$ random variables with $\mu \in \mathbb{R}$. We want to test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$. Consider the following two one-sided testing problems

$$\begin{aligned} H_{0,A} : \mu = 0 \quad &\text{versus} \quad H_{1,A} : \mu > 0 \\ \text{and } H_{0,B} : \mu = 0 \quad &\text{versus} \quad H_{1,B} : \mu < 0. \end{aligned}$$

Let $\phi_{A,\eta}(\boldsymbol{x})$ and $\phi_{B,\eta}(\boldsymbol{x})$ denote the most powerful tests of size $\eta \in (0, 1)$ for $H_{0,A}$ and $H_{0,B}$, respectively. Then, for testing H_0 versus H_1,

(A) $\phi(\boldsymbol{x}) = \phi_{A,\eta}(\boldsymbol{x}) + \phi_{B,\eta}(\boldsymbol{x})$ is a test of size η

(B) $\phi(\boldsymbol{x}) = \phi_{A,\eta}(\boldsymbol{x})\, \phi_{B,\eta}(\boldsymbol{x})$ is a test of size 2η

(C) $\phi(\boldsymbol{x}) = \dfrac{1}{2} \max\{\phi_{A,\eta}(\boldsymbol{x}), \phi_{B,\eta}(\boldsymbol{x})\}$ is a test of size $\frac{\eta}{2}$

(D) $\phi(\boldsymbol{x}) = \phi_{A,\eta}(\boldsymbol{x}) + \phi_{B,\eta}(\boldsymbol{x})$ is a test of size 2η

30. Let ϕ denote the probability density function of the standard normal distribution. Let f_θ, for $\theta \in \{0, 1\}$, be defined as

$$f_\theta(x) = \begin{cases} \phi(x) & \text{if } \theta = 0, \\ \frac{1}{2}\phi\left(\frac{x-1}{2}\right) & \text{if } \theta = 1. \end{cases}$$

Assume that $X_1, \ldots, X_n$ are independent and identically distributed from the density $f_\theta(x)$. Which of the following is a sufficient statistic for θ?

(A) $\sum_{i=1}^{n} X_i$

(B) $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$

(C) $\sum_{i=1}^{n} X_i^2$

(D) $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} \mathbf{1}(|X_i| \geq 2)\right)$