GROUP A

1. Let $A_n = ((a_{ij}))$ be the $n \times n$ matrix defined by

$$a_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1, \\ 1 & \text{if } |i-j| = 1, \\ 2 & \text{if } i = j. \end{cases}$$

Find the determinant of A_n for $n \geq 1$.

2. Consider a random permutation of the eight numbers $1, 2, \ldots, 8$. Compute the probability that no two adjacent numbers in this permutation have a product which is odd. Give justification for your computations.

3. Identify, with justification, all cumulative distribution functions F that satisfy $F(x) = F(x^{2023})$ for every $x \in \mathbb{R}$.

GROUP B

4. A six-faced fair die is rolled repeatedly till 1 appears. Let X be the total number of rolls and Y be the number of times 6 appeared in these X rolls.

(a) Find $\mathrm{E}[Y|X = x]$.

(b) Find $\mathrm{E}[Y]$.

5. Suppose $X_1, \ldots, X_n$ $(n \geq 2)$ are independent and identically distributed observations from a distribution having probability density function

$$f_\theta(x) = \begin{cases} e^{-(x-\theta)} & \text{if } x \geq \theta, \\ 0 & \text{if } x < \theta, \end{cases}$$

where $\theta \in \mathbb{R}$. Let

$$\psi(\theta) = \int_1^\infty f_\theta(x)dx.$$

Define $\widehat{\theta}_n = \min\{X_1, \ldots, X_n\}$. Consider $\psi(\widehat{\theta}_n)$ as an estimator of $\psi(\theta)$ and let $B_n(\theta)$ denote the associated bias.

(a) Show that $B_n(\theta) > 0$ for every $\theta < 1$ and $B_n(\theta) = 0$ for every $\theta \geq 1$.

(b) Show that $\lim_{n\to\infty} B_n(\theta) = 0$ for every $\theta < 1$.

6. Suppose that two observations X_1 and X_2 are drawn at random from a distribution with the following probability density function

$$f_\theta(x) = \begin{cases} \dfrac{1}{2\theta} & \text{if } 0 \leq x \leq \theta \text{ or } 2\theta \leq x \leq 3\theta, \\ \\ 0 & \text{otherwise,} \end{cases}$$

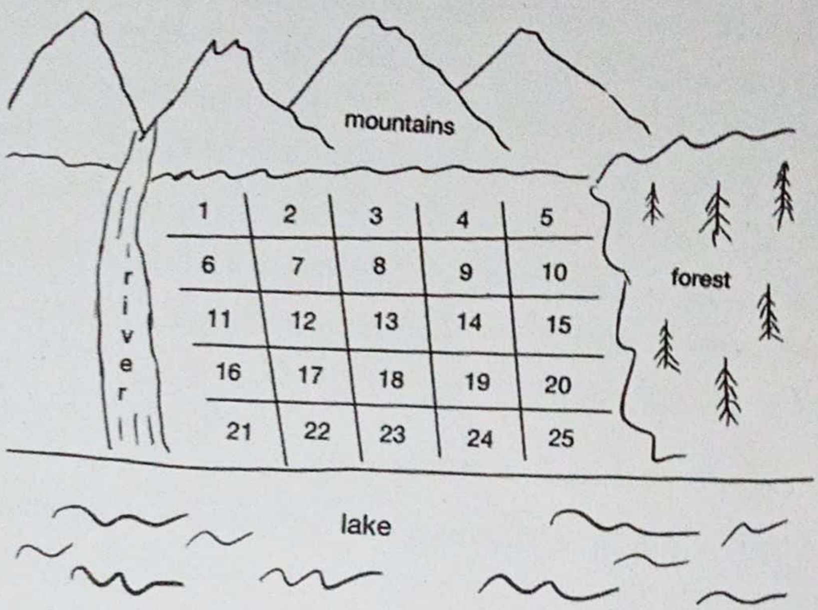
where $\theta > 0$. Determine the maximum likelihood estimator of θ for each of the following observed values of X_1 and X_2.

(a) $X_1 = 7$ and $X_2 = 9$.

(b) $X_1 = 4$ and $X_2 = 9$.

(c) $X_1 = 5$ and $X_2 = 9$.

7. Mr. X wants to compare five different varieties of wheat. He has access to a piece of land which has been divided into 25 smaller plots as shown in the figure below.



Mr. X has asked for your help in his venture.

(a) Give an allocation of the wheat varieties to the plots so that you are able to compare them. Justify your allocation rule. Clearly state all the assumptions you make.

(b) Once the crop is harvested, how would you analyze the data and compare the different varieties of wheat?

8. Let $X_1, \ldots, X_n$ be independent and identically distributed Bernoulli(θ) random variables where $\theta \in [0,1]$. An unbiased estimator of θ is the sample mean $\overline{X}_n$. However, a statistician feels that the sample size n is too small, and decides to increase the sample size. In order to do so, he records the observed values of the data points, $\{x_1, \ldots, x_n\}$, and then selects a random sample of size $m = kn$ with replacement from $\{x_1, \ldots, x_n\}$, where k is a positive integer. The values in the observed sample are recorded as $\{Y_1, \ldots, Y_m\}$. The statistician proposes the new estimator $T = \frac{1}{2}(\overline{Y}_m + \overline{X}_n)$, where $\overline{Y}_m$ is the sample mean of $Y_1, \ldots, Y_m$.

(a) Show that T is unbiased for θ.

(b) Is T a better estimator of θ than $\overline{X}_n$? Justify your answer.

9. To test whether the heights of siblings are correlated, a researcher devised the following plan: She identified a random sample of n families with at least two adult male children. For the ith family, suppose that X_i and Y_i are the heights of the first and second male child, respectively. Assume that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent bivariate normal random vectors with parameters $(\mu, \mu, \sigma^2, \sigma^2, \rho)$, where μ and σ^2 are known from previous studies. She is interested in testing the null hypothesis $H_0 : \rho = 0$ against the alternative $H_1 : \rho = 0.5$.

   Unfortunately, due to a mistake in the questionnaire, she was only able to observe (U_i, V_i) for each i, where $U_i = \max(X_i, Y_i)$ and $V_i = \min(X_i, Y_i)$.

   (a) Based on the observed sample, obtain the test statistic corresponding to the most powerful test of H_0 against H_1.

   (b) Find a critical value so that the size of the test converges to 0.05 as $n \to \infty$.