2017

BOOKLET NO. TEST CODE : **PSA**

Forenoon

Questions : 30	**Time : 2 hours**

Write your Registration Number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answer sheet.

For each question, there are four suggested answers of which exactly one is correct. For each question indicate your choice of the correct answer by darkening the appropriate oval (●) completely on the answer sheet.

4 marks are allotted for each correct answer,
0 mark for each incorrect answer, and
1 mark for each unattempted question.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET ONLY.
YOU ARE NOT ALLOWED TO USE CALCULATORS IN ANY FORM.

STOP! WAIT FOR THE SIGNAL TO START.

ROUGH WORK

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant function satisfying $f(x+2) = f(x)$ for every real x. Then $\lim_{x\to\infty} f(x)$

   (A) does not exist.

   (B) exists and equals $+\infty$ or $-\infty$.

   (C) exists and is finite.

   (D) may or may not exist depending on f.

2. Let A be a square matrix with real entries such that $A^2 = A$. Then

   (A) $(A+I)^{10} = I + 1024A$.

   (B) $(A+I)^{10} = I + 511A$.

   (C) $(A+I)^{10} = I + 1023A$.

   (D) $(A+I)^{10} = I + 512A$.

3. Let A be a 2×2 matrix such that $\mathrm{trace}(A) = \det(A) = 3$. What is $\mathrm{trace}(A^{-1})$?

   (A) 1/2 (B) 3 (C) 1 (D) 1/3

ROUGH WORK

4. Let A be a nonzero 2×2 matrix such that $A^3 = 0$. Then which of the following statements is always false?

   (A) $\mathrm{trace}(A - A^2) = 0$

   (B) $\mathrm{trace}(A + A^2) = 0$

   (C) $\det(I + A) = 0$

   (D) $A^2 = 0$

5. The vertices A, B, C, D of a square are to be coloured with one of three colours red, blue, or green such that adjacent vertices get different colours. What is the number of such colourings?

   (A) 18 (B) 12 (C) 20 (D) 24

6. Let $S \subset \mathbb{R}$. Consider the statement: "If f is a continuous function from S to S, then $f(x) = x$ for some x." This statement is true if S equals

   (A) $[0, 1]$. (B) $(0, 1]$. (C) $\mathbb{R}$. (D) $[-3, -2] \cup [2, 3]$.

ROUGH WORK

7. The coordinates of the point at which the tangent of the curve $y = x^2 - 1$ makes an angle of $45°$ with the positive X-axis are

(A) $(\frac{\sqrt{5}-1}{2}, \frac{1-\sqrt{5}}{2})$. (B) $(\frac{2}{3}, -\frac{5}{9})$. (C) $(\frac{3}{4}, -\frac{7}{16})$. (D) $(\frac{1}{2}, -\frac{3}{4})$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then f is

(A) continuous everywhere but not differentiable at 0.

(B) differentiable everywhere and its derivative is continuous.

(C) discontinuous at 0.

(D) differentiable everywhere and its derivative is discontinuous at 0.

9. If α, β, and γ are the roots of $x^3 - px + q = 0$, then what is the determinant of

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{pmatrix} ?$$

(A) $p^2 + 6q$ (B) 1 (C) p (D) 0

ROUGH WORK

10. Suppose the rank of

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 3 \\ a & b & b & 1 \end{pmatrix}$$

is 2 for some real numbers a and b. What is the value of b ?

(A) $\frac{1}{3}$ (B) 3 (C) 1 (D) $\frac{1}{2}$

11. What is the infinite sum $x + 2x^2 + 3x^3 + \cdots$ for $|x| < 1$?

(A) $\frac{x}{1-x^2}$ (B) $\frac{x}{(1+x)^2}$ (C) $\frac{1}{(1-x)^2}$ (D) $\frac{x}{(1-x)^2}$

12. A function $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = f(-x)$ for all x, has left derivative 5 at $x = 0$. Then, the right derivative of f at $x = 0$

(A) exists and equals -5.

(B) may or may not exist.

(C) does not exist.

(D) exists and equals 5.

ROUGH WORK

13. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{k=0}^{\lfloor |x| \rfloor} \frac{1}{2^k} \, ,$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. Then,

(A) f is a left continuous function which is not right continuous.

(B) f is a continuous function.

(C) f is neither left continuous nor right continuous.

(D) f is a right continuous function which is not left continuous.

14. A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 0. Suppose that

$$f(x) < f(0) < f(y) \text{ for all } x < 0 < y.$$

Then, what is the set of all possible values that $f'(0)$ can take?

(A) $\{0\}$ (B) $[0, \infty)$ (C) $(0, \infty)$ (D) $\mathbb{R}$

15. Suppose that the events A, B, and C are pairwise independent such that each of them occurs with probability p. Assume that all three of them cannot occur simultaneously. What is $P(A \cup B \cup C)$?

(A) $1 - (1 - p)^3$ (B) p^3 (C) $3p(1 - p)$ (D) $3p$

ROUGH WORK

16. Let $X_1, X_2, \ldots$ be a sequence of independent random variables distributed exponentially with mean 1. Suppose that N is a random variable, independent of the X_i-s, that has a Poisson distribution with mean $\lambda > 0$. What is the expected value of $X_1 + X_2 + \cdots + X_{N^2}$?

(A) N^2 (B) $\lambda + \lambda^2$ (C) λ^2 (D) $1/\lambda^2$

17. For which value of k does the inequality

$$\mathrm{Var}(X - Y) \geq |\sigma_X - \sigma_Y|^k$$

hold for all choices of random variables X and Y with finite variances σ_X^2 and σ_Y^2, respectively?

(A) 3 (B) 4 (C) 1 (D) 2

18. Consider a diagnostic test for a disease that gives the correct result with probability 0.9, independently of whether the subject being diagnosed has the disease or not. Suppose that 20% of the population has the disease. For a particular subject, given that the test indicates presence of the disease, what is the conditional probability that the subject has the disease?

(A) $\frac{9}{10}$ (B) $\frac{9}{13}$ (C) $\frac{1}{2}$ (D) $\frac{1}{5}$

ROUGH WORK

19. Let X_1, X_2, X_3, X_4 be independent standard normal random variables. Then what is the distribution of

$$\frac{(X_1 + X_2 + X_3 + X_4)^2}{(X_1 - X_2 + X_3 - X_4)^2}\,?$$

(A) $F_{4,4}$ (B) χ^2_4 (C) $F_{1,1}$ (D) $F_{2,2}$

20. Suppose that $X \sim$ Uniform $(0, 1)$ and $Y \sim$ Bernoulli $(1/4)$, independently of each other. Let $Z = X + Y$. Then,

(A) the distribution of Z is symmetric about 1.

(B) Z has a probability density function.

(C) $E(Z) = 5/4$.

(D) $P(Z \leq 1) = 1/4$.

21. Assume that the events A and B are such that $A \subset B$ and $P(B) > 0$. For every event E, define $\mathbf{1}_E$ to be the random variable which takes value 1 or 0 depending on whether E occurs or not, respectively. Then, $\mathrm{Cov}(\mathbf{1}_A, \mathbf{1}_B)$ equals

(A) $P(A|B)P(B^c)$.

(B) $P(A)P(B^c)$.

(C) $P(A)P(B)$.

(D) $P(B)P(A^c)$.

ROUGH WORK

22. Suppose that X is chosen uniformly from $\{1, 2, \ldots, 100\}$, and given $X = x$, Y is chosen uniformly from $\{1, 2, \ldots, x\}$. What is $P(Y = 30)$?

(A) $\frac{1}{100}$

(B) $\frac{70}{100} \times \frac{1}{30}$

(C) $\frac{1}{100} \times \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30}\right)$

(D) $\frac{1}{100} \times \left(\frac{1}{30} + \cdots + \frac{1}{100}\right)$

23. Three numbers are chosen at random from $\{1, 2, ..., 10\}$ *without* replacement. What is the probability that the minimum of the chosen numbers is 3 or their maximum is 7 ?

(A) $\frac{13}{40}$ (B) $\frac{19}{60}$ (C) $\frac{3}{10}$ (D) $\frac{11}{40}$

24. Let X be a random variable taking values 1, 2, and 3 with respective probabilities $\frac{3}{7}$, $\frac{2}{7}$, and $\frac{2}{7}$. Find the set of values of α for which $E|X - \alpha|$ is minimized.

(A) $[1, 2]$ (B) $\{\frac{13}{7}\}$ (C) $\{2\}$ (D) $\{1\}$

ROUGH WORK

25. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed normal random variables with mean θ and variance θ, where $\theta > 0$. Then,

(A) $\sum_{i=1}^{n} X_i$ is sufficient for θ.

(B) $\sum_{i=1}^{n} X_i^2$ is sufficient for θ.

(C) $\sum_{i=1}^{n} \left(X_i + X_i^2\right)$ is sufficient for θ.

(D) neither $\sum_{i=1}^{n} X_i$ nor $\sum_{i=1}^{n} X_i^2$ is sufficient for θ.

26. Let X be normally distributed with mean μ and variance $\sigma^2 > 0$. What is the variance of e^X ?

(A) $e^{\mu+\sigma^2}(e^{\sigma^2} - 1)$

(B) $e^{2\mu+2\sigma^2}(e^{2\sigma^2} - 1)$

(C) $e^{2\mu+2\sigma^2}(e^{\sigma^2} - 1)$

(D) $e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

27. Consider data from a single replication of a 2^4 factorial experiment. Assume that all interactions of orders higher than two are negligible. What is the error degrees of freedom in the ANOVA table?

(A) 16 (B) 2 (C) 12 (D) 5

ROUGH WORK

28. For $n \geq 2$, let $X_1, \ldots, X_n$ be independent and identically distributed normal random variables with mean 0 and variance σ^2. Consider the most powerful test for the null hypothesis $H_0 : \sigma = 2$ against the alternative $H_1 : \sigma = 1$, at significance level α for some $0 < \alpha < 1$. What is the power of this test?

    In what follows, F_m denotes the cumulative distribution function of a χ^2 random variable with m degrees of freedom, and $\chi^2_{m,\beta}$ satisfies the equation $F_m(\chi^2_{m,\beta}) = 1 - \beta$, for every $m \geq 1$ and $\beta \in (0, 1)$.

    (A) $1 - F_{n-1}(\frac{1}{4}\chi^2_{n-1,\alpha})$

    (B) $F_n(4\chi^2_{n,1-\alpha})$

    (C) $F_{n-1}(4\chi^2_{n-1,1-\alpha})$

    (D) $1 - F_n(\frac{1}{4}\chi^2_{n,\alpha})$

29. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed random variables from a distribution which depends on a parameter θ. For which of the following distributions is the maximum likelihood estimator of θ not unbiased?

    (A) Normal with mean θ and variance 1 where $\theta \in \mathbb{R}$

    (B) Bernoulli with parameter $\theta \in [0, 1]$

    (C) Poisson with mean $\theta \geq 0$

    (D) Normal with mean 0 and variance θ^2 where $\theta > 0$

30. Suppose that $X_1, \ldots, X_n$ are independent observations from a uniform distribution on $[0, \theta]$, where $\theta \in \mathbb{N}$. Then, the maximum likelihood estimator of θ

    (A) may not exist in some cases.

    (B) is $\lfloor X_{(n)} \rfloor$, where $\lfloor a \rfloor$ = greatest integer $\leq a$.

    (C) is $\lceil X_{(n)} \rceil$, where $\lceil a \rceil$ = smallest integer $\geq a$.

    (D) is $\lfloor X_{(n)} \rfloor$, if $\big(X_{(n)} - \lfloor X_{(n)} \rfloor\big) < 1/2$, otherwise it is $\lceil X_{(n)} \rceil$.