PSB-MOCK

April 23, 2024

Group A

1. Consider a sequence $\{a_n : n \geq 1\}$ of real numbers , where

$$a_{n+1} = \frac{3}{2}a_n - \frac{1}{2}a_{n-1} \ \forall n > 1$$

(a) Show that the sequence converges .

(b) Following part (a) , what is the limiting value of the sequence ?

2. For any two events A and B , show that
$(P(A \cap B))^2 + (P(A \cap B^C))^2 + (P(A^C \cap B))^2 + (P(A^C \cap B^C))^2 \geq \frac{1}{4}$.

3. Find all continuously differentiable functions f from the real line to the real line satisfying

$$(f(x))^2 = \int_0^x [f(t)^2 + f'(t)^2]dt + 2016,$$

for all real x.

Group B

4. Let X be a non-negative random variable such that

$$E(\sum_{n=1}^{\infty} X^n) < \infty....(*).$$

(a) Show that X can not be uniformly distributed over (0,1).

(b) Show that $P(X \geq 1) = 0$.

(c) Give example of a continuous random variable that satisfies (*).

5. Suppose X_1, X_2 and X_3 are positive valued i.i.d. non-degenerate random variables with finite variances.
(a) Define $Y = X_1 X_2$ and $Z = X_2 X_3$. Prove that $0 < \rho < \frac{1}{2}$, where ρ is the correlation coefficient between Y and Z.
(b) Prove that $E\left[\frac{X_1+X_2}{\sqrt{X_1^2+X_2^2+X_3^2}}\right] < \frac{2}{\sqrt{3}}$.

6. Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables having distribution function F_θ. Suppose there exists a positive integer m such that $g(X_1, X_2, ..., X_m)$ is unbiased for θ and $E[g(X_1, X_2, .., X_m)^2] < \infty$.
(a) Propose an unbiased estimator of θ which involves the whole sample of size n.
(b) Find the variance of the above estimator.
(c) Prove that if there exists an umvue of θ for $n > m$, the variance of the umvue must converge to 0 an $n \to \infty$

7. Based on 58 pairs of (x,y), it was observed that the sample means and the slope of the least squares regression line of y on x are :

$$\bar{x} = 16, \bar{y} = 14, b_{yx} = 1.2$$

(a) Subsequently, it was found that a pair of (x,y) was not recorded, and it was (16,14). Obtain the least square regression line of y on x based on all 59 pairs of data.
(b)Now we wish to include (16,12) in the dataset. Find the slope of the least squares regression line of y on x based on the 60 pairs of data.

8. Let P be an $n \times n$ non singular matrix such that $I + P + P^2 + ... + P^n$ is a null matrix.
(a) Find the form of the inverse of P
(b) What can be said about the eigen values of P ?

9. Suppose that a sample of size n is drawn using SRSWR from a finite population of N units, where $N > n$ and $N \geq 3$. Let $\bar{y}$ denote the sample mean. Now, let us assume that one variate value y_1 corresponding to one unit is known and consequently a simple random sample of size n without replacement are now drawn from the remaining N-1 units; denote the sample mean of the study variables corresponding to the n selected units as $\bar{y}_0$. Consider the following two estimators of the population total as given by

$$t_1 = N\bar{y}$$

$$t_2 = (N-1)\bar{y}_0 + y_1$$

Prove that
(a) t_2 is unbiased for the population total
(b) $Var(t_1) \geq Var(t_2)$