PSB-MOCK

April 15, 2024

Group A

1. Let $T = \{(x, y, z) \in \mathbb{R}^3 : 5x^2 + y^2 + z^2 + 4xy + 2xz = 0\}$.Show that T is a subspace of $\mathbb{R}^3$,and find the basis of T.

2. Consider all permutations of the integers 1,2,....,100. In how many of those permutations will the 25th number be the minimum of the first 25 numbers and the 50th number will be the minimum of the first 50 numbers?

3. (a) Draw the graph of $f(x) = x^3 - 3x$.
(b) From the graph in (a), or otherwise, find the set of real numbers q such that $x^3 - 3x + q = 0$ has three distinct real roots.

Group B

4. (a) Suppose that there are 5 pairs of shoes in a closet and four shoes are taken out at random . What is the probability that , among the four which are taken out , there is at least one complete pair ?

(b)Two identical independent components having lifetime T_1 and T_2 , respectively, are connected in a parallel system . Suppose that the distributions of both T_1 and T_2 are exponential initially with mean $\frac{1}{\lambda}$. But whenever one component fails, the lifetime distribution of the remaining component changes to exponential with mean $\frac{1}{\alpha}$.If T denotes the overall lifetime of the system, find $P(T \geq t)$ for any $t > 0$.

5. Suppose $\{(X_1, Y_1), ..., (X_n, Y_n)\}$ is a random sample from a bivariate normal distribution with $E(X_i) = E(Y_i) = 0$,$Var(X_i) = Var(Y_i) = 1$ and unknown $Corr(X_i, Y_i) = \rho \in (-1, 1)$, for all $i = 1, 2, .., n$. Define $W_n = \frac{1}{n} \sum\limits_{i=1}^{n} X_i Y_i$.
(a) Is W_n an unbiased estimator of ρ? Justify your answer.
(b) For large n, obtain an approximate level $(1 - \alpha)$ two sided confidence interval for ρ,where $0 < \alpha < 1$.

6. Let $p \in (0, \frac{1}{2})$ be unknown. There are two coins with probabilities of heads p and 1-p. One of the two coins is picked at random. This coin is tossed 10 times independently. Let $X_i = 1$ if the i^{th} toss results in a head, and $X_i = 0$ otherwise, for i=1,2,..,10.

(a) Are $X_1, X_2, ..., X_{10}$ identically distributed? Are they independent ? Justify.

(b) Find the Maximum Likelihood Estimator of p based on $X_1, X_2, ..., X_{10}$.

7. Consider a randomised block design with b treatments , arranged in b blocks, each of size b-1, such that every treatment except the i^{th} treatment occurs exactly once in the i^{th} block, i=1,..,b. Let τ_i denote the effect of the i-th treatment for each i.

(a) Show that every pairwise contrast of the treatment effects is estimable.

(b) Find the Best Linear Unbiased Estimator (BLUE) of $\tau_i - \tau_j$, for $i \neq j$, $i, j \in \{1, 2, ..., b\}$.Also find its Variance.

8. In a linear model $Y = A\beta + \epsilon$, $E(\epsilon) = 0, D(\epsilon) = \sigma^2 I$, $\underset{\sim}{\beta}' = (\beta_1, \beta_2, ..., \beta_p)$. Let $C_1, C_2, ..., C_p$ denote the column vectors of the matrix A. Prove that

(a) β_1 is estimable if and only if C_1 does not belong to the column space spanned by $C_2, ..., C_p$.

(b) $\lambda_1\beta_1 + \lambda_2\beta_2, \lambda_1 \neq 0, \lambda_2 \neq 0$, is estimable if and only if C_1 does not belong to the vector space spanned by $\lambda_2 C_2 - \lambda_1 C_1, C_3, ..., C_p$.

9. Let $X_1, X_2, ..., X_n$ be independent and identically distributed having the discrete uniform distribution on $\{1, 2, ..., \theta\}$, where $\theta \in$A={2,3,4,5,...}.

(a) Given $\theta_0 \in A$,and $0 < \alpha < 1$,find a $level - \alpha$ likelihood ratio test for testing

$$H_0 : \theta \leq \theta_0 \quad against \qquad H_1 : \theta \geq \theta_0$$

(b) Show that the largest order statistic is not complete.