# Isi Mock 3 PSB

April 20, 2024

## Group A

1. Suppose $V$ is the space of all $n \times n$ matrices with real elements. Define $T : V \to V$ by $T(A) = AB - BA$, $A \in V$, where $B \in V$ is a fixed matrix. Show that for any $B \in V$ (a) T is linear; (b) T is not one-one; (c) T is not onto.

2. Let $\sum\limits_{n=1}^{\infty} b_n x^n$ be a power series with radius of convergence $R$. Define the sequence $y_n$ as follows:

$$y_0 = 0; y_n = \theta y_{n-1} + b_n;\ for\ n \geq 1,\ \theta \neq 0.$$

Show that the sequence $\left\{ \frac{y_n - b_n}{\theta^n} : n \geq 1 \right\}$ converges if $|\theta| > \frac{1}{R}$.

3. Consider a random arrangement of 20 boys and 16 girls in a line. Let $X$ be the number of boys with girls on both sides, and $Y$ be the number of girls with boys on both sides. Find $E(X + Y)$.

## Group B

1. Suppose two teams play a series of games, each producing a winner and a loser, until one team has won two more games than the other. Let $G$ be the total number of games played. Assume each team has a chance of 0.5 to win each game, independent of the results of the previous games. $(a)$ Find the probability distribution of G. $(b)$ Find the expected value of G.

2. Let $X_1, X_2, .., X_n$ be *iid* random variables with one of two probability density functions $f(x|\theta), \theta = 0, 1$. If $\theta = 0$, then,

$$f(x|\theta) = \begin{cases} 1 & if\ 0 < x < 1, \\ 0 & otherwise, \end{cases}$$

while if $\theta = 1$, then,

$$f(x|\theta) = \begin{cases} \frac{1}{2\sqrt{x}} & if\ 0 < x < 1, \\ 0 & otherwise. \end{cases}$$

$a)$ Find the maximum likelihood estimator $\hat{\theta_n}$ of $\theta$.
$b)$ Show that $\lim\limits_{n\to\infty} P_{\theta=0}\left(\hat{\theta_n} = 0\right) = 1$.

3. Let $X_1 \sim Geo\,(p_1)$ and $X_2 \sim Geo\,(p_2)$ be independent random variables. We are interested in testing the null hypothesis $H_0 : p_1 = p_2$ against the alternative $H_1 : p_1 < p_2$. Intuitively, it is clear that we should reject if $X_1$is large, but unfortunately we cannot compute a cutoff because the distribution of $X_1$ under $H_0$ depends on the unknown (common) value of $p_1$and $p_2$.
$a)$ Let $Y = X_1 + X_2$. Find the conditional distribution of $X_1|Y = y$ when $p_1 = p_2$.
$b)$ Based on the result obtained in $a)$, derive a level 0.05 test for $H_0$ against $H_1$that rejects $H_0$ when $X_1$ is large.

4. Suppose die A has 4 red faces and 2 green faces while die B has 2 red faces and 4 green faces. Assume that both the dice are unbiased. An experiment is started with the toss of an unbiased coin. If the toss results in a Head, then die A is rolled repeatedly while if the toss of the coin results in a Tail, then die B is rolled repeatedly. For $k = 1, 2, 3, \cdots$, define

$$X_k = \begin{cases} 1 & if\ k^{th}\ role\ of\ the\ die\ results\ in\ red\ face \\ 0 & otherwise \end{cases}$$

(a) Find the probability mass function of $X_k$. (b) Calculate $\rho(X_1, X_7)$.

5. Let $X_1, X_2, Z$ be independent RVs such that $X_i \sim N(0,1)$,i=1,2 and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. Let, $Y_i = Z|X_i|$,i=1,2. Show that $Y_1, Y_2$ are $N(0,1)$ variables. Are $Y_1, Y_2$ independent?

6. A straight line regression $E(y) = \alpha + \beta x$ is to be fitted using four observations. Assume$Var(y|x) = \sigma^2$ for all $x$. The values of $x$ at which observations are to be made lie in the closed interval $[-1, 1]$. The following choices of the values of $x$ where observations are to be made are available: (a) two observations each at $x = -1$ and $x = 1$, (b) one observation each at $x = -1$ and $x = 1$ and two observations at $x = 0$, (c) one observation each at $x = -1, -\frac{1}{2}, \frac{1}{2}, 1$. If the interest is to estimate the slope with least variance, which of the above strategies would you choose and why?