Quiz-2 Answer Key

Q1)

Here $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

$$q_1' = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$r_{11} = \left\|q_1'\right\| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$$

$$q_1 = \frac{1}{\left\|q_1'\right\|} \cdot q_1' = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$r_{12} = q_1^T \cdot a_2 = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} = 3$$

$$q_2' = a_2 - r_{12} \cdot q_1 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -2.5 \\ 2.5 \\ 2.5 \\ -2.5 \end{bmatrix}$$

$$r_{22} = \left\|q_2'\right\| = \sqrt{(-2.5)^2 + 2.5^2 + 2.5^2 + (-2.5)^2} = \sqrt{25} = 5$$

$$q_2 = \frac{1}{\left\|q_2'\right\|} \cdot q_2' = \frac{1}{5} \cdot \begin{bmatrix} -2.5 \\ 2.5 \\ 2.5 \\ -2.5 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$$

$$r_{13} = q_1^T \cdot a_3 = [\ 0.5\ \ 0.5\ \ 0.5\ \ 0.5\ ] \times \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} = 2$$

$$r_{23} = q_2^T \cdot a_3 = [\ -0.5\ \ 0.5\ \ 0.5\ \ -0.5\ ] \times \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} = -2$$

$$q_3' = a_3 - r_{13} \cdot q_1 - r_{23} \cdot q_2 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} -2 \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} +2 \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$r_{33} = \left\|q_3'\right\| = \sqrt{2^2 + (-2)^2 + 2^2 + (-2)^2} = \sqrt{16} = 4$$

$$q_3 = \frac{1}{\left\|q_3'\right\|} \cdot q_3' = \frac{1}{4} \cdot \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$$

$$Q = \left[q_1, q_2, q_3\right] = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 \end{bmatrix}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

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checking $Q \times R = A$?

$$Q \times R = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

and $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

Q2)

2) $w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$

We apply Gram-Schmidt orthogonalization as follows. The first step is to define $u_1 = w_1$. Before defining u_2, we must compute

$$u_1^T w_2 = w_1^T w_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = 3+4 = 7$$

$$u_1^T u_1 = w_1^T w_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1+1 = 2$$

next we define

$$u_2 = w_2 - \frac{u_1^T w_2}{u_1^T u_1} u_1$$

$$= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \frac{7}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 7/2 \\ 7/2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 1/2 \\ 2 \end{bmatrix}$$

By G-S orthogonalization, $\{u_1, u_2\}$ is an orthogonal basis for the span of the vectors w_1 and w_2.

Q3)

3. $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$ (Prithwi)

Ch. polynomial of A is :

$x^3 - \text{trac}(A)\cdot x^2 + \left[\text{co-factor of } 1 + \text{co-factor of } 0 + \text{co-factor of } 2\right]\cdot x - \det A.$

$= x^3 - x^2 - 5x - 3$

To find the Eigen values, we consider that Ch. polynomial of $A = 0$.

$x^3 - x^2 - 5x - 3 = 0.$

$(x+1)(x^2 - 2x - 3) = 0$

$x = -1, 3, -1.$ are the eigenvalues of A.

The algebraic multiplicity of -1 is 2 and the algebraic multiplicity of 3 is 1.

The geometric multiplicity of -1 is 2.

The geometric multiplicity of 3 is 1.

The eigenvectors of -1 are $\longrightarrow \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ $\longrightarrow AX = -1\cdot X \Rightarrow AX + X = 0.$

$\Rightarrow (A + I)\cdot X = 0.$

$\Rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 3 & 0 & 3 \\ 2 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$X = x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$

The eigenvectors of 3 is $\rightarrow$

$\rightarrow AX = 3X$

$\Rightarrow (A - 3I)X = 0.$

$\Rightarrow \begin{pmatrix} -2 & 0 & 2 \\ 3 & -4 & 3 \\ 2 & 0 & -2 \end{pmatrix} X = 0 \Rightarrow X = x_1 \begin{pmatrix} 1 \\ 3/2 \\ 1 \end{pmatrix}$

Q4)

(4) $f_1(x) = 1, \quad f_2(x) = \sin x \quad f_3(x) = \cos x.$

$$\langle f_1, f_2\rangle = \int_{-\pi}^{\pi} \sin x \, dx = 0 = \langle f_2, f_1\rangle \quad 0.5$$

$$\langle f_2, f_3\rangle = \int_{-\pi}^{\pi} \sin x \cos x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin 2x \, dx = 0 = \langle f_3, f_2\rangle$$

0.5

$$\langle f_3, f_1\rangle = \int_{-\pi}^{\pi} \cos x \, dx = 0 = \langle f_1, f_3\rangle. \quad 0.5$$

$$\langle f_1, f_1\rangle = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

$$\|f_1\| = \sqrt{\langle f_1, f_1\rangle} = \sqrt{2\pi} \quad 0.5 \qquad \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\|f_2\| = \sqrt{\int_{-\pi}^{\pi} \sin^2 x \, dx} = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2}\left(1 - \cos 2x\right) dx} \quad 0.5$$

$$= \sqrt{\frac{1}{2}\left(2\pi\right)} = \sqrt{\pi}$$

$$\|f_3\| = \sqrt{\int_{-\pi}^{\pi} \cos^2 x \, dx} = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2}\left(1 + \cos 2x\right) dx} \quad 0.5$$

$$= \sqrt{\frac{1}{2}\left(2\pi\right)} = \sqrt{\pi}$$

0.5 Thus, defining $f_1' = \frac{1}{\sqrt{2\pi}}, \quad f_2' = \frac{\sin x}{\sqrt{\pi}} \quad f_3' = \frac{\cos x}{\sqrt{\pi}}$

$$= \sqrt{\frac{1}{2}(2\pi)} = \sqrt{\pi}$$

$$\|f_3\| = \sqrt{\int_{-\pi}^{\pi} \cos^2 x \, dx} = \sqrt{\int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos 2x)\, dx} \qquad 0.5$$

$$= \sqrt{\frac{1}{2}(2\pi)} = \sqrt{\pi}$$

0.5 Thus, defining $f_1' = \frac{1}{\sqrt{2\pi}}$, $f_2' = \frac{\sin x}{\sqrt{\pi}}$ $f_3' = \frac{\cos x}{\sqrt{\pi}}$

gives an orthonormal set of functions

$\{f_1', f_2', f_3'\}$. 0.5

$\langle f_i, f_j \rangle = \langle f_j, f_i \rangle$ because functions commute

1 under multiplication. It should be mentioned.

Otherwise explicit calculation should be given.

Q5)

→ 5 (a) For orthogonal Matrix, $\underset{\text{Method I}}{\boxed{AA^T = 1}}$ or $\underset{\text{Method III}}{\boxed{A^T = A^{-1}}}$ or $\underset{\text{Method II}}{\boxed{\det(A) = \pm 1}}$

M I

$$AA^T = \begin{pmatrix} 1/3 & 1/2 & 1/5 \\ 1/3 & -1/2 & 1/5 \\ -1/3 & 0 & 2/5 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & -1/3 \\ 1/2 & -1/2 & 0 \\ 1/5 & 1/5 & 2/5 \end{pmatrix}$$

$$= \begin{pmatrix} 361/900 & -89/900 & -7/225 \\ -89/900 & 361/900 & -7/225 \\ -7/225 & -7/225 & 61/225 \end{pmatrix} \neq \mathbb{1}$$

Not orthogonal

M II

$$\det(A) = \frac{1}{3}\left(\frac{-1}{\cancel{2}} \times \frac{\cancel{2}}{5}\right) - \frac{1}{2}\left(\frac{1}{3} \times \frac{2}{5} - \frac{1}{5} \times \left(\frac{-1}{3}\right)\right) + \frac{1}{5}\left(-\left(\frac{-1}{2}\right)\left(\frac{-1}{3}\right)\right)$$

$$= -\frac{1}{15} - \frac{1}{2} \times \frac{3}{15} - \frac{1}{30}$$

$$= -\frac{2}{30} - \frac{3}{30} - \frac{1}{30} = -\frac{6}{30} = -1/5 \neq \pm 1$$

M III

$$A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 5/6 & 5/6 & 5/3 \end{pmatrix} \neq A^T \implies A \text{ is not orthogonal.}$$

In any method, deduct 1 marks if numerical mistake is made.

(b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be orthogonal

then $\det(A) = \pm 1$. (1)

(b) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be orthogonal.

then $\det(A) = \pm 1$. 1

Also, $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ 1

Case I: $\det(A) = +1$ then $d = a$ and $c = -b$

0.5 Thus, $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

Case II: $\det(A) = -1$, then,

0.5 $\begin{bmatrix} -d & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$\Rightarrow \quad d = -a \qquad b = c$

Thus, $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$