What about systems $AX = Y$? non-homogenous systems.

→ While $AX = 0$ always has a trivial solⁿ, systems $AX = Y$ for $Y \neq 0$ need not have a solⁿ.

How to find solutions for $AX = Y$, $Y \neq 0$?

→ Form the augmented matrix A' of the system $AX = Y$. A' is the $m \times (n+1)$ matrix whose 1^{st} n columns are the columns of A and whose last column is Y.

$$A'_{ij} = A_{ij} \quad \forall\, j \leq n$$

$$A'_{i(n+1)} = y_i \, .$$

$$\begin{tikzcd}
A \arrow[d, "\text{sequence of elementary row operations (same for both } A \,\&\, A')"] & A' \arrow[d] & Y \arrow[d] \\
R & R' & Z
\end{tikzcd}$$

Then, $R' = \left[\begin{array}{c|c} R & Z_{m \times 1} \end{array} \right]$, $Z = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix}$

$AX = Y$ and $RX = Z$ are equivalent and hence have same solutions.

Whether $RX = Z$ has any solutions? To determine all the sol^{n}s if any exist.

If R has r non-zero rows, with leading non-zero entry of row i occurring in column k_i, $i = 1, \ldots, r$, then the first r eq^{n}s of $RX = Z$ effectively express $x_{k_1}, \ldots, x_{k_r}$ in the terms of the $(n-r)$ remaining x_j and the scalars $z_1, \ldots, z_r$. The last $(m-r)$ eq^{n}s are:

$$\begin{aligned} 0 &= z_{r+1} \\ &\vdots \\ 0 &= z_m \end{aligned}$$

and accordingly the cond^n for the system to have a sol^n is $z_i = 0$ for $i > r$. If this cond^n is satisfied, all sol^{n}s to the system are found as in the homogenous case, by assigning arbitrary values to $(n-r)$ of the x_j and then computing x_{k_i} from the i^{th} eq^n.

<u>Example</u>: F be a field of $\mathbb{Q}$ and

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}. \quad \text{Solve for } AX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

We perform a sequence of row operations on the augmented matrix A' which row-reduces A:

$$\begin{bmatrix} 1 & -2 & 1 & y_1 \\ 2 & 1 & 1 & y_2 \\ 0 & 5 & -1 & y_3 \end{bmatrix} \xrightarrow{2-2\otimes 1} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & (y_2-2y_1) \\ 0 & 5 & -1 & y_3 \end{bmatrix}$$

$$\xrightarrow{3-1} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 5 & -1 & (y_2-2y_1) \\ 0 & 0 & 0 & (y_3-y_2+2y_1) \end{bmatrix}$$

$$\xrightarrow{2 \cdot 1/5} \begin{bmatrix} 1 & -2 & 1 & y_1 \\ 0 & 1 & -1/5 & 1/5(y_2-y_1) \\ 0 & 0 & 0 & (y_3-y_2+2y_1) \end{bmatrix}$$

$$\xrightarrow{1+2\otimes 2} \begin{bmatrix} 1 & 0 & 3/5 & 1/5(y_1+2y_2) \\ 0 & 1 & -1/5 & 1/5(y_2-2y_1) \\ 0 & 0 & 0 & (y_3-y_2+2y_1) \end{bmatrix}$$

Condⁿ that $AX=Y$ has a solⁿ is

$$2y_1-y_2+y_3=0$$

and if scalars y_i satisfy this condⁿ, all solⁿs are obtained by assigning a value c to x_3 & then computing

$$x_1 = -\frac{3}{5}c + \frac{1}{5}(y_1+2y_2)$$

$$x_2 = \frac{1}{5}c + \frac{1}{5}(y_2-2y_1)$$

Lecture 3:
Matrix multiplication

Suppose B is an $n\times p$ matrix over a field F with rows $\beta_1, \dots, \beta_n$ and that from B we construct a matrix C with rows $\gamma_1, \dots, \gamma_m$ by forming certain linear combinations:

$$\gamma_i = A_{i1}\beta_1 + A_{i2}\beta_2 + \dots + A_{in}\beta_n \quad \text{——} (1.4)$$

The rows of C are determined by the ~~$m\times n$~~ mn scalars A_{ij} which are themselves the entries of an $m\times n$ matrix A.
From (1.4),

$$(C_{i1} \; \dots \; C_{ip}) = \sum_{r=1}^{n} (A_{ir}B_{r1} \dots A_{ir}B_{rp}),$$

entries of C: $C_{ij} = \sum_{r=1}^{n} A_{ir}B_{rj}$.

Def^n: Let A be an $m\times n$ matrix over the field F and let B be an $n\times p$ matrix over F. The product AB is the $m\times p$ matrix C whose i,j entry is

$$C_{ij} = \sum_{r=1}^{n} A_{ir}B_{rj} .$$

Example: (a) Consider $\begin{bmatrix} 5 & -1 & 2 \\ 0 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{bmatrix}$

Here, $\gamma_1 = (5 \; -1 \; 2) = 1\cdot(5 \; -1 \; 2) + 0\cdot(15 \; 4 \; 8)$

$\gamma_2 = (0 \; 7 \; 2) = -3\cdot(5 \; -1 \; 2) + 1\cdot(15 \; 4 \; 8)$.

(b) $\begin{bmatrix} 0 & 6 & 1 \\ 9 & 12 & -8 \\ 12 & 62 & -3 \\ 3 & 8 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 5 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 1 \\ 3 & 8 & -2 \end{bmatrix}$

$\gamma_3 = 5\cdot(0 \; 6 \; 1) + 4\cdot(3 \; 8 \; -2)$

B is an $n \times p$ matrix, $B = [B_1, \ldots, B_p]$,

$$B_j = \begin{bmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{bmatrix}, \quad 1 \leq j \leq p.$$ B_j is $1 \times n$ matrix. column matrix.

Check that $AB = [AB_1, \ldots, AB_p]$.

<u>Thm</u>: If A, B, C are matrices over the field F such that the products BC and $A(BC)$ are defined, then so are the products AB, $(AB)C$, and

$$(AB)C = A(BC).$$

Proof: ———

Remark: For a square matrix A, A^n is well-defined

$$A^p A^q A^r = A^s A^t A^u \quad \text{for all } p+q+r = s+t+u = n.$$

$A(BC) = (AB)C \longrightarrow$ linear combinations of linear combinations of the rows of C are again linear combinations of the rows of C.

If $B \xrightarrow[\text{row operations}]{\text{elementary}} C$, then each row of C is a linear combination of the rows of B, and so $\exists$ a matrix A s.t. $AB = C$.

(There can be many such A's in general.)

Def^n: An $m\times m$ matrix is said to be an elementary matrix if it can be obtained from the $m\times m$ identity matrix $\mathbb{1}_{m\times n}$ by means of a single elementary row operation.

Example: 2×2 elementary matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix},$$

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \text{ for } c \neq 0, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \text{ for } c \neq 0.$$

Thm: Let e be an elementary row operation and ~~let~~ E be the $m\times n$ elementary matrix $E = e(\mathbb{1})$. Then, for every $m\times n$ matrix A,

$$e(A) = EA.$$

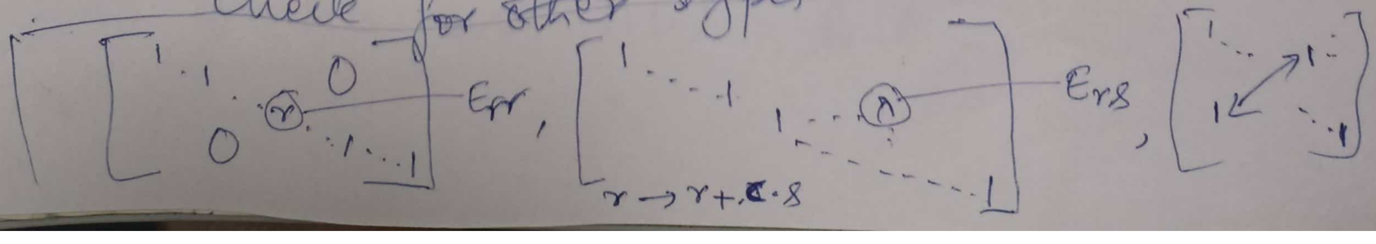
Proof: Type (ii)

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ \delta_{rk} + c\,\delta_{sk}, & i = r \end{cases}$$

(To replace row r with row r + c·row s)

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A_{ik}, & i \neq r \\ A_{rj} + cA_{sj}, & i = r. \end{cases}$$

Check for other types.



Corollary: Let A and B be $m \times n$ matrices over the field F. Then B is row-equivalent to A if and only if $B = PA$, where P is a product of $m \times n$ elementary operation.

1.1 Invertible matrices.

Def^n: Let A be an $n \times n$ matrix over the field F. An $n \times n$ matrix B such that $BA = \mathbb{1}$ is called a <u>left inverse</u> of A; an $n \times n$ matrix B such that $AB = \mathbb{1}$ is called a <u>right inverse</u> of A. If $AB = BA = \mathbb{1}$ then B is called a two-sided inverse of A and A is said to be invertible.

<u>Lemma</u>: If A has a left inverse B and a right inverse C, then $B = C$.

Proof: $B = B\mathbb{1} = BAC = \mathbb{1}C = C$.

<u>Thm</u>: Let A and B be $n \times n$ matrices over the field F.

(i) If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

(ii) If both A and B are invertible, so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

Corollary: A product of invertible matrix is invertible.

Theorem: An elementary matrix is invertible.

Proof: ______

Thm. If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible.

(ii) A is row-equivalent to $\mathbb{1}_{n \times n}$.

(iii) A is product of elementary operations.

Proof: ______

Thm: For an $n \times n$ matrix A, the following are equivalent.

(i) A is invertible.

(ii) The homogenous system $AX=0$ has only the trivial solⁿ.

(iii) The system of eqⁿs $AX=Y$ has a solⁿ X for each $n \times 1$ matrix Y.

Proof: ______

$$AX = Y$$
$$X^T A^T = Y^T$$

Column-equivalent

Column-reduced echelon matrix

~~Column~~ Elementary column operations: $A\underline{E}$

Corollary: A square matrix with either a left inverse or right inverse is invertible.

Proof: $A_{n\times n}$. Suppose, left inverse of A exists, $BA = \mathbb{1}$. Then, $AX = 0$ has only the trivial sol^{n}, because $X = \mathbb{1}X = B(AX)$. $\therefore$ A is invertible.

If A has a right inverse, $AC = \mathbb{1}$. Then C has a left inverse & is therefore invertible. It then follows $A = C^{-1}$, so A is invertible w/ inverse C.

$$\begin{tikzcd} A \arrow[d] \arrow[r, phantom, "\text{sequence of e.}"] & \mathbb{1} \arrow[d] \\ \mathbb{1} & A^{-1} \end{tikzcd}$$

$AX = Y$

$RX = Z$,

if $R = PA$

then $Z = PY$.

$\downarrow$

invertible matrix.

Lecture 4 Vector spaces

Def^n. A vector space (or linear space) consists of the following: Vector space over the field

1. a field F of scalars;
2. a set V of objects, called vectors;
3. a rule (or operation), called vector addition, which associates with each pair of vectors $\vec{\alpha}, \vec{\beta} \in V$ a vector $\vec{\alpha}+\vec{\beta} \in V$, called the sum of $\vec{\alpha}$ & $\vec{\beta}$, in such a way that
   (a) addition is commutative, $\vec{\alpha}+\vec{\beta} = \vec{\beta}+\vec{\alpha}$;
   (b) addition is associative, $\vec{\alpha}+(\vec{\beta}+\vec{\gamma}) = (\vec{\alpha}+\vec{\beta})+\vec{\gamma}$;
   (c) $\exists$ a unique vector $\vec{0} \in V$, called the zero vector, s.t. $\vec{\alpha}+\vec{0} = \vec{\alpha} \;\forall\, \vec{\alpha} \in V$;
   (d) for each $\vec{\alpha} \in V$ $\exists$ a unique vector $-\vec{\alpha} \in V$ s.t. $\vec{\alpha}+(-\vec{\alpha}) = 0$.
4. a rule, called scalar multiplication, which associates with each scalar $c \in F$ & $\vec{\alpha} \in V$ a vector $c\vec{\alpha} \in V$, called the product of c & $\vec{\alpha}$ s.t.
   (a) $1\vec{\alpha} = \vec{\alpha} \;\forall\, \vec{\alpha} \in V$;
   (b) $(c_1c_2)\vec{\alpha} = c_1(c_2\vec{\alpha})$;
   (c) $c(\vec{\alpha}+\vec{\beta}) = c\vec{\alpha}+c\vec{\beta}$;
   (d) $(c_1+c_2)\vec{\alpha} = c_1\vec{\alpha}+c_2\vec{\alpha}$.

A vector space is a composite object consisting of a field, a set of 'vectors', & two operations w/ certain properties.

Examples

① The n-tupple space, F^n. Let F be any field. Let V be the set of all n-tupples $\bar{\alpha} = (x_1, \dots, x_n)$ of scalars $x_i \in F$. If $\bar{\beta} = (y_1, y_2, \dots, y_n)$ w/ $y_i \in F$, the sum of $\bar{\alpha}$ & $\bar{\beta}$ is defined by

$$\bar{\alpha} + \bar{\beta} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \quad \text{—— (2.1)}$$

The product of a scalar c and vector $\bar{\alpha}$ is defined by

$$c\bar{\alpha} = (cx_1, \dots, cx_n) \quad \text{—— (2.2)}$$

② The space of $m \times n$ matrices, $F^{m\times n}$. Let F be any field and let m & n be +ve integers. Let $F^{m\times n}$ be the set of all $m\times n$ matrices over the field F.

$\bar{A}, \bar{B} \in F^{m\times n}$ then

$$(\bar{A} + \bar{B})_{ij} = A_{ij} + B_{ij}.$$

$c \in F$, $\bar{A} \in F^{m\times n}$ then

$$(cA)_{ij} = cA_{ij}.$$

③ The space of functions from a set to a field. F be field, S be any non-empty set. V be the set of all f^ns from the set S into F.

For $\bar{f}, \bar{g} \in V$,

$$(f+g)(s) = f(s) + g(s).$$

for $c \in F$, $f \in V$,

$$(cf)(s) = cf(s).$$

(4) The space of polynomial $f^{\underline{n}}s$ over a field F. Let F be a field and let V be the set of all $f^{\underline{n}}s$ f from F into F which have the rule of the form

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n,$$

where $c_0, c_1, \ldots, c_n \in F$ are independent of x.

(5) The field $\mathbb{C}$ of complex nos. $\longrightarrow$ a vector space over the field $\mathbb{R}$ of real nos.

From the def$^{\underline{n}}$ of a vector space, we observe: ~~that~~ If for a scalar $c \in F$ and vector $\bar{\alpha} \in V$ we have $c\bar{\alpha} = 0$ then either $c = 0$ or $\bar{\alpha} = \bar{0}$.

For any $\bar{\alpha} \in V$, $-\bar{\alpha} \in V$ since $\bar{0} \in V$ and

$$\bar{0} = 0\bar{\alpha} = (1-1)\alpha = 1\cdot\bar{\alpha} + (-1)\cdot\bar{\alpha} = \bar{\alpha} + (-1)\bar{\alpha},$$

$$(-1)\bar{\alpha} = -\bar{\alpha}.$$

For any $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4 \in V$, $(\bar{\alpha}_1 + \bar{\alpha}_2) + (\bar{\alpha}_3 + \bar{\alpha}_4)$
$= [\bar{\alpha}_2 + (\bar{\alpha}_1 + \bar{\alpha}_3)] + \bar{\alpha}_4.$