Tutorial Quiz-1 Answer Key

Below are solutions for Tutorial quiz section B.

Q1)

The Gauss-Jordan Method for Computing the Inverse

We can perform row operations on A and I simultaneously by constructing a "super-augmented matrix "

$$[A \quad | \quad I]$$

.

Then, perfoming elementary row operations to reduce A to I we will obtain:

$$[I \quad | \quad A^{-1}]$$

.

Basically, reduce A to identity matrix, and what we obtain on the right side is inverse.

Step 2

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$$\left[\begin{array}{cccc|cccc} 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_3}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{1}$$

$$\tag{2}$$

$$\overset{R_2 - 2R_1}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & -6 & 2 & 0 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{3}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & -6 & 2 & 0 & 1 & -2 & 0 \end{array}\right] \tag{5}$$

$$\tag{6}$$

$$\xrightarrow{R_3+R_2 \text{ and } R_4-3R_2} \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -9 & 5 & 0 & 1 & -2 & -3 \end{array}\right] \tag{7}$$

$$\tag{8}$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & -9 & 5 & 0 & 1 & -2 & -3 \end{array}\right] \tag{9}$$

$$\tag{10}$$

$$\xrightarrow{R_4+9R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 9/2 & 1 & -2 & 3/2 \end{array}\right] \tag{11}$$

$$\tag{12}$$

$$\xrightarrow{2R_4} \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{array}\right] \tag{13}$$

Step 3

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{array}\right] \xrightarrow{R_3+\frac{1}{2}R_4 \text{ and } R_2+R_4} \left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 9 & 2 & -4 & 4 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{array}\right] \tag{14}$$

$$\tag{15}$$

$$\xrightarrow{R_2-R_3 \text{ and } R_1-3R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & -15 & -3 & 7 & -6 \\ 0 & 1 & 0 & 0 & 4 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{array}\right] \tag{16}$$

$$\tag{17}$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -11 & -2 & 5 & -4 \\ 0 & 1 & 0 & 0 & 4 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 5 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 9 & 2 & -4 & 3 \end{array}\right] \tag{18}$$

Step 4

Inverse is:

$$\begin{bmatrix} -11 & -2 & 5 & -4 \\ 4 & 1 & -2 & 2 \\ 5 & 1 & -2 & 2 \\ 9 & 2 & -4 & 3 \end{bmatrix}$$

Q2)

Given a matrix A_1 we have that $A_1^T = A_1^T$ then the statement is true for $n = 1$.

Suppose the statement is true for some $n \geq 1$ then given $A_1, A_2, \ldots, A_n$ of the same size we have that $(A_1 + A_2 + \ldots + A_n)^T = A_1^T + A_2^T + \ldots + A_n^T$.

Let $A_1, A_2, \ldots, A_n, A_{n+1}$ be matrices of the same size. Then we have that $(A_1 + A_2 + \ldots + A_n + A_{n+1})^T = ((A_1 + A_2 + \ldots + A_n) + A_{n+1})^T$.

By Theorem 3.4 (b) we have that $((A_1 + A_2 + \ldots + A_n) + A_{n+1})^T = (A_1 + A_2 + \ldots + A_n)^T + A_{n+1}^T$.

Hence since $(A_1 + A_2 + \ldots + A_n)^T = A_1^T + A_2^T + \ldots + A_n^T$ we have that $(A_1 + A_2 + \ldots + A_n + A_{n+1})^T = A_1^T + A_2^T + \ldots + A_n^T + A_{n+1}^T$.

Thus the statement is true for all $n \geq 1$.

Q3)

(3) Find row reduced echelon form of M.

$$M = \begin{pmatrix} b & c \\ 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix} \xrightarrow{R_2 \to R_2 - bR_1} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{c}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

So,

$E_3 E_2 E_1 M = I$ where E_3, E_2, E_1 are elementary matrices.

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{c} \end{pmatrix}$$

$$M = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$$

Section A should be fairly easy to prove.

Q1) for reference

Step 1 1 of 3

Let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary vectors and assume that $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$. Since we know that norm is non-negative function, by squaring both sides of the equation we have that previous equation is equivalent to

$$\begin{aligned}
& \|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2 \\
\iff & (\mathbf{u}+\mathbf{v})(\mathbf{u}+\mathbf{v}) = (\mathbf{u}-\mathbf{v})(\mathbf{u}-\mathbf{v}) \\
\iff & \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} \\
\iff & \|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2 = \|\mathbf{u}\|^2-2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2 \quad (\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u} \text{ and } \mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2 \text{ for all vectors } \mathbf{u}, \mathbf{v} \\
\iff & 2\mathbf{u}\cdot\mathbf{v} = -2\mathbf{u}\cdot\mathbf{v} \quad (\ \mathbf{u}\cdot\mathbf{v} \text{ is real number }) \\
\iff & \mathbf{u}\cdot\mathbf{v} = 0 \\
\iff & \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}
\end{aligned}$$