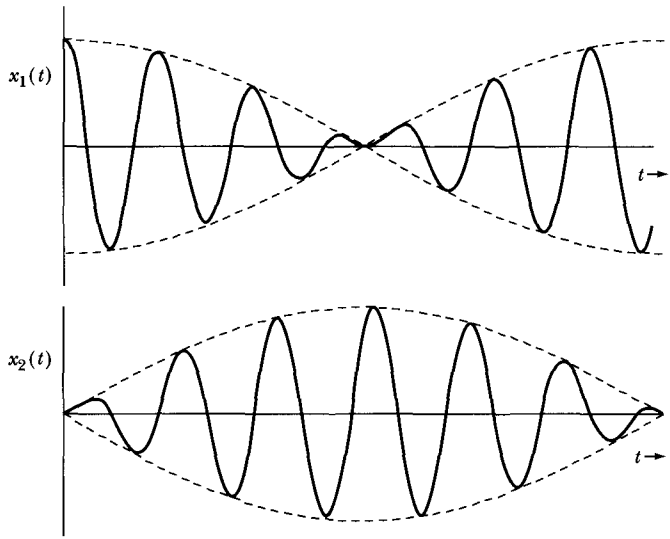

FIGURE 12-4 The solutions for $x_1(t)$ and $x_2(t)$ have a high frequency component (ω_0) that oscillates inside a slowly varying component ($\varepsilon\omega_0$). Energy is transferred back and forth between the two oscillators.

Similarly,

$$x_2(t) = (D \sin \varepsilon\omega_0 t)\sin \omega_0 t \qquad \textbf{(12.26b)}$$

Because ε is small, the quantities $D \cos \varepsilon\omega_0 t$ and $D \sin \varepsilon\omega_0 t$ vary slowly with time. Therefore, $x_1(t)$ and $x_2(t)$ are essentially sinusoidal functions with slowly varying amplitudes. Although only x_1 is initially different from zero, as time increases the amplitude of x_1 decreases slowly with time, and the amplitude of x_2 increases slowly from zero. Hence, energy is transferred from the first oscillator to the second. When $t = \pi/2\varepsilon\omega_0$, then $D \cos \varepsilon\omega_0 t = 0$, and all the energy has been transferred. As time increases further, energy is transferred back to the first oscillator. This is the familiar phenomenon of *beats* and is illustrated in Figure 12-4. (In the case illustrated, $\varepsilon = 0.08$.)

12.4 General Problem of Coupled Oscillations

In the preceding sections, we found that the effect of coupling in a simple system with two degrees of freedom produced two characteristic frequencies and two modes of oscillation. We now turn our attention to the general problem of coupled oscillations. Let us consider a conservative system described in terms of a set of generalized coordinates q_k and the time t. If the system has n degrees of freedom, then $k = 1, 2, \ldots, n$. We specify that a configuration of stable equilibrium exists for the system and that at equilibrium the generalized coordinates have values q_{k0}. In such a configuration, Lagrange's equations are satisfied by

$$q_k = q_{k0}, \qquad \dot{q}_k = 0, \qquad \ddot{q}_k = 0, \qquad k = 1, 2, \ldots, n$$

Every nonzero term of the form $(d/dt)(\partial L/\partial \dot{q}_k)$ must contain at least either $\dot{q}_k$ or $\ddot{q}_k$, so all such terms vanish at equilibrium. From Lagrange's equation, we therefore have

$$\left.\frac{\partial L}{\partial q_k}\right|_0 = \left.\frac{\partial T}{\partial q_k}\right|_0 - \left.\frac{\partial U}{\partial q_k}\right|_0 = 0 \tag{12.27}$$

where the subscript 0 designates that the quantity is evaluated at equilibrium.

We assume that the equations connecting the generalized coordinates and the rectangular coordinates do not explicitly contain the time; that is, we have

$$x_{\alpha,i} = x_{\alpha,i}(q_j) \quad \text{or} \quad q_j = q_j(x_{\alpha,i})$$

The kinetic energy is thus a homogeneous quadratic function of the generalized velocities (see Equation 7.121):

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k \tag{12.28}$$

Therefore, in general,

$$\left.\frac{\partial T}{\partial q_k}\right|_0 = 0, \quad k = 1, 2, \ldots, n \tag{12.29}$$

and hence, from Equation 12.27, we have

$$\left.\frac{\partial U}{\partial q_k}\right|_0 = 0, \quad k = 1, 2, \ldots, n \tag{12.30}$$

We may further specify that the generalized coordinates q_k be measured from the equilibrium positions; that is, we choose $q_{k0} = 0$. (If we originally had chosen a set of coordinates q'_k such that $q'_{k0} \neq 0$, we could always effect a simple linear transformation of the form $q_k = q'_k + \alpha_k$ such that $q_{k0} = 0$.)

The expansion of the potential energy in a Taylor series about the equilibrium configuration yields

$$U(q_1, q_2, \ldots, q_n) = U_0 + \sum_k \left.\frac{\partial U}{\partial q_k}\right|_0 q_k + \frac{1}{2}\sum_{j,k} \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_0 q_j q_k + \cdots \tag{12.31}$$

The second term in the expansion vanishes in view of Equation 12.30, and—without loss of generality—we may choose to measure U in such a way that $U_0 \equiv 0$. Then, if we restrict the motion of the generalized coordinates to be small, we may neglect all terms in the expansion containing products of the q_k of degree higher than second. This is equivalent to restricting our attention to simple harmonic oscillations, in which case only terms quadratic in the coordinates appear. Thus,

$$U = \frac{1}{2}\sum_{j,k} A_{jk} q_j q_k \tag{12.32}$$

where we define

$$A_{jk} \equiv \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_0 \tag{12.33}$$

Because the order of differentiation is immaterial (if U has continuous second partial derivatives), the quantity A_{jk} is symmetrical; that is, $A_{jk} = A_{kj}$.

We have specified that the motion of the system is to take place in the vicinity of the equilibrium configuration, and we have shown (Equation 12.30) that U must have a minimum value when the system is in this configuration. Because we have chosen $U = 0$ at equilibrium, we must have, in general, $U \geq 0$. It should be clear that we must also have $T \geq 0$.*

Equations 12.28 and 12.32 are of a similar form:

$$\boxed{\begin{aligned} T &= \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k \\ U &= \frac{1}{2}\sum_{j,k} A_{jk}q_jq_k \end{aligned}} \qquad (12.34)$$

The quantities A_{jk} are just numbers (see Equation 12.33); but the m_{jk} may be functions of the coordinates (see Equation 7.119):

$$m_{jk} = \sum_\alpha m_\alpha \sum_i \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}$$

We can expand the m_{jk} about the equilibrium position with the result

$$m_{jk}(q_1, q_2, \ldots, q_n) = m_{jk}(q_{l0}) + \sum_l \left.\frac{\partial m_{jk}}{\partial q_l}\right|_0 q_l + \cdots \qquad (12.35)$$

We wish to retain only the first nonvanishing term in this expansion; but, unlike the expansion of the potential energy (Equation 12.31), we cannot choose the constant term $m_{jk}(q_{l0})$ to be zero, so this leading term becomes the constant value of m_{jk} in this approximation. This is the same order of approximation as that used for U, because the next higher order term in T would involve the cubic quantity $\dot{q}_j\dot{q}_kq_l$ and the next higher order term in U would contain $q_jq_kq_l$. In the small oscillation approximation, T should be treated similarly to U, and just like U is normally expanded to order q^2, one needs to expand T to order $\dot{q}^2$, and m_{jk} is evaluated at equilibrium. Thus, in Equation 12.34, the m_{jk} and the A_{jk} are $n \times n$ arrays of *numbers* specifying the way the motions of the various coordinates are coupled. For example, if $m_{rs} \neq 0$ for $r \neq s$, then the kinetic energy contains a term proportional to $\dot{q}_r\dot{q}_s$, and a coupling exists between the rth and sth coordinate. If, however, m_{jk} is *diagonal*, so that† $m_{jk} \neq 0$ for $j = k$ but vanishes otherwise, then the kinetic energy is of the form

$$T = \frac{1}{2}\sum_r m_r\dot{q}_r^2$$

*That is, both U and T are *positive definite* quantities, in that they are always positive unless the coordinates (in the case of U) or the velocities (in the case of T) are zero, in which case they vanish.

†If a diagonal element of m_{jk} (say, m_{rr}) vanishes, then the problem can be reduced to one of $n - 1$ degrees of freedom.

where m_{rr} has been abbreviated to m_r. Thus, the kinetic energy is a simple sum of the kinetic energies associated with the various coordinates. As we see below, if, in addition, A_{jk} is diagonal so that U is also a simple sum of individual potential energies, then each coordinate behaves in an uncomplicated manner, undergoing oscillations with a single, well-defined frequency. The problem is therefore to find a coordinate transformation that simultaneously diagonalizes both m_{jk} and A_{jk} and thereby renders the system describable in the simplest possible terms. Such coordinates are the *normal coordinates.*

The equations of motion of the system with kinetic and potential energies given by Equation 12.34 are obtained from Lagrange's equation

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0$$

But because T is a function only of the generalized velocities and U is a function only of the generalized coordinates, Lagrange's equation for the kth coordinate becomes

$$\frac{\partial U}{\partial q_k} + \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} = 0 \tag{12.36}$$

From Equations 12.34, we evaluate the derivatives:

$$\left.\begin{aligned} \frac{\partial U}{\partial q_k} &= \sum_j A_{jk} q_j \\ \frac{\partial T}{\partial \dot{q}_k} &= \sum_j m_{jk} \dot{q}_j \end{aligned}\right\} \tag{12.37}$$

The equations of motion then become

$$\boxed{\sum_j (A_{jk} q_j + m_{jk} \ddot{q}_j) = 0} \tag{12.38}$$

This is a set of n second-order linear homogeneous differential equations with constant coefficients. Because we are dealing with an oscillatory system, we expect a solution of the form

$$q_j(t) = a_j e^{i(\omega t - \delta)} \tag{12.39}$$

where the a_j are real amplitudes and where the phase δ has been included to give the two arbitrary constants (a_j and δ) required by the second-order nature of each of the differential equations.* (Only the real part of the right-hand side is to be considered.) The frequency ω and the phase δ are to be determined by the equations of motion. If ω is a real quantity, then the solution (Equation 12.39) represents oscillatory motion. That ω is indeed real may be seen by the following physical argument. Suppose that ω contains an imaginary part $i\omega_i$ (in which ω_i is real). This produces terms of the form $e^{\omega_i t}$ and $e^{-\omega_i t}$ in the expression

*This is entirely equivalent to our previous procedure of writing $x(t) = B\exp(i\omega t)$ (see Equations 12.2) with B allowed to be complex. In Equations 12.9, we exhibited the requisite arbitrary constants as real amplitudes by using $\exp(i\omega t)$ and $\exp(-i\omega t)$ rather than by incorporating a phase factor as in Equation 12.39.

of q_j. Thus, when the total energy of the system is computed, $T + U$ contains factors that increase or decrease monotonically with the time. But this violates the assumption that we are dealing with a conservative system: therefore, the frequency ω must be a real quantity.

With a solution of the form given by Equation 12.39, the equations of motion become

$$\boxed{\sum_j (A_{jk} - \omega^2 m_{jk})\, a_j = 0} \tag{12.40}$$

where the common factor $\exp[i(\omega t - \delta)]$ has been canceled. This is a set of n linear, homogeneous, *algebraic* equations that the a_j must satisfy. For a nontrivial solution to exist, the determinant of the coefficients must vanish:

$$|A_{jk} - \omega^2 m_{jk}| = 0 \tag{12.41}$$

To be more explicit, this is an $n \times n$ determinant of the form

$$\begin{vmatrix} A_{11} - \omega^2 m_{11} & A_{12} - \omega^2 m_{12} & A_{13} - \omega^2 m_{13} \cdots \\ A_{12} - \omega^2 m_{12} & A_{22} - \omega^2 m_{22} & A_{23} - \omega^2 m_{23} \cdots \\ A_{13} - \omega^2 m_{13} & A_{23} - \omega^2 m_{23} & A_{33} - \omega^2 m_{33} \cdots \\ \vdots & \vdots & \vdots \end{vmatrix} = 0 \tag{12.42}$$

where the symmetry of the A_{jk} and m_{jk} has been explicitly included.

The equation represented by this determinant is called the **characteristic equation** or **secular equation** of the system and is an equation of degree n in ω^2. There are, in general, n roots we may label ω_r^2. The ω_r are called the **characteristic frequencies** or **eigenfrequencies** of the system. (In some situations, two or more of the ω_r can be equal; this is the phenomenon of **degeneracy** and is discussed later.) Just as in the procedure for determining the directions of the principal axes for a rigid body, each of the roots of the characteristic equation may be substituted into Equation 12.40 to determine the ratios $a_1{:}a_2{:}a_3{:}\cdots{:}a_n$ for each value of ω_r. Because there are n values of ω_r, we can construct n sets of ratios of the a_j. Each of the sets defines the components of the n-dimensional vector $\mathbf{a}_r$, called an **eigenvector** of the system. Thus $\mathbf{a}_r$ is the eigenvector associated with the eigenfrequency ω_r. We designate by a_{jr} the jth component of the rth eigenvector.

Because the principle of superposition applies for the differential equation (Equation 12.38), we must write the general solution for q_j as a linear combination of the solutions for each of the n values of r:

$$q_j(t) = \sum_r a_{jr} e^{i(\omega_r t - \delta_r)} \tag{12.43}$$

Because it is only the *real* part of $q_j(t)$ that is physically meaningful, we actually have*

$$q_j(t) = \mathrm{Re} \sum_r a_{jr} e^{i(\omega_r t - \delta_r)} = \sum_r a_{jr} \cos(\omega_r t - \delta_r) \tag{12.44}$$

*Notice here, unlike the example of weak coupling described in Section 12.3 (Equation 12.26), the real part of $q_j(t)$ has to be explicitly taken so that the $q_j(t)$ in Equation 12.44 is not the same as the $q_j(t)$ in Equation 12.43. But here and elsewhere, because of their close relationship, we use the same symbol (e.g., $q_j(t)$) for convenience.

The motion of the coordinate q_j is therefore compounded of motions with each of the n values of the frequencies ω_r. The q_j evidently are not the normal coordinates that simplify the problem. We continue the search for normal coordinates in Section 12.6.

EXAMPLE 12.1

Find the characteristic frequencies for the case of the two masses connected by springs of Section 12.2 by means of the general formalism just developed.

Solution. The situation is that shown in Figure 12-1. The potential energy of the system is

$$\begin{aligned} U &= \frac{1}{2}\kappa x_1^2 + \frac{1}{2}\kappa_{12}(x_2 - x_1)^2 + \frac{1}{2}\kappa x_2^2 \\ &= \frac{1}{2}(\kappa + \kappa_{12})x_1^2 + \frac{1}{2}(\kappa + \kappa_{12})x_2^2 - \kappa_{12}x_1x_2 \end{aligned} \tag{12.45}$$

The term proportional to x_1x_2 is the factor that expresses the coupling in the system. Calculating the A_{jk}, we find

$$\left.\begin{aligned} A_{11} &= \left.\frac{\partial^2 U}{\partial x_1^2}\right|_0 = \kappa + \kappa_{12} \\ A_{12} &= \left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_0 = -\kappa_{12} = A_{21} \\ A_{22} &= \left.\frac{\partial^2 U}{\partial x_2^2}\right|_0 = \kappa + \kappa_{12} \end{aligned}\right\} \tag{12.46}$$

The kinetic energy of the system is

$$T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 \tag{12.47}$$

According to Equation 12.28,

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{x}_j\dot{x}_k \tag{12.48}$$

Identifying terms between these two expressions for T, we find

$$\left.\begin{aligned} m_{11} &= m_{22} = M \\ m_{12} &= m_{21} = 0 \end{aligned}\right\} \tag{12.49}$$

Thus, the secular determinant (Equation 12.42) becomes

$$\begin{vmatrix} \kappa + \kappa_{12} - M\omega^2 & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} - M\omega^2 \end{vmatrix} = 0 \tag{12.50}$$

This is exactly Equation 12.5, so the solutions are the same (see Equations 12.7 and 12.8) as before:

$$\omega = \sqrt{\frac{\kappa + \kappa_{12} \pm \kappa_{12}}{M}}$$

The eigenfrequencies are

$$\omega_1 = \sqrt{\frac{\kappa + 2\kappa_{12}}{M}}, \qquad \omega_2 = \sqrt{\frac{\kappa}{M}}$$

The results of the two procedures are identical.

12.5 Orthogonality of the Eigenvectors (Optional)*

We now wish to show that the eigenvectors $\mathbf{a}_r$ form an orthonormal set. Rewriting Equation 12.40 for the sth root of the secular equation, we have

$$\omega_s^2 \sum_k m_{jk} a_{ks} = \sum_k A_{jk} a_{ks} \tag{12.51}$$

Next, we write a comparable equation for the rth root by substituting r for s and interchanging j and k:

$$\omega_r^2 \sum_j m_{jk} a_{jr} = \sum_j A_{jk} a_{jr} \tag{12.52}$$

where we have used the symmetry of the m_{jk} and A_{jk}. We now multiply Equation 12.51 by a_{jr} and sum over j and also multiply Equation 12.52 by a_{ks} and sum over k:

$$\left.\begin{aligned} \omega_s^2 \sum_{j,k} m_{jk} a_{jr} a_{ks} &= \sum_{j,k} A_{jk} a_{jr} a_{ks} \\ \omega_r^2 \sum_{j,k} m_{jk} a_{jr} a_{ks} &= \sum_{j,k} A_{jk} a_{jr} a_{ks} \end{aligned}\right\} \tag{12.53}$$

The right-hand sides of Equations 12.53 are now equal, so subtracting the first of these equations from the second, we have

$$(\omega_r^2 - \omega_s^2) \sum_{j,k} m_{jk} a_{jr} a_{ks} = 0 \tag{12.54}$$

We now examine the two possibilities $r = s$ and $r \neq s$. For $r \neq s$, the term $(\omega_r^2 - \omega_s^2)$ is, in general, different from zero. (The case of degeneracy, or multiple roots, is discussed later.) Therefore the sum must vanish identically:

$$\sum_{j,k} m_{jk} a_{jr} a_{ks} = 0, \quad r \neq s \tag{12.55}$$

*Section 12.5 may be omitted without losing physical understanding. This highly mathematical section is included for completeness. The method used here is a generalization of the steps used in Section 11.6 for the inertia tensor.

For the case $r = s$, the term $(\omega_r^2 - \omega_s^2)$ vanishes and the sum is indeterminate. The sum, however, cannot vanish identically. To show this, we write the kinetic energy for the system and substitute the expressions for $\dot{q}_j$ and $\dot{q}_k$ from Equation 12.44:

$$\begin{aligned} T &= \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k \\ &= \frac{1}{2}\sum_{j,k} m_{jk}\left[\sum_r \omega_r a_{jr}\sin(\omega_r t - \delta_r)\right]\left[\sum_s \omega_s a_{ks}\sin(\omega_s t - \delta_s)\right] \\ &= \frac{1}{2}\sum_{r,s}\omega_r\omega_s \sin(\omega_r t - \delta_r)\sin(\omega_s t - \delta_s)\sum_{j,k} m_{jk}a_{jr}a_{ks} \end{aligned}$$

Thus, for $r = s$, the kinetic energy becomes

$$T = \frac{1}{2}\sum_r \omega_r^2 \sin^2(\omega_r t - \delta_r)\sum_{j,k} m_{jk}a_{jr}a_{kr} \tag{12.56}$$

We note first that

$$\omega_r^2 \sin^2(\omega_r t - \delta_r) \geq 0$$

We also know that T is positive and can become zero only if all the velocities vanish identically. Therefore,

$$\sum_{j,k} m_{jk}a_{jr}a_{kr} \geq 0$$

Thus, the sum is, in general, positive and can vanish only in the trivial instance that the system is not in motion—that is, that the velocities vanish identically and $T \equiv 0$.

We previously remarked that only the ratios of the a_{jr} are determined when the ω_r are substituted into Equation 12.40. We now remove this indeterminacy by imposing an additional condition on the a_{jr}. We require that

$$\sum_{j,k} m_{jk}a_{jr}a_{kr} = 1 \tag{12.57}$$

The a_{jr} are then said to be *normalized.* Combining Equations 12.55 and 12.57, we may write

$$\boxed{\sum_{j,k} m_{kj}a_{jr}a_{ks} = \delta_{rs}} \tag{12.58}$$

Because a_{jr} is the jth component of the rth eigenvector, we represent $\mathbf{a}_r$ by

$$\mathbf{a}_r = \sum_j a_{jr}\mathbf{e}_j \tag{12.59}$$

The vectors $\mathbf{a}_r$ defined in this way constitute an **orthonormal** set; that is, they are *orthogonal* according to the result given by Equation 12.55, and they have been *normalized* by setting the sum in Equation 12.57 equal to unity.

All the preceding discussion bears a striking resemblance to the procedure given in Chapter 11 for determining the principal moments of inertia and the principal axes for a rigid body. Indeed, the problems are mathematically identical, except that we are now dealing with a system with n degrees of freedom.

The quantities m_{jk} and A_{jk} are actually tensor elements, because m and A are two-dimensional arrays that relate different physical quantities,* and as such, we write them as $\{\mathbf{m}\}$ and $\{\mathbf{A}\}$. The secular equation for determining the eigenfrequencies is the same as that for obtaining the principal moments of inertia, and the eigenvectors $\mathbf{a}_r$ correspond to the principal axes. Indeed, the proof of the orthogonality of the eigenvectors is merely a generalization of the proof given in Section 11.6 of the orthogonality of the principal axes. Although we have made a physical argument regarding the reality of the eigenfrequencies, we could carry out a mathematical proof using the same procedure used to show that the principal moments of inertia are real.

12.6 Normal Coordinates

As we have seen (Equation 12.43), the general solution for the motion of the coordinate q_j must be a sum over terms, each of which depends on an individual eigenfrequency. In the previous section, we showed that the vectors $\mathbf{a}_r$ are orthogonal (Equation 12.55) and, as a matter of convenience, we even normalized their components a_{jr} (Equation 12.57) to arrive at Equation 12.58; that is, we have removed all ambiguity in the solution for the q_j, so it is no longer possible to specify an arbitrary displacement for a particle. Because such a restriction is not physically meaningful, we must introduce a constant scale factor α_r (which depends on the initial conditions of the problem) to account for the loss of generality introduced by the arbitrary normalization. Thus,

$$q_j(t) = \sum_r \alpha_r a_{jr} e^{i(\omega_r t - \delta_r)} \tag{12.60}$$

To simplify the notation, we write

$$q_j(t) = \sum_r \beta_r a_{jr} e^{i\omega_r t} \tag{12.61}$$

where the quantities β_r are new scale factors† (now complex) that incorporate the phases of δ_r.

We now define a quantity η_r,

$$\boxed{\eta_r(t) \equiv \beta_r e^{i\omega_r t}} \tag{12.62}$$

so that

$$\boxed{q_j(t) = \sum_r a_{jr}\eta_r(t)} \tag{12.63}$$

The η_r, by definition, are quantities that undergo oscillation at only one frequency. They may be considered as new coordinates, called *normal coordinates*, for the system. The η_r satisfy equations of the form

$$\ddot{\eta}_r + \omega_r^2 \eta_r = 0 \tag{12.64}$$

*See the discussion in Section 11.7 concerning the mathematical definition of a tensor.

†There is a certain advantage in normalizing the a_{jr} to unity and introducing the scale factors α_r and β_r rather than leaving the normalization unspecified. The a_{jr} are then independent of the initial conditions, and a simple orthonormality equation results.

There are n independent such equations, so the equations of motion expressed in normal coordinates become completely separable.

EXAMPLE 12.2

Derive Equation 12.64 directly by using Lagrange's equations of motion.

Solution. We note that from Equation 12.63

$$\dot{q}_j = \sum_j a_{jr}\dot{\eta}_r$$

and from Equation 12.34 we have, for the kinetic energy,

$$\begin{aligned} T &= \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k \\ &= \frac{1}{2}\sum_{j,k} m_{jk}\left(\sum_r a_{jr}\dot{\eta}_r\right)\left(\sum_s a_{ks}\dot{\eta}_s\right) \\ &= \frac{1}{2}\sum_{r,s}\left(\sum_{j,k} m_{jk}a_{jr}a_{ks}\right)\dot{\eta}_r\dot{\eta}_s \end{aligned}$$

The sum in the parentheses is just δ_{rs}, according to the orthonormality condition (Equation 12.58). Therefore,

$$T = \frac{1}{2}\sum_{r,s}\dot{\eta}_r\dot{\eta}_s\delta_{rs} = \frac{1}{2}\sum_r \dot{\eta}_r^2 \tag{12.65}$$

Similarly, from Equations 12.34 we have for the potential energy,

$$\begin{aligned} U &= \frac{1}{2}\sum_{j,k} A_{jk}q_jq_k \\ &= \frac{1}{2}\sum_{r,s}\left(\sum_{j,k} A_{jk}a_{jr}a_{ks}\right)\eta_r\eta_s \end{aligned}$$

The first equation in Equation 12.53 is

$$\begin{aligned} \sum_{j,k} A_{jk}a_{jr}a_{ks} &= \omega_s^2\sum_{j,k} m_{jk}a_{jr}a_{ks} \\ &= \omega_s^2\delta_{rs} \end{aligned}$$

so the potential energy becomes

$$U = \frac{1}{2}\sum_{r,s}\omega_s^2\eta_r\eta_s\delta_{rs} = \frac{1}{2}\sum_r \omega_r^2\eta_r^2 \tag{12.66}$$

Using Equations 12.65 and 12.66, the Lagrangian is

$$L = \frac{1}{2}\sum_r(\dot{\eta}_r^2 - \omega_r^2\eta_r^2) \tag{12.67}$$

and Lagrange's equations are

$$\frac{\partial L}{\partial \eta_r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_r} = 0$$

or

$$\ddot{\eta}_r + \omega_r^2 \eta_r = 0$$

as found in Equation 12.64.

Thus, when the configuration of a system is expressed in normal coordinates, both the potential and kinetic energies become simultaneously diagonal. Because it is the off-diagonal elements of $\{\mathbf{m}\}$ and $\{\mathbf{A}\}$ that give rise to the coupling of the particles' motions, it should be evident that a choice of coordinates that renders these tensors diagonal uncouples the coordinates and makes the problem completely separable into the independent motions of the normal coordinates, each with its particular normal frequency.*

The foregoing has been a mathematical description of the methods used to determine the characteristic frequencies ω_r and to describe the coordinates η_r of the normal mode motion. The actual application of the method can be summarized by several statements:

1. Choose generalized coordinates and find T and U in the normal Lagrangian method. This corresponds to using Equations 12.34.
2. Represent A_{jk} and m_{jk} as tensors in $n \times n$ arrays, and use Equation 12.42 to determine the n values of eigenfrequencies ω_r.
3. For each value of ω_r, determine the ratios $a_{1r} : a_{2r} : a_{3r} : \cdots : a_{nr}$ by substituting into Equation 12.40:

$$\sum_j (A_{jk} - \omega_r^2 m_{jk})\, a_{jr} = 0 \qquad \textbf{(12.68)}$$

4. If needed, determine the scale factors β_r (Equation 12.60) from the initial conditions.
5. Determine the normal coordinates η_r by appropriate linear combinations of the q_j coordinates that display oscillations at the single eigenfrequency ω_r. The description of motion for this single normal coordinate η_r is called a normal mode. The general motion (Equation 12.63) of the system is a complicated superposition of the normal modes.

We now apply these steps in several examples.

EXAMPLE 12.3

Determine the eigenfrequencies, eigenvectors, and normal coordinates of the mass–spring example in Section 12.2 by using the procedure just described. Assume $\kappa_{12} \approx \kappa$.

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*The German mathematician Karl Weierstrass (1815–1897) showed in 1858 that the motion of a dynamical system can always be expressed in terms of normal coordinates.