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2022113019

Linear Algebra Assignment 4 : —

Ref: cuemath.com/calculus/zero-vector/

1. $\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$, $c.\bar{\alpha} = -c\bar{\alpha}$, defined on $\mathbb{R}^n$

$\because$ $\bar{\alpha}, \bar{\beta}$ have been defined over $\mathbb{R}^n$,

$\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$

$\bar{\beta} = (\beta_1, \ldots, \beta_n)$

Proving the axioms:— $\mathbb{R}$ is the field of scalars defined over arithmetic add^n & multiplication
$\mathbb{R}^n$ is the set of objects

① Commutativity:

$(\alpha_1, \ldots, \alpha_n) \oplus (\beta_1, \ldots, \beta_n)$
$= (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$

$(\beta_1, \ldots, \beta_n) \oplus (\alpha_1, \ldots, \alpha_n)$
$= (\beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n)$

$\because \bar{\alpha} \oplus \bar{\beta} \neq \bar{\beta} \oplus \bar{\alpha}$

$\therefore$ commutativity does not hold.

② Associativity :-

$\big((\alpha_1, \ldots, \alpha_n) \oplus (\beta_1, \ldots, \beta_n)\big) \oplus (v_1, \ldots, v_n)$
$= (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n) \oplus (v_1, \ldots, v_n)$

$(\alpha_1, \ldots, \alpha_n) \oplus [(\beta_1, \ldots, \beta_n) \oplus (v_1, \ldots, v_n)]$
$= (\alpha_1, \ldots, \alpha_n) \oplus (\beta_1 - v_1, \ldots, \beta_n - v_n)$
$= (\alpha_1 - \beta_1 + v_1, \ldots, \alpha_n - \beta_n + v_n)$

$\bar{\alpha} \oplus (\bar{\beta} \oplus \bar{v}) \neq (\bar{\alpha} \oplus \bar{\beta}) \oplus \bar{v}$

Associativity does not hold.

③ [$\because$ commutativity does not hold, the def^n of additive identity becomes
$\bar{\alpha} + \bar{0} = \bar{\alpha}$, $\bar{0} + \bar{\alpha} = \bar{\alpha}$]

$(\alpha_1, \ldots, \alpha_n) \oplus (0, \ldots, 0) = (\alpha_1, \ldots, \alpha_n)$
$(0, \ldots, 0) \oplus (\alpha_1, \ldots, \alpha_n) = (-\alpha_1, \ldots, -\alpha_n)$

$\because$ Additive identity does not exist

④ Additive Inverse :-

∵ Additive inverse is defined for additive identity, additive inverse doesnt exist

⑤ Multiplicative identity :

$1 \cdot (\alpha_1, \ldots, \alpha_n) = (-\alpha_1, \ldots, -\alpha_n)$ {defined for field of $\mathbb{R}$}

Does not hold

check if

⑥ $(c_1 c_2) \cdot \alpha = c_1(c_2 \cdot \alpha)$

$(c_1 c_2)\alpha = (-c_1 c_2 \alpha_1, \ldots, -c_1 c_2 \alpha_n)$

$c_1(c_2\alpha) = c_1(-c_2\alpha_1, \ldots, -c_2\alpha_n)$

$= (c_1 c_2 \alpha_1, \ldots, c_1 c_2 \alpha_n)$

Does not hold.

⑦ check if $c(\alpha \oplus \beta) = c\alpha \oplus c\beta$

$c \cdot (\alpha \oplus \beta) = c \cdot ((\alpha_1, \ldots, \alpha_n) \oplus (\beta_1, \ldots, \beta_n))$

$= c(\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$

$= (c\beta_1 - c\alpha_1, \ldots, c\beta_n - c\alpha_n)$

$c\alpha \oplus c\beta = c(\alpha_1, \ldots, \alpha_n) \oplus c(\beta_1, \ldots, \beta_n)$

$= (-c\alpha_1, \ldots, -c\alpha_n) \oplus (-c\beta_1, \ldots, -c\beta_n)$

$= (c\beta_1 - c\alpha_1, \ldots, c\beta_n - c\alpha_n)$

LHS = RHS, hence ~~[illegible]~~ axiom holds.

check if

⑧ $(c_1 + c_2) \cdot \alpha = c_1 \cdot \alpha \oplus c_2 \cdot \alpha$

$(c_1 + c_2)\alpha = ((c_1 + c_2)\alpha_1, \ldots, (c_1 + c_2) \cdot \alpha_n)$

~~[illegible]~~

~~$c_1\alpha \oplus c_2 \cdot \alpha = (c_1\alpha_1, \ldots$~~

~~From ⑦, $c_1 \cdot \alpha \oplus c_2 \cdot \alpha = (c\beta_1 - c\alpha$~~

$c_1 \cdot \alpha \oplus c_2 \cdot \alpha = (-c_1\alpha_1, \ldots, c_2\alpha_n) \oplus (-c_2\alpha_1, \ldots, -c_2\alpha_n)$

$= (c_2\alpha_1 - c_1\alpha_1, \ldots, c_2\alpha_n - c_1\alpha_n)$

∴ Axiom does not hold

Thus only one axiom holds.

2. Notation used:
If a is a vector, we write $\vec{a}$, and when we ~~talk~~ write $\bar{a}$, we mean the complex conjugate.

$(f+g)(t) = f(t) + g(t)$, $t \in \mathbb{F}$
$(c \cdot f)(t) = c f(t)$

Checking whether V is a vector space

~~① Commutativity:~~

$$\boxed{\begin{aligned} &(f+g)(-t) \text{ is in } V \text{ because} \quad \text{(in general)} \\ &(f+g)(-t) = f(-t) + g(-t) \\ &= f^*(t) + g^*(t) \\ &= (f+g)^*(t) \end{aligned}}$$

① Commutativity:
$(f+g)(t) = f(t) + g(t) = g(t) + f(t)$
$= (g+f)(t)$
Holds true

② Associativity:
$((f+g)+h)(t) = (f+g)(t) + h(t)$
$= f(t) + g(t) + h(t)$
$= f(t) + (g(t) + h(t))$
$= f(t) + (g+h)(t)$
$= (f + (g+h))(t)$

Hence holds true

③ Defining a function $Z(t)$
s. that $Z(t) = Z(-t) = 0$ $[0 \in \mathbb{F}$ (scalar)$]$
$\because V$ is the set of all such functions,
$Z(t) \in V$

$\therefore$ Zero vector exists

④ Defining a function $g(t) = (-f)(t)$
$\forall f$ s. that $f(t) + g(t) = Z(t) = g(t) + f(t)$
We have to prove $g(t) \in V$
$g(-t) = -f(-t) = [(-f)(t)]^* = ((-f)^* t)$
$= g^*(t)$

Hence, $g \in V$, additive inverse exists

⑤ checking first if $(c \cdot f)(t) \in V$

$\cancel{(cf)(t) = c(f) \; cf}$

$(cf)(-t) = c f(-t) = c f^*(t) = (cf^*)t$

Hence $cf(-t) \in V$

If $c = 1$,

$1 \cdot f(t) = f(t)$

$\therefore 1 \in \mathbb{F}$, is multiplicative identity

$= (c_1 c_2 f)(t)$

⑥ $(c_1 c_2)(f)(t) = c_1 c_2 f(t) = (c_1 (c_2 f))t$

$\therefore$ Holds true

$\cancel{(c \cdot (f+g))(t) = (cf+g)\,t = (cf + cg)(t) = cf(t) + cg(t)}$

⑦ $(c \cdot (f+g))(t) = c(f+g)(t)$

$= c(f(t) + g(t))$

$= cf(t) + cg(t) = (cf + cg)(t)$

$\therefore$ B Holds true

⑧ $((c_1 + c_2) f)(t) = (c_1 + c_2) f(t)$

$= c_1 f(t) + c_2 f(t)$

$= (c_1 f + c_2 f)(t)$

$\therefore$ Holds true

$\because$ ① - ⑧ all hold true, $\mathbb{V}$ is a vector space.

(b) An example of f in V not real valued could be

$f(t) = e^{it}$ even $\{ f(t) = i \; \forall t, f(-t) = i \}$

$f(-t) = e^{-it}$

$\therefore$ It satisfies the given constraints classmate

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3. P: non empty W [$W \subseteq V$] is a subspace

Q: $\forall\ \alpha, \bar{\beta} \in W, c \in F,\ c\alpha + \beta \in W$

Prove $P \leftrightarrow Q$

$P \to Q$:

If W is a subspace,

$\alpha \in W,\ c\alpha \in W$ [For closure to hold]

$c\alpha, \beta \in W,\ c\alpha + \beta \in W$ [For closure of vector add^n]

Hence proved.

$Q \to P$:

$c\alpha + \beta \in W$

RTP W is a subspace.

Subspace needs to be vector space.

We can simplify things, because properties of scalar multiplication need not be checked [except closure] because they are defined over V & do not change in F. Similarly, commutativity and associativity over + need not be checked.

For $c = -1,\ \vec{\beta} = \vec{\alpha}$, $c\alpha + \beta \in W \Rightarrow -\vec{\alpha} + \vec{\alpha} \in W$

$\Rightarrow \vec{0} \in W$

Additive identity exists.

Also, $c = -1, \vec{\beta} = \vec{0}$, $c\alpha + \beta \in W \Rightarrow -\vec{\alpha} \in W$

Inverse exists,

~~W~~ Hence proved.

$\therefore P \to Q$ and $Q \to P$

$\Rightarrow P \leftrightarrow Q$ [Hence proved]