Classical Mechanics

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Abstract

Classical Mechanics course taught at IIIT-H in Winter 2024. These notes are not substitute for the books mentioned in the class. If you find any mistake please inform me in class or send me an email.

Contents

1 Elementary Principles

1.1 Single Particle Mechanics

Position vector of a single particle is $\vec{r}(t)$. Velocity is

$$\vec{v} = \frac{d\vec{r}}{dt}\,, \tag{1}$$

linear momentum

$$\vec{p} = m\vec{v}\,, \tag{2}$$

where m the mass of the particle. If $\vec{F}$ is the force acting on it then from Newtons's second law

$$\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}} = \frac{d}{dt}(m\vec{v})\,. \tag{3}$$

The reference frame in which this law is valid is called the *inertial* or *Galilean frame*. The notion of such a frame is ideal. Unless otherwise mentioned, we will take the earth as the inertial frame. Acceleration is

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}\,. \tag{4}$$

If $\vec{F} = 0$ then $\vec{p}$ =constant:- *in the absence of a force, linear momentum is conserved.* This is conservation of linear momentum.

The work done by $\vec{F}$ to move the particle from point 1 to point 2 is

$$W_{12} = \int_1^2 \vec{F}.d\vec{r} = m\int_1^2 \frac{d\vec{v}}{dt}.\vec{v}dt = \frac{m}{2}(v^2 - v_1^2) = T_2 - T_1 \tag{5}$$

where $v_{1,2}$ are velocities and $T_{1,2}$ are the kinetic energies at the two points.

Conservative force: if the work done is independent of the path and depends only on the end points then it is called a conservative force. In this case, you can take the particle from point 1 to point 2 and then back to point 1 and the total work done is zero

$$\int_1^2 \vec{F}.d\vec{r} + \int_2^1 \vec{F}.d\vec{r} = \oint \vec{F}.d\vec{r} = 0 \tag{6}$$

From the properties of vector analysis it can be shown that the above is true if

$$\vec{F} = -\vec{\nabla}V\,, \tag{7}$$

where $V \equiv V(\vec{r})$ is a scalar function of position coordinates. The function is known as the *potential energy* or simply *potential.* Force is unique if a constant is added to the potential, *i.e.,* $\vec{F} = -\vec{\nabla}(V + \text{const.})$, therefore absolute value of a potential is immaterial – what is physical is the potential difference.

For a conservative force, the work done is

$$W_{12} = \int_1^2 \vec{F}.d\vec{r} = -\int_1^2 \vec{\nabla}V.d\vec{r} = V_1 - V_2\,. \tag{8}$$

We have already found $W_{12} = T_2 - T_1$. Hence we get

$$T_2 - T_1 = V_1 - V_2 \Rightarrow T_1 + V_1 = T_2 + V_2 = E. \tag{9}$$

In other worrds, thotal energy $E = T + V$ is conserved in case of conservative field. Example: gravity (when air resistance negleted).

1.2 System of many Particles

1.2.1 Constraints

Consider a system of particles. Two types of forces act on the i^{th} particle (where $i = 1, 2, 3...$ etc) – an external force $\vec{F}_i^e$, and interaction force $\vec{F}_{ji}$ due to the j^{th} particle. Newton's law for this system is

$$m_i \frac{d^2 \vec{r}_i}{dt^2} = F_i^e + \sum_{j \neq i} F_{ji} \,. \tag{10}$$

In most practical problems, the above set of equations is not enough to solve the system as there exists *constraints* that limit the motion. Constraints are there in single particle motion also. For example: in motion of a particle in a straight line along the x axis, there are constraints $y = 0$, $z = 0$. The motion of a particle on the surface of a sphere of radius R is subject to the constraint that the magnitude of the position vector $r = R$. In case of of a gas inside a ballon, the motion of the molecules is constrained by the surface of the ballon. Another example is: the motion of system of two particles where the distance between the particles is constant.

Constraints can be expressed in terms of equations (holonomic) or some inequality (non-holonomic). In this course we will mostly discuss constraints of the first kind.
Examples: particle moving in a straight line along the X axis – the constraints are

$$y = 0 \,, \; z = 0 \,.$$

Motion of a particle on a surface of a sphere of radius R

$$x^2 + y^2 + z^2 = R^2 \,,$$

Motion of a two particle system such that their distance is always ℓ

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = \ell^2 \,.$$

There are motion where more than one constraints exist – in these cases more than one equations will also exist.

1.2.2 Generalized coordinates

Number of independent coordinates required to describe a system of N particles not subjected to any constraints is $3N$: (x_1, y_1, z_1) for 1st particle, (x_2, y_2, z_2) for 2nd particle..and so on. This $3N$ coordinates is called the *degrees of freedom* of the system. If k holonomic constraints exist, then there are k equations of constraints that can be used to solve for k coordinates and there will be $3N - k$ degrees of freedom.

For example: equation of constraints for a particle at the surface of a sphere of radius R is $x^2 + y^2 + z^2 = R^2$. In this case if two coordinates x, y are determined then z is given as $z = \sqrt{R^2 - x^2 - y^2}$. Hence the degrees of freedom of motion of a particle on surface of a sphere is 3.1 - 1 = 2. An alternative is to introduce spherical polar coordinates (R, θ, ϕ)

$$x = R \sin\theta \cos\phi \,, \quad y = R \sin\theta \sin\phi \,, \quad z = R \cos\theta \tag{11}$$

Since the R is constant for the motion, it can be described by two independent coordinates θ, ϕ. What we have done in the above equation is that we have expressed the three old coordinates (x, y, z) in terms of two new independent coordinates θ, ϕ. These are called the *generalized* coordinates – a set of independent coordinates that completely specifies a particle or a system of particles. In this course we will denote generalized coordinates by q – they are functions of time. In this example two generalized coordinates are $q_1 = \theta, q_2 = \phi$.

So in a system of N particles with k constraints, there exist $3N-k$ generalized coordinates $q_1, q_2, ...q_{3N-k}$, and one can always express the old coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2)....(x_N, y_N, z_N)$ in terms of the generalized coordinates

$$
\begin{aligned}
x_1 &= f_1(q_1, q_2...q_{3N-k}, t)\,, \\
y_1 &= f_2(q_1, q_2...q_{3N-k}, t)\,, \\
z_1 &= f_3(q_1, q_2...q_{3N-k}, t)\,, \\
... &= ...\,, \\
... &= ...\,, \\
... &= ...\,, \\
z_N &= f_{3N}(q_1, q_2...q_{3N-k}, t)\,,
\end{aligned}
$$

If t is absent in the above functions then constraints are independent of time – scleronomic constraints. We will mostly deal with these types. These equations can be inverted to solve for the generalized coordinates.

The time derivative of generalized coordinates are called

$$
\text{generalized velocity:} \dot{q}_i = \frac{dq_i}{dt}\,. \tag{12}
$$

The coordinates q_i and the generalized velocity $\dot{q}_i$ are independent of each other since at time $t = 0$, we can chose $q_i(0)$ and $\dot{q}_i(0)$ independently.

1.2.3 Configuration Space

The instantaneous configuration is given in terms of a set of $n = 3N - k$ generalized coordinates $q_i \equiv q_1, q_2, q_3........q_n$. You can consider a coordinate system of n axes which are perpendicular to each other – this n-dimensional hyperspace called the configuration space. As the system evolves in time, it makes a *path* in this configuration space. Cartesian coordinate space is then 3-dimensional configuration space. From now on we will always refer to generalized coordinates and configuration space to describe dynamical system.

2 Lagrangian formulation

Solving mechanics problem using Newton's 2nd law involve the following problems: *i)* you have to know all the external forces – may not always straightforward, also vectors are difficult to deal with compared to scalars *ii)* you have to know the forces of constraints, for example friction, inter-particle interactions – not easy *iii)* you have to implement all the constraints *iv)* Newton's 2nd law is not invariant under coordinate transformation – for example if you go from Cartesian to spherical. To address these we go to a new formulation of classical mechanics, called Lagrangian mechanics.

2.1 Hamilton's Principle

Lagrangian formulation is based on *Hamilton's* prnciple, also known as the *least action principle.* Please read *Feynman Lectures on Physics, Vol-2, Chapter 19* for an excellent introduction to this topic. Suppose a dynamical system with n-generalized coordinates has to go from of a point $A \equiv \{q_1(t_A), q_2(t_A).....q_n(t_A)\}$ to another point $B \equiv \{q_1(t_B), q_2(t_B).....q_n(t_B)\}$ between time t_A to t_B. It is observed that of all infinitely many paths, the Nature choses the one for which the time integral between t_A, t_B of the kinetic energy minus the potential is extremum (either minimum of maximum). This is known as the *Hamilton's principle* or the *least action principle.* The integral

$$S = \int_{t_A}^{t_B} \Big(T(q_i, \dot{q}_i, t) - V(q_i, t) \Big) dt = \int_{t_A}^{t_B} L(q_i, \dot{q}_i, t) dt \tag{13}$$

is called the *action.* So Hamilton's principle simply means that during time evolution of dynamical system, the actual path is the one for which the action is extremum, *i.e.,*

$$\delta S = \delta \int_{t_A}^{t_B} L(q_i, \dot{q}_i, t) dt = 0 \,. \tag{14}$$

The *functional* (function of a function is called a functional)

$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i, t) \,. \tag{15}$$

It is a function of generalized coordinates $q_i(t)$, generalized velocity $\dot{q}_i$, and time. Here time t is the parameter of the path followed by the system.

2.2 Variational Calculas

Given Hamilton's principle (14) we want to find the actual path. This is mathematical problem in calculus of variation – it goes like this: suppose you are given a integral of a functional $f(y(x), y'(x), x)$, where $y'(x) = dy/dx$

$$J = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx \,.$$

The problem is to find the path $y = y(x)$ for which J is either maximum or minimum. The solution to the problem is that the actual path is the one for which the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \,,$$

is satisfied. We derive the equation for our dynamical problem in the next section.

An example: the calculus of variation can be used to solve one of the most famous problem in physics, the *brachistochrone* problem – please see *example 6.2* in *Classical Dynamics* by Thornton and Marion.