A-1 Answer Key

Question-1 (6 marks)

For each of the part, in case of correct conversion using the Augmented matrix method, award full 3 marks. There are roughly 6 steps for conversion in either part. Cut 0.5 marks per calculation mistake.

Q1] (a) $A = \begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$

Augmented matrix $\Rightarrow [A \,\vdots\, I]$

$$\downarrow$$

$$[I \,\vdots\, A^{-1}]$$

Final inverse matrix does not exist

↳ Because the left side cannot be reduced to I

[det $(A) = 0$ can also be a valid argument, give marks for both]

(b) $A = \begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix}$

final inverse

$$\begin{bmatrix} 1/a & 0 & 0 \\ -1/a^2 & 1/a & 0 \\ 1/a^3 & -1/a^2 & 1/a \end{bmatrix}$$

Question-2 (2 marks)

Given A is nonsingular and symmetric, show that $A^{-1} = (A^{-1})^T$.

Since A is nonsingular, A^{-1} exists. Since $I = I^T$ and $AA^{-1} = I$,

$$AA^{-1} = (AA^{-1})^T.$$

Since $(AB)^T = B^T A^T$,

$$AA^{-1} = (A^{-1})^T A^T.$$

Since $AA^{-1} = A^{-1}A = I$, we rearrange the left side to obtain

$$A^{-1}A = (A^{-1})^T A^T.$$

Since A is symmetric, $A = A^T$, and we can substitute this into the right side to obtain

$$A^{-1}A = (A^{-1})^T A.$$

From here, we see that

$$A^{-1}A(A^{-1}) = (A^{-1})^T A(A^{-1})$$

$$A^{-1}I = (A^{-1})^T I$$

$$A^{-1} = (A^{-1})^T,$$

Award full 2 marks if all steps are followed along with correct properties being stated. In case that properties are not stated everywhere, cut 0.5-1 marks depending on the number of such occurences.

Question-3 (5 marks)

0.5*6 marks for finding each of the matrices correctly.

0.5 mark for stating A^7=A

1.5 mark for correct manipulations to calculate A^2015

Question-4 (3 marks)

Proof by contradiction, take trace on both sides and disprove the claim.

Cut 0.5-1 marks if vague/missing arguments.

In case of any other solutions provided, give marks based on corectness.

Reference Link: https://yutsumura.com/matrix-xy-yx-never-be-the-identity-matrix/

Question-5 (5 marks)

2 things are required to be proved for the 1st part:

1) The vectors are orthogonal → 0.5*3 for all 3 checking <ui,uj≥0 for all i,j.

2) They form basis of R^3 → 1 mark for showing that the vectors are spanning

For the 2nd part, if all calculations are correct, award 2.5 marks. Cut 0.5 mark per calculation mistake.

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = \omega$$
$$x_1 + x_2 + x_3 = 4$$
$$x_1 - x_2 + x_3 = 5$$
$$x_1 - 2x_3 = 6$$
$$\hookrightarrow x_1 = 5, x_2 = x_3 = -\frac{1}{2}$$

Question-6 (4 marks)

For upper-triangular matrix, we only need to concern with proving that the resulting multiplication would lead to matrix C having cij=0 for all i>j:

> **Definition.** A $n \times n$ matrix A is upper triangular if $a_{ij} = 0$ for $i > j$.

Now consider A, B upper triangular and $C = AB$. Then consider C_{ij} entry from C with $i > j$, given by

$$\begin{aligned} C_{ij} &= \sum_{k=1}^{n} a_{ik} b_{kj} \\ &= \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^{n} a_{ik} b_{kj} \\ &= 0 + 0 = 0, \end{aligned}$$

since

$$\begin{aligned} k \le i - 1 &\implies a_{ik} = 0 \\ k \ge i &\implies b_{kj} = 0 \end{aligned}$$

for this particular combination of (i, j). We conclude that $C_{ij} = 0$ if $i > j$ hence C is upper triangular.

The above proof should be used as reference, more elaborate reasoning is expected from the students in their answers.