2.4 Conservation theorems

2.4.1 Conjugate/Canonical/generalized momentum

Let us consider a system in motion under a force that can be derived from a potential which is a function of position only. Moreover, lets describe the system in Cartesian $x_i = (x, y, z)$ coordinates – example can be motion of a particle under gravity. In this example, the first term of the (17) is

$$\frac{\partial L}{\partial x_i} = \frac{\partial T}{\partial x_i} - \frac{\partial V(x,y,z)}{\partial x_i} = F_{x_i}\,,$$

and from the second term we get

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}} - \frac{\partial V(x,y,z)}{\partial \dot{x}}\right), \\ &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\sum \frac{m^2}{2}(\dot{x}_i^2)\right) = \frac{d}{dt}(m_i \dot{x}_i) = \frac{d}{dt}(p_i)\,, \end{aligned}$$

This gives the Newton's law, and you obtain the x_i^{th} linear momentum from the Lagrangian as

$$m\dot{x}_i = \frac{\partial L}{\partial \dot{x}_i}\,.$$

This result can be generalized to define the *generalized momentum* or the *canonical momentum* or *conjugate momentum* for the coordinate q_i as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{18}$$

If q_i are not the Cartesian coordinates then p_i may not have the dimension of linear momentum. Furthermore, for velocity dependent potentials, even in case of Cartesian coordinate, q_i are not identical to mechanical momentum. An example is the motion of a charged particle in electromagnetic field, the Lagrangian is

$$L = \frac{m^2}{2}\dot{r}^2 - Q\phi(\vec{r}) + Q\vec{A}(\vec{r}).\dot{\vec{r}},$$

where $\vec{r}$ is the position vector of the particle, Q is the charge, and $\vec{A}$ is the magnetic vector potential. In this example the x-component of the mechanical momentum of the particle is $m\dot{x}$, but the generalized momentum is

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + QA_x\,.$$

If a Lagrangian does not contain a given coordinate q_i but it contains the corresponding velocity $\dot{q}_i$ then the coordinate is said to be *cyclic* or *ignorable.* Then in EL EoM $\partial L \partial q_i = 0$ and it becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \Longrightarrow p_i = \text{constant}\,,$$

so, *generalized momentum corresponding to generalized coordinate is conserved.* In the previous example, of both ϕ and $\vec{A}$ are independent of r then

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + QA_x = \text{constant}$$

2.4.2 Energy conservation

Another conservation theorem is obtained for Lagrangian that has no explicit time dependence, *i.e.,*

$$\frac{\partial L}{\partial t} = 0\,.$$

so the total time deribvative of the Lagrangian is

$$\begin{aligned}\frac{dL}{dt} &= \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\,, \\ &= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\,, \\ &= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) = \sum_i \frac{d}{dt}\Big(p_i\dot{q}_i\Big)\,.\end{aligned}$$

which gives

$$\frac{d}{dt}\left(\sum_i p_i\dot{q}_i - L\right) = 0\,, \tag{19}$$

i.e., the function inside the parenthesis is a constant of motion. It is called the energy function

$$h(q_i, p_i; t) = \sum_i p_i\dot{q}_i - L\,. \tag{20}$$

It can be shown that if L has no explicit time dependence and the potential is dependent of position coordinate only then the energy function is equal to the total energy of the system.

Examples: Simple pendulum, double pendulum, coupled pendulum, etc

3 Motion under Central Force

This chapter is and application of the Lagrangian formulation developed in the previous chapters. We will start by reviewing some basic concepts of rotational motion

3.1 Rotational Motion: Review

A particle rotates about an axis in a circle. If the radius vector $\vec{r}$ makes an angle $d\theta$ in time dt then the instantaneous angular velocity is

$$\omega = \frac{d\theta}{dt} = \dot{\theta}\,. \tag{21}$$

The linear velocity is

$$v = \frac{d}{dt}(r d\theta) = r\omega\,. \tag{22}$$

The direction of ω is along the axis and determined by the right-hand rule, and direction of v is perpendicular to the $\vec{r}$. So we can write the above relation in vectorial form as

$$\vec{v} = \vec{\omega} \times \vec{r}\,. \tag{23}$$

This relates the angular velocity with the linear velocity. As directions are determined by right-hand rule, you can also write

$$\vec{\omega} = \vec{v} \times \vec{r}\,. \tag{24}$$

The angular momentum of the particle about O is

$$\vec{\mathbb{L}} = \vec{r} \times \vec{p}, \tag{25}$$

and the *torque* or moment of force about the same origin is

$$\vec{N} = \vec{r} \times \vec{F} = \vec{r} \times \dot{\vec{p}}, \tag{26}$$

In linear motion rate of change of momentum can be equated to force. In rotational motion, the rate of change of angular momentum is given as

$$\begin{aligned} \frac{d}{dt}\vec{\mathbb{L}} = \frac{d}{dt}(\vec{r} \times \vec{p}) &= \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}, \\ &= m\dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \dot{\vec{p}} = \vec{N} \end{aligned} \tag{27}$$

So the rate of change of angular momentum is equal to the applied torque − *in the absence of any torque, the angular momentum is conserved.* This is the conservation of angular momentum.

3.2 Equivalent one-body problem

Consider motion of two masses m_1 and m_2 where the only force is their interaction potential. The distance between the mass points is not fixed. The dof of the system is six. One may chose to describe the system by three coordinates $\vec{r}_1$ for m_1 and three coordinates $\vec{r}_2$ for m_2. Then the Lagrangian of the system is

$$L = \frac{m_1}{2}\dot{r}_1^2 + \frac{m_2}{2}\dot{r}_2^2 - U(r)\,,$$

where $\dot{r}_{1,2}^2 = |\dot{\vec{r}}_{1,2}|^2$, and $r = |\vec{r}_1 - \vec{r}_2|$ is the distance between them. Out of the six dof, three describes translations and rest of the three describes rotation. Since overall translational motion of the system is unimportant, we shift the origin of the coordinate system to the center of mass (CoM) $\vec{R}$. Then

$$m_1\vec{r}_1 + m_2\vec{r}_2 = (m_1 + m_2)\vec{R} = 0\,,$$

the RHS is zero since we have shifted the origin of the coordinate system to the CoM. The above equation along with $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$\vec{r}_1 = \frac{m_2}{m_1 + m_2}\vec{r}\,, \qquad \vec{r}_2 = -\frac{m_1}{m_1 + m_2}\vec{r}\,.$$

After substituting $\vec{r}_{1,2}$ in terms of $\vec{r}$, the Lagrangian becomes

$$L = \frac{1}{2}\mu\dot{r}^2 - U(r)\,, \tag{28}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}\,. \tag{29}$$

is called the reduced mass of the system. This is now a one-body problem of a particle of mass μ in a central force potential $U(r)$. The dof of the system is now 3.