LA Assignment 7

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[Ref: Hoffmann & Kunze]

1. Let N be the null space for T.
Let the basis for N be $\{\alpha_1, \ldots, \alpha_k\}$
there are vectors $\{\alpha_{k+1}, \ldots, \alpha_n\}$ in V such that
$\{\alpha_1, \ldots, \alpha_k, \ldots, \alpha_n\}$ is a basis for V

We know, $T\alpha_1, \ldots, T\alpha_n$ span the range of T
but since $T\alpha_j = 0, j \le k$, $T\alpha_{k+1}, \ldots, T\alpha_n$ span the range of T.
Checking if $T\alpha_{k+1}, \ldots, T\alpha_n$ are independent, suppose we have scalar c_i ($c_i \in \mathbb{F}$) s. that

$$\sum_{i=k+1}^{n} c_i (T\alpha_i) = 0$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} c_i \alpha_i\right) = 0$$

By definition of null space,
the vector $\vec{\alpha}$ (let) $= \sum_{i=k+1}^{n} c_i \alpha_i$ must belong in N. — ①

$[T\vec{\alpha} = 0]$

But if $\vec{\alpha} \in N$,
$\vec{\alpha}$ can be represented as linear combination of the basis vectors $\alpha_1, \alpha_2, \ldots, \alpha_k$

$\therefore \exists\, b_1, b_2, \ldots, b_k \in \mathbb{F}$ s.that

$$\vec{\alpha} = \sum_{i=1}^{k} b_i \alpha_i \quad — ②$$

From ① & ②

$$\sum_{i=k+1}^{n} c_i \alpha_i - \sum_{i=1}^{k} b_i \alpha_i = 0$$

$\because$ we know $\{\alpha_1, \ldots, \alpha_k\}$ and $\{\alpha_{k+1}, \ldots, \alpha_n\}$ are independent

$\therefore b_1 = b_2 = \ldots = b_k = c_{k+1} = \ldots = c_n = 0$

$\therefore T\alpha_{k+1}, T\alpha_{k+2}, \ldots, T\alpha_n$ are both linearly independent & span the entire range of T

$\therefore T\alpha_{k+1}, \ldots, T\alpha_n$ are basis for range of T

If r is the rank of T, & $T\alpha_{k+1}, \ldots, T\alpha_n$ form a basis for the range of T,

$r = n - k$. — ③

∵ k is the nullity of T, & $n = \dim(V)$

Using ③,

$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V)$

[Hence proved]

2. Let

$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\left[\because (\dim(V) = n)\right.$

$\mathcal{B}' = \{\beta_1, \ldots, \beta_m\}$ $\quad (\dim(W) = m)$

be the ordered bases for V & W respectively.

For each p, q ; $1 \le p \le m$; $1 \le q \le n$; $p, q \in \mathbb{Z}^+$

we define a linear transformation $E^{p,q}$ from V into W by

$$E^{p,q}(\alpha_i) = \begin{cases} 0 & i \neq q \\ \beta_p & i = q \end{cases}$$

$$= \delta_{iq} \beta_p$$

We know that there is a unique linear transformation from V to W that satisfy these given conditions.

Checking whether the 'mn' transformations $E^{p,q}$ forms a basis for $L(V, W)$

Let T be a linear transformation from V into W.

For each j, $1 \le j \le n$, let

$A_{1j}, \ldots, A_{mj}$ be the coordinates of $T\alpha_j$ in the ordered basis $\mathcal{B}'$, ie

$$T\alpha_j = \sum_{p=1}^{m} A_{pj} \beta_p \quad \text{— ①}$$

We wish to show that

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q} \quad — ②$$

Let U be the linear transformation in the right hand member of ②. Then, $\forall j$,

$$U\alpha_j = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}(\alpha_j)$$

$$= \sum_{p} \sum_{q} A_{pq} \delta_{jq} \beta_p$$

$$= \sum_{p=1}^{m} A_{pj} \beta_p$$

$$= T\alpha_j \quad \{\text{from } ①\}$$

$\therefore U = T$ [we have also already noted that linear transformations are unique]

~~Now~~

$\because T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}$,

$E^{p,q}$ spans $L(V,W)$.

If $U = \sum_{p} \sum_{q} A_{pq} E^{p,q}$ is the zero transformation

$U\alpha_j = 0 \quad \forall j$, so

$$\sum_{p=1}^{m} A_{pj} \beta_p = 0 \quad — ③$$

~~β_p forms ... is~~

~~But we already know ... independent set~~

But $\because \beta_p \in B'$ (basis of W)

Each of β_p is linearly independent.

$\therefore A_{pj} = 0$ for ③ to hold $\forall p, j$

$\therefore E^{p,q}$ is independent & also spans $L(V,W)$

$\therefore E^{p,q}$ is the basis for $L(V,W)$

But, we know, $E^{p,q}$ is a mn transformation

$\therefore \dim(L(V,W)) = mn = \dim(V) \times \dim(W)$.

Also, since m, n are finite, mn is also finite. $\therefore L(V,W)$ is finite dimensional.