Tutorial-5

1 Basis and Dimensions

1. Consider the vectors in $\mathbb{R}^4$ defined by

$$\alpha_1 = (-1, 0, 1, 2), \alpha_2 = (3, 4, -2, 5), \alpha_3 = (1, 4, 0, 9)$$

   (a) What is the dimension of the subspace W of $\mathbb{R}^4$ spanned by the three given vectors? Find a basis for W and extend it to a basis $\mathcal{B}$ of $\mathbb{R}^4$.

   (b) Use a basis $\mathcal{B}$ of $\mathbb{R}^4$ as in (a) to characterize all linear transformations $T : \mathbb{R}^4 \to \mathbb{R}^4$ that have the same null space W. What can you say about the rank of such a T? What is therefore the precise condition on the values of T on $\mathcal{B}$?

2. Let V be a vector space with $v = (v_1, v_2, \ldots, v_n)$ as one of its basis. Now we define a new list $\widetilde{v}$ by subtracting from each vector of v (except the first one) its preceding vector, i.e.

$$\widetilde{v} = (v_1, v_2 - v_1, v_3 - v_2, \ldots, v_n - v_{n-1})$$

   Prove that $\widetilde{v}$ is also a basis of V .
   Hint: Using the properties of v as a basis, you have to show the two properties satisfy for $\widetilde{v}$ — it is linearly independent, and it spans V .

3. Let $P_4(F)$ be the vector space of all polynomials of degree at most 4 over field (F). And let p denote a basis of this vector space.

   (a) Find the dimension of $P_4(F)$.

   (b) **Prove or refute**: There is a valid basis $p = (p_0, p_1, p_2, p_3, p_4)$ of $P_4(F)$ such that none of the polynomials p_0, p_1, p_2, p_3, p_4 has degree 3.

4. Let V be a subspace of $\mathbb{R}^n$. Suppose that $\mathbb{B} = \{v_1, v_2, \ldots, v_k\}$ is a basis of the subspace V. Prove that every basis of V consists of k vectors in V.

2 Linear Transformation

1. Let $T : \mathbb{R}^3 \to P_2$ be a linear transformation, where P_2 is the vector space of polynomials in x with real coefficients having degree at most 2, given by

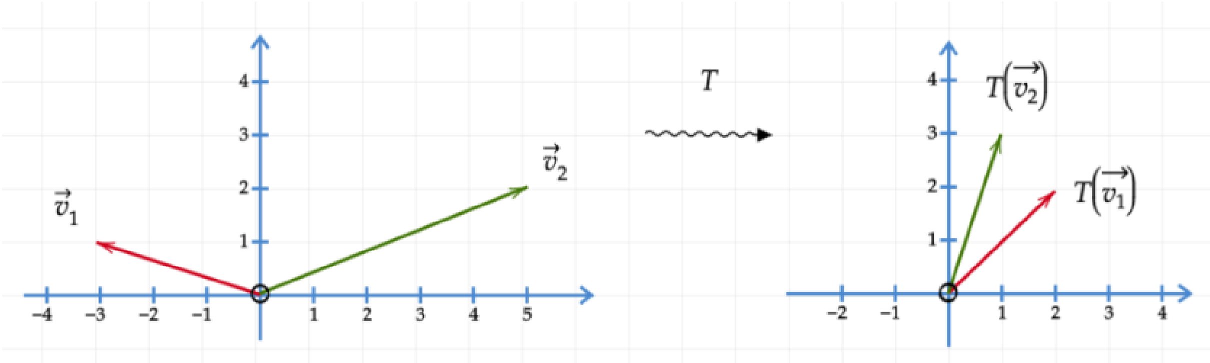
$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a - b)\, x^2 + cx + (a + b + c)$$

Let $\tau = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$ and $\Omega = (x+1, x^2 - x, x^2 + x - 1)$ be the respective bases. Find $[T]_\tau^\Omega$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Let $\tau = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ and $\Omega = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ be ordered basis for $\mathbb{R}^2$. Suppose $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find $[T]_\tau^\Omega$.

3. Let U and V be finite dimensional vector spaces over a scalar field F. Consider a linear transformation $T : U \to V$. Prove that if $dim(U) > dim(V)$, then T cannot be injective (one-to-one).

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$be a linear transformation such that it maps the vectors v_1, v_2 as indicated in the figure below.



Find the matrix representation A of the linear transformation T.

5. Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$.

   (a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?

   (b) Interpret the operation of T geometrically.

   (c) What is the matrix of T in the ordered basis $\mathbb{B} = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1, 2)$ and $\alpha_2 = (1, -1)$?

   (d) Prove that for every real number c the operator $(T - cI)$ is invertible.

6. Let $T, U \in L(V, V)$ be linear operators on the finite dimensional vector space V . Prove that the rank of the composition UT is less than or equal to the minimum of the ranks of T and U.

7. Let V be the vector space over $\mathbb{R}$ of all real polynomial functions p of degree at most 2. For any fixed $a \in \mathbb{R}$ consider the shift operator $T : V \to V$ with $(Tp)(x) = p(x + a)$. Explain why T is linear and find the range and null space of T. Write down the matrix of T with respect to the ordered basis $\mathbb{B} = \{1, x, x^2\}$.