Linear Transformations

Def^n Let V & W be vector spaces over the field F. A linear transformation from V into W is a f^n T from V into W, i.e., $T: V \to W$, s.t.

$$T(c\bar{\alpha}+\bar{\beta}) = c(T\bar{\alpha}) + T\bar{\beta}$$

$\forall\ \bar{\alpha}, \bar{\beta} \in V$ and $\forall\ c \in F$.

Example: If V is any vector space, the identity transformation I, defined by $I\bar{\alpha} = \bar{\alpha}$, is a linear transformation from V into V. The zero transformation 0, defined by $0\bar{\alpha} = 0$, is a linear transformation from V into V.

Remark: If T is a linear transformation $T: V \to W$, then $T(0) = 0$.

$$T(0) = T(0+0) = T(0) + T(0) \Rightarrow T(0) = 0.$$

Linear transformation preserves linear combinations; i.e., $T: V \longrightarrow W$

$\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in V, \quad c_1, c_2, \ldots, c_n \in F$

$$T(c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_n\bar{\alpha}_n) = c_1 T(\bar{\alpha}_1) + c_2 T\bar{\alpha}_2 + \ldots + c_n T\bar{\alpha}_n$$

<u>Thm</u>. Let V be a finite-dim. vector space over the field F. Let $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ be an ordered basis for V. Let W be a vector space over the same field F. Let $\{\bar{\beta}_1, \ldots, \bar{\beta}_n\}$ be any vectors in W. Then there is precisely one linear transformation $T: V \longrightarrow W$ s.t. $T\bar{\alpha}_j = \bar{\beta}_j \quad j = 1, \ldots, n$.

Proof: First we prove $\exists\ T: V \to W$ w/ $T\bar{\alpha}_j = \bar{\beta}_j$.

Given $\bar{\alpha} \in V$, $\exists$ unique n-tupple $(x_1, \ldots, x_n)$ s.t. $\bar{\alpha} = x_1\bar{\alpha}_1 + \ldots + x_n\bar{\alpha}_n$.

For $\bar{\alpha}$, we define

$$T\bar{\alpha} = x_1\bar{\beta}_1 + \ldots + x_n\bar{\beta}_n.$$

Then T is a well-defined rule for associating w/ each vector $\bar{\alpha} \in V$ a vector $T\bar{\alpha} \in W$.

From def^n: $T\bar{\alpha}_j = \bar{\beta}_j$ for each j.

To check if T is linear, let

$$\bar{\beta} = y_1\bar{\alpha}_1 + \ldots + y_n\bar{\alpha}_n \in V, \; c \in F.$$

Now

$$c\bar{\alpha} + \bar{\beta} = (cx_1 + y_1)\bar{\alpha}_1 + \ldots + (cx_n + y_n)\bar{\alpha}_n$$

& so by def^n

$$T(c\bar{\alpha} + \bar{\beta}) = (cx_1 + y_1)\beta_1 + \ldots + (cx_n + y_n)\bar{\beta}_n.$$

On the other hand,

$$c(T\bar{\alpha}) + T\bar{\beta} = c\sum_{i=1}^{n} x_i\bar{\beta}_i + \sum_{i=1}^{n} y_i\beta_i$$

$$= \sum_{i=1}^{n}(cx_i + y_i)\bar{\beta}_i$$

Thus, $T(c\bar{\alpha} + \bar{\beta}) = cT\bar{\alpha} + T\bar{\beta}$.

If U is a linear transformation $T: V \to W$ w/ $U\bar{\alpha}_i = \bar{\beta}_j$, $j = 1, \ldots, n$, then for $\bar{\alpha} = \sum_{i=1}^{n} x_i\bar{\alpha}_i$

~~then for~~ we have,

$$U\bar{\alpha} = U\left(\sum_{i=1}^{n} x_i \alpha_i\right)$$

$$= \sum_{i=1}^{n} x_i (U\bar{\alpha}_i)$$

$$= \sum_{i=1}^{n} x_i \bar{\beta}_i,$$

so that U is exactly the rule T which we defined above. This shows that the linear transformation T w/ $T\bar{\alpha}_j = \bar{\beta}_j$ is unique.

$T : V \longrightarrow W$ Image of T is a subspace of W

Def^n: Let V & W be vector spaces over the field F & let T be a linear trans. from V into W. The null space of T is the set of all vectors $\bar{\alpha} \in V$ s.t. $T\bar{\alpha} = 0$.

If V is fin-dimensional, the rank of T is the dimension of the range of T & the nullity of T is the dim. of the null space of T.

<u>Thm.</u> Let V & W be vector spaces over the field F & $T: V \to W$ be a linear transformation. Suppose V is fin. dim. Then,

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

<u>Thm.</u> If A is an $m \times n$ matrix w/ entries in the field F, then

$$\text{row rank}(A) = \text{column rank}(A).$$

<u>Thm.</u>

The Algebra of linear transformations

Thm. Let V & W be vector spaces over the field F. Let $T: V \to W$, $U: V \to W$ be linear transformations. The f^n $(T+U)$ defined by

$$(T+U)(\bar{\alpha}) = T(\bar{\alpha}) + U(\bar{\alpha})$$

is a linear transformation $(T+U): V \to W$.

If $c \in F$, the f^n (cT) defined by

$$(cT)(\bar{\alpha}) = c(T\bar{\alpha})$$

is a linear transformation $(cT): V \to W$.

The set of all linear transformations from V into W, together w/ the add^n & scalar multiplication defined above, is a vector space over the field F.

- The space of linear transformations $T: V \to W$ to be denoted as $L(V, W)$.

__Thm__. Let V be an n-dim. vector space over the field F, and let W be an m-dim vector space over F. Then the space $L(V,W)$ is finite dim. & has dim. mn ($= \dim V \times \dim W$).

Proof. Let $\mathcal{B} = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ and $\mathcal{B}' = \{\bar{\beta}_1, \ldots, \bar{\beta}_m\}$ be ordered bases for V & W, resp. For each pair of integers (p,q) with $1 \leq p \leq m$ & $1 \leq q \leq n$, we define a linear transformation $E^{p,q}$ from V into W by

$$E^{p,q}(\bar{\alpha}_i) = \begin{cases} 0, & \text{if } i \neq q \\ \bar{\beta}_p, & \text{if } i = q \end{cases}$$

$$= \delta_{iq} \bar{\beta}_p.$$

A/c to a theorem earlier, $\exists$ a unique linear transformation from $V \to W$ satisfying these cond$\underline{\text{n}}$s. The claim is ~~that~~ the mn transformations $E^{p,q}$ form a basis for $L(V,W)$.

Thm. Let V, W, & Z be vector spaces over the field F. Let $T: V \to W$ & $U: W \to Z$ be linear transformations. Then the composed f^n UT defined by $(UT)(\bar{\alpha}) = U(T(\bar{\alpha}))$ is a linear trans.

$$UT: V \longrightarrow Z.$$

Proof: $(UT)(c\bar{\alpha}+\bar{\beta}) = U(T(c\bar{\alpha}+\bar{\beta}))$

$$= U(cT\bar{\alpha}+T\bar{\beta})$$

$$= c\,UT\bar{\alpha} + UT\bar{\beta}$$

$$= c(UT)(\bar{\alpha}) + (UT)(\bar{\beta}).$$

Def^n: If V is a vector space over the field F, a linear operator on V is a linear transformation from V to V.

$L(V,V)$ has a 'multiplication' defined on it by composition. Suppose $T: V \to V$ and $U: V \to V$ for $T, U \in L(V,V)$ are distinct, UT & TU are well-defined. However, in general $UT \neq TU$. $\underbrace{T \circ T \circ \ldots \circ T}_{n \text{ times}} = T^n$.

For $T \neq 0$, we define $T^0 = \mathbb{1}$.

Lemma: Let V be a vector space over the field F; let $U, T_1, T_2 \in L(V,V)$; let $c \in F$.

(a) $\mathbb{1}U = U\mathbb{1} = U$;

(b) $U(T_1 + T_2) = UT_1 + UT_2$;
$(T_1 + T_2)U = T_1U + T_2U$;

(c) $c(UT_1) = (cU)T_1 = U(cT_1)$.

Proof: (a) obvious.

(b) $[U(T_1+T_2)](\bar{\alpha}) = U[(T_1+T_2)(\bar{\alpha})]$

$$= U(T_1\bar{\alpha} + T_2\bar{\alpha})$$

$$= U(T_1\bar{\alpha}) + U(T_2\bar{\alpha})$$

$$= (UT_1)(\bar{\alpha}) + (UT_2)(\bar{\alpha})$$

so that $U(T_1+T_2) = UT_1 + UT_2$.

$$[(T_1+T_2)U](\bar{\alpha}) = (T_1+T_2)(U\bar{\alpha})$$

$$= T_1(U\bar{\alpha}) + T_2(U\bar{\alpha})$$

$$= (T_1U)(\bar{\alpha}) + (T_2U)(\bar{\alpha})$$

so that $(T_1+T_2)U = T_1U + T_2U$.

Note that the proofs of these two distributive laws do not use the fact that T_1 & T_2 are linear, and the proof of the 2nd one does not use the fact that U is linear either.

(c) exercise.

For which linear operators $T : V \to V$ does there exist a linear operator T^{-1} s.t. $TT^{-1} = T^{-1}T = \mathbb{1}$?

F^n $T : V \to W$ is called invertible if $\exists$ a f^n $U : W \to V$ s.t. UT is the identity f^n on V & TU is the identity f^n on W. If T is invertible, the f^n U is unique & is denoted by T^{-1}.

Furthermore, T is invertible iff

1. T is 1:1, i.e., $T\bar{\alpha} = T\bar{\beta} \Rightarrow \bar{\alpha} = \bar{\beta}$.
2. T is onto, i.e., range$(T) = W$.

<u>**Thm.**</u> Let V & W be vector spaces over the field F. Let $T : V \to W$ be a linear trans. If T is invertible, then the inverse f^n T^{-1} is a linear transformation from W to V, i.e., $T^{-1} : W \to V$.

Proof: When T is one-one and onto, there is uniquely determined inverse f^n T^{-1} which maps W onto V s.t. $T^{-1}T$ is the identity f^n on V, and TT^{-1} is the identity f^n on W. We now prove here that if T is a linear f^n that is invertible, the the inverse T^{-1} is also linear.

Let $\bar{\beta}_1, \bar{\beta}_2 \in W$, $c \in F$. We need to show, $T^{-1}(c\bar{\beta}_1 + \bar{\beta}_2) = cT^{-1}\bar{\beta}_1 + T^{-1}\bar{\beta}_2$

Let $\bar{\alpha}_i = T^{-1}\bar{\beta}_i \ \forall i \in \{1,2\}$, i.e., $\bar{\alpha}_i \in V$ is the unique vector s.t. $T\bar{\alpha}_i = \bar{\beta}_i$.

$\because$ T is linear,

$$T(c\bar{\alpha}_1 + \bar{\alpha}_2) = cT\bar{\alpha}_1 + T\bar{\alpha}_2$$
$$= c\bar{\beta}_1 + \bar{\beta}_2.$$

$\therefore$ $c\bar{\alpha}_1 + \bar{\alpha}_2 \in V$ is the unique vector which is sent by T into $c\bar{\beta}_1 + \bar{\beta}_2$, & so

$$T^{-1}(c\bar{\beta}_1 + \bar{\beta}_2) = c\bar{\alpha}_1 + \bar{\alpha}_2$$
$$= cT^{-1}\bar{\beta}_1 + T^{-1}\bar{\beta}_2$$

& T is linear.

For invertible linear transformations:

$T: V \to W$, $U: W \to Z$,

we have invertible linear transformation $UT: V \to Z$ and $(UT)^{-1} = T^{-1}U^{-1}$. Verification of $(UT)^{-1} = T^{-1}U^{-1}$ requires that $T^{-1}U^{-1}$ is both a left and a right inverse of UT.

If T is linear, then $T(\bar{\alpha} - \bar{\beta}) = T\bar{\alpha} - T\bar{\beta}$. Hence, $T\bar{\alpha} = T\bar{\beta}$ iff $T(\bar{\alpha} - \bar{\beta}) = 0$.

$\rightarrow$ verifies that T is 1:1.

A linear transformation T is non-singular if $T\bar{\gamma} = 0$ implies $\bar{\gamma} = 0$, i.e., if the null space of T is $\{0\}$.

T is 1:1 iff T is non-singular.

<u>Thm</u>: Linear trans. $T: V \to W$. T is non-singular iff T carries each linearly independent subset of V onto a linearly indep. subset of W.

A linear transformation may be non-singular w/o being onto and maybe onto w/o being non-singular.

<u>Thm</u>. Let V & W be finite-dim. vector spaces over the field F s.t. $\dim V = \dim W$. If $T: V \longrightarrow W$ is a linear transf., the following are equiv.

(i) T is invertible.

(ii) T is non-singular.

(iii) T is onto, i.e., $\text{range}(T) = W$.

(iv) If $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ is a basis for V, then $\{T\bar{\alpha}_1, \ldots, T\bar{\alpha}_n\}$ is a basis for W.

(v) There is some basis $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ for V s.t. $\{T\bar{\alpha}_1, \ldots, T\bar{\alpha}_n\}$ is a basis for W.