5 Hamiltonian Mechanics

Following two applications of Lagrangian mechanics (central force and coupled oscillations), we are again back to further formulation of classical mechanics. We have already seen the advantages of Lagrangian mechanics in solving mechanics problems. Unfortunately, there is nothing physical that the Lagrangian functional can be associated with. This chapter we develop Hamiltonian mechanics that is based on a functional H, called the *Hamiltonian* which will be functions of a set of n generalized coordinates q_i and a set of n generalized momentum p_i. Interestingly, in many physical problem $H \equiv H(q_i, p_i; t)$ is the total energy of the system. The Hamiltonian formulation paves the way for another important formulation of classical mechanics, the powerful *Hamilton-Jacobi formulation*. More importantly Hamiltonian formulation is key to almost all branches of modern physics, including *quantum mechanics*, *statistical mechanics*, *particle physics*, *astrophysics*, *accelerator physics*, etc.

5.1 Hamilton's EoM

In the Lagrangian mechanics, the Lagrangian functional $L(q_i, \dot{q}_i; t)$ are functions of $2n$ independent variables: these are the n generalized coordinates q_i and the n generalized variables $\dot{q}_i$. For the n independent generalized coordinates, there are n 2nd order differential equations requiring $2n$ initial conditions, say at $t = 0$.

$$\{q_1(t=0) \cdots q_n(t=0), \quad \dot{q}_1(t=0) \cdots \dot{q}_n(t=0)\}\,.$$

In the Hamiltonian formalism, there are again $2n$ independent variables, but there are 1st oder differential equations of the $2n$ variables. Therefore, as in the case of the Lagrangian mechanics, we still need $2n$ inditial conditions to solve the problem. The choice of the $2n$ variables are

- – n variables are the n generalized coordinates $q_i = \{q_1, q_2, \cdots\cdots q_n\}$
- – the rest n variables are n generalized momentum $p_i = \{p_1, p_2, \cdots\cdots p_n\}$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\,.$$

The set (q_i, p_i) are called *canonically conjugate* variables and they constitute a $2n$ dimensional *phase space*.

To obtain the EoMs, consider time derivative of the Lagrangian

$$\begin{aligned}
\frac{dL}{dt} &= \sum_i \frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial t}\,, \\
&= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{dq_i}{dt} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial t}\,, \\
&= \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t}\,,
\end{aligned}$$

So

$$\begin{aligned}
\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) &= -\frac{\partial L}{\partial t}\,, \\
\frac{dH}{dt} &= -\frac{\partial L}{\partial t}\,,
\end{aligned} \tag{56}$$

where

$$\boxed{H \equiv H(q_i, p_i, t) = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \sum_i \dot{q}_i p_i - L(q_i, \dot{q}_i, t)} \tag{57}$$

The functional $H(q_i, p_i, t)$ is called the Hamiltonian of the system. Unlike the Lagrangian, the Hamiltonian is a function of the generalized coordinates and generalized momentum.

If L does not depend explicitly on time then $\frac{\partial L}{\partial t} = 0$ and

$$\frac{dH}{dt} = 0\,, \quad H \text{ is conserved}$$

Recall that it is identical to the energy function h we discussed in chapter 2.

Since $L \equiv L(q_i, \dot{q}_i, t)$ we can write

$$\begin{aligned} dL &= \frac{\partial L}{dq_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt\,, \\ &= \dot{p}_i dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt\,, \text{we used EL EoM and} \quad p_i = \frac{\partial L}{\partial \dot{q}_i} \end{aligned}$$

From the expression $H \equiv H(q_i, p_i, t) = \sum \dot{q}_i p_i - L$ we get

$$\begin{aligned} dH &= p_i d\dot{q}_i + \dot{q}_i dp_i - dL\,, \\ &= p_i d\dot{q}_i + \dot{q}_i dp_i - \left(\dot{p}_i dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt \right) \\ &= \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} \end{aligned} \tag{58}$$

Since $H \equiv H(q_i, p_i, t)$

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t}\,, \tag{59}$$

Comparing the last two equation we get the Hamilton's EoMs

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{aligned}} \tag{60}$$

Example: A particle moving in a straight line under a potential $U(x)$. The Lagrangian is

$$L(x, \dot{x}) = T - U = \frac{1}{2} m\dot{x}^2 - U(x)\,.$$

To set up the Hamiltonian we find the generalized momentum

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\,.$$

So the Hamiltonian of the system is

$$H = p\dot{x} - L = \frac{p^2}{m} - \left(\frac{p^2}{2m} - U(x) \right) = \frac{p^2}{2m} + U(x)\,.$$

Here, the first term is the kinetic energy and the second term is the potential energy. So Hamiltonian is the total energy functional. The two Hamilton's EoMs are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}\,, \quad \dot{p} = -\frac{\partial H}{\partial x} = -\frac{dU}{dx} = F(x)\,.$$

So the first equation is the Newton's definition of momentum. The second equation gives the Newton's second law of motion.

Example: A mass m is constrained to move on a frictionless surface of a vertical cone $\rho = cz$ under uniform gravitational field. To obtain the Hamiltonian for the system, we write the Lagrangain

$$L = \frac{1}{2}m\left[\dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2\right] - mgz\,,$$

$$\frac{1}{2}m\left[(c^2+1)\dot{z}^2 + (cz\dot{\phi})^2\right] - mgz\,.$$

The generalized potential

$$p_z = \frac{\partial L}{\partial \dot{z}} = m(c^2+1)\dot{z}\,, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mc^2z^2\dot{\phi}\,.$$

Hence the Hamiltonian is

$$H = \frac{1}{2m}\left[\frac{p_z^2}{(c^2+1)} + \frac{p_\phi^2}{c^2z^2}\right] + mgz\,.$$

It may seem from the above examples that there is no benefit in using the Hamiltonian formulation over the Lagrangian formulation. You will see in the future that in advanced modern theories such as quantum mechanics, statistical mechanics, particle physics, the Hamiltonian formalism provides distinct advantages. Here we will see the elegance of the Hamiltonian formalism in a special scenario – ignorable coordinates. Let say we have a system with two coordinates q_1, q_2. Their conjugate momentum are p_1 and p_2. So the Hamiltonian is

$$H = H(q_1, q_2, p_1, p_2)\,.$$

If q_2 is ignorable then $p_2 = k$ is a constant. So the Hamiltonian is

$$H = H(q_1, p_1, k)\,.$$

So the Hamiltonian depends on just two variables. Let see the same system from the Lagrangian point of view. The generalized velocities are $\dot{q}_1$ and $\dot{q}_2$. Since, q_2 is ignorable, the Lagrangian is

$$L = L(q_1, \dot{q}_1, \dot{q}_2)\,.$$

So unlike the Hamiltonian, the Lagrangian depends on three variables.

5.2 Time dependence of H

The total time derivative of Hamiltonian $H(q_i, \dot{q}_i, t)$ is given by

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}\,.$$

where the terms within the square bracket vanishes due to Hamilton's EoM (60). The physical significance of the results is that unless the Hamiltonian has explicit time dependence, it is constant in time. This is the energy conservation that we obtained at the end of chapter 2.

5.3 Phase space

For a mechanical system with n dof, the Hamiltonian is described in terms of $2n$ coordinates, the generalized coordinates $q_i \equiv \{q_1, q_2, ...q_n\}$ and canonical momentum $p_i \equiv \{p_1, p_2, ...p_n\}$. The set (q_i, p_i) constitute a $2n$-dimensional coordinate system called the *phase space*. At a given instant, a point in phase space is a set of $2n$ coordinates that uniquely define the state of the system. As the system evolves, it traces points in the phase space that gives the *trajectory* of the system in the phase space. The most interesting aspect of this line of thinking is that if the position and momentum of a system then with these initial conditions the entire system is solved using the Hamilton's EoMs.

Lets take an example of how to construct phase space. Consider a one-dimensional simple harmonic oscillator defined by coordinate x and momentum p. The Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = A\,.$$

Let say the oscillation is started at $x = A$. The total energy of the system is

$$\frac{1}{2}m\omega^2 A^2\,.$$

Since total energy is conserved, we can write

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2\,.$$

We can rewrite this as

$$\frac{x^2}{A^2} + \frac{p^2}{(m\omega A)^2} = 1\,.$$

This equation correspond to an ellipse in the $x-p$ phase space as shown in figure 7. The oscillator starts at $x = A$ when the momentum $p = 0$. The restoring force brings it towards $x = 0$ and the momentum starts to increase. When $x = 0$ the momentum is maximum $p = -m\omega A$. It then overshoots the equilibrium and reaches $x = -A$ when p is again zero. This cycle goes on.

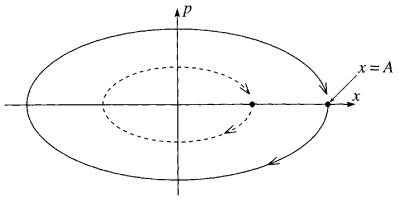

Figure 7: Phase space of a one-domensional harmonic oscillator. The outer circle is for an oscillator with initial condition $x = A$. The inner circle is for initial condition $x = A/2$.

Two things about the phase space trajectories are noteworthy – the arrow indicates the direction of time, and phase space trajectories never cross each other. The reason why phase space trajectories do not cross each other is that each point in phase space uniquely determines the state of the system and the Hamiltonian determines the future evolution. From a given point the system can only evolve in one direction.