12 THE SPECIAL THEORY OF RELATIVITY

12.1 Introduction

In the centuries following publication of the *Principia*, Newtonian dynamics was accepted whole-heartedly not only because of its enormous success in explaining planetary motion but also in accounting for all motions commonly encountered on the Earth. Physicists and mathematicians (often the same people) created elegant reformulations of Newtonian physics and introduced more powerful analytical and calculational techniques, but the foundations of Newtonian physics were assumed to be unassailable. Then, on June 30 1905, Albert Einstein presented his special theory of relativity in his publication *The Electrodynamics of Moving Bodies*. The English translation, available on the web, is reprinted from *Relativity: The Special and General Theory,* Albert Einstein, Methuen, London (1920). The original publication is *Zur Elektrodynamik bewegter Körper*, Annalen der Physik 17 (1905). Einstein's paper transformed our fundamental view of space, time, and measurement.

The reason that Newtonian dynamics went unchallenged for over two centuries is that although we now realize that it is only an approximation to the laws of motion, the approximation is superb for motion with speed much less than the speed of light, $c \approx 3 \times 10^8$ m/s. Relativistic modifications to observations of a body moving with speed v typically involve a factor of v^2/c^2. Most familiar phenomena involve speeds $v \ll c$. Even for the high speed of an Earth-orbiting satellite, $v^2/c^2 \approx 10^{-10}$. There is one obvious exception to this generalization about speed: light itself. Thus, it is hardly surprising that the problems that triggered Einstein's thinking concerned not mechanics but light, problems that grew out of Einstein's early fascination with Maxwell's electromagnetic theory—the theory of light.

12.2 The Possibility of Flaws in Newtonian Physics

The German physicist and philosopher Ernst Mach first pointed out the possibility of flaws in Newtonian thought. Although Mach proposed no changes to Newtonian dynamics, his analysis impressed the young Einstein and was crucial in the revolution shortly to come. Mach's 1883 text *The Science of Mechanics* incorporated the first incisive critique of Newton's ideas about dynamics. Mach carefully analyzed Newton's explanation of the dynamical laws, taking care to distinguish between definitions, derived results, and statements of physical law. Mach's approach is now widely accepted; our discussion of Newton's laws in Chapter 2 is very much in Mach's spirit.

The Science of Mechanics raised the question of the distinction between absolute and relative motion. According to Mach, the fundamental weakness in Newtonian dynamics was Newton's conception of space and time. Newton avowed that he would forgo abstract speculation ("I do not make hypotheses") and deal only with observable facts, but he was not totally faithful to this resolve. In particular, consider the following description of time that appears in the *Principia*. (The excerpt is condensed.) *Absolute, true and mathematical time, of itself and by its own true nature, flows uniformly on, without regard to anything external. Relative, apparent and common time is some sensible and external measure of absolute time estimated by the motions of bodies, whether accurate or inequable, and is commonly employed in place of true time; as an hour, a day, a month, a year.*

Mach commented "it would appear as though Newton in the remarks cited here still stood under the influence of medieval philosophy, as though he had grown unfaithful to his resolve to investigate only actual facts." Mach went on to point out that since time is necessarily measured by the repetitive motion of some physical system, for instance the pendulum of a clock or the revolution of the Earth about the Sun, then the properties of time must be connected with the laws that describe the motions of physical systems. Simply put, Newton's idea of time without clocks is metaphysical; to understand the properties of time we must observe the properties of clocks. As a prescient question, we might inquire whether a time interval observed on a moving clock has the same value as the interval observed on a clock at rest. A simple question? Yes indeed, except that the idea of absolute time is so natural that the eventual consequences of Mach's critique, the relativistic description of time, still comes as something of a shock to students of science.

There are similar weaknesses in the Newtonian view of space. Mach argued that since position in space is determined using measuring rods, the properties of space can be understood only by investigating the properties of meter sticks. For example, does the length of a meter stick observed while it is moving agree with the length of the same meter stick at rest? To understand space we must look to nature, not to Platonic ideals.

Mach's special contribution was to examine the most elemental aspects of Newtonian thought, to look critically at matters that might seem too simple to discuss, and to insist that correctly understanding nature means turning to experience rather than invoking mental abstractions. From this point of view, Newton's assumptions about space and time must be regarded merely as postulates. Newtonian mechanics follows from these postulates, but other assumptions are possible and from them different laws of dynamics could follow.

Mach's critique had no immediate effect but its influence was eventually profound. The young Einstein, while a student at the Polytechnic Institute in Zurich in the period 1897–1900, was much attracted by Mach's work and by Mach's insistence that physical concepts be defined in terms of observables. However, the most urgent reason for superseding Newtonian physics was not Mach's critique but Einstein's recognition that there were inconsistencies in interpreting the results of Maxwell's electromagnetic theory, notwithstanding that Maxwell's theory was considered the crowning achievement of classical physics.

The crucial event that triggered the theory of special relativity and decisively altered physics is generally taken to be the Michelson–Morley experiment, though it is not clear precisely what role this experiment actually played in Einstein's thinking. Nevertheless, most treatments of special relativity take it as the point of departure and we shall follow this tradition.

12.3 The Michelson–Morley Experiment

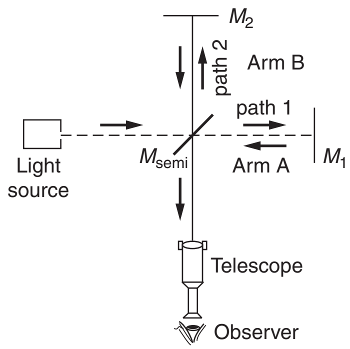
The problem that Michelson attacked was to detect the effect of the Earth's motion on the speed of light. Briefly, Maxwell's electromagnetic theory (1861) predicted that electromagnetic disturbances in empty space would propagate at 3×10^8 m/s—the speed of light. The evidence was overwhelming that light consisted of electromagnetic waves, but there was a serious conceptual difficulty.

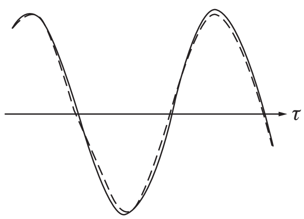
The only waves then known to physics propagated in matter—solid, liquid, or gas. A sound wave in air, for example, consists of alternate regions of higher and lower pressure propagating with a speed of 330 m/s, somewhat less than the speed of molecular motion. The speed of mechanical waves in a metal bar is higher, typically 5000 m/s. The speed of sound increases with the rigidity of the material or the strength of the "spring forces" between neighboring atoms.

Electromagnetic wave propagation seemed to be fundamentally different. By analogy with mechanical waves in matter, electromagnetic waves were assumed to propagate through space as vibrations in a medium called the *ether* that supported electromagnetic wave propagation. Unfortunately, the ether had to possess contradictory properties; immensely rigid to allow light to propagate at 3×10^8 m/s while so insubstantial that it did not interfere with the motion of the planets.

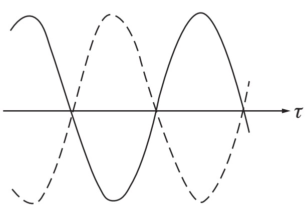
One consequence of the ether hypothesis is that the speed of light should depend on the observer's motion relative to the ether. Maxwell suggested an astronomical experiment to detect this effect. The motion of the planet Jupiter through space relative to the Earth should affect the speed with which its light reaches us. The periodic eclipses of the moons of Jupiter create a clock. The clock should appear to periodically advance or fall behind, as the speed of light increases or decreases as Jupiter approaches to and recedes from the Earth. The effect turned out to be too small to be measured accurately. Nevertheless, Maxwell's proposal was historically important: it stimulated Albert A. Michelson, a young U.S. Navy officer at Annapolis, to invent a laboratory experiment for measuring the Earth's motion through the ether.

The following explanation of the Michelson–Morley experiment assumes some familiarity with optical interference. If you do not yet know about interference, you can skip the description and take the conclusion on faith: the speed of light is always the same, regardless of the relative motion of the source and the observer.

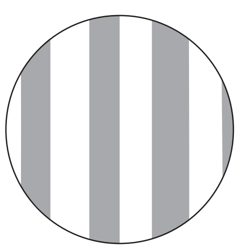




(a) in phase



(b) 180° out of phase



Michelson's apparatus was an optical interferometer. As shown in the drawing, light from a source is split into two beams by a semi-silvered mirror M_{semi} that reflects half the light and transmits half. Half of the beam from the light source travels straight ahead on path 1, passing through M_{semi} until it is reflected by mirror M_1. It then returns to mirror M_{semi}, and half is reflected to the observer. The remainder of the beam from the light source is the half that is reflected by M_2, along path 2. It is reflected by mirror M_2, which directs it to the observer after passing again through M_{semi}. Thus beams 1 and 2 each have 1/4 the intensity of the initial beam.

If beams 1 and 2 travel the same distance, they arrive at the observer in phase so that their electric fields add. The observer sees light. However, if the path lengths differ by *half* a wavelength, the fields arrive out of phase and cancel so that no light reaches the observer. In practice, the two beams are slightly misaligned and the observer sees a pattern of bright and dark interference fringes.

If the length of one of the arms is slowly changed, the fringe pattern moves. Changing the difference in path lengths by one wavelength shifts the pattern by one fringe.

The motion of the Earth through the ether should cause a difference between the times for light to transit the two arms of the interferometer, just as if there were a small change in the distance. The difference in transit time depends on the orientation of the arms with respect to the velocity of the Earth through the ether.

We suppose that the laboratory moves through the ether with speed v and that arm A lies in the direction of motion while arm B is perpendicular. According to the ether hypothesis, an observer moving toward the source of a light signal with speed v will observe the signal to travel with speed $c + v$, while for motion away from the source the speed is $c - v$.

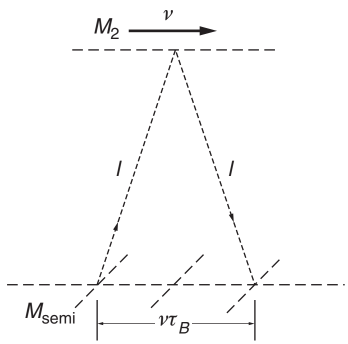
If the length of the arms from the partially silvered mirror M_{semi} to their ends is l, then the time interval for the light to go from M_{semi} to M_1 and return along arm A is τ_A, where

$$\tau_A = \frac{l}{c+v} + \frac{l}{c-v} = \frac{2l}{c}\left(\frac{1}{1 - v^2/c^2}\right).$$

Because $v^2/c^2 \ll 1$, we can simplify the result using the Taylor series expansion in Note 1.3: $1/(1-x) = 1 + x + x^2 + \cdots$. Letting $x = v^2/c^2$ we have

$$\tau_A \approx \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right).$$

Arm B is perpendicular to the motion so the speed of light is not affected by the motion. However, there is nevertheless a time delay due to the motion of M_{semi} as the light traverses the arm. Denoting the round trip time by τ_B, then during that interval M_{semi} moves a distance $v\tau_B$.



Consequently, the light travels along the hypotenuse of the right triangles shown in the sketch, and the distance traveled is $2\sqrt{l^2 + (v\tau_B/2)^2}$.

Consequently,

$$\tau_B = \frac{2}{c}\sqrt{l^2 + (v\tau_B/2)^2},$$

which gives

$$\tau_B = \frac{2l}{c}\frac{1}{\sqrt{1 - v^2/c^2}}.$$

Using the approximation $1/\sqrt{1-x} = 1 + (1/2)x + (1/8)x^2 + \cdots$, and keeping the first term, we have

$$\tau_B = \frac{2l}{c}\left(1 + \frac{1}{2}v^2/c^2\right).$$

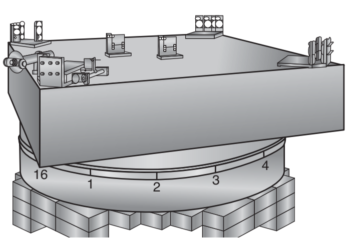
The difference in time for the two paths is

$$\Delta\tau = \tau_A - \tau_B \approx \frac{l}{c}\left(\frac{v^2}{c^2}\right).$$

The frequency of light ν is related to its wavelength λ and the speed of light by $\nu = c/\lambda$. The interference pattern shifts by one fringe for each cycle of delay. Consequently, the number of fringe shifts caused by the time difference is

$$N = \nu\Delta T = \frac{l}{\lambda}\left(\frac{v^2}{c^2}\right).$$

The orbital speed of the Earth around the Sun gives $v/c \approx 10^{-4}$. Taking the path length $l = 1.2$ m, and using sodium light for which $\lambda = 590 \times 10^{-9}$ m, Michelson predicted a fringe shift of $N = 0.02$. In his initial attempt in 1881, Michelson searched for a fringe shift as the rotation of the Earth changed the direction of motion through the ether, but could detect none to within experimental accuracy.



In 1887 Michelson repeated the experiment in collaboration with the chemist Edward Morley using an apparatus mounted on a granite slab 35 cm thick that floated on mercury and could be continuously rotated. The path length was extended by a factor of 10 using repeated reflections between the mirrors. However, again no fringe shift. The Michelson–Morley experiment has been refined and repeated over the years but no effect of motion through the ether has ever been detected. We are forced to recognize that the speed of light is unaffected by motion of the observer through the ether. Ironically, Michelson, who conceived and executed the experiment for which he is famous, viewed it as a failure. He set out to see the effect of motion through the ether but could not detect any.

Various attempts to explain the null result of the Michelson–Morley experiment introduced such complexity as to threaten the foundations of electromagnetic theory. One attempt was the hypothesis proposed by the

Irish physicist G.F. FitzGerald and the Dutch physicist H.A. Lorentz that motion of the Earth through the ether caused a shortening of one arm of the Michelson interferometer (the "Lorentz–FitzGerald contraction") by exactly the amount required to eliminate the fringe shift. Other theories that involved such artifacts as drag of the ether by the Earth were even less productive. The elusive nature of the ether remained a troubling enigma.

12.4 The Special Theory of Relativity

It is an indication of Einstein's genius that the troublesome problem of the ether pointed the way not to complexity and elaboration but to a simplification that unified the fundamental concepts of physics. Einstein regarded the difficulty with the ether not as a fault in electromagnetic theory but as an error in basic dynamical principles. He presented his ideas in the form of two postulates, prefacing them with a note on simultaneity and how to synchronize clocks.

12.4.1 Synchronizing Clocks

Before presenting his theory of space and time, Einstein considered the elementary process of comparing measurements of time by different observers having identical clocks. For the measurements to agree, the clocks must be *synchronized*—they must be adjusted to agree on the time of a single event. In Newtonian physics, if a flash of light occurs, the flash arrives simultaneously at all synchronized clocks, wherever their locations.

The Newtonian procedure would work if the speed of light were infinite or so large that it could be regarded as infinite. However, if one accepts that signals can propagate no faster than the speed of light, the procedure is wrong in principle. For instance, a signal from the Moon to the Earth takes about one second. One might attempt to synchronize a clock on the Moon with a clock on the Earth by advancing the Moon clock by one second. With this adjustment, the Moon clock would always appear to agree with the Earth clock. However, for the observer on the Moon, the Earth clock would always lag the Moon clock by two seconds. Thus the clocks would be synchronized for one observer but not the other.

Einstein proposed a simple procedure for synchronizing clocks so that all observers agree on the time of an event. Observer A sends observer B a signal at time T_A. Observer B notes that the signal arrives at time T_B on the local clock. B immediately sends a signal back to A who detects it at time $T'_A = T_A + \Delta T$. The clocks are synchronized if *B's* clock reads $T_B = T_A + \Delta T/2$. Interpreting the times reported by different observers requires knowing their positions, but everyone would agree on the time of an event.

Einstein thought about time measurements in terms of railway clocks at different stations for which light propagation times are of no practical importance. Today, Einstein's procedure for synchronizing clocks is crucially important: it is essential for comparing atomic clocks in international time standards laboratories, as well as for keeping the Internet synchronized and for maintaining the voltage–current phase across the national power grid.

12.4.2 The Principle of Relativity

The special theory of relativity rests on two postulates. The first, known as the principle of relativity, is that the laws of physics have the same form with respect to all inertial systems. In Einstein's words: "The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion." The principle of relativity was hardly novel; Galileo is credited with first pointing out that there is no dynamical way to determine whether one is moving uniformly or is at rest, and Newton gave it rigorous expression in his dynamical laws in which acceleration, not velocity, is paramount. If the principle of relativity were not true, energy and momentum might be conserved in one inertial system but not in another. The principle of relativity played only a minor role in the development of classical mechanics: Einstein elevated it to a keystone of dynamics. He extended the principle to include not only the laws of mechanics but also the laws of electromagnetic interaction and *all* the laws of physics. Furthermore, in his hands the principle of relativity became a powerful tool for discovering the correct form of physical laws.

We can only guess at the sources of Einstein's inspiration, but they must have included the following consideration. If the speed of light were not a universal constant, that is, if the ether could be detected, then the principle of relativity would fail; a special inertial frame would be singled out, the one at rest in the ether. However, the form of Maxwell's equations, as well as the failure of any experiment to detect motion through the ether, cause us to conclude that the speed of light is independent of the motion of the source. Our inability to detect absolute motion, either with light or with Newtonian dynamics, forces us to accept that absolute motion has no role in physics.

The Universal Speed

The second postulate of relativity is that the speed of light is a universal constant, the same for all observers. "Any ray of light moves in the stationary system of co-ordinates with the determined speed c, whether the ray be emitted by a stationary or by a moving body." Einstein argued that because the speed of light c predicted by electromagnetic theory involves no reference to a medium, then no matter how we measure the speed of light the result will *always* be c, independent of our motion. This is in contrast to the behavior of sound waves, for example, where the observed speed of the wave depends on the motion of the observer through the medium. The idea of a universal speed was indeed a bold hypothesis, contrary to all previous experience and, for many of Einstein's contemporaries, defying common sense. But common sense can be a poor guide. Einstein once quipped that common sense consists of the prejudices one learns before the age of eighteen.

Rather than regarding the absence of the ether as a paradox, Einstein saw that the concept of a universal speed preserved the simplicity of the principle of relativity. His view was essentially conservative; he insisted on maintaining the principle of relativity that the ether would destroy. The urge toward simplicity appeared to be fundamental to Einstein's personality. The special theory of relativity was the simplest way to preserve the unity of classical physics.

To summarize, the postulates of special relativity are: *The laws of physics have the same form in all inertial systems. The speed of light in empty space is a universal constant, the same for all observers regardless of their motion.*

These postulates require us to revise our ideas about space and time, and this has immediate consequences for physics. The mathematical expression of kinematics and dynamics in the special theory of relativity is embodied in the Lorentz transformation—a simple prescription for relating events in different inertial systems.

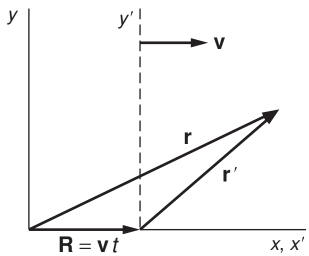
12.5 Transformations

In the world of relativity, a transformation is a set of equations that relate observations in one coordinate system to observations in another. As you will see, the logic of special relativity is reasonably straightforward and the mathematics is not arcane. Nevertheless, the reasoning is likely to seem perplexing because of the underlying question "Isn't this a peculiar way to do physics?" The answer is "Yes! This is a most peculiar way to do physics!" Rather than examining forces, conservation laws, dynamical equations, and other staples of Newtonian physics, Einstein merely discussed how things look to different observers.

Einstein was the first person to use transformation theory to discover new physical behavior, in particular, to create the theory of special relativity. From two simple assumptions, he derived a new way to look at space and time and discovered a new system of dynamics.

Special relativity can be written with all the elegance of a beautiful mathematical theory but its most attractive attribute is that it not only looks beautiful, it works beautifully. The theory of special relativity is among the most carefully studied theories in physics and its predictions have always been correct within experimental error.

The heart of special relativity is the Lorentz transformation, but to introduce Einstein's approach let us first look at the corresponding procedure for Newtonian physics where the transformation is known as the Galilean transformation.



12.5.1 The Galilean Transformation

We will frequently refer to observations in two standard inertial systems: $S = (x, y, z, t)$ and $S' = (x', y', z', t')$. S' moves with respect to S at speed v in the x direction. Alternatively, S moves with respect to S' at speed v in the negative x direction. For convenience, we take the origins to coincide at $t = 0$, and take the x and x' axes to be parallel.

If a particular point in space has coordinates $\mathbf{r} = (x, y, z)$ in S, the coordinates in S' are $\mathbf{r}' = (x', y', z')$. These are related by

$$\mathbf{r}' = \mathbf{r} - \mathbf{R},$$

where

$$\mathbf{R} = \mathbf{v}t.$$

Since $\mathbf{v}$ is in the x direction, we have

$$\begin{aligned} x' &= x - vt, \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned} \tag{12.1}$$

The fourth equation $t' = t$, listed merely for completeness, is taken for granted in Newtonian dynamics, and follows immediately from the Newtonian concept of "ideal" time.

Equations (12.1) are known as the *Galilean transformation*. Because the laws of Newtonian mechanics hold in all inertial systems, the form of the laws is unaffected by this transformation. More concretely, there is no way to distinguish between systems on the basis of the motion they predict. The following example illustrates what this means.

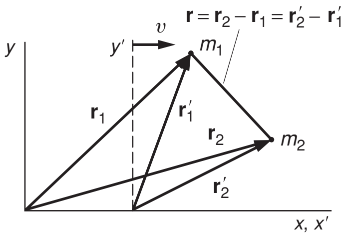
Example 12.1 Applying the Galilean Transformation

Consider how we might discover the law of force between two isolated bodies from observations of their motion. For example, the problem might be to discover the law of gravitation from data on the elliptical orbit of one of Jupiter's moons. If m_1 and m_2 are the masses of the moon and of Jupiter, respectively, and if $\mathbf{r_1}$ and $\mathbf{r_2}$ are their positions relative to an observer on the Earth, we have

$$\begin{aligned} m_1\ddot{\mathbf{r}}_1 &= \mathbf{F}(r) \\ m_2\ddot{\mathbf{r}}_2 &= -\mathbf{F}(r), \end{aligned}$$

where we assume that the force $\mathbf{F}$ between the bodies is a central force that depends only on the separation $r = |\mathbf{r}_2 - \mathbf{r}_1|$. From our observations of $\mathbf{r}_1(t)$ we can evaluate $\ddot{\mathbf{r}}_1$, from which we obtain the value of $\mathbf{F}$. Suppose the data reveal that $\mathbf{F}(r) = -Gm_1m_2\hat{\mathbf{r}}/\mathbf{r^2}$.

Now let us look at the problem from the point of view of an observer in a spacecraft that is moving with constant speed far from the Earth.



According to the principle of relativity this observer must obtain the same force law as the earthbound observer. The situation is represented in the drawing. x, y is the earthbound system, x', y' is the spacecraft system, and $\mathbf{v}$ is the relative velocity of the two systems. Note that the vector $\mathbf{r}$ from m_1 to m_2 is the same in both coordinate systems.

In the x', y' system the observer sees that the moon is accelerating at rate $\ddot{\mathbf{r}}'_1$ and concludes that the force is

$$\mathbf{F}'(r) = m_1\ddot{\mathbf{r}}'_1.$$

A fundamental property of the Galilean transformation is that acceleration is unaltered. Here is the formal proof: because $\dot{\mathbf{v}} = 0$, we have

$$\begin{aligned}\mathbf{r}_1 &= \mathbf{r}'_1 + \mathbf{v}t\\ \dot{\mathbf{r}}_1 &= \dot{\mathbf{r}}'_1 + \mathbf{v}\\ \ddot{\mathbf{r}}_1 &= \ddot{\mathbf{r}}'_1.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbf{F}'(r') &= m_1\ddot{\mathbf{r}}'_1\\ &= m_1\ddot{\mathbf{r}}_1\\ &= \mathbf{F}(r)\\ &= -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}.\end{aligned}$$

The law of force is identical in the two systems. This is what we mean when we say that two inertial systems are equivalent. If the form of the law, or the value of G, were not identical we could make a judgment about the speed of a coordinate system in empty space by investigating the law of gravitation in that system. The inertial systems would not be equivalent.

Example 12.1 is almost trivial because the force depends on the separation of the two particles, a quantity that is unchanged (invariant) under the Galilean transformation. All forces in Newtonian physics are due to interactions between particles, interactions that depend on the *relative* coordinates of the particles. Consequently, they are invariant under the Galilean transformation.

What happens to the equation for a light signal under the Galilean transformation? The following example shows the difficulty.

Example 12.2 Describing a Light Pulse by the Galilean Transformation

At $t = 0$ a pulse of light is emitted from the origin of the S system, and travels along the x axis at speed c. The equation for the location of the pulse along the x axis is $x = ct$.

In the S' system, the equation for the wavefront along the x' axis is

$$\begin{aligned} x' &= x - vt \\ &= (c - v)t, \end{aligned}$$

where v is the relative velocity of the two systems. The speed of the pulse in the S' system is

$$\frac{dx'}{dt} = c - v.$$

But this result is contrary to the postulate that the speed of light is always c, the same for all observers.

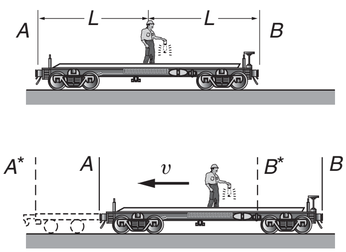
Because the Galilean transformation is incompatible with the principle that the speed of light is always c, our task is to find a transformation that is compatible. Before undertaking this, it is useful to think carefully about the nature of measurement.

The Galilean transformation relates the spatial coordinates of an event measured by observers in two inertial systems moving with relative speed v. By an "event" we mean the unique values of a set of coordinates in space and time. Physically meaningful measurements invariably involve more than a single event. For instance, measuring the length of a rod involves placing the rod along a calibrated scale such as a meter stick, and recording the position at each end. Consequently, length involves two measurements. If the rod is at rest along the axis in the S system, the coordinates of its end points might be x_a and x_b, where $x_b = x_a + L$. According to an observer in the S' system, the x' coordinates are given by Eq. (12.1): $x'_a = x_a - vt$ and $x'_b = x_b - vt$. Since $x_b = x_a + L$, we have $L = x_b - x_a$ and $L' = x'_b - x'_a = L$. The two observers agree on the length.

In this simple exercise in measurement we have made a natural assumption: the measurements are made *simultaneously*. This is not important in the S system because the rod is at rest. However, in the S' system the rod is moving. If the end points were recorded at different times, the value for L' would have been incorrect. We have used the Galilean assumption $t' = t$, which implies that if measurements are simultaneous in one coordinate system they are simultaneous in all coordinate systems. This would be the case if the speed of light were infinite, but the finite speed of light profoundly affects our idea of simultaneity. We therefore digress briefly to examine the nature of simultaneity.

12.6 Simultaneity and the Order of Events

We have an intuitive idea of simultaneity: two events are simultaneous if their time coordinates have the same value. However, as the following example shows, events that are simultaneous in one coordinate system are not necessarily simultaneous when observed in a different coordinate system.



Example 12.3 Simultaneity

A railwayman stands at the middle of a flatcar of length $2L$. He flicks on his lantern and a light pulse travels out in all directions with the velocity c.

Light arrives at the ends of the car after a time interval L/c. In this system, the flatcar's rest system, the light arrives simultaneously at the end points A and B.

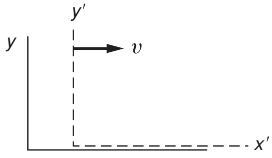
Now let us observe the same situation from a frame moving to the right with velocity v. In this frame the flatcar moves to the left with velocity v. As observed in this frame the light still has velocity c, according to the second postulate of special relativity. However, during the transit time, A moves to A^* and B moves to B^*. It is apparent that the pulse arrives at B^* before A^*; the events are not simultaneous in this frame.

Just as events that are simultaneous in one inertial system may not be simultaneous in another, it can be shown that events that are spatially coincident— having the same coordinates in space—in one system may not appear to be coincident in another. We shall show later that two events can be classified as either *spacelike* or *timelike*. For spacelike events it is impossible to find a coordinate system in which the events coincide in space, though there is a system in which they are simultaneous in time. For timelike events it is impossible to find a coordinate system in which the events are simultaneous in time, though there is a system in which they coincide in space.

At this point we need a systematic way to solve the problem of relating observations made in different inertial systems in a fashion that obeys the principle of relativity. This task constitutes the core of special relativity.

12.7 The Lorentz Transformation

The failure of the Galilean transformation to satisfy the postulate that the speed of light is a universal constant constituted a profound dilemma. Einstein solved the dilemma by introducing a new transformation law for relating the coordinates of events as observed in different inertial systems. He introduced a system designed to ensure that a signal moving at the speed of light in one system would be observed to move at the same speed in the other, irrespective of the relative motion. Such a "fix" took some courage because to alter a transformation law is to alter the fundamental relation between space and time.



Let us refer once more to our standard systems, the S system (x, y, z, t), and the S' system (x', y', z', t'). The system S' moves with velocity v along the positive x axis, and the origins coincide at $t = t' = 0$. We take the most general transformation relating the coordinates of a given event in the two systems to be of the form

$$x' = Ax + Bt \tag{12.2a}$$

$$y' = y \tag{12.2b}$$

$$z' = z \tag{12.2c}$$

$$t' = Cx + Dt. \tag{12.2d}$$

Some comments on Eqs. (12.2): the transformation equations are linear because a nonlinear transformation could yield an acceleration in one system even if the velocity were constant in the other. Further, we leave the y' and z' axes unchanged, by symmetry.

Here is one model to justify the assumption that $y' = y$ and $z' = z$: consider two trains on parallel tracks. Each train has an observer holding a paint brush at the same height in their system, say at 1 m above the floor of the train. Each train is close to a wall. The trains approach at relative speed v, and each observer holds the brush to the wall, leaving a stripe. Observer 1 paints a blue stripe and observer 2 paints a yellow stripe.

Suppose that observer 1 sees that the height of observer 2 has changed, so that the blue stripe is below the yellow stripe. Observer 2 would have to see the same phenomenon except that it is now the yellow stripe that is below the blue stripe. Because their conclusions are contradictory they cannot both be right. Since there is no way to distinguish between the systems, the only conclusion is that both stripes are at the same height. We conclude that distance perpendicular to the direction of motion is unchanged by the motion of the observer.

We can evaluate the four constants A, B, C, D in Eqs. (12.2) by comparing coordinates for four events. These could be:

(1) The origin of S is observed in S':
$S : (x = 0, t)$; $S' : (x' = -vt', t')$.
From Eqs. (12.2a) and (12.2d), $-vt' = 0 + Bt$, and $t' = 0 + Dt$.
Consequently, $B = -vD$.

(2) The origin of S' is observed in S:
$S' : (x' = 0, t')$; $S : (x = +vt, t)$.
From Eq. (12.2a), $0 = Avt + Bt$.
Consequently, $B = -vA$, and using result (1), if follows that
$D = A$.

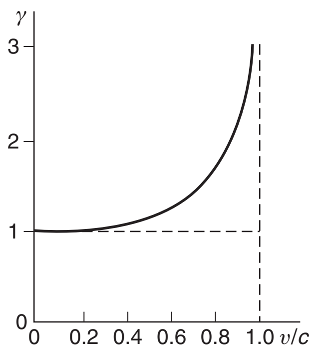
(3) A light pulse is emitted from the origin at $t = 0, t' = 0$ and is observed later along the x and x' axes.
$S : (x = ct, t)$; $S' : (x' = ct', t')$.
From Eqs. (12.2a) and (12.2d), $ct' = ctA + Bt$, and $t' = ctC + Dt$
Using $D = A$ and $B = -vA$, it follows that
$C = -(v/c^2)A$.

(4) A light pulse is emitted from the origin at $t = 0, t' = 0$ and is observed later along the y axis in the S system. In S, $x = 0, y = ct$, but in S' the pulse has both x' and y' coordinates.

$S : (x = 0, y = ct, t)$; $S' : (x' = -vt', y' = \sqrt{(ct)^2 - (-vt')^2}, t')$.

From Eqs. (12.2b) and (12.2d), $ct = \sqrt{(ct')^2 - (-vt')^2}$ and $t' = Dt$ which give

$D = 1/\sqrt{1 - v^2/c^2}$.



(We selected the positive sign for the square root because otherwise t and t' would have opposite signs.) The factor $1/\sqrt{1 - v^2/c^2}$ occurs so frequently that it is given a special symbol:

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Note that $\gamma \geq 1$ and that as $v \to c$, $\gamma \to \infty$.

Substituting our results in Eqs. 12.2 yields

$$x' = \gamma(x - vt) \tag{12.3a}$$

$$y' = y \tag{12.3b}$$

$$z' = z \tag{12.3c}$$

$$t' = \gamma\left(t - vx/c^2\right). \tag{12.3d}$$

The transformation from S' to S can be found by letting $v \to -v$:

$$x = \gamma(x' + vt') \tag{12.4a}$$

$$y = y' \tag{12.4b}$$

$$z = z' \tag{12.4c}$$

$$t = \gamma\left(t' + vx'/c^2\right). \tag{12.4d}$$

Equations (12.3) and (12.4) are the prescription for relating the coordinates of an event in different inertial systems so as to satisfy the postulates of special relativity. They are called the *Lorentz transformation* after the physicist Hendrik Lorentz who first wrote them, though in a very different context.

The Lorentz transformation equations have a straightforward physical interpretation. The factor γ is a scaling factor that ensures that the speed of light is the same in both systems. The factor vt in Eq. (12.3a) reveals that system S' is moving in the positive x direction, with speed v. The factor vx/c^2 in Eq. (12.3d) is a little more subtle. The clock synchronization algorithm requires that the time registered on a clock be corrected for the transit time τ_{transit} from the event point. If the point is moving with speed v, then the transit time correction must be adjusted correspondingly. The additional distance traveled is $d = v\tau_{\text{transit}}$, where $\tau_{\text{transit}} = x/c$. Hence, the time in Eq. (12.3d) needs to be corrected by the quantity $d/c = vx/c^2$.

In the limit $v/c \to 0$ (or alternatively $c \to \infty$), where $\gamma \to 1$, the Lorentz transformation becomes identical to the Galilean transformation. However, in the general case, the Lorentz transformation requires a rethinking of the concepts of space and time.

Before looking into the consequences of this rethinking, let us examine how the Lorentz transformation demonstrates why Michelson's experiment had to give a null result.

12.7.1 Michelson–Morley Revisited

With the Lorentz transformation in hand we can understand why the Michelson–Morley experiment failed to display any fringe shift as the apparatus was rotated. We introduce again the two reference systems $S : (x, y, z, t)$ and the system $S' : (x', y'z', t')$ moving with relative speed v along x. Their origins coincide at $t = t' = 0$. A pulse of light is emitted at $t = 0$ in the S system and spreads spherically. The locus of the pulse is given by $x^2 + y^2 + z^2 = (ct)^2$. We leave it as an exercise to show that the Lorentz transformation, Eqs. (12.3), predicts that in the "moving" S' system the locus of the pulse is given by $x'^2 + y'^2 + z'^2 = (ct')^2$. Observers in the two frames see the same phenomenon: a pulse spreading in space with the speed of light c. There is no trace of a reference to their relative speed v.

The Michelson–Morley experiment was designed to show the difference in the speed of light between directions parallel and perpendicular to the Earth's motion but according to the second postulate of the special theory of relativity—the speed of light is the same for all observers—there should be none. The Lorentz transformation shows explicitly that there is none.

12.8 Relativistic Kinematics

Because the principles of special relativity require us to rethink basic ideas of measurement and observation, they have important consequences for dynamics. The goal for the rest of this chapter is to learn how the principles of special relativity are employed to relate measurements in different inertial systems. The motivation for this is to some extent practical: relativistic kinematics is essential in areas of physics ranging from elementary particle physics to cosmology and also to technologies such as the global positioning system. More fundamentally, the study of relativistic transformations leads to new physics, most famously the relation $E = mc^2$, and to an elegant unified approach to dynamics and electromagnetic theory.

Predictions of the Lorentz transformation often defy intuition because we lack experience moving at speeds comparable to the speed of light. Two surprising predictions are that a moving clock runs slow and a moving meter stick contracts. These follow from the Lorentz transformation of time and space intervals. We will also derive these results by geometric arguments that may help provide intuition about this unexpected behavior.

Caution: in the discussion to follow, *either* S or S' may be the rest system for an observer, and in addition there is the possibility of introducing other systems. We need to be clear not only about the physical phenomena taking place but the system from which it is being observed.

12.8.1 Time Dilation

A clock is at rest in S at some location x. The clock's rate is determined by the interval τ_0 between its ticks. The problem is to find the corresponding interval observed in the S' system, in which the clock is moving with speed $-v$.

Successive ticks of the clock in the rest system S are

$$\begin{aligned} &\text{tick 1 (event 1)}: \quad t \\ &\text{tick 2 (event 2)}: \quad t+\tau_0. \end{aligned}$$

The corresponding times observed in the moving system S' are, from Eq. (12.3d),

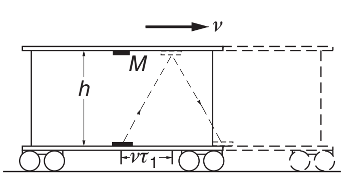
$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ t' + \tau_0' &= \gamma(t + \tau_0 - vx/c^2). \end{aligned}$$

Subtracting, we obtain

$$\tau_0' = \gamma\tau_0. \tag{12.5}$$

Because $\gamma \geq 1$, the time interval observed in the moving system is longer than in the clock's rest system. Thus, the moving clock runs slow. As $v \to c$, time stands still.

This result, known as *time dilation*, is hardly intuitive and so it may be instructive to derive it by a different approach.



Let us consider an idealized clock in which the timing element consists of two parallel mirrors with a light pulse bouncing between them. (Our discussion follows *Introduction to Electrodynamics*, David J. Griffiths, Prentice Hall, Upper Saddle Ridge New Jersey, 1999.)

Each round trip of the light constitutes a clock tick. The clock is mounted vertically on a railway car that moves with speed v, as shown. An observer on the railway car monitors the rate of the ticks. If the distance between the mirrors is h, then the time interval between ticks is

$$\tau_0 = 2h/c.$$

In this calculation, the railway car is the rest system S for the clock.

An observer on the ground system S' also monitors the rate of ticks of the clock. S' is moving at speed $-v$ with respect to the rest system on the railway car. For this observer, the time interval for the light, up or down, is $\tau_1 = \sqrt{h^2 + (v\tau_1)^2}/c$. Solving for τ_1, the roundtrip time τ_0' is

$$\tau_0' = 2\tau_1 = (2h/c)\frac{1}{\sqrt{1 - v^2/c^2}}.$$

Recalling that $\gamma = 1/\sqrt{1 - v^2/c^2}$, we have

$$\tau_0' = \gamma\tau_0$$

in agreement with Eq. (12.5).

Example 12.4 The Role of Time Dilation in an Atomic Clock

Possibly you have looked through a spectroscope at the light from an atomic discharge lamp. Each line of the spectrum is the light emitted when an atom makes a transition between two of its internal energy states. The lines have different colors because the frequency ν of the light is proportional to the energy change ΔE in the transition. If ΔE is of the order of electron-volts, the emitted light is in the optical region ($\nu \approx 10^{15}$ Hz). There are some transitions, however, for which the energy change is so small that the emitted radiation is in the microwave region ($\nu \approx 10^{10}$ Hz). These microwave signals can be detected and amplified with available electronic instruments. Since the oscillation frequency depends almost entirely on the internal structure of the atom, the signals can serve as a frequency reference to govern the rate of an atomic clock. Atomic clocks are highly stable and relatively immune to external influences.

Each atom radiating at its natural frequency serves as a miniature clock. The atoms are frequently in a gas and move randomly with thermal velocities. Because of their thermal motion, the clocks are not at rest with respect to the laboratory and the observed frequency is shifted by time dilation.

Consider an atom that is radiating its characteristic frequency ν_0 in the rest frame. We can think of the atom's internal harmonic motion as akin to the swinging motion of the pendulum of a grandfather clock: each cycle corresponds to a complete swing of the pendulum. If the period of the swing is τ_0 seconds in the rest frame, the period in the laboratory is $\tau = \gamma\tau_0$. The observed frequency in the laboratory system is

$$\begin{aligned} \nu &= \frac{1}{\tau} = \frac{1}{\gamma\tau_0} = \frac{\nu_0}{\gamma} \\ &= \nu_0\sqrt{1 - \frac{v^2}{c^2}}. \end{aligned}$$

The shift in the frequency is $\Delta\nu = \nu - \nu_0$. If $v^2/c^2 \ll 1$, $\gamma \approx 1 - \frac{1}{2}v^2/c^2$, and the fractional change in frequency is

$$\frac{\Delta\nu}{\nu_0} = \frac{\nu - \nu_0}{\nu_0} = -\frac{1}{2}\frac{v^2}{c^2}. \tag{1}$$

A handy way to evaluate the term on the right is to multiply numerator and denominator by the mass of the atom M:

$$\frac{\Delta \nu}{\nu_0} = -\frac{\frac{1}{2}Mv^2}{Mc^2}.$$

$\frac{1}{2}Mv^2$ is the kinetic energy due to thermal motion of the atom. This energy increases with the temperature of the gas, and according to our treatment of the ideal gas in Section 5.9

$$\frac{1}{2}M\bar{v^2} = \frac{3}{2}kT,$$

where $\bar{v^2}$ is the average squared velocity, $k = 1.38 \times 10^{-23}$ J/deg is Boltzmann's constant, and T is the absolute temperature.

In the atomic clock known as the hydrogen maser, the reference frequency arises from a transition in atomic hydrogen. M is close to the mass of a proton, 1.67×10^{-27} kg, and using $c = 3 \times 10^8$ m/s, we find

$$\begin{aligned}\frac{\Delta \nu}{\nu} &= \frac{\frac{3}{2}kT}{Mc^2} = -\frac{\frac{3}{2}(1.38 \times 10^{-23})T}{(1.67 \times 10^{-27})(9 \times 10^{16})}\\ &= 1.4 \times 10^{-13}T.\end{aligned}$$

At room temperature, $T = 300$ K (300 degrees on the absolute temperature scale $\approx 27\,^\circ$C), we have

$$\frac{\Delta \nu}{\nu} = -4.2 \times 10^{-11}.$$

This is a sizable effect in modern atomic clocks. In order to correct for time dilation to an accuracy of 1 part in 10^{13}, it is necessary to know the temperature of the hydrogen atoms to an accuracy of 1 K. However, if one wishes to compare frequencies to parts in 10^{15}, the absolute temperature must be known to within a millikelvin, a much harder task.

The creation of techniques to cool atoms to the microkelvin regime has opened the way to a new generation of atomic clocks. These clocks, operating at optical rather than microwave frequencies, have achieved a stability greater than 1 part in 10^{17}—equivalent to a difference of about 1 second over the age of the Earth.

12.8.2 Length Contraction

A rod at rest in S has length L_0. What is the length observed in the system S' that is moving with speed $-v$ along the direction of the rod?

The rod lies along the x axis and its ends are at x_a and x_b, where $x_b = x_a + L_0$. The measurement involves two events but because the rod is at rest in S the times are unimportant, so we can take the observations in S to be simultaneous at time t. The length in S is found from the coordinates of two events:

$$\begin{aligned}&\text{event 1} : (x_a, t)\\&\text{event 2} : (x_b, t).\end{aligned}$$

The length observed in the rest frame S is $x_b - x_a = L_0$. The problem is to find the length observed in system S' where the rod is moving with speed $-v$.

A natural, but *wrong!*, approach to finding the coordinates in S' would be to use Eq. (12.3a) to find values for x'_b and x'_a and subtract. This would give $L'_0 = x'_b - x'_a = \gamma L_0$. The result is wrong because the times for the two events in S' are not identical, as can be seen from Eq. (12.3d).

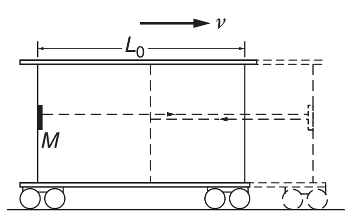
Meaningful measurements of the dimensions of a moving object must be made simultaneously. We must therefore find the correspondence between values of x' and x at the same time t' in the S' system. This is readily accomplished by applying the Lorentz transformation to relate events in S to those in S'. Equation (12.4a) gives $x = \gamma(x' - vt')$. Consequently,

$$\begin{aligned}x_b &= \gamma(x'_b - vt')\\x_a &= \gamma(x'_a - vt').\end{aligned}$$

Subtracting, we obtain $x_b - x_a = \gamma(x'_b - x'_a)$, so that $L_0 = \gamma L'$ and

$$L' = L_0/\gamma = \sqrt{1 - v^2/c^2}\,L_0. \tag{12.6}$$

The rod appears to be contracted. As $v \to c$, $L' \to 0$. The contraction occurs along the direction of motion only: if the rod lay along the y axis, we would use the transformation $y' = y$ and conclude that $L' = L_0$.



As in the case of time dilation, we have a non-intuitive result. This, too, can be understood using a geometrical argument.

An observer on a train could measure the length of the train car L_0 by bouncing light between mirrors at each end and measuring the roundtrip time τ_0:

$$\tau_0 = 2\frac{L_0}{c}.$$

The observer on board concludes that the length of the car is

$$L_0 = \frac{c}{2}\tau_0. \tag{12.7}$$

An observer on the ground also measures the length of the car L' as the train goes by at speed $+v$ by measuring the time for a pulse to make a roundtrip between the ends. As seen by the ground observer, the time τ_+ for the pulse to travel from the rear mirror to the front is longer than L'/c, because the front mirror moves slightly ahead during the transit time. The distance traveled is $L' + v\tau_+$. Consequently, $\tau_+ = (L' + v\tau_+)/c$ so that $\tau_+ = L'/(c - v)$. Similarly, the time for the return trip is $\tau_- = L'/(c + v)$.

The roundtrip for the light pulse is

$$\tau_0' = \tau_+ + \tau_- = L'\left(\frac{1}{c-v} + \frac{1}{c+v}\right) = \frac{2L'}{c}\left(\frac{1}{1-v^2/c^2}\right).$$

Consequently,

$$L' = \frac{c}{2}\tau_0'(1 - v^2/c^2).$$

Comparing this with Eq. (12.7), we have

$$L' = L_0\frac{\tau_0}{\tau_0'}(1 - v^2/c^2).$$

Taking the value of τ_0/τ_0' from Eq. (12.5), we have

$$L' = L_0\sqrt{1 - v^2/c^2}.$$

Because $L' < L_0$, the ground observer sees the length of the car contracted by the factor $\sqrt{1 - v^2/c^2}$.

12.8.3 Proper Time and Proper Length

We introduced the symbols τ_0 and L_0 to denote time and length intervals observed in the rest frame of the events. These quantities are referred to as *proper*: τ_0 is the *proper time* and L_0 is the *proper length*.

Proper time τ is the time measured by a clock in its own rest system, which might for example be a clock carried aboard a spacecraft. According to Eq. (12.5), a time interval $\Delta t'$ measured in a moving frame is always *greater* than the proper time interval $\Delta\tau$:

$$\Delta t' = \gamma\Delta\tau = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} \geq \Delta\tau.$$

Similarly, proper length is the length of an object measured in its own rest frame, for example a meter stick carried aboard a spacecraft. According to Eq. (12.6), the length L' measured in a moving frame is always *less* than the proper length L_0:

$$L' = \frac{L_0}{\gamma} = \sqrt{1 - v^2/c^2}L_0 \leq L_0.$$

12.8.4 Are Relativistic Effects Real?

Time and distance are such intuitive concepts that it may be difficult, at least at first, to accept that the predictions of special relativity are "real" in the familiar sense of physical reality. We shall look at some examples where time dilation and length contraction unquestionably occur. Paradoxes immediately come to mind, for instance the pole-vaulter paradox: a farmer has a barn with a door at each end. A pole-vaulter runs through the barn gripping a horizontal pole longer than the barn. The farmer wants to slam the doors with the pole inside. The farmer instructs the runner to go so fast that the length contraction permits the pole to fit.

The moment the runner is inside, the farmer slams the doors. The paradox is that from the runner's point of view, the pole is unchanged but the length of the barn has contracted. Rather than making the task of fitting in the barn easier, running makes it harder!

The paradox hinges on the difference between Newtonian and relativistic concepts of simultaneity. The runner will not agree that the doors were both shut at the same time, and it will be left as a problem to show that from the runner's point of view the pole was never totally in the barn.

The first dramatic experimental demonstration of time dilation occurred in an early study of cosmic rays. The experiment also demonstrated that although time dilation and length contraction appear to be fundamentally different phenomena, they are essentially two sides of the same coin.

Example 12.5 Time Dilation, Length Contraction, and Muon Decay

The negatively charged muon (symbol μ^-) is an elementary particle related to the electron: it carries one unit of negative charge, same as the electron, and it has a positively charged antiparticle μ^+ analogous to the positron, the electron's antiparticle. The muon differs from the electron most conspicuously in its mass, which is about 205 times the electron's mass, and in being unstable. Electrons are totally stable, but the muon decays into an electron and two neutrinos.

The decay of the muon is typical of radioactive decay processes: if there are $N(0)$ muons at $t = 0$, the number at time t is

$$N(t) = N(0)e^{-t/\tau}$$

where τ is a time constant characteristic of the decay. It is easy to show that the average time before a given muon decays is τ, and so τ is known as the "lifetime" of the particle. For muons, $\tau = 2.2$ μs. (Caution: the symbol μ stands for "micro", 10^{-6}, as well as for muon. One needs to keep one's wits about symbols in physics.) If the muons travel with speed v, the average distance they travel before decaying is $\langle L\rangle = v\tau$.

Muons were discovered in research on cosmic rays. They are created at high altitudes by high energy protons streaming toward the Earth. The protons are quickly lost in the atmosphere by collisions, but the muons continue to sea level with very little loss. The early experiments ran into a paradox. If one assumes that the muons travel at high speed, close to the speed of light, then the maximum average distance that they travel before decay should be no bigger than $\langle L\rangle = c\tau$. Consequently, after traveling distance L, the flux of muons should be decreased by a factor of at least $\exp -L/\langle L\rangle$. For a 2.2 μs lifetime, $\langle L\rangle = 660$ m. In the initial experiment (B. Rossi and D. B. Hall, Physical Review, 59, 223 (1941)), the flux was monitored on a mountain top in Colorado and at a site 2000 m below. The flux was expected to decrease by a factor of $\exp -2000\text{ m}/660\text{ m} = 0.048$. However, the observed loss ratio was much smaller. The paradox was that the muons behaved as if on their journey to Earth they lived for about three times their known lifetime.

The resolution of the paradox was the realization that the muons actually lived that long. The quantity γ was determined from measurements of the muon energy. When time dilation was taken into account, the lifetimes were calculated to increase by a factor close to the observed value.

The concept of proper time is another way to look at this. The muons carry their own "clock" that determines their decay rate. Their clock, in the muon rest frame, measures proper time, so the decay rate measured by a ground observer is longer. Using modern particle accelerators, muons can be created with much higher energies than obtained with cosmic rays, leading to correspondingly larger values of γ. In one experiment (R. M. Casey, *et al.*, Phys. Rev. Letters, 82, 1632 (1997)) the lifetime was extended so much that useful signals could be observed for up to 440 μs, 200 times the muon lifetime. Do moving clocks "really" run slow? The answer depends on how you wish to interpret the experiment. In a coordinate system moving with the muons (in the muon rest system), the particles decay with their natural decay rate. However, in this system the muons "see" that the thickness of the atmosphere is smaller than seen by a ground-based observer. Lorentz contraction reduced the path length from 2000 m to $2000/\gamma$ m. The fraction of muons that penetrated through is the same as if the problem were viewed from a ground-based coordinate system.

We see that once we accept the postulates of relativity we are forced to abandon the intuitive idea of simultaneity. Nevertheless, the Lorentz transformation, which embodies the postulates of relativity, allows us to calculate the times of events in two different systems.

Example 12.6 An Application of the Lorentz Transformation

A light pulse is emitted at the center of a railway car $x = 0$ at time $t = 0$. How do we find the time of arrival of the light pulse at each end of the railway car, which has length $2L$? The problem is trivial in the rest frame. The two events are

$$\text{Event 1: The pulse arrives at end A} \begin{cases} x_1 = -L \\ t_1 = \dfrac{L}{c} = T \end{cases}$$

$$\text{Event 2: The pulse arrives at end B} \begin{cases} x_2 = L \\ t_2 = \dfrac{L}{c} = T. \end{cases}$$

To find the time of the events in system S' moving with respect to the railway car, we apply the Lorentz transformation for the time coordinates.

Event 1:

$$\begin{aligned} t_1' &= \gamma\left(t_1 - \frac{vx_1}{c^2}\right) \\ &= \gamma\left(T + \frac{vL}{c^2}\right) \\ &= \frac{1}{\sqrt{1 - v^2/c^2}}\left(T + \frac{v}{c}T\right) \\ &= T\sqrt{\frac{1 + v/c}{1 - v/c}}. \end{aligned}$$

Event 2:

$$\begin{aligned} t_2' &= \gamma\left(t_2 - \frac{vx_2}{c^2}\right) \\ &= T\sqrt{\frac{1 - v/c}{1 + v/c}}. \end{aligned}$$

In the moving system, the pulse arrives at B (event 2) earlier than it arrives at A, as we anticipated.

Simultaneity is not a fundamental property of events; it depends on the coordinate system. Is it possible to find a coordinate system in which any two events are simultaneous? The following example proves what was asserted in Section 12.6: there are two classes of events. For two given events, we can find either a coordinate system in which the events are simultaneous in time or one in which the events occur at the same point in space—but not both.

Example 12.7 The Order of Events: Timelike and Spacelike Intervals

Two events A and B on the x axis have the following coordinates in S:

Event A : $(x_A,\ t_A)$; Event B : $(x_B,\ t_B)$.

The distance between the events is $L = x_B - x_A$ and the time T separating the events is $T = t_B - t_A$.

The distance between the events in the $x',\ y'$ system, as described by the Lorentz transformation, is

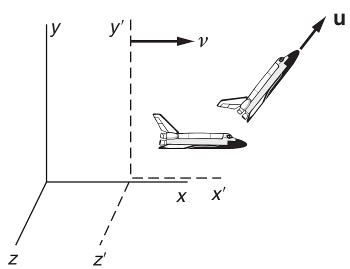
$$L' = \gamma(L - vT).$$

$$T' = \gamma\left(T - \frac{vL}{c^2}\right).$$

Because v is always less than c, it follows that if $L > cT$ then L' is always positive, while T' can be positive, negative, or zero. Such an interval is called *spacelike*, since it is impossible to choose a system in which the events occur at the same place, though it is possible for them to be simultaneous, namely, in a system moving with $v = c^2T/L$. On the other hand, if $L < cT$, then T' is always positive and the events can never appear to be simultaneous, but L' can be positive, negative, or zero. The interval is then known as *timelike*, because it is impossible to find a coordinate system in which the events occur at the same time.

12.9 The Relativistic Addition of Velocities

The closest star is more than four light years away and our galaxy is roughly 10 000 light years across. Consequently, any method for traveling faster than light could be priceless for galactic exploration. Toward this goal, suppose we build a spaceship, the Starship Sophie, that can achieve a speed of $0.900c$. The crew of the Sophie then launches a second ship, Starship Surprise, that can reach $0.800c$. According to Newtonian rules, the Surprise should fly away at $1.700c$. Let's see what happens relativistically.



We designate our rest system (x, y, z, t) by S and spaceship Sophie's system (x', y', z', t') by S'. S' moves with velocity v along the x axis. The velocity of the Surprise, as observed from the Sophie, is $\mathbf{u}' = (u'_x, u'_y)$. Our task is to find the velocity $\mathbf{u}$ of the Surprise that we observe in our rest system S.

From the definition of velocity, in S' we have

$$\begin{aligned}
u'_x &= \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'} \\
u'_y &= \lim_{\Delta t' \to 0} \frac{\Delta y'}{\Delta t'} \\
u'_z &= \lim_{\Delta t' \to 0} \frac{\Delta z'}{\Delta t'}.
\end{aligned}$$

The corresponding components in S are

$$\begin{aligned}
u_x &= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \\
u_y &= \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\
u_z &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}.
\end{aligned}$$

The problem is to relate displacements and time intervals in S to those in S'. From the Lorentz transformation Eqs. (12.3) we have

$$\begin{aligned}
\Delta x &= \gamma(\Delta x' + v\Delta t') \\
\Delta y &= \Delta y' \\
\Delta z &= \Delta z' \\
\Delta t &= \gamma\left(\Delta t' + (v/c^2)\Delta x'\right).
\end{aligned}$$