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Linear Algebra Assignment 3

Q1. For the product of matrices B & C, BC to be defined,
# of columns of B = # of rows of C
Let B be of order $n \times p$ & C be of order $p \times q$
s.that $n, p, q \in \mathbb{Z}^+$; $n, p, q \geq 1$; n, p, q are finite

For the product AB of matrices A & B to be defined,
# of columns of A = # of rows of B
$\because$ B is of order $n \times p$, A must be of order $m \times n$
for some $m \geq 1$, $m \in \mathbb{Z}^+$, m finite.

$\therefore$ $A_{m\times n}$, $B_{n\times p}$, $C_{p\times q}$ are the matrices.

$$[A(BC)]_{ij} = \sum_{r=1}^{n} A_{ir} (BC)_{rj} \quad ; \boxed{i, j \in \mathbb{Z}^+ \\ 1 \leq i \leq m, \ 1 \leq j \leq q}$$

$$= \sum_{r=1}^{n} A_{ir} \sum_{s=1}^{p} B_{rs} C_{sj}$$

$$= \sum_{r=1}^{n} \sum_{s=1}^{p} A_{ir} B_{rs} C_{sj}$$

At this juncture, we ~~must~~ note the following,
$\because$ n, p are both finite, the double summation has finite terms hence, it converges to a finite value.
We also note that the individual summations in a double summation are interchangeable so long as the summation converges to a finite value.

$$\therefore \sum_{r=1}^{n} \sum_{s=1}^{p} A_{ir} B_{rs} C_{sj} = \sum_{s=1}^{p} \sum_{r=1}^{n} A_{ir} B_{rs} C_{sj}$$

$$= \sum_{s=1}^{p} \left(\sum_{r=1}^{n} A_{ir} B_{rs} \right) C_{sj}$$

$$= \sum_{s=1}^{p} (AB)_{rs} C_{sj}$$

$$= ((AB)C)_{ij}$$

$$\because [A(BC)]_{ij} = [(AB)C]_{ij}$$

$$\forall i, j \in Z,\ 1 \le i \le m,\ 1 \le j \le q \quad \text{——} \ (1)$$

$\because$ We know order of $A(BC)$ & $(AB)C$ is $m \times q$. Using that, we ~~note~~ conclude from (1) that every element in $A(BC)$ is exactly equal to every element in $(AB)C$.

Hence,

$$A(BC) = (AB)C.$$

2. Let the three elementary row operations be:

① e_1 : Non-zero scalar multiplication of a row

② e_2 : ~~Addition~~ Replacement of some r^{th} $[1 \leq r \leq m$ for a $m \times n$ matrix$]$ row ~~to itself~~ with the addition of itself plus ~~scalar~~ a non-zero scalar times another row s of the same $m \times n$ matrix $[1 \leq s \leq m]$

③ e_3 : Swapping 2 rows

$E = e(I_{m \times m})$

And $I_{m \times m} = \{\delta_{ij}, 1 \leq i, j \leq m\}$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{[Kroneker Delta Function]}$$

$$\Rightarrow I_{ij} = \delta_{ij} \quad , \quad 1 \leq i, j \leq m$$

~~We also use,~~

RTP $e(A) = EA$ $\forall$ matrices A of order $m \times n$

We examine each operation separately [the operation is done on row r, $1 \leq r \leq m, r \in \mathbb{Z}$]

① $E_1 = e_1(I_{m \times m})$

$$E_{ij} = e_1(I_{ij}) \quad 1 \leq i, j \leq m ; i, j \in \mathbb{Z}$$
$$= e_1(\delta_{ij})$$
$$= \begin{cases} c\delta_{ij} & i = r \\ \delta_{ij} & i \neq r \end{cases}$$

LHS: $e_1(A_{jk}) = \begin{cases} cA_{jk} & j = r \\ A_{jk} & j \neq r \end{cases} \quad ; 1 \leq k \leq n, \; k \in \mathbb{Z}^+$

RHS:

$$(EA)_{ik} = \sum_{j=1}^{m} E_{ij} A_{jk}$$

$$= \begin{cases} c\sum_{j=1}^{m} \delta_{ij} A_{jk} & i = r \\ \sum_{j=1}^{m} \delta_{ij} A_{jk} & i \neq r \end{cases}$$

All terms where $i \neq j$ will become 0 because of the δ_{ij} function

$$\therefore (EA)_{ik} = \begin{cases} cA_{ik} & i = r \\ A_{ik} & i \neq r \end{cases} \quad 1 \leq i \leq m, \ i \in \mathbb{Z}^+$$

$$\Rightarrow (E_1 A)_{jk} = \begin{cases} cA_{jk} & j = r \\ A_{jk} & j \neq r \end{cases} \quad 1 \leq j \leq m, \ j \in \mathbb{Z}^+$$

$\therefore$ LHS = RHS

$\Rightarrow [e_1(A)]_{jk} = [(E_1 A)]_{jk}$

$\forall j,k$

$1 \leq j \leq m, 1 \leq k \leq n$

And order of $E_1 A$ and $e_1(A)$ will both be $m \times n$

~~$\therefore \forall j,k, [e_1(A)] = E$~~

Hence, $e_1(A) = E_1 A$

② $E_{ij} = e_2(I_{ij})$ [same i, j]

$= e_2 \delta_{ij}$

$$= \begin{cases} \delta_{ij} & i \neq r \\ \delta_{ij} + c\delta_{sj} & i = r, \ c \neq 0, \ r \neq s, \ 1 \leq s \leq m \end{cases}$$

RTP: $e_2(A)_{jk} = [EA]_{jk}$

LHS: $e_2(A_{jk}) = \begin{cases} A_{jk} & j \neq r \\ A_{jk} + cA_{sk} & j = r, \ c \neq 0, \ r \neq s, \ 1 \leq s \leq m \end{cases}$

RHS: $[E_2 A]_{ik} = \sum_{j=1}^{m} E_{ij} A_{jk}$

$$= \begin{cases} \sum_{j=1}^{m} \delta_{ij} A_{jk} & i \neq r \\ \sum_{j=1}^{m} (\delta_{ij} + c\,\delta_{sj}) A_{jk}, & i = r \end{cases}$$

$$= \begin{cases} A_{ik} & i \neq r \\ \sum_{j=1}^{m} \delta_{rj} A_{jk} + c \sum_{j=1}^{m} \delta_{sj} A_{jk}, & i = r \end{cases}$$

$$= \begin{cases} A_{ik} & i \neq r \\ A_{rk} + c A_{sk}, & i = r, c \neq 0 \end{cases}$$

~~$\therefore$ LHS = RHS~~ $[E_2 A]_{jk} = \begin{cases} A_{jk} & j \neq r \\ A_{jk} + c A_{sk}, & j = r, c \neq 0 \end{cases}$

$\therefore$ LHS = RHS $\forall$ $1 \leq j \leq m$, $1 \leq k \leq n$, $j, k \in \mathbb{Z}$

$\because$ $E_2 A$ and $e_2(A)$ are both of $m \times n$ size

$\therefore$ $E_2 A = e_2(A)$

③ $E_{ij} = e_3(I_{ij})$ [same i, j]

$= e_3\, \delta_{ij}$

$$= \begin{cases} \delta_{ij} & i \neq r, s \\ \delta_{sj} & i = r \\ \delta_{rj} & i = s \end{cases}, s \neq r, \ 1 \leq s \leq m, s \in \mathbb{Z}$$

RTP $e_3(A) = \cancel{E_3(A)}\ E_3 A$

LHS: $[e_3(A)]_{jk} = \begin{cases} A_{jk} & j \neq r, s \\ A_{sk} & j = r \\ A_{rk} & j = s \end{cases}$

RHS: $(E_3 A)_{ik} = \sum_{j=1}^{m} E_{ij} A_{jk}$

$= \begin{cases} \sum_{j=1}^{m} \delta_{ij} A_{jk}, & i \neq r, s \\ \sum_{j=1}^{m} \delta_{sj} A_{jk}, & i = r \\ \sum_{j=1}^{m} \delta_{rj} A_{jk}, & i = s \end{cases}$

$= \begin{cases} A_{ik} & i \neq r, s \\ A_{sk} & i = r \\ A_{rk} & i = s \end{cases}$ $\neq$ $(E_3 A)_{jk} = \begin{cases} A_{jk} & j \neq r, s \\ A_{sk} & j = r \\ A_{rk} & j = s \end{cases}$

$\therefore$ LHS = RHS

$\because$ orders of $e_3(A)$ and $E_3 A$ are both $m \times n$

$\therefore$ $E_3 A = e_3(A)$

Hence, $\forall A_{m\times n}$, $EA = e(A)$ for e_1, e_2, e_3 {the elementary row operations}

[Hence proved]

Q3. ~~...~~

{From the previous assignment} We know that every matrix can be represented in its ~~row equivalent form~~ row-reduced echelon form. Let $R_{n\times n}$ be the row-reduced echelon form of $A_{n\times n}$

$R = E_1 E_2 \ldots E_k A$ —— (1)

where E_i, $1 \leq i \leq k$, $i \in \mathbb{Z}^+$, are elementary matrices.

{Using Q2, $E_1 A$ is row equivalent to A

$E_2 A$ is row equivalent to $E_1 A$, subsequently row equivalent to A}

Thus, we can represent any row equivalent matrix in this form.

Since each elementary matrix has an inverse,

$$A = E_k^{-1} \cdots E_2^{-1} F_1^{-1} R \quad \text{—— ②}$$

(a) → (b)

A is invertible

let us assume 2 invertible matrices P & Q

$$(PQ)(Q^{-1}P^{-1}) = (Q^{-1}P^{-1})(PQ) = I$$

Hence their product is also invertible.

Using ①, ~~and the~~ R is ~~nothing but~~ a product of invertible matrices [elementary matrices & A]

⇒ R is invertible.

R is an invertible row reduced echelon matrix.

Let us assume it has at least one zero row.

Let the no. of zero rows be k, $1 \le k \le n$ [$P_1, P_2, \dots P_k$ be the rows]

∵ R is invertible, R^{-1} exist s.t. that

$$RR^{-1} = R^{-1}R = I$$

$$(RR^{-1})_{il} = \sum_{j=1}^{n} R_{ij} R_{jl} \qquad 1 \le i, j, l \le n$$

When $i = P_t$ $\quad 1 \le t \le k$

$$(RR^{-1})_{P_t l} = \sum_{j=1}^{n} R_{P_t j} R_{jl} \qquad 1 \le i, j, l \le n$$

but $R_{P_t j} = 0 \ \forall \ 1 \le j \le n$,

$\Rightarrow (RR^{-1})_{P_t l} = 0$ but there are no constraints on l

$\Rightarrow (RR^{-1})_{P_t l} = 0 \ \forall \ 1 \le l \le n$

(P+) { w.l.o.g. extendable to k zero rows }

$\therefore$ (RR^{-1}) has a zero row, & is hence not equal to I. $\therefore$

$\therefore$ (RR^{-1}) can have no zero rows.

Being row reduced echelon & square of size $n \times n$,

R must ~~have~~ be I

$\because$ R was row equivalent to A,

A is row equivalent to $I_{n\times n}$

(a) $\rightarrow$ (b) proved.

~~(b) $\rightarrow$ (c)~~

~~If I = BA , where B is product of elementary new operations,~~

~~B is left inverse of A.~~

~~Similarly~~

(b) $\rightarrow$ (c)

If A is row equivalent to $I_{n\times n}$; using (2)

$A = (E_k^{-1} \cdots E_2^{-1} E_1^{-1}) R$

R is row equivalent echelon matrix.

Since I is a row-reduced echelon matrix, that is row equivalent to A,

$R = I$.

$\therefore A = (E_k^{-1} \cdots E_2^{-1} E_1^{-1}) I$

~~$\therefore A = P$, P is product of elementary matrices~~

$E_k^{-1}, \cdots, E_2^{-1}, E_1^{-1}$ are all elementary matrices

$\therefore$ A is product of elementary matrices.

$\therefore$ (b) $\rightarrow$ (c) proved.

(c) → (a)

(c) A is a product of elementary matrices

let $A = PI$ [where $P = E_1 E_2 \cdots E_k$] k is finite

∵ E_t $1 \le t \le k$, is always invertible,

~~∃ E_t^{-1} ∴ ∃ $Q = E_t^{-1}$ $1 \le t \le k$ s. that $PQ = QP =$~~

Product of invertible matrices is also invertible

⇒ P^{-1} exist. $A = PI = P = IP$ [I is commutative]

$A = PI$	$A = IP$
⇒ $P^{-1}A = I$	⇒ $AP^{-1} = I$
⇒ P^{-1} is left inverse of A	P^{-1} is right inverse of A.

∵ L. Inverse = R. Inverse

A is invertible

∴ (c) → (a) proved.

∵ (a) → (b), (b) → (c), (c) → (a) } are proved. (a) ↔ (b) ↔ (c) has been proved

4. $A_{n \times m}$ matrix, $B_{n \times 1}$ vector with real entries

$AX = B$ ($X \in R^m$).

AX = B has a unique solution, ~~check~~

that $n \ge m$

If $m > n$,

no. of variables > # of equations

because X is of order $m \times 1$ for $AX = B$'s order to be preserved. There are m unknowns, and n equations [AX will be of order $n \times 1$ which when equated with B gives n equations.

$\Rightarrow$ no solution

~~Hence~~

If $n > m$

The # of variables < # of equation

Let m non-zero rows ~~equations~~ be ~~represented~~ ~~as e1, e2,~~ chosen from n rows of A

They give rise to m equations, and m variables, thereby giving rise to a unique sol^n as long as it satisfies the other '$n-m$' equations.

~~If $n = m$~~

If $n = m$,

# of variables = # of equations

A unique sol^n is obtained

$\Rightarrow$ If $n \geq m$,

So long as the equations are continuous, and the augmented matrix $A' = [A|B]$ is row reducible to a row-reduced echelon matrix with at least m non-zero rows, we get a unique sol^n.

In other words, we are guaranteed that $m > n$ cannot provide a unique sol^n. ~~because of augmented~~

$\Rightarrow$ If a unique sol^n exists to $AX = B$, $n \geq m$.

$n \geq m$ does not guarantee a unique sol^n, $\therefore$ not sufficient for it, but is necessary for it

[Hence proved].