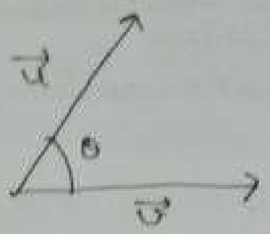
Linear Algebra

Inner product:



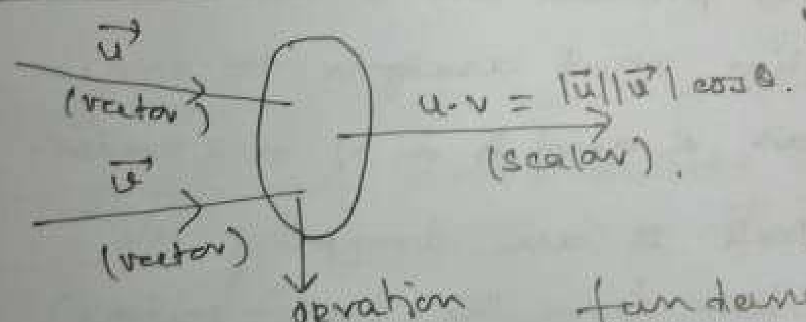
→ An old reference to a dot product of two vectors. If we have two vectors $\vec{u}$ and $\vec{v}$. what is then dot product between the two vectors?

$$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta.$$

↳ is it a scalar or a vector

→ The answer to this question is scalar. Sometimes it is also known as scalar product between two vectors.

So what is the take away?



Two vectors are giving you scalar. The output is fundamentally different from the input

Let us generalize the situation:

Now we start taking about vectors in vector space. We are not restricting ourselves to vectors that we are leaving in Physics!

[Here vector is an abstract concept ($\vec{\alpha} \in V$)]

where $\bar{\alpha}$ is a vector in the vector space V over the field F. $(V, \oplus) \to$ vector space defⁿ:

$\to$ vector addition.

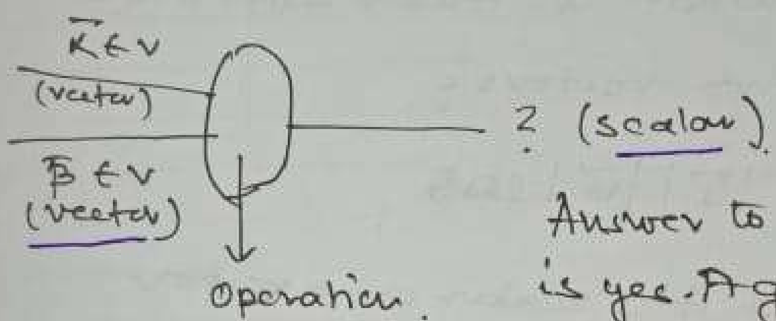
$\to$ scalar multiplication

the field $(F, +, \cdot)$

$\to$ scalar addition

$\to$ multiplication within scalar

Can we have this in general?



Answer to this question is yes. A generalization of dot product can be possible. ~~Or we~~.
This is known as innerproduct. We can say that dot product is an example of inner product.

Defⁿ: An inner product on a vector space V is an operation. that assigns to any pair of vectors α and β in V. a scalar.

$\langle \alpha, \beta \rangle$ (Note I am dropping the bar from the top of vectors)

~~In other words,~~ Then what is an (inner. product space)?

Is there are something called inner. product space? $\longrightarrow$ Yes $\longrightarrow$ Then what it is

Def^n: An inner product space is a vector space V over the field F together with an inner product that is a map

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow F.$$

[Takes two vectors α, β as an input from $V \times V$ (cartesian product) and gives you a scalar from the field F on which the vector space V is defined]

that satisfies following properties for all vectors $\alpha, \beta, \gamma \in V$ and all scalars $a, b \in F$.

(i) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ (conjugate symmetry)

(ii) $\langle a\alpha + b\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle$ (linearity in first arg)

(iii) $\langle \alpha, \alpha \rangle \geq 0$ and $\langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0$. (positive definiteness)

Now verify that these properties hold for ordinary dot products between two vectors $\vec{u}$ and $\vec{v}$ i.e. $\vec{u} \cdot \vec{v}$.

Another example in this context:

Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times 1}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{2\times 1}$ be two vectors in $\mathbb{R}^2$. Show that

$$\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$$

is an inner product.

Solⁿ:

Ex: Let f and g be two elements in the vector space of all continuous functions on the closed interval [a, b]. Show that

$$\langle f, g\rangle = \int_a^b f(x)\,g(x)\,dx$$

defines an inner product.

Solⁿ: We have,

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$
$$= \int_a^b g(x)\,f(x)\,dx = \langle g, f\rangle$$

$$\langle a'f + b'g, h\rangle = \int_a^b \left(a'f(x) + b'g(x)\right) h(x)\,dx.$$
$$= a'\int_a^b f(x)\,h(x)\,dx + b'\int_a^b f(x)\,h(x)\,dx$$
$$= a'\langle f, h\rangle + b'\langle f, h\rangle.$$

Finally, $\langle f, f\rangle = \int_a^b (f(x))^2\,dx \geq 0$ and it follows from a theorem of calculus that since f is continuous. $\langle f, f\rangle = \int_a^b (f(x))^2$ $= 0$ iff f is a zero function.

(As zero vector is orthogonal to all vectors)

(Prove orthogonal set of vectors are linearly independent) → X

Solⁿ: Now in this problem, the underlying field is the set of real numbers $\mathbb{R}$.

So then the properties we have to show for two vectors $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$ are :

i) $\langle u, v \rangle = \langle v, u \rangle$

ii) ~~$\langle u,$~~ $\langle au + bv ; w \rangle = a \langle u, w \rangle + b \langle v, w \rangle$

iii) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = \bar{0}$.

(i) $\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$

$= \cancel{2v_1u_1}\ 2v_1u_1 + 3v_2u_2$ ($\because \mathbb{R}$ is commutative)

$= \langle v, u \rangle$ (Proved)

(ii)

$\langle au + bv, w \rangle = 2(au_1 + bv_1)w_1 + 3(au_2 + bv_2)w$

$= 2au_1w_1 + 3a_2u_2w_2 + 2bv_1w_1 + 3bv$

$= a(2u_1w_1 + 3u_2w_2) + b(2v_1w_1 + v_2w_2)$

$= a\langle u, w \rangle + b \langle v, w \rangle$

(iii) $\langle u,u\rangle = 2u_1u_1 + 3u_2u_2$

$= 2u_1^2 + 3u_2^2 \geq 0$

It will be equal to zero.

i.e $2u_1^2 + 3u_2^2 = 0 \Rightarrow u_1 = u_2 = 0 \Rightarrow$

$u = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \bar{0}$.

Interestingly this problem can be generalized to $\rightarrow$ a much bigger one.

If $w_1, w_2, \ldots, w_n$ are positive scalars and

$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ are vectors in $\mathbb{R}^n$, then.

$\langle u,v\rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nv_nu_n$.

$= \sum_i w_iu_iv_i$

defines an inner dot product on $\mathbb{R}^n$, called the weighted dot product. (If any of the weights w_i is negative or zero it does not define an inner product).

Example: Let A be a symmetric positive definite $n\times n$ matrix and let u, v be vectors in $\mathbb{R}^n$ over the field $\mathbb{R}$. Show that

$\langle u,v\rangle = u^T A v$

defines an inner product.

$$[A|0] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

$$= [I_n | 0]$$

Thus the reduced echelon form of A

(d) $\Rightarrow$ (e) $\Rightarrow$ (a)

↳ It can be shown.

(f) $\Leftrightarrow$ (g)

rank (A) = n.

rank (A) + nullity (A) = n.

$\Rightarrow$ nullity (A) = 0.

(f) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (h)

~~As~~ rank of A = n, the reduced row echelon form ~~with~~ has n leading elem 1.

$\therefore$ The reduced row echelon for is I_n

(d) $\Rightarrow$ (c) $\boxed{Ax = 0}$ has only the trivial solⁿ: i.e $x = 0$.

$$\underset{n\times n}{[P_1 \; P_2 \; \cdots \; P_n]} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$x_1 P_1 + x_2 P_2 + \cdots + x_n P_n = 0$

$\Rightarrow x = 0$ ∫. B $\Rightarrow$ The column vectors are L.I.

Thm: Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of A. If B_i is a basis for eigenspace E_{λ_i} then $B = B_1 \cup B_2 \cup \dots \cup B_k$ (i.e the total collection of bases vectors of all the eigen spaces) is linealy independent.

Proof: Let $B_i = \{u_{i1}, u_{i2} \cdots u_{in_i}\}$ $i = 1, \dots, k$.

~~$B_2 = \{$~~

we have to show that $B = \{u_{11} \cdots u_{1n_1}, u_{21} \cdots u_{2n_2}, \dots \dots u_{k1}, \dots u_{kn_k}\}$ is linealy independent.

suppose some nontrival combination of these vectors is zero vector, say

$$(c_{11}u_{11} + \dots + c_{1n_1}u_{1n_1}) + (c_{21}u_{21} + \dots + c_{2n_2}u_{2n_2}) + \dots + (c_{k1}u_{k1} + \dots + c_{kn_k}u_{kn_k}) = 0$$

Denoting the sums in the paranthesis as $x_1, x_2, \dots, x_k$

$$x_1 + x_2 + \dots + x_k = 0$$

Now each x_i is in $E_{\lambda_i} \rightarrow$ so it is ~~an~~ either eigen vector to λ_i or it is zero.

Length Distance and Orthogonality

Let u and v are two vectors in an inner product space V.

1) The length (or norm) of v is $\|v\| = \sqrt{\langle v, v\rangle}$
2) The distance between u and v is $d(u,v) = \|u - v\|$
3) u and v are orthogonal if $\langle u, v\rangle = 0$.

(Also in $\mathbb{R}^n$, the vector of unit length 1 is called a unit vector.)

An orthogonal set of vectors in an inner product space V is the set of $\{v_1, v_2, \dots, v_n\}$ of vectors from V such that $\langle v_i, v_j\rangle = 0$ whenever $v_i \neq v_j$

An orthonormal set of vectors is then an orthogonal set of vectors. An orthogonal basis for a subspace W of V is just a basis for W that is an orthogonal set. Similarly, an orthonormal basis for a subspace W of V is a basis for W that is an ortho-normal set.

The Gram Schmidt process:

Our target is to find an orthogonal basis for a subspace W of $\mathbb{R}^n$. The idea is to begin with arbitary basis $\{x_1, x_2, \dots, x_n\}$ and we want to orthogonalize it.

Take an example:

Let $W = \text{span}\{x_1, x_2\}$

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Construct an athogonal basis for W.

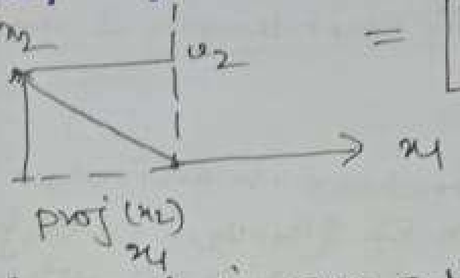
** (problem)

Solⁿ: Let $v_1 = x_1$.

$v_2 = \text{perp}_{x_1}(x_2) = x_2 - \text{proj}_{x_1}(x_2)$.

$= x_2 - \left(\dfrac{x_1 \cdot x_2}{x_1 \cdot x_1}\right) x_1$

(By construction v_1, v_2 are athogonal)



$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{-2}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\{v_1, v_2\}$ is an athogonal set of vectors in W. $\{v_1, v_2\}$ is also a basis for W.

A generalization to Gram Schmidt

Let $\{x_1, x_2, \cdots, x_k\}$ be a basis for a subs W of and define.

$x_1 \cdot x_2$ — ✗

$\langle v_1, x_2 \rangle$ — ✓

$v_1 = x_1. \longrightarrow (1)$

$$v_2 = x_2 - \underbrace{\left(\frac{\langle v_1, x_2\rangle}{\langle v_1, v_1\rangle}\right) v_1}_{\text{perp}_{x_1}(x_2)} \longrightarrow (2)$$

(Check $\langle v_1 \cdot v_2 \rangle = \langle v_1, x_2 \rangle - \dfrac{\langle v_1, x_2\rangle}{\langle v_1 \cdot v_1 \rangle}\cancel{\langle v_1 \cdot v_1\rangle}$

$= 0$)

$$v_3 = x_3 - \frac{\langle v_1 \cdot x_3\rangle}{\langle v_1 \cdot v_1\rangle} v_1 - \frac{\langle v_2 \cdot x_3 \rangle}{\langle v_2 \cdot v_2\rangle} v_2 \longrightarrow (3)$$

$\text{perp}_{\{v_1, v_2\}}(x_3)$

⋮

$$v_k = x_k - \frac{\langle v_1 \cdot x_k \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle v_2, x_k \rangle}{\langle v_2, v_2 \rangle} v_2 - \cdots - \frac{\langle v_{k-1} \cdot x_k \rangle}{\langle v_{k-1} \cdot v_{k-1} \rangle} v_{k-1}. \rightarrow (k)$$

$\{v_1, v_2, \ldots, v_k\}$ is an orthogonal basis for W.

Now how we are going to convert the newly formed orthogonal set of basis vectors into an orthonormal state.

$$\left\{ e_1 = \frac{v_1}{\| v_1 \|}, e_2 = \frac{v_2}{\| v_2 \|}, \ldots, e_k = \frac{v_k}{\| v_k \|} \right\}$$

→ change to inner product **

Class — 2 → step — 0

The properties that come immediately from the defination of inner product.

For arbitary vectors u, v, w and scalars a, b,

1) $\langle 0, u \rangle = \langle u, 0 \rangle = 0$

2) $\langle u, av + bw \rangle = \bar{a} \langle u, v \rangle + \bar{b} \langle u, w \rangle$.

Orthogonality in $\mathbb{R}^n$ Step 1

We consider $\mathbb{R}^n$ as the vector space. We define the ~~inner~~ inner product between two vectors for u and v in $\mathbb{R}^n$ as,

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \longrightarrow (1)$$

$$\therefore u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$ is called an orthogonal set iff all pairs of distinct vectors in the set are orthogonal — i.e if

$$v_i \cdot v_j = 0 \text{ whenever } i \neq j \text{ for } i, j = 1, 2, \ldots, k$$

Example:

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{aligned} v_1 \cdot v_2 &= 2\cdot 0 + 1\cdot 1 + 1\cdot -1 = 0 \\ v_2 \cdot v_3 &= 0\cdot 1 + 1\cdot -1 + 1\cdot 1 = 0 \\ v_1 v_3 &= 2\cdot 1 + 1\cdot -1 + (-1)\cdot 1 = 0 \end{aligned} \right.$$

If $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal set of non zero vectors in $\mathbb{R}^n$, then vectors are linearly independent

Solⁿ: If $c_1, c_2, \ldots c_k$ are scalars such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

$$(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) \cdot v_i = 0 \cdot v_i = 0$$

$\Rightarrow c_1(u_1 \cdot u_i) + \cdots + c_i(u_i \cdot u_i)$

$+ \cdots + c_k(u_k \cdot u_i) = 0.$

Since $\{u_1, u_2, \ldots, u_k\}$ is an orthogonal set of vectors dot products are zero except $u_i \cdot u_i$

$\Rightarrow c_i(u_i \cdot u_i) = 0 \quad (\because (u_i \cdot u_i) \neq 0 \quad \because u_i \neq 0)$

$\Rightarrow c_i = 0$

So we must have $\{u_1, u_2, \ldots, u_k\}$ are linearly independent set of vectors.

Def^n: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis of W that is orthogonal set.

Eg: $u_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

Thm ~~Prob~~: Let $\{u_1, u_2, \ldots, u_k\}$ be an orthogonal basis ~~of~~ for a subspace W of $\mathbb{R}^n$ and let w be any vector in W. Then the unique scalars $c_1, c_2, \ldots, c_k$ such that

$$w = c_1 u_1 + \cdots + c_k u_k$$

are given by, $\boxed{c_i = \frac{w \cdot u_i}{u_i \cdot u_i}}$ for $i = 1, \ldots, k$.

Proof: Since $\{v_1, v_2, \ldots, v_k\}$ is a basis for W, then any vector $w \in W$ can be written as a linear combination of these vectors.

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$w \cdot v_i = (c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) \cdot v_i$$

$$= c_1 (v_1 \cdot v_i) + c_2 (v_2 \cdot v_i) + \cdots + c_k (v_k \cdot v_i)$$

$$= c_i (v_i \cdot v_i)$$

$$\Rightarrow c_i = \frac{w \cdot v_i}{v_i \cdot v_i} \Big\} .$$

$\because v_i \cdot v_i \neq 0.$
$(v_i \neq 0)$

Find the coordinate of $w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the orthogonal basis $v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

Sol^n: $c_1 = \frac{w \cdot v_1}{v_1 \cdot v_1} = \frac{2+2-3}{4+1+1} = \frac{+1}{6}.$

$$c_2 = \frac{w \cdot v_2}{v_2 \cdot v_2} = \frac{0+2+3}{0+1+1} = \frac{5}{2}$$

$$c_3 = \frac{w \cdot v_3}{v_3 \cdot v_3} = \frac{1-2+3}{1+1+1} = \frac{2}{3}$$

Thus $w = \frac{1}{6} v_1 + \frac{5}{2} v_2 + \frac{2}{3} v_3.$

set if it is an orthog... vectors ... orthonormal basis for a the ... subspace W of $\mathbb{R}^n$ is a basis of W ... is an orthonormal set.

Eg: Show that $S = \{q_1, q_2\}$ is an orthonormal set in $\mathbb{R}^3$ if

$$q_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

We show that,

$$q_1 \cdot q_2 = \cancel{1/\sqrt{18}} - \cancel{2/\sqrt{18}} + \cancel{1/\sqrt{18}} = 0$$

$$q_1 \cdot q_1 = 1/3 + 1/3 + 1/3 = 1$$

$$q_2 \cdot q_2 = 1/6 + 4/6 + 1/6 = 1$$

Find an orthogonal basis for the subspace W of $\mathbb{R}^3$ given by,

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0 \right\}$$

Sol^n:

$$\left.\begin{aligned} x - y + 2z &= 0 \\ y &= x + 2z. \\ x &= \cancel{z} + y - 2z \end{aligned}\right\} \begin{bmatrix} y - 2z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

The orthogonal decomposition of Theorem

Let W be a subspace of $\mathbb{R}^n$ and let v be a vector in $\mathbb{R}^n$. Then there are unique vectors w in W and $w^\perp$ in $W^\perp$ such that

$$v = w + w^\perp.$$
$$= \operatorname{proj}_W(v) + \operatorname{perp}_W(v).$$

→ Step 3

Orthogonal Complements.

Let W be a subspace of $\mathbb{R}^n$. We say a vector v in $\mathbb{R}^n$ is orthogonal to W if v is orthogonal to every vector in W.

The set of all vectors that are orthogonal to ~~W~~ W is called orthogonal complement of W denoted by $W^\perp$.

$$W^\perp = \{ v \in \mathbb{R}^n : v \cdot w = 0 \text{ for all } w \text{ in } W \}.$$

Thm: Let W be a subspace of $\mathbb{R}^n$.

Then a) $W^\perp$ is a subspace of $\mathbb{R}^n$

b) $(W^\perp)^\perp = W$

c) $W \cap W^\perp = \{0\}$

d) If $W = \operatorname{span}(w_1, w_2, \ldots, w_k)$ then v is in $W^\perp$ iff $v \cdot w_i = 0 \;\forall i = 1, 2, \ldots, k$

Proof: Since $0 . w = 0 \ \forall w \in W$
$\Rightarrow 0 \in W^{\perp}$ (non empty)

Let $u, v \in W^{\perp}$ $c \in F$

$$u \cdot w = v \cdot w = 0 \quad \forall w \in W.$$

$$(u+v) . w = u \cdot w + v \cdot w = 0 + 0 = 0$$

$$\Rightarrow u + v \in W^{\perp} \longrightarrow (1).$$

$$(cu) \cdot w = c(u \cdot w) = c(0) = 0$$

$$\Rightarrow cu \in W^{\perp}.$$

Hence $W^{\perp}$ is a subspace.

** Recall that in $\mathbb{R}^2$ the projection of ~~a vector u~~ a vector v onto a non zero vector u is given by,

$$\text{proj}_u(v) = \left(\frac{u \cdot v}{u \cdot u}\right) u$$

[Recall the concept of projection]

$$v = \text{proj}_u(v) + \text{perp}_u(v)$$

$\longrightarrow$ we use this formula in the problem.

Let W be a subspace of $\mathbb{R}^n$ and let $\{u_1, u_2, \dots, u_n\}$ be an orthogonal basis for W. For any vector v in $\mathbb{R}^n$, the orthogonal projection of ~~v~~ v onto W is defined as.

$$\text{proj}(v) = \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right) u_1 + \dots + \left(\frac{u_n \cdot v}{u_n \cdot u_n}\right) u_n.$$

... orthogonal

(ii) If $w \in W$ and $x \in$ ... This implies that then $w.x = 0$

$w \in (W^\perp)^\perp$ $\therefore$ $W \subseteq (W^\perp)^\perp$. By theorem we can write $v = w + w^\perp$ for unique vectors $w \in W$ and $w^\perp$ in $W^\perp$.

$$0 = v.w^\perp = (w + w^\perp).w^\perp$$

$$(\because v \in V) \quad = w.w^\perp + w^\perp.w^\perp$$

$$= 0 + w^\perp.w^\perp = w^\perp.w^\perp$$

$$\Rightarrow w^\perp = 0 \quad \therefore v = w + w^\perp \in W.$$

$$(W^\perp)^\perp \subseteq W \quad \therefore (W^\perp)^\perp = W.$$

$\text{proj}_W(v) =$

$=$

$\text{perp}_W(v)$

Prob: Let W be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$ and let $v = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of v onto W.

~~$\text{proj}_W(v) = \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right) u_1 + \cdots + \left(\right.$~~

Solⁿ: We already know from the previous example of constructing the orthogonal basis for W that

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ are}$$

orthogonal.

$$\therefore u_1 \cdot v = 2, \; u_2 \cdot v = -2, \; \left| \begin{matrix} u_1 \cdot u_1 = 2 \\ u_2 \cdot u_2 = 3 \end{matrix} \right\}$$

$$\mathrm{proj}_W(v) = \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right)u_1 + \left(\frac{u_2 \cdot v}{u_2 \cdot u_2}\right)u_2$$

$$= \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$\mathrm{perp}_W(v) = v - \mathrm{proj}_W(v)$$

$$= \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix} \longrightarrow$$

(Find for B)

**) Apply the Gram Schmidt process to the conv... an orthonormal basis for the subspace $W = $ ... of $\{x_1, x_2, x_3\}$ of $\mathbb{R}^4$, where

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Sol^n: We note that $\{x_1, x_2, x_3\}$ is linear independent set and forms a basis of $\mathbb{R}^4$, W.

Let $v_1 = x_1$, then by Gram Schmidt we calculate the component of x_2 orthog... to $W_1 = \text{span}\{v_1\}$

$$v_2 = \text{perp}_{W_1}\{x_2\}$$

$$= x_2 - \text{proj}_{W_1}\{x_2\}$$

$$= x_2 - \frac{\langle v_1 \cdot x_2 \rangle}{\langle v_1 \cdot v_1 \rangle} v_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{2}{4}\right) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

We now find the component of x_3 orthogo... to $W_2 = \text{span}\{v_1, v_2\}$

$$v_3 = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

a1 =

a2

Thus

Note: ...

wi...

by

the

Sol:

Let

The corresponding orthonormal basis

$$q_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \qquad q_3 = \frac{v_3}{\|v_3\|}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix} \qquad = \begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

Thus $\{q_1, q_2, q_3\}$ forms a basis of $\mathbb{R}^3$.

Prob: Construct an orthogonal basis for P_2 with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x) g(x)\, dx.$$

by applying Gram Schmidt process to the basis $\{1, x, x^2\}$.

Sol^n: Let $x_1 = 1$, $x_2 = x$, $x_3 = x^2$

$v_1 = x_1$

$$\langle v_1 \cdot v_1\rangle = \int_{-1}^{1} dx = x\Big]_{-1}^{+1} = 2$$

$$\langle v_1 \cdot x_2\rangle = \int_{-1}^{1} x\, dx = \frac{x^2}{2}\Big]_{-1}^{1} = 0$$

$$\therefore v_2 = x_2 - \frac{\langle v_1, x_2\rangle}{\langle v_1, v_1\rangle} v_1 = x - \frac{0}{2}(1) = x.$$

$$\langle v_2, v_2\rangle = \frac{2}{3}.$$

$$\langle v_1, x_3\rangle = \frac{2}{3}, \quad \langle v_2, x_3\rangle = 0$$

$$v_3 = x_3 - \frac{\langle v_1, x_2\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle v_2, x_3\rangle}{\langle v_2, v_2\rangle} v_2$$

$$= x^2 - 1/3.$$

Thus the new orthogonal basis for $\mathbb{P}_2$ is $\{1, x, x^2 - 1/3\}$.

<u>Cauchy-Schwartz Inequality</u>

Let u and v be vectors in the inner product space V. Then.

$$|\langle u, v\rangle| \le \|u\|\|v\|$$

with equality holding iff u and v are scalar multiples of each other.

<u>Proof:</u> If $u=0$, then the inequality is actually equality

$$\because \quad |\langle 0, v\rangle| = 0 = \|0\|\|v\| \longrightarrow (1).$$

If $u \neq 0$. Let W be the subspace of V spanned by u.

$$\therefore \quad \mathrm{proj}_W(v) = \frac{\langle u, v\rangle}{\langle u, u\rangle} u \text{ and}$$

$$\mathrm{perp}_W(v) = v - \mathrm{proj}_W(v) \longrightarrow (2)$$

$$\|v\|^2 = \|\mathrm{proj}_W(v) + (v - \mathrm{proj}_W(v))\|^2$$

$$= \|\mathrm{proj}_W(v) + \mathrm{perp}_W(v)\|^2$$

$$= \|\mathrm{proj}_W(v)\|^2 + \|\mathrm{perp}_W(v)\|^2 \Big(\because$$

inner product between them is 0

It follows that

$$\|proj_W(v)\|^2 \leq \|v\|^2.$$

$$\|proj_W(v)\|^2 = \left\langle \frac{\langle u \cdot v\rangle}{\langle u \cdot u\rangle} u, \frac{\langle u \cdot v\rangle}{\langle u \cdot u\rangle} u \right\rangle$$

$$= \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)^2 \langle u \cdot u\rangle = \frac{\langle u,v\rangle^2}{\langle u,u\rangle} = \frac{\langle u,v\rangle^2}{\|u\|^2}$$

$$\therefore \frac{\langle u,v\rangle^2}{\|u\|^2} \leq \|v\|^2$$

$$\Rightarrow \boxed{\langle u,v\rangle^2 \leq \|u\|^2 \|v\|^2} \Rightarrow \langle u,v\rangle \leq \|u\| \|v\|$$

The equality holds iff $\boxed{perp_W(v) = 0}$

<u>Triangle Inequality</u>

Let u and v be vectors in an inner product ~~space~~ space ~~IS~~ V. Then.

$$\|u+v\| \leq \|u\| + \|v\|$$

<u>Proof:</u>

$$\|u+v\|^2 = \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$$

$$\leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2.$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$= (\|u\| + \|v\|)^2$$

Taking the square root,

$$\|u+v\| \leq \|u\| + \|v\| \quad \Big\} \quad \text{(Proved)}$$

Thm: Let $\{q_1, q_2, \ldots, q_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let w be any vector in W. Then

$$w = (w\cdot q_1)\cdot q_1 + (w\cdot q_2)\cdot q_2 + \cdots + (w\cdot q_n)\cdot q_n.$$

Thm: The columns of an $m\times n$ matrix Q forms an orthonormal set iff $Q^TQ = I_n$

Proof: We need to show that,

$$(Q^TQ)_{ij} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j. \end{cases} \longrightarrow (1)$$

Let q_i be the i-th column of Q (hence it is i th row of Q^T). Since the (i,j) th entry ~~is the~~ of Q^TQ is the dot product of the i-th row of Q^T and j th column of Q,

$$(Q^TQ)_{ij} = q_i\cdot q_j \quad \text{(by matrix multiplication)} \longrightarrow (2)$$

Now the columns of Q forms an orthonormal set iff

$$q_i\cdot q_j = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$$

which is by eq^n (2)

$$g_{ia}(Q^TQ)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

This completes the proof.

Def^n: An $n \times n$ matrix Q whose columns form an orthonormal set is called an orthogonal matrix

Thm: A square matrix Q is orthogonal iff $Q^{-1} = Q^T$

Proof: Since Q is orthogonal iff $Q^TQ = I$

$\rightarrow (1)$.

This is true iff Q is invertible and $Q^T = Q^{-1}$ (inverse is unique).

Show that the following matrices are orthogonal and find their inverses.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Sol^n: The columns of A are standard basis vectors for $\mathbb{R}^3$. (These are orthonormal)

Hence A is orthogonal,

$$A^{-1} = A^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Defn:

Let us consider an $n \times n$ matrix A. A scalar λ is called an eigen value of A if there is a non zero vector x such that

$$Ax = \lambda x \longrightarrow (1)$$

Such vector x is called the eigen vector of A corresponding to eigen value λ.

Prob: Show that $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and find the corresponding eigen value.

Soln:
$$Ax = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4x$$

The eigenvalue of A is 4.

Defn: Let A be an $n \times n$ matrix. Let λ be the eigen value of A. The collection of all eigen vectors corresponding to λ, together with the zero vector is called the eigen space of λ, and is denoted by

Ex.

The eigen values of a square matrix A are precisely the solutions λ of the equation.

$$\det(A-\lambda I)=0$$

When we expand $\det(A-\lambda I)$ we get a polynomial in λ called the characteristic polynomial of λ.

$\det(A-\lambda I)=0 \longrightarrow$ characteristic polynomial.

Steps: (Let A be an $n \times n$ matrix)

1. Compute the characteristic polynomial $\det(A-\lambda I)$ of A

2. Find the eigenvalues of A by solving the characteristic equation $\det(A-\lambda I)=0$ for λ.

3. For each eigenvalue ~~of~~ λ, find the null space of the matrix $A-\lambda I$. This is the eigen-space E_λ. The non zero vectors of which are the eigen vectors of A corresponding to λ.

4. Find a basis for each eigen space.

Prob: Find the eigen values [and] corresponding eigenspaces of.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}.$$

Sol^n: The characteristic polynomial

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2.$$

To find the eigen-values, we need to solve the characteristic eq^n $\det(A-\lambda I) = 0$

Sol^{n}s: $\lambda = 1, 1, 2.$

$2 \to$ A.M

$1 \to$ A.M

Algebric multiplicity 2 (A.M)

The eigen vectors corresponding to $\lambda_1 = \lambda_2 = 1$ we find the null space

$$A - \lambda I = A - 1I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$[A - I \mid 0] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

eigenvalue of A^{-1} with corresponding eigenvector x

c) If A is invertible, then for any integer n, λ^n is eigenvalue of A^n with corresponding eigen vector x.

Sol^n:

(a) We know, $Ax = \lambda x$. $\longrightarrow (1)$.

Assume that the result is true for $n=k$,

$$A^k x = \lambda^k x.$$

$$\therefore A^{k+1}(x) = A(A^k x) = \lambda^k (Ax)$$

$$= \lambda^k (\lambda x) = \lambda^{k+1} x.$$

Therefore by induction it is true for all integer $n \geq 1$.

(b) $Ax = \lambda x$

$$\Rightarrow (A^{-1}A)x = \lambda (A^{-1}x)$$

$$\Rightarrow Ix = \lambda (A^{-1}x)$$

$$\Rightarrow A^{-1}x = \frac{1}{\lambda} x.$$

Thus $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Prob: Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$

Sol^n: Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ $x = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Eigenvalues of A are $\lambda_1 = -1, \lambda_2 = 2$
with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
and $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore$ $\left.\begin{aligned} A v_1 &= -v_1 \\ A v_2 &= 2 v_2 \end{aligned}\right\}$ since $\{v_1, v_2\}$ form a basis

we can rewrite x as a linear combi
of v_1 and v_2.

$\therefore x = 3 v_1 + 2 v_2$.

$$A^{10} x = 3 A^{10} v_1 + 2 A^{10} v_2$$

$$= 3(-1)^{10} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2 (2)^{10} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2051 \\ 4093 \end{bmatrix}.$$

Thm: Suppose the $n \times n$ matrix A has eigenvectors $v_1, v_2, \ldots, v_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. If x is a vector in $\mathbb{R}^n$, that can be expressed as a linear combination of these eigenvectors — say

$$\therefore x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

then for any integer k,

$$A^k x = c_1 \lambda_1^k e_1 + c_2 \lambda_2^k e_2 + \cdots + c_m \lambda_m^k e_m$$

Thm: Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $e_1, e_2, \ldots, e_m$. then $e_1, e_2, \ldots, e_m$ are linearly independent

Proof: we will prove by contradiction.
let us assume that $e_1, e_2, \ldots, e_m$ are linearly dependent. let us say e_{k+1} be the first vector that can be expressed. In other words, $e_1, e_2, \ldots, e_k$ are L.I, but there are scalars $c_1 \ldots c_k$ such that

$$e_{k+1} = c_1 e_1 + \cdots + c_k e_k \longrightarrow (1)$$

Multiplying both the sides by eq^n (1)

$$A e_{k+1} = c_1 A e_1 + \cdots + c_k A e_k$$

$$\lambda_{k+1} e_{k+1} = c_1 \lambda_1 e_1 + c_2 \lambda_2 e_2 + \cdots + c_k \lambda_k e_k \longrightarrow (2)$$

Multiplying both the sides of eq^n (1) by λ_{k+1}, we get,

$$\lambda_{k+1} e_{k+1} = c_1 \lambda_{k+1} e_1 + c_2 \lambda_{k+1} e_2 + \cdots + c_k \lambda_{k+1} e_k \longrightarrow (3)$$

Subtracting (2)

$$0 = c_1(\lambda_1 - \lambda_{k+1})v_1 + c_2(\lambda_2 - \lambda_{k+1})v_2 + \dots + c_k(\lambda_k - \lambda_{k+1})v_k$$

the linear independence of $v_1, v_2, \dots, v_k \Rightarrow$

$$c_i(\lambda_i - \lambda_{k+1}) = 0 \text{ for } i = 1, 2, \dots, k.$$

Since the eigenvalues λ_i are all distinct, the terms $(\lambda_i - \lambda_{k+1}) \neq 0$

Hence $c_1 = c_2 = \dots = c_k = 0 \Rightarrow v_{k+1} = c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$ which is impossible since eigenvector v_{k+1} can't be zero.

Thus we have contradiction.

Hence our assumption that $v_1, v_2, \dots, v_m$ are l. dependent is false. $v_1, v_2, \dots, v_m$ are linearly independent.

Similar Matrices:

Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible matrix $P (n \times n)$ such that $P^{-1}AP = B$. If A is similar to B, we with $A \sim B$.

If $A \sim B$, $\Rightarrow A = PBP^{-1}$ or $AP = PB$.

Prob: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$

then $A \sim B$, since

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

Thus $AP = PB$, with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Them: Let A, B and C be $n \times n$ matrices

a) $A \sim A$

b) If $A \sim B$, then $B \sim A$

c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof:

(a) $A = I^{-1} A I$.

$\Rightarrow A \sim A$ (reflexive)

(b) If $A \sim B$

$B = P^{-1} A P$ for some invertible matrix P.

$\Rightarrow PBP^{-1} = A$. Setting $Q = P^{-1}$

$\Rightarrow Q^{-1} B Q = A$. $\Rightarrow B \sim A$.

(iii) $A \sim B$

$P^{-1}AP = B \longrightarrow (1)$ (for some invertible matrix P)

$B \sim C$

$Q^{-1} B Q = C. \longrightarrow (2)$

(for some invertible matrix Q)

$C = Q^{-1}(P^{-1}AP)Q$

$= (PQ)^{-1} A (PQ)$. PQ is invertible matrix

$\therefore A \sim C.$

$\therefore \sim$ is a transitive relation.

Thm: Let A and B be $n \times n$ matrices with $A \sim B$, then.

(i) $\det A = \det B$

(ii) A and B have the same characteristic polynomial.

Soln: (i) Taking determinants both sides of the eqn

$B = P^{-1}AP$ (since $A \sim B$)

$\det B = \det(P^{-1}AP)$

$= \det(P^{-1}) \det(A) \det(P)$

$(\because PP^{-1} = I)$ $= \frac{1}{\cancel{\det(P)}} \det(A) \cancel{\det(P)}$

$= \det(A)$

(ii) The characteristic polynomial of B is.

$\det(B-\lambda I) = \det(P^{-1}AP - \lambda I)$

$= \det(P^{-1}AP - \lambda P^{-1}IP)$

$= \det(P^{-1})\det(A-\lambda I)\det(P)$

$= \det(A-\lambda I)$

= the characteristic polynomial of the matrix A.

Diagonalization:

Def^n: An $n \times n$ matrix A is diagonalizable if there is a diagonal matrix D such that A is similar to D — that is if there is an invertible matrix P ~~such that~~ $(n \times n)$ such that

$$\boxed{P^{-1}AP = D}$$

Thm: Let A be an $n \times n$ matrix. Then A is diagonalizable iff A has a n-linearly independent eigen vector.

Proof: Suppose that A is similar to the diagonal matrix D

$$\Rightarrow P^{-1}AP = D.$$

$$\Rightarrow AP = PD.$$

Let the columns of P be $p_1, p_2, \ldots, p_n$ and let the entries of D be $\lambda_1, \lambda_2, \ldots$

Then.

$$A\left[p_1\ p_2 \cdots p_n\right] = \left[p_1\ p_2 \cdots p_n\right]\begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & & \\ 0 & 0 & \cdots \end{bmatrix} \rightarrow (1)$$

$$\left[Ap_1, Ap_2, \ldots, Ap_n\right]$$

$$= \left[\lambda_1 p_1\ \lambda_2 p_2 \cdots \lambda_n p_n\right] \rightarrow (2)$$

Equating the columns we get,

$$Ap_1 = \lambda_1 p_1,\ Ap_2 = \lambda_2 p_2 \ldots Ap_n = \lambda_n$$

which proves that column ~~sp~~ vectors of P are eigenvectors of A whose correspon eigenvalues are the diagonal entries of D. Since P is invertible it columns linearly independent.

Conversely, if A has n linearly independent eigenvectors $p_1, p_2, \ldots, p_n$ with correspon eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively the

$$Ap_1 = \lambda_1 p_1,\ Ap_2 = \lambda_2 p_2 \ldots Ap_n = \lambda_n p_n.$$

This implies eq^n(2) (above) which is equivalent to eq^n (1). Consequently if we take P to be $n \times n$ matrix with columns $p_1, p_2 \cdots, p_n$, then eq^n (1) becomes $AP = PD$. Since columns of P are linearly independent the fundamental theorem of invertible matrices implies P is invertible **

Hence $P^{-1}AP = D$, that is A is diagonalizable.

Prob: (1) If possible find a matrix P that diagonalizes.

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Sol^n: Eigen-values are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -2$.

For $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -2$

For $\lambda_1 = \lambda_2 = 0$, E_0 has basis $p_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $p_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ For $\lambda_3 = -2$, E_{-2} has basis $p_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$.

All these vectors are linearly independent, Since the eigenvalues λ_j are distinct, any of one of the factors of λ_i are the eigen vectors then they are linear independent. Then ⇒

eq^n(4) is a linear dependence relation which is a contradiction.

we conclude that eq^n(3) must be trivial i.e all the coeff are zero. Hence B is linearly independent.

<u>Thm:</u> If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

<u>Proof:</u> Let $v_1, v_2, \ldots, v_n$ be the eigenvectors corresponding to n distinct eigenvalues of A. Hence they are linearly independent. If they are linearly independent then A is diagonalizable.

<u>Thm:</u> Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$. The following statements are equivalent

a) A is diagonizable

b) The union B of the bases of the eigenspaces of A contains n-vectors.

c) the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity. [Resource] (To be uploaded in moodle)

Prob: Compute

A^{10} if $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

Solⁿ: The eigen values of A are $\lambda_1 = -1$, $\lambda_2 = 2$ with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

A is diagonalizable and $P^{-1}AP = D$

$$P = [v_1 \; v_2] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore D^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}.$$

$$\left[\begin{aligned} A^k &= PD^kP^{-1} \\ A^{k+1} &= A^k A \\ &= PD^k(P^{-1}P)DP^{-1} \\ &= PD^{k+1}P^{-1} \end{aligned} \right]$$

$A^n = (P^{-1}A$

$A^n = (PDP^{-1})^n =$

$$A^n = PD^nP^{-1}$$

$$A^{10} = \begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix}.$$

Thm: Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of A. If B_i is a basis for eigenspace E_{λ_i} then $B = B_1 \cup B_2 \cup \dots \cup B_k$ (i.e. the total collection of bases vectors of all the eigen spaces) is linearly independent.

Proof: Let $B_i = \{v_{i1}, v_{i2} \dots v_{in_i}\}$ $i = 1, \dots, k$.

~~$B_2 = \{$~~

we have to show that $B = \{v_{11} \dots v_{1n_1}, v_{21} \dots v_{2n_2}, \dots \dots v_{k1}, \dots v_{kn_k}\}$ is linearly independent.

Suppose some nontrivial combination of these vectors is zero vector — say

$$(c_{11}v_{11} + \dots + c_{1n_1}v_{1n_1}) + (c_{21}v_{21} + \dots + c_{2n_2}v_{2n_2}) + \dots + (c_{k1}v_{k1} + \dots + c_{kn_k}v_{kn_k}) = 0$$

Denoting the sums in the parenthesis as $x_1, x_2, \dots, x_k$

$$x_1 + x_2 + \dots + x_k = 0$$

Now each x_i is in $E_{\lambda_i} \rightarrow$ so it is ~~ama~~ either eigen vector to λ_i or it is zero.

$$P = [P_1 \; P_2 \; P_3] = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Furthermore;

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D.$$

② If possible find a matrix P that diagonlizes

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}.$$

** Fundamental Theorem of Invertible Matrices :

Let A be an $n \times n$ matrix. The following statements are equivalent.

a) A is invertible

b) $Ax = b$ has an unique sol^n for any $b \in \mathbb{R}^n$.

c) $Ax = 0$ has only the trivial sol^n.

d) the reduced row echolon form of A is I_n.

e) A is the product of elementary matrices

f) rank of A is n.

g) nullity of A is I_n.

h) The column vec of A are linearly indep (a)⇒(b) If A is an $n\times n$ invertible matrix then the system of eq^n $Ax=b$ has an unique sol^n $x=A^{-1}b$ for any $b\in\mathbb{R}^n$.

Verification: $A(A^{-1}b)=(AA^{-1})b$

$= Ib = b$.

($\because$ A is invertible)

$\therefore$ $A^{-1}b$ satisfies the eq^n $Ax=b$.

Let y be another sol^n.

$\therefore$ $Ay=b$.

$A^{-1}(Ay)=A^{-1}b \Rightarrow (A^{-1}A)y=A^{-1}b$.

$\Rightarrow \boxed{y=A^{-1}b}$ y is the same sol^n and

hence unique.

(b)⇒(c) The homogeneous eq^n $Ax=0$ has only at least one sol^n $x=0$. Hence (b) implies $x=0$ must be the solution

(c) ⇒ (d)

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=0\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=0\\ \vdots\\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=0 \end{array}\right\}$$

and we are assuming the sol^n is

$$\left.\begin{array}{l} x_1=0\\ x_2=0\\ x_n=0 \end{array}\right\}$$