the RHS is zero since we have shifted the origin of the coordinate system to the CoM. The above equation along with $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$\vec{r}_1 = \frac{m_2}{m_1 + m_2}\vec{r}\,, \qquad \vec{r}_2 = -\frac{m_1}{m_1 + m_2}\vec{r}\,.$$

After substituting $\vec{r}_{1,2}$ in terms of $\vec{r}$, the Lagrangian becomes

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\,, \tag{28}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}\,. \tag{29}$$

is called the reduced mass of the system. This is now a one-body problem of a particle of mass μ in a central force potential $U(r)$. The dof of the system is now 3.

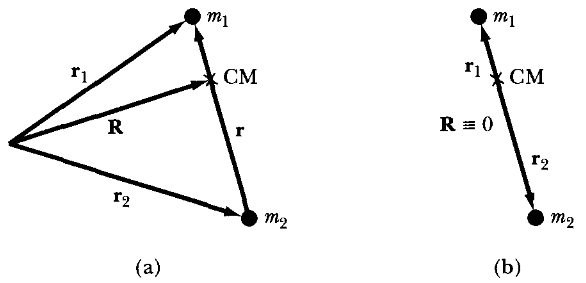

Figure 1: Frames to describe two-body system under central force

3.3 Integrals of motion

The force due to $U(r)$ acts along $\vec{r}$ so the from (27), the angular momentum is conserved

$$\vec{\mathbb{L}} = \vec{r} \times \vec{p} = \text{const.}\,.$$

As shown in the figure the angular momentum is perpendicular to the plane that contains the $\vec{r}$ and the $\vec{p}$ vector – so the (axis of the) plane is constant. So, in the CoM frame two-body problem with central (conservative force) reduces to a one body problem confined to a plane. There is also an obvious spherical symmetry of the system. So the dof of the system is actually 2, and we can use spherical coordinates r and θ to describe the system. The Lagrangian can be written as

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)\,.$$

Here, θ is a cyclic coordinate, so the corresponding conjugate momentum p_θ is constant in time

$$\dot{p}_\theta = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(\mu r^2\dot{\theta}) = \frac{\partial L}{\partial \theta} = 0\,.$$

where in the last line we have used EL EoM discussed in the previous chapter. So we have the angular momentum as our first *constant of motion*

$$\ell = \mu r^2\dot{\theta} = \text{const.}\,. \tag{30}$$

The ℓ can both be positive or negative. This is actually the EoM for the θ coordinate and the solution as a function of time is monotonic.

In a short time interval dt the radius vector of the system sweeps an area

$$dA = \frac{1}{2} r \cdot r d\theta = \frac{1}{2} r^2 d\theta \,,$$

Dividing by the time interval, we obtain the *areal velocity*

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{\ell}{2\mu} \,. \tag{31}$$

which is constant in time. This is known as the *Kepler's second law* of planetary motion which states that *the radius vectors of planets sweeps equal area in unit time at any point in their orbits.* This law was empirically discovered by Kepler.

In terms of the constant ℓ, the Lagrangian of the system can be rewritten as

$$L = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{\ell^2}{\mu^2 r^2} - U(r) \,. \tag{32}$$

This Lagrangian does not explicitly depend on time so the total energy of the system

$$E = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{\ell^2}{\mu^2 r^2} + U(r) = \text{const.} \tag{33}$$

This is another integral of motion. We can write

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{\ell^2}{2mr^2}\right)} \Longrightarrow t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - U(r) - \frac{\ell^2}{2mr^2}\right)}}$$

From this one can in principle solve r as a function of time. But the integral on the RHS are difficult to perform. Instead of solving r and θ as a function of time, we solve the equation of orbits *i.e.,* $r \equiv r(\theta)$.

3.4 Classifications of orbits

Before solving the orbits for a given potential $U(r)$ lets classify the orbits. The radial velocity $\dot{r} = 0$ then from (33) we get

$$E = \frac{1}{2} \frac{\ell^2}{\mu^2 r^2} + U(r) \,.$$

This is quadratic in r and has two roots, say $r = r_{\max}$ and $r = r_{\min}$. Radial velocity vanishes at the turning point of the motion – so the motion is between the annular region $r = r_{\max}$ and $r = r_{\min}$. For example the motion of the earth around the sun has a max and min radius vector. For some specific combinations of E, ℓ, μ and $U(r)$ it is possible to obtain same values for the two roots – this is an example of circular motion – the radial velocity is zero all the time.

For motions that have both $r = r_{\max}$ and $r = r_{\min}$ are called bounded. There are two types of orbits in bounded motions – open and closed. Closed orbits are those that are repeated after the radius vector has made a finite number of excursion between $r_{\max}$ and $r_{\min}$. In other words close on itself. But orbits that do not close on itself, are called open orbits. An example of open orbit is shown in figure 2.

Orbits can also be examined in the following way. We rewrite (33)

$$\mu \dot{r}^2 = E - V'(r) \,, \tag{34}$$

by defining *effctive potential*

$$V' = \frac{1}{2} \frac{\ell^2}{\mu^2 r^2} + U(r) \,.$$

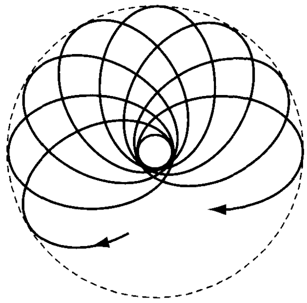

Figure 2: Open orbit

The second term $\ell^2/2\mu^2 r^2$ is called the *centrifugal forve*, though it is not a force in the usual sense. In figure 3 we plot $V'(r)$ against r for

$$U(r) = -\frac{k}{r}\,,$$

which correspond to the inverse square force

$$F = -\frac{\partial}{\partial r}U(r) = -\frac{k}{r^2}\,,$$

like the gravitational force. In the In figure 3 we also have four values of the total energy E_1, E_3, E_3, E_4.

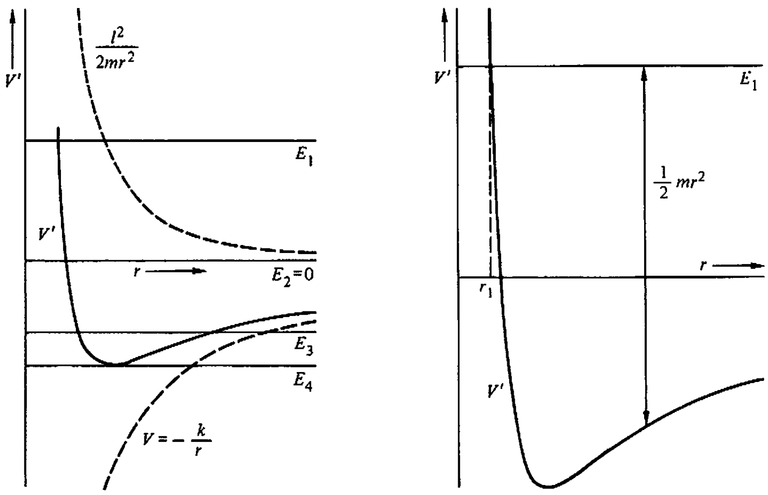

Figure 3: Equivalent one dimensional potential for inverse square law.

For $E = E_1$ the radius vector can not be smaller than r_1 otherwise V' is greater than the total energy and hence the velocity is imaginary following (34). A similar situation is obtained for $E = E_2$. In these cases motion is unbounded as shown in the left figure . For $E = E_3$, the motion is bounded between r_1 and r_2 as shown in the right figure . For $E = E_4$ the motion is circular.

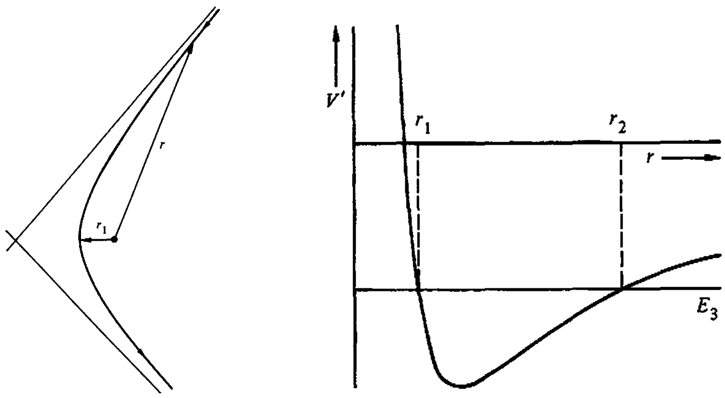

Figure 4: Bound and unbound motion.

3.5 Equations of motion

As mentioned in the previous section, our interest is to obtain the equation for the orbits *i,e,.* a functional dependance bewtween θ and r. We note that

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr = \frac{\ell}{\mu r^2 \dot{r}}dr\,.$$

So from (33) we get

$$\theta(r) = \int \frac{\pm(\ell/r^2)dr}{\sqrt{2\mu\Big(E - U - \frac{\ell^2}{2\mu^2 r^2}\Big)}}\,. \tag{35}$$

This is the solution for θ in terms of r. To obtain the solution for r in terms of time we use the El EoM

$$\frac{d}{dt}\left(\frac{\partial L}{\partial r}\right) - \frac{\partial L}{\partial r} = 0\,,$$

which gives

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r)\,. \tag{36}$$

In the above equation, if we replace $\dot{\theta}$ in terms of ℓ we get the EoM for r in terms of time. However, in central force problems, the equation of orbit *i.e.,* $r \equiv r(\theta)$ is of interests. Therefore we convert the above equation as a differential equation for r in terms of the θ. To this end, we cast this in a suitable form by change of variable

$$u = \frac{1}{r}\,.$$

We compute

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} = -\frac{\mu}{\ell}\dot{r}\,.$$

In the last line angular $\ell = \mu r^2\dot{\theta}$ has been used. The second derivative of the above is

$$\frac{du^2}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{\ell}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{\ell}\dot{r}\right) = -\frac{\mu}{\ell\dot{\theta}}\ddot{r} = -\frac{\mu^2}{\ell^2}r^2\ddot{r}\,.$$

which can be written as

$$\ddot{r} = -\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}\,.$$