Date
Page

LA Assignment 1 (H2)

2023113019

1\. Let $A = \begin{pmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{pmatrix}$

Taking augmented matrix $A|I$ (for gauss jordan)

Case I: $a \neq 0$

$$\left[\begin{array}{ccc|ccc} 0 & a & 0 & 1 & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - d/a\, R_1} \left[\begin{array}{ccc|ccc} 0 & a & 0 & 1 & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & 0 & 0 & -d/a & 0 & 1 \end{array}\right]$$

let it be $\boxed{e_1}$

let $e_1 = R_3 \to R_3 - \frac{d}{a} R_1$

$e_1(A) = A'$ s.that A' has a zero row

$A \sim A'$

but A' is non invertible

$\Rightarrow A$ is non invertible.

Hence for $a \neq 0$, A is non invertible.

Case 2: $a = 0$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right)$$

Here the first row of A is in itself zero row,
thus A is not invertible.

$\therefore$ For both $a \neq 0$ & $a = 0$ $[\mathbb{C}]$, the matrix is not invertible.

(b) $A = \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$

Taking augmented matrix $A|I$

For $a \neq 0$:

$$\left(\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to \frac{1}{a} R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right)$$

① $R_2 \to R_2 - R_1$
② $R_2 \to R_2/a$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & -1/a^2 & 1/a & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right)$$

① $R_3 \to R_3 - R_2$
② $R_3 \to R_3 (1/a)$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & -1/a^2 & 1/a & 0 \\ 0 & 0 & 1 & 1/a^3 & -1/a^2 & 1/a \end{array}\right) = (I|B)$$

~~Assuming $A \cdot A^{-1} = I \Rightarrow A^{-1}$~~ ~~$AX = I \Rightarrow X = A \Rightarrow A^{-1} = I$~~

$AX = I \Rightarrow X = A^{-1}$

$\Rightarrow A^{-1} = XI$

We know $(A|I)X = (I|B)$

$\Rightarrow (I|A^{-1}) = (I|B)$

$\Rightarrow A^{-1} = B$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ -1/a^2 & 1/a & 0 \\ 1/a^3 & -1/a^2 & 1/a \end{pmatrix}}$$

Case 2: $a = 0 \rightarrow$ A has zero row & is non-invertible

PTO

3. $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

$A^3 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$ (*)

$A^{2015} = (A^{2013})(A^2)$

$= (A^3)^{371}(A^2)$

$= (-I)^{371}(A^2)$

$= (-1)^{371}(I)^{371}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}}$ Ans.

(*) $A^4 = A^3 \cdot A = -IA = -A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

$A^5 = A^3 \cdot A^2 = -A^2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$A^6 = (A^3)^2 = (-I)^2 = (-1)^2 \cdot I^2$
$= I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^7 = A^6 \cdot A = A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

classmate
Date
Page

1. Let there exist matrices $X_{n\times n}$, $Y_{n\times n}$
[They must have same size ~~because~~ because they are $n\times n$ ∴ square & for matrix multiplication to be valid]

let us assume
$XY - YX = I$
$\therefore (XY - YX)_{ij} = \begin{cases} 1 & \forall i = j \\ 0 & \forall i \neq j \end{cases}$

∵ We know $(XY)_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}$ $\Rightarrow XY_{ii} = \sum_{k=1}^{n} x_{ik} y_{ki}$

$(YX)_{ij} = \sum_{k} y_{ik} x_{kj}$ $\Rightarrow YX_{ii} = \sum_{k=1}^{n} y_{ik} x_{ki}$

$tr(XY) = \sum_{i=1}^{n} (XY)_{ii}$ [defining tr: transpose as a function on a matrix]

$= \sum_{i=1}^{n} \sum_{k=1}^{n} x_{ik} y_{ki}$

w.l.o.g.; $tr(YX) = \sum_{i=1}^{n} \sum_{k=1}^{n} y_{ik} x_{ki}$

$= \sum_{i=1}^{n} \sum_{k=1}^{n} x_{ki} y_{ik}$

∵ the series converges, we can replace ~~the~~ inter $\Sigma\Sigma$
∴ we have,

$tr(YX) = \sum_{k=1}^{n} \sum_{i=1}^{m} x_{ik} y_{ki}$

$\therefore tr(XY) = tr(YX)$
$\therefore tr(XY) - tr(YX) = 0$
$tr(XY - YX) = 0$
[∵ $tr(A) - tr(B) = tr(A - B)$]
$\Rightarrow tr(I) = 0$
But $tr(I) = \sum_{i=1}^{n} I_{ii} = n \neq 0$

Thus, by contradiction, our assumption is wrong.
Hence, no such matrix exists.

classmate
Date
Page

5. Given,

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad w = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

To show, $B = \{u_1, u_2, u_3\}$ ~~$\{u_1, u_2, u_3\}$~~ is orthogonal basis for $\mathbb{R}^3$ [Representing here elements as 3-tuple]

i.e

i) $\langle u_1, u_2 \rangle = \langle u_2, u_3 \rangle = \langle u_3, u_1 \rangle = 0$ [Orthogonal]

ii) $\sum_{i=1}^{3} \alpha_i u_i = (0, 0, 0)$ $\forall \alpha_i \in \mathbb{R}$ [linearly Independent]

$\Rightarrow \alpha_i = 0$ $\forall i = 1, 2, 3$

iii) for any $(x, y, z) \in \mathbb{R}^3$ $\exists \alpha_1, \alpha_2, \alpha_3$

s.that $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = (x, y, z)$ [Spanning]

Proof:

i) $\langle u_1, u_2 \rangle = 1.1 + 1.(-1) + 1.(0)$
$= 0$

$\langle u_2, u_3 \rangle = 1.1 + (-1).1 + 0.(-2) = 0$

$\langle u_3, u_1 \rangle = 1.1 + 1.1. + 1.(-2) = 0$

Hence B is orthogonal

ii) Let $\alpha_1, \alpha_2, \alpha_3$ s.that

$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = (0, 0, 0)$

$$\begin{bmatrix} \alpha_1 \\ \alpha_1 \\ \alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ -\alpha_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 \\ \alpha_3 \\ -2\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 - \alpha_2 + \alpha_3 \\ \alpha_1 - 2\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\alpha_1 = 2\alpha_3$

$\therefore 3\alpha_3 + \alpha_2 = 0$ & $3\alpha_3 - \alpha_2 = 0$

$\Rightarrow \alpha_2 = 0 \Rightarrow 3\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \Rightarrow \alpha_1 = 0$

classmate

Date
Page

$\therefore \alpha_1, \alpha_2, \alpha_3 = 0$

Hence B is linearly independent.

③ Orthogonal vectors span the subspace, hence ① → ③

$\therefore B = \{u_1, u_2, u_3\}$ is an orthogonal basis for

$$\therefore w = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 - \alpha_2 + \alpha_3 \\ \alpha_1 - 2\alpha_3 \end{bmatrix}$$

But we can also use

$w = c_1 u_1 + c_2 u_2 + c_3 u_3$ [orthogonal basis]

$c_i = \dfrac{w \cdot u_i}{u_i \cdot u_i}$

$$c_1 = \frac{4+5+6}{1+1+1} = 5$$

$$c_2 = \frac{4-5}{1+1} = -\frac{1}{2}$$

$$c_3 = \frac{4+5-12}{1+1+4} = \frac{-3}{6} = -1/2$$

$\therefore$ Coordinate of w wrt B is $(5, -1/2, -1/2)$

6. Let the 2 upper rectangular be $A_{n\times n}$ & $B_{n\times n}$ [for square + for product to be defined, size must be like this]

$A_{ij} = 0 \; \forall \, i > j$ {upper triangular}

$B_{ij} = 0 \; \forall \, i > j$

Let $C = AB$

$\Rightarrow C_{ij} = (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

but $a_{i,k} = 0 \ \forall \ k < i$

$b_{k,j} = 0 \ \forall \ k > j$

$\Rightarrow a_{ik} \cdot b_{kj} = 0 \ \forall \ k > j$ or $k < i$

$\because$ For $i > j$, we have

Case I: $k \leq j \Rightarrow k < i \Rightarrow a_{ik} b_{kj} = 0$

Case II: $k > j \Rightarrow a_{ik} \cdot b_{kj} = 0$

~~Thus $C_{ij} = \Sigma$~~

Thus, for $i > j$, $a_{ik} b_{kj} = 0 \ \forall \ k$

Thus $C_{ij} = \Sigma a_{ik} b_{kj} = 0 \ \forall \ i > j$

$\therefore$ By def^n,

C_{ij} is upper triangular matrix.

[Hence proved]