Lemma: If A is an $m \times n$ matrix over F and B, C are $n \times p$ matrices over F then

$$A(dB+C) = d(AB)+AC \text{ for each scalar } d \in F.$$

Proof:
$$\begin{aligned}[A(dB+C)]_{ij} &= \sum_k A_{ik}(dB+C)_{kj} \\ &= \sum_k (d A_{ik} B_{kj} + A_{ik} C_{kj}) \\ &= d \sum_k A_{ik} B_{kj} + \sum_k A_{ik} C_{kj} \\ &= d(AB)_{ij} + (AC)_{ij} \\ &= [d(AB)+AC]_{ij}.\end{aligned}$$

Similarly, one can show that

$(dB+C)A = d(BA)+CA$, if matrix sums & products are defined.

Thm. Let V be a vector space over the field F. The intersection of any collection of subspaces of V is a subspace of V.

Proof: Let $\{W_a\}$ be a collectⁿ of subspaces of V, and let $W = \bigcap_a W_a$ be their intersection. Recall that W is defined as the set of all elements belonging to every W_a. Since each

W_a is a subspace, each contains the zero vector. Thus the zero vector is in the intersection W, and W is non-empty. Let α & β be vectors in W and let c be a scalar. By definition of W, both α & β ~~be vectors in W and~~ belong to each W_α, and because each W_a is a subspace, the vector $(c\alpha+\beta) \in W_\alpha$ $\forall \alpha$. Thus, $(c\alpha+\beta)$ is again in W. ~~Thus, W is a~~ And, W is a subspace of V.

From aforementioned theorem it follows that if S is any collection of vectors in V, then there is a smallest subspace of V which contains S, i.e., a subspace which contains S and which is contained in every ~~subspace~~ other subspace containing S.

Def: Let S be a set of vectors in a vector space V. The subspace spanned by S is defined to be the intersection W of all subspaces of V which contains S. When S is a finite set of vectors, $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, we shall simply call W the subspace spanned by the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$.

<u>Theorem</u>. The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

Proof. W be the subspace spanned by S. Each linear combⁿ

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_m\alpha_m$$

of vectors $\alpha_1, \alpha_2, \dots, \alpha_m \in S$ is clearly in W. Thus, W contains the set L of all linear combⁿs. of vectors in S. On the other hand, the set L contains S and is non-empty. If $\alpha, \beta \in L$ then α is linear combⁿ

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_m\alpha_m$$

of vectors $\alpha_i \in S$, and β is linear combⁿ.

$$\beta = y_1\beta_1 + y_2\beta_2 + \dots + y_n\beta_n.$$

of vectors $\beta_j \in S$. For each scalar c,

$$c\alpha + \beta = \sum_{i=1}^{m} (cx_i)\alpha_i + \sum_{j=1}^{n} y_j\beta_j.$$

$c\alpha + \beta \in L$. $\therefore$ L is subspace of V.

⋮

Def^n If $S_1, S_2, \ldots, S_k$ are subsets of a vector space V, the set of all sums

$$\bar{\alpha}_1 + \bar{\alpha}_2 + \ldots + \bar{\alpha}_k$$

of $\bar{\alpha}_i \in S_i$ is called the sum of the subsets $S_1, S_2, \ldots, S_k$ and is denoted by

$$S_1 + S_2 + \ldots + S_k$$

or by $\displaystyle\sum_{i=1}^{k} S_i$.

If $W_1, W_2, \ldots, W_k$ are subspaces of V, then the sum

$$W = W_1 + W_2 + \ldots + W_k$$

is easily seen to be a subspace of V which contains each of the subspaces W_i. From this it follows that W is the subspace spanned by the union of $W_1, W_2, \ldots, W_k$.

Example. Let F be a subfield of $\mathbb{C}$.
Suppose, $\bar{\alpha}_1 = (1,2,0,3,0)$,

$\bar{\alpha}_2 = (0,0,1,4,0)$,

$\bar{\alpha}_3 = (0,0,0,0,1)$.

A vector $\bar{\alpha}$ is in the subspace W of F^5 spanned by $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$ if and only if $\exists\ c_1, c_2, c_3 \in F$ s.t.

$$\bar{\alpha} = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + c_3\bar{\alpha}_3 .$$

Thus, W consists of all vectors of the form $\bar{\alpha} = (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3)$, where c_1, c_2, c_3 are arbitray scalars in F.

Alternatively,

W can be described as the set of all 5-tupples $\bar{\alpha} = (x_1, x_2, x_3, x_4, x_5)$ w/ $x_i \in F$ s.t.

$$x_2 = 2x_1$$
$$x_4 = 3x_1 + 4x_3 .$$

Thus $(-3,-6,1,-5,2) \in W$, whereas $(2,4,6,7,8)$ is not.

Example. Let F be a subfield of $\mathbb{C}$, and let V be the vector space of all 2×2 matrices over F. Let W_1 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \quad x, y \in F \text{ are arbitrary}.$$

Then W_1 & W_2 are subspaces of V.
Also, $V = W_1 + W_2$
because $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$.
The subspace $W_1 \cap W_2$ consists of all matrices of the form $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$.

Bases and Dimension

Def^n Let V be a vector space over F. A subset S of V is said to be <u>linearly dependent</u> (or, <u>dependent</u>) if $\exists$ distinct vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in S$ and scalars $c_1, c_2, \ldots, c_n \in F$, not all of which are $\bar{0}$, s.t.

$$c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_n\bar{\alpha}_n = \bar{0}.$$

A set which is not linearly dependent is called <u>linearly independent</u>. If the set S contains only finitely many $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in V$, we sometimes say that $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ are dependent (or independent) instead of saying S is dependent (or independent).

The following are easy consequences:

1. Any set which contains a linearly dependent set is linearly dependent.
2. Any subset of linearly independent set is linearly independent.
3. Any set which contains the $\bar{0}$ is linearly dependent.

4. A set S of vectors is linearly independent iff each finite subset of S is linearly independent, i.e., iff for any distinct vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in S$, $c_1\bar{\alpha}_1 + \ldots + c_n\bar{\alpha}_n = 0$ implies each $c_i = 0$.

<u>Def^n</u>: Let V be a vector space. A <u>basis</u> for V is a linearly independent set of vectors in V which spans the space V. The space V is finite-dimensional if it has a finite basis.

Example: Let F be a subfield of $\mathbb{C}$.
In F^3 the vectors

$$\bar{\alpha}_1 = (3, 0, -3),$$
$$\bar{\alpha}_2 = (-1, 1, 2),$$
$$\bar{\alpha}_3 = (4, 2, -2),$$
$$\bar{\alpha}_4 = (2, 1, 1)$$

are linearly dependent, since

$$2\bar{\alpha}_1 + 2\bar{\alpha}_2 - \alpha_3 + 0 \cdot \bar{\alpha}_4 = 0.$$

The vectors

$$\bar{\varepsilon}_1 = (1, 0, 0),$$
$$\bar{\varepsilon}_2 = (0, 1, 0),$$
$$\bar{\varepsilon}_3 = (0, 0, 1)$$

are linearly dependent.

Example: Consider F^n over F. $S \subset F^n$ contains

$$\bar{\varepsilon}_1 = (1, 0, 0, \ldots, 0),$$
$$\bar{\varepsilon}_2 = (0, 1, 0, \ldots, 0),$$
$$\vdots$$
$$\bar{\varepsilon}_n = (0, 0, 0, \ldots, 1).$$

Let $x_1, x_2, \ldots, x_n \in F$ and put $\bar{\alpha} = x_1\bar{\varepsilon}_1 + x_2\bar{\varepsilon}_2 + \ldots + x_n\bar{\varepsilon}_n$.
Then, $\bar{\alpha} = (x_1, x_2, \ldots, x_n)$.
This shows that $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_n$ span F^n.
$\because$ $\bar{\alpha} = \bar{0}$ iff $x_1 = x_2 = \ldots = x_n = 0$, $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_n$ are linearly independent. Set $\underbrace{S = \{\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_n\}}_{\text{Standard basis of } F^n}$ is a basis of F^n.

Example: P be an $n \times n$ invertible matrix over F. $P_1, \ldots, P_n$, the columns of P, form a basis for the space of column matrices, $F^{n \times 1}$. If X is a column matrix, $PX = x_1 P_1 + \ldots + x_n P_n$.
$\because$ $PX = 0$ has only the trivial sol^n $X = 0$, $\{P_1, \ldots, P_n\}$ is a linearly independent set.
Spans $F^{n \times 1}$: Let Y be a column matrix. If $X = P^{-1} Y$, then $Y = PX$, i.e.,

$$Y = x_1 P_1 + \ldots + x_n P_n.$$

So, $\{P_1, \ldots, P_n\}$ is a basis for $F^{n \times 1}$.

Example: Let F be a subfield of $\mathbb{C}$.
V be the space of polynomial f^{ns} over F,

$$f(x) = c_0 + c_1 x + \ldots + c_n x^n.$$

Let $f_k(x) = x_k$, $k = 0, 1, 2, \ldots$ The (infinite) set $\{f_0, f_1, f_2, \ldots\}$ is a basis for V. Clearly, the set spans V, because the f^n f (above) is $f = c_0 f_0 + c_1 f_1 + \ldots + c_n f_n$.

<u>Thm</u>. Let V be a vector space which is spanned by a finite set of vectors $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$. Then any independent set of vectors in V is finite and contains no more than m elements.

Proof: Show that any set of vectors with more than m elements is linearly dependent.

$\because$ $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$ spans V, $\exists\ A_{ij} \in F$

s.t. $\bar{\alpha}_j = \sum_{i=1}^{m} A_{ij} \bar{\beta}_i \quad \forall\ \bar{\alpha}_j \in V$.

Consider $S = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$ be set of n distinct vectors. For any $x_1, x_2, \ldots, x_n \in F$,

$$x_1\bar{\alpha}_1 + x_2\bar{\alpha}_2 + \ldots + x_n\bar{\alpha}_n = \sum_{j=1}^{n} x_j \bar{\alpha}_j$$

$$= \sum_j x_j \sum_i A_{ij}\bar{\beta}_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij} \cdot x_j\right)\bar{\beta}_i$$

$\because n > m$, $\sum_{j=1}^{n} A_{ij} x_j = 0$ has a non-trivial sol.^{n} for x_j's, i.e., not all x_j's are 0.

$\Rightarrow$ S is linearly dependent.

Corollary: If V is a finite-dimensional vector space, ~~then~~ any bases of V have the same (finite) number of elements.

Corollary: Let V be a finite-dim. vector space & let $\dim V = n$. Then, n is cardinality of basis &

(a) any subset of V which contains more than n ~~linear~~ vectors is linearly dependent.

(b) no subset of V which contains less than n vectors can span V.

Lemma: Let S be a linearly independent subset of a vector space V. Suppose $\beta \in V$ is not in a subspace spanned by S. Then the set obtained by adjoining β to S is linearly dependent.

__Thm.__ If W is a subspace of a finite-dim. vector space V, every linearly independent subset of W is finite and is part of a (finite) basis for W.

Corollary: If W is a proper subspace of a finite-dim. V, then W is finite-dim. and $\dim W < \dim V$.

Corollary: Let $A_{n \times n}$ over F. Suppose the row vectors of A form a linearly independent set of vectors in F^n. Then A is invertible.

__Thm.__ If W_1 & W_2 are finite-dim subspaces of a vector space V, then $W_1 + W_2$ is finite-dim &

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

Coordinates

A basis $\mathcal{B}$ in an n-dim space V enables introduction of coordinates in V analogous to the 'natural coordinates' x_i of a $\bar{\alpha} = (x_1, \ldots, x_n) \in F^n$. Then, the coordinates of $\bar{\alpha} \in V$ relative to $\mathcal{B}$ will be scalars which serve to express $\bar{\alpha}$ as a linear combination of the vectors in the basis. The natural coordinates of $\bar{\alpha} \in F^n$ is defined by $\bar{\alpha}$ and the std basis for F^n. If $\bar{\alpha} = (x_1, \ldots, x_n) = \sum_i x_i \varepsilon_i$ and $\mathcal{B}$ is the std. basis for F^n, how are the coordinates of $\bar{\alpha}$ determined by $\mathcal{B}$ & $\bar{\alpha}$?

Def^n: If V is a finite-dimensional vector space, an ordered basis for V is a finite sequence of vectors which is linearly independent and spans V.

If the sequence $\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n$ is an ordered basis for V, then the set $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\}$ is a basis for V. The ordered basis is the set, together w/ the specified ordering. W/ slight abuse of notation: $\mathcal{B} = \{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\}$ is an ordered basis for V.

V is a finite-dim vector space over F and $\mathcal{B} = \{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\}$ is an ordered basis for V. For $\overline{\alpha} \in V$, $\overline{\alpha} = \sum_{i=1}^{n} x_i \overline{\alpha}_i$ for some unique n-tupple $(x_1, x_2, \ldots, x_n)$. It's unique because if $\overline{\alpha} = \sum_{i=1}^{n} y_i \overline{\alpha}_i$, then

$$\overline{\alpha} - \overline{\alpha} = \sum_{i=1}^{n} (x_i - y_i) \overline{\alpha}_i = \overline{0} \Rightarrow x_i = y_i \ \forall i.$$

x_i is called i^{th} coordinate of $\overline{\alpha}$ relative to an ordered basis $\mathcal{B}$.

If $\overline{\alpha} = \sum x_i \overline{\alpha}_i$ & $\overline{\beta} = \sum y_i \overline{\alpha}_i$,

then $\overline{\alpha} + \overline{\beta} = \sum_i \underbrace{(x_i + y_i)}_{i^{th} \text{ coordinate of } \overline{\alpha}+\overline{\beta} \text{ w.r.t } \mathcal{B}} \overline{\alpha}_i$

i^{th} coordinate of $c\bar{\alpha}$ is cx_i w.r.t. $\mathcal{B}$.

Note that every n-tupple $(x_1, \ldots, x_n) \in F^n$ is the n-tupple of coordinates of some vector in V, namely the vector $\sum_{i=1}^{n} x_i \bar{\alpha}_i$.

I.e., each ordered basis for V determines a one-to-one correspondence

$$\bar{\alpha} \longrightarrow (x_1, \ldots, x_n)$$

b/w the set of all vectors in V & the set of all n-tupples in F^n. This correspondence has the property that the correspondent of $(\bar{\alpha} + \bar{\beta})$ is the sum in F^n of the correspondents of $\bar{\alpha}$ & $\bar{\beta}$, and that the correspondent of $c\bar{\alpha}$ is the product in F^n of the scalar c & the correspondent of $\bar{\alpha}$.

Coordinate matrix of $\bar{\alpha}$ w.r.t. the ordered basis $\mathcal{B}$, $[\alpha]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

$$\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i .$$

Suppose V is n-dim. and
$\mathcal{B} = \{\bar{\alpha}_1, \bar{\alpha}_2, \dots \bar{\alpha}_n\}$ and $\mathcal{B}' = \{\bar{\alpha}_1', \dots, \bar{\alpha}_n'\}$
are two ordered bases for V. There are unique scalars P_{ij} s.t.

$$\bar{\alpha}_j' = \sum_{i=1}^{n} P_{ij} \alpha_i \quad , \quad 1 \le j \le n.$$

Let $x_1', \dots, x_n'$ be the coordinates of a given $\bar{\alpha}$ in the ordered basis $\mathcal{B}'$. Then

$$\bar{\alpha} = x_1' \bar{\alpha}_1' + \dots + x_n' \bar{\alpha}_n$$

$$= \sum_{j=1}^{n} x_j' \bar{\alpha}_j'$$

$$= \sum_{j=1}^{n} x_j' \sum_{i=1}^{n} P_{ij} \bar{\alpha}_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (P_{ij} x_j') \right) \bar{\alpha}_i$$

$$\bar{\alpha} = \sum_{i} \left(\sum_{j} P_{ij} x_j' \right) \bar{\alpha}_i$$

Then $\bar{\alpha} = \sum_i x_i \bar{\alpha}_i$, $(x_1, \dots, x_n)$ being coordinates w.r.t. $\mathcal{B}$,

$$x_i = \sum_j P_{ij} x_j' , \quad 1 \le i \le n.$$

Let $P_{n\times n}$ whose i,j entry is the scalar P_{ij}, and let X and X' be the coordinate matrices of the vector $\bar{\alpha} \in V$ in the ordered bases $\mathcal{B}$ & $\mathcal{B}'$. Then,

$$x_i = \sum_{j=1}^{n} P_{ij} x_j' \quad \forall i \in \{1, \dots, n\}$$

can be expressed as $X = PX'$.

Since ~~$\mathcal{B}$~~ & $\mathcal{B}'$ are linearly independent sets, $X = 0$ iff $X' = 0$. This implies, P is invertible. Hence, $X' = P^{-1}X$.

I.e.,

$$[\bar{\alpha}]_{\mathcal{B}} = P[\bar{\alpha}]_{\mathcal{B}'}$$

$$[\bar{\alpha}]_{\mathcal{B}'} = P^{-1}[\bar{\alpha}]_{\mathcal{B}}$$

<u>Thm</u>. Let V be an n-dim vector space over F. Let $\mathcal{B}$ & $\mathcal{B}'$ be two ordered bases ~~vector spaces~~ of V. Then there is a unique, necessarily invertible, $n\times n$ matrix P over F s.t.

(i) $[\bar{\alpha}]_{\mathcal{B}} = P[\bar{\alpha}]_{\mathcal{B}'}$

(ii) $[\bar{\alpha}]_{\mathcal{B}'} = P^{-1}[\bar{\alpha}]_{\mathcal{B}}$

$\forall \bar{\alpha} \in V$. Columns P_j of P are $P_j = [\bar{\alpha}_j']_{\mathcal{B}}$, for $j \in \{1, \dots, n\}$.

Thm. Suppose P is an $n \times n$ invertible matrix over F. Let V be an n-dim vector space over F, and let $\mathcal{B}$ be an ordered basis of V. Then there is a unique ordered basis $\mathcal{B}'$ of V s.t.

(i) $[\bar{\alpha}]_{\mathcal{B}} = P[\bar{\alpha}]_{\mathcal{B}'}$

(ii) $[\bar{\alpha}]_{\mathcal{B}'} = P^{-1}[\bar{\alpha}]_{\mathcal{B}}$

$\forall \bar{\alpha} \in V$.

$$\bar{\alpha} = \sum_{j=1}^{n} x_j' \alpha_j' = \sum_{i=1}^{n} x_i \bar{\alpha}_i$$

$$\bar{\alpha}_j' = \sum_{i=1}^{n} P_{ij} \bar{\alpha}_i$$

Example: $\mathbb{R}$ be real field & $\theta \in \mathbb{R}$ is fixed.

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\mathcal{B}' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\} \subset \mathbb{R}^2$$

$$[\bar{\alpha}]_{\mathcal{B}'} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(X' = P^{-1}X)$$