Linear Algebra (UG1, Spring 2023)

Re-exam [10+10 marks]; Time: 60+60 mins (+15 mins)

April 29, 2023

Notations are from class lectures unless stated otherwise. Each step of the proof should be clear. Appropriate reasoning for your claims are must.

Part A

1. (4 marks) Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are unique vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

2. (2 marks) The system of equations
$$\begin{aligned} x + y + z &= 6 \\ x + 4y + 6z &= 20 \\ x + 4y + \lambda z &= \phi. \end{aligned}$$
Find the values of λ and ϕ for which this system of equations has no solutions.

3. (2 marks) Compute the reduced row echelon form of the following matrix.
$$\begin{bmatrix} 2 & 0 & -6 \\ 0 & 1 & 2 \\ 3 & 6 & -2 \end{bmatrix}$$

4. (1 marks) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$ holds, where N and R denotes the nullity and the range of the linear transformation T.

5. (1 marks) Give an example of two linear transformations T and U for which $N(T) = N(U)$ and $R(T) = R(U)$ holds, where N and R denotes the nullity and the range of the linear transformation T as well as U.

Part B

1. (5 marks) Let V be a finite dimensional vectior space defined over the field $\mathbb{R}$ of real numbers. Let $T : V \to V$ is a linear transformation such that $\mathbf{Rank}(T) = \mathbf{Rank}(T^2)$, where $T^2 = T \circ T$. Then show that

   - $\mathbf{Ker}(T) = \mathbf{Ker}(T^2)$.
   - $\mathbf{Range}(T) = \mathbf{Range}(T^2)$.
   - $\mathbf{Ker}(T) \cap \mathbf{Range}(T) = \{0\}$.
   - $\mathbf{Ker}(T^2) \cap \mathbf{Range}(T^2) = \{0\}$.

2. (5 marks) Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o, be the subset of odd functions, $f(-x) = -f(x)$.

   - Prove that V_e and V_o are subspaces of V.
   - Prove that $V_e + V_0 = V$.
   - Prove that $V_e \cap V_o = \{0\}$.