2.3 Euler-Lagrange equation of motion

If $q_i(t)$ correspond ti the actual path then, for all neighborhood paths, no matter how close, the action S will increase. The neighborhood paths are parametrized as

$$q_i(t,\alpha) = q_i(t) + \alpha\eta_i(t)\,, \tag{16}$$

where $\eta_i(t)$ are arbitrary, but continuous and single valued in the interval $t_A - t_B$. The conditions for extremum for S is that when that when action is calculated for the neighborhood paths (16), it changes only at the second order in α and the first derivative

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = 0\,.$$

Differentiating (13) w.r.to α we get

$$0 = \left.\frac{dS}{d\alpha}\right|_{\alpha=0} = \int_{t_A}^{t_B} \left.\left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha}\right)\right|_{\alpha=0} dt\,.$$

Note that the derivative with respect to α evaluated at $\alpha = 0$, but for the simplicity of writing we omit $|_{\alpha=0}$ in each step until the end. The second term in the integral can be written as

$$\int_{t_A}^{t_B} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha} dt = \int_{t_A}^{t_B} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial^2 q_i}{\partial t \partial \alpha} dt\,.$$

Now, integrating by parts we get

$$\int_{t_A}^{t_B} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial^2 q_i}{\partial t \partial \alpha} dt = \left.\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \alpha}\right|_{t_A}^{t_B} - \int_{t_A}^{t_B} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \alpha}\right) dt\,.$$

Since all the paths go through the end points $q_i(t_A)$ and $q_i(t_B)$, and the end points are always fixed, the first terms is zero. Hence we get

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = \int_{t_A}^{t_B} \left.\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \alpha}\right|_{\alpha=0} dt = \int_{t_A}^{t_B} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i dt\,.$$

Since q_i are all independent, their variations $\eta_i(x) = dq_i/d\alpha$ are independent of each other. From the "fundamental lemma" of calculus it implies that the terms in the parenthesis vanish for each i,

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0\,. \tag{17}$$

This is called the Euler-Lagrange equation of motion (EL EoM). Note that if there are n generalized coordinate, then there are n EL EoMs. Unlike the Newton's 2nd law, this is a second order differential equations involving only scalar quantities. For a system with n dof, there are n such equations. Hence solutions would involve two integrals, and $2n$ constant of integration. These constants can be determined from the initial conditions.

Few comments regarding the EL Eom (17):

- In the last step of the derivation q_i are assumed independent. This is valid only for holonomic constraints. For non-holonomic constraints (which we will not discuss in this course) q_i are not all independent. But the EL EoM can be extended using Lagrange multiplier methods.
- In the definition (15), the potential is function of coordinates only. For velocity dependent potenatils $Vq_i, \dot{q}_i, t$, the EL EoM is modified.
- To a Lagrangian (15), you can add total time derivative $dF(q_i,t)/dt$ of any arbitrary function $F(q_i,t)$ and it would lead to same equation of motion.