$(x+iy \rightarrow x-iy$

Unitary and Hermitian Matrices

The conjugate transpose of a complex matrix A denoted by A^* is given by $A^* = \bar{A}^T$. $\rightarrow$ [conjugate transp ose]

(where the entries of $\bar{A}$ are the corresponding complex conjugates of the entries of A) $\rightarrow$ If the matrix is Real then $A^* = A^T$ ✓

Prob: Find the conjugate transpose of the matrix $A = \begin{bmatrix} 3+7i & 0 \\ 2i & 4-i \end{bmatrix}$ $\bar{A} = \begin{bmatrix} \overline{3+7i} & \bar{0} \\ \overline{2i} & \overline{4-i} \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 3-7i & 0 \\ -2i & 4+i \end{bmatrix}$$

$$A^* = \bar{A}^T = \begin{bmatrix} 3-7i & -2i \\ 0 & 4+i \end{bmatrix}$$

Properties of the Conjugate Transpose.

If A and B are complex matrices and k is a complex number then the following properties are true.

1) $(A^*)^* = A$
2) $(A+B)^* = A^* + B^*$
3) $(kA)^* = \bar{k} A^*$
4) $(AB)^* = B^* A^*$

Unitary Matrix:

(For real matrix a matrix will be called as unitary if

$$UU^* = U^*U = I$$

(but we know, $UU^{-1} = U^{-1}U = I$

Inverse is unique !!

In that case $U^{-1} = U^T$.

Incase of complex matrices, we have the property that

$$U^{-1} = U^* \longrightarrow \text{unitary matrices}$$

A complex matrix is called unitary if $U^{-1} = U^*$

Prob: Show that the following matrix is unitary

$$A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

Check out $\longrightarrow$

$$AA^* = \frac{1}{4}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A^*A = I$

$\Rightarrow A^* = A^{-1}$

$\therefore$ ↳ unitary

Result: An $n \times n$ matrix (complex) A is unitary iff its row (or column) vectors form an orthonormal set in $\mathbb{C}^n$

$$A = \begin{bmatrix} 1/2 & \frac{1+i}{2} & -\frac{1}{2} \\ -i/\sqrt{3} & i/\sqrt{3} & 1/\sqrt{3} \\ 5i/2\sqrt{15} & \frac{3+i}{2\sqrt{15}} & \frac{4+3i}{2\sqrt{15}} \end{bmatrix}$$

Let $\vec{r_1} = \left(\frac{1}{2}, \frac{1+i}{2}, -1/2\right)$

$\vec{r_2} = \left(-i/\sqrt{3}, i/\sqrt{3}, 1/\sqrt{3}\right)$

$\vec{r_3} = \left(5i/2\sqrt{15}, \frac{3+i}{2\sqrt{15}}, \frac{4+3i}{2\sqrt{15}}\right)$

Length of $\vec{r_1}$

$$\|\vec{r_1}\| = (\vec{r_1} \cdot \vec{r_1})^{1/2}$$

$$= \sqrt{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)}$$

$$= \sqrt{\frac{1}{4} + \frac{2}{4} + \frac{1}{4}}$$

$$= \sqrt{1} = 1 \longrightarrow \text{they are of unit length}$$

Take any two row vectors and take the dot product (inner product) —

$$\bar{r}_1 \cdot \bar{r}_2 = \left(\frac{1}{2}\right)\overline{\left(\frac{-i}{\sqrt{3}}\right)} + \left(\frac{1+i}{2}\right)\overline{\left(\frac{i}{\sqrt{3}}\right)} + \left(-\frac{1}{2}\right)\overline{\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \cancel{\frac{i}{2\sqrt{3}}} - \cancel{\frac{i}{2\sqrt{3}}} + \cancel{\frac{1}{2\sqrt{3}}} - \cancel{\frac{1}{2\sqrt{3}}} = 0$$

Similarly we can see

$$\vec{r}_1 \cdot \bar{r}_3 = \bar{r}_2 \cdot \bar{r}_3 = \bar{r}_1 \cdot \bar{r}_2 = 0.$$

$$\{\|r_i\| = 1\} \quad i = 1, 2, 3.$$

$\{r_1, r_2, r_3\} \longrightarrow$ O.N set

↳ forms an orthonormal set.

Hermitian Matrices

Def^n: A square matrix A is Hermitian if $A = A^*$

[For symmetric matrices it is easy to check whether it is hermitian or not]

$$A = \begin{bmatrix} a_1 + a_2 i & b_1 + b_2 i \\ c_1 + c_2 i & d_1 + d_2 i \end{bmatrix}$$

$$A^* = \bar{A}^T = \begin{bmatrix} \overline{a_1 + a_2 i} & \overline{c_1 + c_2 i} \\ \overline{b_1 + b_2 i} & \overline{d_1 + d_2 i} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 - a_2 i & c_1 - c_2 i \\ b_1 - b_2 i & d_1 - d_2 i \end{bmatrix}$$

For A to be hermitian, $A = A^*$

$$\cancel{a_1} + a_2 i = \cancel{a_1} - a_2 i$$

Real $\Rightarrow a_2 = 0$ SLY $\rightarrow d_2 = 0$

$$c_1 - c_2 i = b_1 + b_2 i$$

$$c_1 + c_2 i = b_1 - b_2 i$$

$$A = \begin{bmatrix} a_1 & b_1 + b_2 i \\ b_1 - b_2 i & d_1 \end{bmatrix}$$

Real

$c_1 = b_1,\ c_2 = -b_2$

$\Rightarrow$ (1) The entries of the main diagonals are real.

(2) The entry a_{ij} of the ith row and jth column is the complex conjugate of the entry a_{ji} (jth row and ith column)

Theorem: If A is a hermitian matrix, then its eigenvalues are real numbers.

Problem:

which of the following are Hermitian matrices

(a) $\begin{bmatrix} 1 & 3-i \\ 3+i & i \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 3-2i \\ 3-2i & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$

Proof: Let λ be the eigen value of the matrix A and

$v = \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \\ a_n + b_n i \end{bmatrix}$ is the e-vector corresponding

to the eigenvalue λ.

The eigen equation is,

$\boxed{Av = \lambda v} \longrightarrow (1)$

$$v^* A v = v^* (\lambda v)$$

$$= \lambda (v^* v)$$

$$= \lambda \{a_1^2 + b_1^2 + a_2^2 + b_2^2 + \dots + a_n^2 + b_n^2\} \longrightarrow \text{number !!}$$

x

$$(v^* A v)^*$$

$$= v^* A^* (v^*)^* \quad [A^* = A]$$

$$= v^* A v \longrightarrow (1)$$

$x^* = x \longrightarrow$ (number) $\longrightarrow$ conjugate of the no. = ~~eq~~ the no.

$\Rightarrow$ the no is a real number

$\Rightarrow$ λ is a real number.

Find out the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$$

e.v of A are -1, 6 and -2

find out e. vectors.

<u>Theorem</u>: If A is a $n \times n$ matrix (Hermitian) then eigen vectors corresponding to distinct eigen values are orthogonal

<u>Proof</u>:

Proof: Let v_1 and v_2 be two eigen vectors corresponding to two distinct e. values λ_1 and λ_2

$$\left.\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \end{aligned}\right\}$$

$$(Av_1)^* v_2 = v_1^* A^* v_2 = v_1^* Av_2$$

$$= \lambda_2 v_1^* v_2 \quad \longrightarrow (1)$$

($\because A \rightarrow$ hermitian)

($\lambda_1 \rightarrow$ real)

$$(Av_1)^* v_2 = (\lambda_1 v_1)^* v_2$$

$$= v_1^* \lambda_1 v_2 = \lambda_1 v_1^* v_2 \quad \longrightarrow (2)$$

From (1) and (2)

$$\lambda_1 u_1^* u_2 - \lambda_2 u_1^* u_2 = 0$$

$$\Rightarrow (\lambda_1 - \lambda_2) u_1^* u_2 = 0$$

$$\because \lambda_1 \neq \lambda_2 \Rightarrow u_1^* u_2 = 0$$

$\Rightarrow$ u_1 and u_2 are orthogonal.

Diagonalization of Hermitian Matrix

Result: If A is an $n \times n$ Hermitian matrix, then A is unitarily diagonalizable.

⇒ You are going to find a unitary matrix P ~~such~~ ~~that~~ $P^* A P$ is a diagonal matrix

Ex: $$A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$$

↳ find the unitary matrix P such that $P^* A P$ is a diagonal matrix.

Can you differentiate a symmetric matrix from a Hermitian matrix

↳ same

difference

↳ different