2023113019
Raunak Seksaria
Assignment. 5
Linear Algebra

1. Let W be the subspace spanned by the set S.
By the def^n of subspace spanned by set S consisting of vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ [~~[illegible]~~ it need not be finite], W contains
$\omega \in W$
$$\omega = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m$$
$$\forall x_i \in F$$
Thus W contains the set L of all linear combinations of $\alpha_i \in S$
The set L contains S, and is therefore obviously not empty [S is non-empty].
$\{\forall \alpha_i \in S, x_j\ 1 \le j \le m, j \ne i, x_i = 1, l \in L \mid l = \alpha_i\}$

Let $\alpha, \beta \in L$, then
$$\alpha = \sum_{i=1}^{m} x_i \alpha_i, \quad \beta = \sum_{i=1}^{m} y_i \beta_i \quad [\alpha_i, \beta_i \in S, \ x_i, y_i \in F]$$

$\therefore$ For each scalar c,
$$c\alpha + \beta = \sum_{i=1}^{m} (cx_i)\alpha_i + \sum_{j=1}^{n} y_j \beta_j$$

As by def^n is also just a linear combo of vectors $\in S$. $\therefore$ L is a subspace of V.

Thus, L is a subspace of V, which contains S, and also any subspace which contains S contains L {the subspace spanned is the smallest subspace containing the set} {ref. Hoffman Kunze}

The two statements above conclude our proof, that L is nothing but the intersection of all subspaces containing S.
$\therefore$ [Hence proved]

2. ~~Now, we know~~, let us assume a subspace ρ, of a finite dimensional vector space V, having a linearly independent subset S_0. If S is a linearly independent subset of ρ, containing S_0, then S is also a linearly independent subset of V.

$\because$ V is finite dimensional, S contains no more than dim V elements.

Extending S_0 to a basis for W:

If S_0 spans W, then S_0 is a basis for W

If S_0 does not span W, we find

$\beta_1 \notin$ Subspace spanned by S_0, $\beta_1 \in W$ | $S_1 = S_0 \cup \{\beta_1\}$ is independent.

[{Ref Hoffman Kunze} Using the property that $S_0 \cup \{\beta\}$ is linearly independent in the given scenario.]

$\therefore$ If S_1 spans W, then S_1 is a basis for W

otherwise we keep adding β_j, $j \geq 0$, $j \in \mathbb{Z}^+$ to obtain $S_m = S_0 \cup \{\beta_1, \beta_2, \ldots, \beta_m\}$ which is a basis for W.

Using this above proven property that every ~~S ∈ W~~ subset of subspace of vector space V is part of some basis set of W, we can say that

$W_1 \cap W_2$ has a finite basis $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$

Let basis of W_1 be $\{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_n\}$

dim $W_1 = k + n$, $k, n \in \mathbb{Z}^+$

Let basis of W_2 be $\{\alpha_1, \alpha_2, \ldots, \alpha_k, \nu_1, \nu_2, \ldots, \nu_m\}$

dim $W_2 = k + m$, $k, m \in \mathbb{Z}^+$

keeping $W_1 \cap W_2 \neq \phi$, and $W_1 - W_2 \neq \phi$ and $W_2 - W_1 \neq \phi$ {in case it is, k, m, or n resp. turn 0} classmate

Date
Page

The subspace $W_1 + W_2$ is spanned by
α_i , $1 \le i \le k$, $i \in \mathbb{Z}$
β_j, $1 \le j \le m$, $\beta \in \mathbb{Z}^+$
v_p, $1 \le p \le n$, $p \in \mathbb{Z}^+$

& these vectors form an independent set.

suppose
$$\sum_i x_i \alpha_i + \sum_j y_j \beta_j + \sum_p z_p v_p = 0$$

$$\Rightarrow \quad -\sum z_p v_p = \sum x_i \alpha_i + \sum y_j \beta_j$$

~~this shows~~ $\Rightarrow$ $\sum z_p v_p \in W_1$ {linear combo of vectors in it}
$\because$ we know, $\sum z_p v_p \in W_2$ $\Rightarrow$ $\sum z_p v_p \in W_1 \cap W_2$
$\Rightarrow$ $\sum z_p v_p = \sum c_i \alpha_i$ $\{\because \sum z_p v_p \in W_1 \cap W_2\}$
for some $c_i \in \mathbb{F}$

since our assumption was
$\overline{\alpha_i}$ & $\overline{v_p}$ $\forall i, p$ acc. to initial condns. is
indepent, $z_p = 0$ $\forall p$.

Thus $\sum x_i \alpha_i + \sum y_j \beta_j = 0$

$\because$ $\overline{\alpha_i}$ & $\overline{\beta_j}$ are also independent $\forall i, j$ acc to initial conds,
$x_i = 0$, $y_j = 0$ $\forall i, j$

Thus, $\{\alpha_1, \alpha_2, \dots \alpha_k, \beta_1, \beta_2, \dots, \beta_m, v_1, v_2 \dots v_n\}$
is a basis for $W_1 + W_2$ and $W_1 + W_2$ is finite dimensional.

Finally,
$$\dim W_1 + \dim W_2 = (k+m) + (k+n)$$
$$= k + (m+k+n)$$
$$= \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

[Hence proved]

3. Let R be a $n \times m$ non-zero row reduced echelon matrix

Let leading non-zero entry of row i occur in column k_i $[1 \le i \le n, i \in \mathbb{Z}^{+0}, 1 \le k_i \le m, k_i \in \mathbb{Z}^{+}]$

$\because$ R is row-reduced echelon

$\{r_{ij} \in R\}$ $r_{pk_i} = 0$ $\forall p \in \mathbb{Z}^{+}, 1 \le p \le n, p \ne i$

Let there be j rows s.that $j \in \mathbb{Z}^{+}, 1 \le j \le m$,

$r_{jk} = 0$ $\forall k \in \mathbb{Z}^{+}, 1 \le k \le n$

$\therefore$ R_1 to R_j are the non-zero rows.

$R_1 = (0, 0, \ldots k_1 \text{ times} \ldots, 1, 0, 0, \ldots)$

$R_2 = (0, \ldots k_2 \text{ times}, \ldots, 1, 0, \ldots)$

$R_j = (0, \ldots k_j \text{ times}, \ldots \ldots, 1, 0)$

Let $x_i \in F$, $1 \le i \le j$, $i \in \mathbb{Z}^{+}$

Putting $\alpha = x_1 R_1 + x_2 R_2 + \ldots + x_j R_j$

Then $\alpha = (0, \ldots k_1 \text{ times}, \ldots, x_1, 0 \ldots k_2 - k_1 + 1 \text{ times} \ldots, x_2, 0, \ldots, x_j, 0 \ldots)$

~~thus they are linearly independent~~

~~they also~~ ~~any element in row~~

~~In other words, we can ass~~ on

If $\alpha = [0]_{1 \times m}$

It would be true if & only if $x_1, x_2, \ldots x_j = 0$ $\forall i \in \mathbb{Z}^{+}, 1 \le i \le j$

Hence, R_1 to R_j are linearly independent.

$\because$ any element in row space of R can be written as linear combination of its rows,

we can say that non-zero row vectors of R form a basis for the row space of R.

[Hence proved]