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2023113019
LA Assignment - 6

1. Firstly, we shall prove that
∃ a row reduced echelon matrix of order $m \times n$ over $\mathbb{F}$ that has W as its row space.

$\therefore \dim W \leq m$, $m \in \mathbb{Z}^+$
We note that W is finite dimensional.

We know that any finite dimensional vector space V has a finite basis, let it be $S = \{\beta_1, \beta_2, \dots, \beta_m\}$
[for $\dim(V) = m$]

$\therefore n(S) = m$, & S is a basis, it by definition spans the vector space V

Similarly for, $\dim W \leq m$,
∃ a set $P = \{\beta_1, \beta_2, \beta_3, \dots, \beta_m\}$ of linearly independent vectors that spans W {where $\beta \in \mathbb{F}^n$}

let the row vectors of a matrix $A_{m \times n}$ be $\beta_1, \beta_2, \dots, \beta_m$ respectively.

[Ref: Hof & Kunze] We know, every matrix has a row reduced echelon form,
$\therefore$ ∃ R | R is row reduced echelon, R is row equivalent to A.

[Ref: Hof & Kunze thing] We know that row equivalent matrices have same row space
We know that the subspace spanned by row vectors of a matrix is the row space of that matrix.
$\therefore$ The row space of R is W.

~~To also prove:~~
~~Let R be any row~~

Let $\rho_1, \rho_2, \dots, \rho_r$ be the non-zero row vectors of R.
~~We know, $\rho_1, \dots, \rho_r$ span W~~

[Ref: Thm 10 Hof & Kunze] We also know, the non-zero row vectors of a row reduced echelon matrix forms a basis for the row space of R [here W]
$\therefore \{\rho_1, \dots, \rho_r\} = B$ ~~is the~~ are the basis vectors for W.

Also, if $\mathcal{S}$ is the basis set for W,

$\forall \alpha \in W,\ \alpha = (x_1, x_2, \ldots, x_n),\ x_i \in F,\ 1 \le i \le n$

$$\alpha = \sum_{i=1}^{r} c_i \rho_i$$

but we also know,

$c_i = x_{k_i}$ [k_i is the column where the leading non-zero entry of row R occurs $\forall i \in \mathbb{Z}^+$, $1 \le i \le r$]

$$\therefore \alpha = \sum_{i=1}^{r} x_{k_i} \rho_i$$

Suppose $\alpha \neq 0 \Rightarrow \exists k_s \in \mathbb{Z}^+,\ 1 \le k_s \le n$ [$s \in \mathbb{Z}^+,\ 1 \le s \le m$] s. that x_{k_s} is the first non-zero element.

$$\therefore \alpha = \sum_{i=1}^{r} x_{k_i} \rho_i,\quad x_{k_s} \neq 0$$

We know, $R_{ij} = 0$ if $i > s$ & $j \le k_s$

Thus,

$$\alpha = (0, \ldots, 0, x_{k_s}, \ldots, x_n)$$

~~Let $k_1, \ldots, k_r \in \mathbb{Z}^+$, that~~

Let $k_1, \ldots, k_r$ be the integers t s. that $\exists$ some $\alpha \neq 0, \alpha \in W$, the first non-zero coordinate of which occurs in column t. Arranging $k_1, \ldots, k_r$ in order $k_1 < k_2 < \ldots < k_r$ [noting $k_1 \neq k_2 \neq \ldots \neq k_r$] ~~(row reduced echelon)~~

For each

$\therefore$ k_s, there will be only one $\rho_s \in R$ s. that k_sth coordinate of $\rho_s = 1$, and k_ith coord. of $\rho_s = 0$ for $i \neq s$.

Hence there will be precisely one R which has W as its row space [Row vectors are unique].

[Hence proved]

classmate

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2. $p(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4$

$p(x) = (a_0, a_1, a_2, a_3, a_4)$

[Representing it as F^5 touple]

$p(1) + p(-1) = 0$

$\Rightarrow (a_0 + a_1 + a_2 + a_3 + a_4) + (a_0 - a_1 + a_2 - a_3 + a_4) = 0$

$\Rightarrow 2(a_0 + a_2 + a_4) = 0$

$\Rightarrow a_0 + a_2 + a_4 = 0$ — 1

$p(2) + p(-2) = 0$

$\Rightarrow (a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4) + (a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4) = 0$

$\Rightarrow a_0 + 4a_2 + 16a_4 = 0$ — 2

2 − 1 :

$\Rightarrow 3a_2 + 15a_4 = 0$

$\Rightarrow a_2 + 5a_4 = 0 \Rightarrow a_2 = -5a_4$ — 3

Using 3 in 1

$a_0 - 4a_4 = 0$

$\Rightarrow a_0 = 4a_4$

$\therefore p(x) = (a_0, a_1, a_2, a_3, a_4)$ — 4

$= (4a_4, a_1, -5a_4, a_3, a_4)$ w.l.o.g

Assuming 3 vectors, $(4, 0, -5, 0, 1) = \alpha$

$\beta = (0, 1, 0, 0, 0)$ & $(0, 0, 0, 1, 0) = v$

We find $c_1\alpha + c_2\beta + c_3 v = 0$, $c_1, c_2, c_3 \in \mathbb{F}$

only if $c_1 = 0, c_2 = 0, c_3 = 0$

$\therefore$ It is linearly independent.

$\because p(x) = c_1\alpha + c_2\beta + c_3 v$ [$\because c_1, c_2, c_3 \in \mathbb{F}$ & a_1, a_3, a_4 in 4 $\in \mathbb{F}$ and are arbitrary]

$\therefore \forall w \in W$ or $\forall x \in F \mid p(x)$ is defined,
$p(x) = c_1\alpha + c_2\beta + c_3\gamma$
$\therefore$ The vectors α, β, γ span the vector space W

$\because \alpha, \beta, \gamma$ span W and are linearly independent, the set

$S = \{(4,0,-5,0,1), (0,1,0,0,0), (0,0,0,1,0)\}$
is the basis for W

(b) $\dim(W) = n(S) = |S| = 3$

(3) $\because v_1, v_2, v_3$ span V $[v_1, v_2, v_3 \in V]$
$\forall v \in V$, $v = c_1v_1 + c_2v_2 + c_3v_3$ $[c_1, c_2, c_3 \in F]$
by definition of span of a set of vectors

$\because v_4 \in V$, $v_4 = c_1'v_1 + c_2'v_2 + c_3'v_3$ $[c_1', c_2', c_3' \in F$ for specific$]$ —— (1)

Now, defining span of S'

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = v'$ $[v' \in V'$ some vector space$]$
using (1)
$v' = c_1v_1 + c_2v_2 + c_3v_3 + c_4(c_1'v_1 + c_2'v_2 + c_3'v_3)$
$v' = (c_1 + c_1'c_4)v_1 + (c_2 + c_2'c_4)v_2 + (c_3 + c_3'c_4)v_3$

$\because c_1, c_2, c_3, c_4$ are all arbitrary,
$c_1 + c_1'c_4 = c_5$ also arbitrary

$\therefore v' = c_5v_1 + c_6v_2 + c_7v_3$ $[c_5, c_6, c_7 \in F$ arbitrary$]$

$\therefore$ v' denotes the same span V
$v = v'$
$\Rightarrow V = V'$
$\therefore S'$ spans V [proved]

~~④ ∵ V, W subspaces, ... $\vec{0} \in V, W$~~

①⑤ Assuming $v \neq 0$ & $0 \in V$ [V subspace]

$\because v \in V$, $-v \in V$ [additive inverse]

$v + (-v) = 0$

Acc. to question, $v + w = 0$

True only when $w = -v \in V$

but we know $V \cap W = \{0\}$

if $-v \in V$ & $(-v) = w \in W$,

then $(-v) \in V \cap W$

$\Rightarrow (-v) = 0$

$\Rightarrow v = 0$ [Contradicting our assumption]

Similarly, assume $w \neq 0$,

$w + (-w) = 0$ $[-w \in W]$

Also $w + v = 0$

True only if $v \in V$, $v = -w \in W$

$\therefore -w \in V \cap W$

$\Rightarrow (-w) = 0$ $[\because V \cap W = \{0\}]$

$\Rightarrow w = 0$ [contradicting our assumption]

Hence $v = 0$ & $w = 0$ [proved]

⑥ $B_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_a\} \Rightarrow \dim(V) = a$

$B_2 = \{\beta_1, \beta_2, \ldots, \beta_b\} \Rightarrow \dim(W) = b$

$\because \dim(V) + \dim(W) = n$

$\Rightarrow a + b = n$

Let basis of $\mathbb{R}^n$ be $\{r_1, r_2, \ldots, r_n\}$

$\dim(\mathbb{R}^n) = n = a + b$

∵ V is a subspace of R^n, basis of V are subsets of basis of R^n

Similarly, basis of W is basis of R^n since linearly independent

We know $B_1 \cap B_2 = \phi$ [∵ $\alpha_i \neq 0, \beta_i \neq 0$

∴ basis of $R^n = \{\alpha_1, \ldots, \alpha_a, \beta_1, \beta_2, \ldots, \beta_b,$

B_n & x_1, & $x_2, \ldots, x_m\}$

But we know, $a + b = n$

$\dim(R^n) = n \Rightarrow$ & $x_i \notin R^n$

∵ $|B_n| = n$

∴ Basis of $R^n = \{\alpha_1, \alpha_2, \ldots \alpha_a, \beta_1, \beta_2, \ldots \beta_b\}$

~~$B_1 \cap B_2 = \phi$ ∵ ...~~ [Hence proved]

c. $x \in R^n$

RTP $x = v + w$, $v \in V$, $w \in W$

Using (a)b, $B_1 \cup B_2 = B_n$

$\Rightarrow V \cup W = R^n$

[Basis vectors span the entire space R^n]

$x \in R^n$

$x = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_a\alpha_a + c_{a+1}\beta_1 + c_{a+2}\beta_2$
$+ \cdots\cdots + c_{a+b}\beta_b$

We know,

$v = \sum c_i \alpha_i \in V$ [$\because \{\alpha_i\}$ is basis for subspace V]

$w = \sum c_k \beta_j \in W$ [$\because \{\beta_j\}$ is basis for subspace W]

$\therefore x = v + w$ [Hence proved]

4d Assuming representation of x is not unique

$\therefore x = v_1 + w_1 = v_2 + w_2$ [$v_1, v_2 \in V$, $w_1, w_2 \in W$
$v_1 \neq v_2$, $w_1 \neq w_2$]

$\because v_2 \in V$, $(-v_2) \in V$; $\because w_1 \in W$, $(-w_1) \in W$

$v_1 + w_1 = v_2 + w_2$

$\Rightarrow v_1 + w_1 + (-w_1) + (-v_2) = v_2 + w_2 + (-w_1) + (-v_2)$

$\Rightarrow v_1 + (-v_2) = w_1 + (-w_2)$

$\because v_1 + (-v_2) \in V$ [closure] & $w_1 + (-w_2) \in W$ [closure]

$v_1 + (-v_2) \quad w_1 + (-w_2) \in V \ \& \in W$

$\Rightarrow \in V \cap W$.

But $V \cap W = \{0\}$

$\therefore v_1 + (-v_2) = 0 = w_1 + (-w_2)$

$\Rightarrow v_1 = v_2 \ , \ w_1 = w_2$

Hence by contradiction. [proved]