Linear Algebra 4 Credits

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- Solving systems of linear equations.
  Row reduction, free variables, row reduced echeleon matrices.
- Vector spaces basics: Def^{n}s, subspaces, bases, dimension.
- Linear transformations: Def^{n}s, Effect of changes of basis on transformation, Rank of transformation.
  Range & Kernel of transformation. Rank-Nullity theorem.
- Determinants: Cofactor expansions, Multilinearity.
  Axiomatic approach. Physical meaning of determinants.
- Eigenvalues and Eigenvectors.
- Diagonalizability & Triangularizability
- Advanced Spectral Theory.

Evaluation (Part I : 12-13 lectures)

<u>50%</u>.
1. Mid-sem (90 mins exam) : ~~20~~ 15 % = 15%*
2. Quiz I (45 mins exam) : 10%
3. Assignments : 15% + 5% = 20%*
4. In class light quizes : 5%

* subject to change (possible)

Textbooks & References:
1. Linear Algebra by Hoffmann & Kunz
2. Algebra by Artin
3. Linear Algebra by Kumaresan
4. Introduction to Linear Algebra by Strang
5. https://textbooks.math.gatech.edu/ilat
6. Linear Algebra by Jänich

Lecture 1: Linear Equations

Fields

Let us first list out properties of addition & multiplication. Consider that F denotes the set of real nos. or the set of complex nos.

1. Addition is commutative, $x + y = y + x$, $\forall\, x, y \in F$.
2. Addition is associative, $x + (y + z) = (x + y) + z$, $\forall\, x, y, z \in F$.
3. $\exists$ unique element 0 (zero) in F s.t. $x + 0 = x$, $\forall\, x \in F$.
4. $\forall\, x \in F$, $\exists$ a unique element $(-x)$ in F s.t. $x + (-x) = 0$.
5. Multiplication is commutative, $x \cdot y = y \cdot x$, $\forall\, x, y \in F$.
6. Multiplication is associative, $x(yz) = (xy)z$, $\forall\, x, y, z \in F$.
7. There is a unique non-zero element 1 (one) in F s.t. $x \cdot 1 = x$ $\forall\, x \in F$.
8. To each non-zero x in F there corresponds a unique element x^{-1} (or $1/x$) in F s.t. $xx^{-1} = 1$.
9. Multiplication distributes over addition; i.e.,
$$x(y + z) = xy + xz, \quad \forall\, x, y, z \in F.$$

A set F of objects $x, y, z, \ldots$ ~~and~~ along with two operations, addition and multiplication, satisfying conditions ①–⑨ above, is called a <u>Field</u>. $(F, +, \cdot)$ algebra

Elements of field $\rightarrow$ scalars or numbers

A ~~subfield~~ $(S, +, \cdot)$ ~~is a~~ ~~subset~~ of a ~~field~~ $F(+,$ ~~s.t.~~ ~~associated~~ ~~operations~~ do not take elements ~~of~~ ~~the~~ ~~subset~~

A subfield $(S, +, \cdot)$ is a subset of a field $(F, +, \cdot)$ in the sense that $S \subseteq F$ and $(S, +, \cdot)$ is also a field.

A subfield of the field $(\mathbb{C},+,\cdot)$ is a set F of complex nos. which itself is a field under usual operations — add. & multip. That is, $0, 1 \in F$, and that if $x, y \in F$, so are $(x+y), -x, xy$, and x^{-1} (if $x \neq 0$). For example, field $(\mathbb{R},+,\cdot)$.

$\mathbb{Z}^+ \rightarrow$ not a ~~field~~ subfield of $(\mathbb{C},+,\cdot)$
$\mathbb{Z} \rightarrow$ not a subfield of $(\mathbb{C},+,\cdot)$

$\Big\}$ $n \in \mathbb{Z}^+$ but $\frac{1}{n} \notin \mathbb{Z}$.

$\mathbb{Q}$ (set of rational nos.) $\rightarrow$ subfield of $(\mathbb{C},+,\cdot)$

Problems:

① Any subfield of $(\mathbb{C},+,\cdot)$ must contain every rational no.

② Set of all complex no. of the form $x+y\sqrt{2}$, where $x, y \in \mathbb{Q}$, is a subfield of $\mathbb{C}$!

[Characteristic of field:

The least n such that the sum of n 1's is 0 is called the characteristic of the field F. If it does n't happen in F, then F is called a field of characteristic zero.]

Systems of Linear Equations

F is a field. We consider the problem of finding n scalars (elements of F) $x_1, x_2, \ldots, x_n$ which satisfy the conditions

$$\left.\begin{aligned} A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n &= y_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \ldots + A_{n2}x_n &= y_n \end{aligned}\right\} \text{———} (1.1)$$

where $y_1, \ldots, y_n$ and A_{ij}, $1 \le i \le m$, $1 \le j \le n$, are given elements of F. (1.1) is called a system of m linear equations in n unknowns.

Any n-tuple $(x_1, \ldots, x_n)$ of elements of F which satisfies each of the eq^{n}s (1.1) is called a solution of the system.
If $y_i = 0$, $1 \le i \le m$, we say that the system is homogeneous, or that each of the eq^{n}s is homogeneous.

- Finding the solutions of a system of linear eq^{n}s.

Technique of elimination.

Example: Consider homogenous system:

$$2x_1 - x_2 + x_3 = 0 \text{ —— } (a1)$$
$$x_1 + 3x_2 + 4x_3 = 0 \text{ —— } (a2)$$

$-2 \times (a2) + (a1) \Rightarrow \quad -7x_2 - 7x_3 = 0$ or, $x_2 = -x_3$.

$3 \times (a1) + (a2) \Rightarrow \quad x_1 = -x_3$.

If (x_1, x_2, x_3) is a solution then $x_1 = x_2 = -x_3$.
Or, any such tripple i.e., $(x, x, -x)$ is a sol^n.
Thus, the set of sol^{n}s consists of all tripples $(x, x, -x)$.

For the general system (1·1), suppose we select m scalars $(c_1, \ldots, c_m)$, multiply the j^{th} eq^n by c_j and then add.. We have

$$(c_1 A_{11} + \ldots + c_m A_{m1}) x_1 + \ldots + (c_1 A_{1n} + \ldots + c_m A_{mn}) x_n$$
$$= c_1 y_1 + \ldots + c_m y_m \quad \text{—— (1.1 a)}$$

→ we call it a <u>linear combination of the eq^{n}s in (1.1)</u>

Any sol^n of the entire system of eq^n of (1.1) will also be a sol^n of (1.1a).

Fundamental idea of the elimination process.

Consider another system of linear eq's:

$$\left.\begin{aligned} B_{11} x_1 + \ldots + B_{1n} x_n &= z_1 \\ &\vdots \\ B_{k1} x_1 + \ldots + B_{kn} x_n &= z_k \end{aligned}\right\} \quad (1.2)$$

where each of the k eq^{n}s is a linear combination of the eq^{n}s in (1.1) then every sol^n of (1.1) is also a sol^n of (1.2). However, it may happen that some sol^{n}s of (1.2) are not sol^{n}s of (1.1).

Two systems of linear eq^{n}s are <u>equivalent</u> if each eq^n in each system is a <u>linear combination</u> of the eq^{n}s in the other system.

<u>**Thm**</u>: Equivalent systems of linear eq^{n}s have exactly the same solutions.

Matrices and Elementary Row Operations

Eq^n (1.1) can be abbreviated as

$$AX = Y$$

where
$$A = \underbrace{\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}}_{\text{representation of a matrix (not a matrix itself)}}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

matrix of coefficients of the system

$$A_{[m \times n]} X_{[n \times 1]} = Y_{[m \times 1]}$$

An $m \times n$ matrix over the field F is a function A from the set of pairs of integers (i,j), $1 \le i \le m$, $1 \le j \le n$, into the field F. Entries of the matrix A are the scalars $A(i,j) = A_{ij}$.

Plan to consider operations on the rows of the matrix A which correspond to forming linear combinations of the eq^{n}s in the system $AX = Y$.

We focus on 3 elementary row operations on an $m \times n$ matrix A over the field F:

1. multiplication of one row of A by a non-zero scalar c
2. replacement of the r^{th} row of A by row plus c times row s, c any scalar and $r \neq s$;
3. interchange of two rows of A.

An elementary row operation is thus a special type of function (rule) e which associated with each $m \times n$ matrix A an $m \times n$ matrix $e(A)$.

1. $e(A)_{ij} = A_{ij}$ if $i \neq r$, $e(A)_{rj} = cA_{rj}$, $c \neq 0$
2. $e(A)_{ij} = A_{ij}$ if $i \neq r$, $e(A)_{rj} = A_{rj} + cA_{sj}$, $r \neq s$
3. $e(A)_{ij} = A_{ij}$ if i is different from both r & s, $e(A)_{rj} = A_{sj}$, $e(A)_{sj} = A_{rj}$.

$\boxed{r, s \text{ constrained by } m.}$

Thm: To each elementary row operation e there corresponds an elementary row operation as e_1, such that $e_1(e(A)) = e(e_1(A)) = A$ for each A. I.e., the inverse operation (function) of an elementary row operation exists and is an elementary row operation of the same type.

Proof: ———

Def^n: If A & B are $m \times n$ matrices over the field F, we say that B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

Remark: Row-equivalence is an equivalence relation.

A binary relation $\sim$ on a set X is said to be an equivalence relation iff it is reflexive, symmetric and transitive. I.e., $\forall\, a, b, c \in X$:

① $a \sim a$ (reflexivity)

② $a \sim b$ iff $b \sim a$ (symmetry)

③ If $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity)

Equivalence class of a under $\sim$, denoted $[a]$ is defined as $[a] = \{x \in X : x \sim a\}$.

Thm: If A & B are row-equivalent $m \times n$ matrices, the homogenous systems of linear equations $AX = 0$ & $BX = 0$ have exactly the same solutions.

Proof: $A = A_0 \to A_1 \to A_2 \to \dots \to A_k = B$.

$$\xrightarrow{e} \quad \xleftarrow[e]{}$$

Example: Let F be the field of rational numbers, and

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}.$$ We perform elementary row operations.

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{① - 2\times ②} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{③ - 2\times ②} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & -2 & -1 & +7 \end{bmatrix}$$

$$\xrightarrow{③/-2} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{② - 3\times ③} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{9\times ③ + ①} \begin{bmatrix} 0 & 0 & 15/2 & -55/2 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix}$$

$$\xrightarrow{① \times 2/15} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{2\times ① + ②} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 1/2 & -7/2 \end{bmatrix} \xrightarrow{-1/2 \times ① + ③} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}$$

Row equivalence of A w/ the final matrix above tells us that the two systems are equivalent, i.e., have the same solutions.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 2x_4 &= 0 \\ x_1 + 4x_2 \qquad\quad - x_4 &= 0 \\ 2x_1 + 6x_2 - x_3 + 5x_4 &= 0 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} x_3 - \tfrac{11}{3}x_4 &= 0 \\ x_1 \qquad + \tfrac{17}{3}x_4 &= 0 \\ x_2 \quad - \tfrac{5}{3}x_4 &= 0. \end{aligned}$$

Def^n: An $m\times n$ matrix R is called <u>row-reduced</u> if

ⓐ the 1^{st} non-zero entry in each non-zero row of R is equal to 1;

ⓑ each column of R which contains the leading nonzero entry of some row has all its other entries 0.

<u>Thm</u>: Every $m\times n$ matrix over the field is row-equivalent to a row-reduced matrix.

Proof: ________ (as an exercise).

Row-Reduced Echelon Matrices

Def^n: An $m\times n$ matrix R is called a row-reduced echelon matrix if:

(a) R is row-reduced;

(b) every row of R which has all its entries 0 occurs below every row which has a non-zero entry;

(c) if rows $1,\dots,r$ are the non-zero rows of R, and if the leading non-zero entry of row i occurs in column k_i, $i=1,\dots,r$, then $k_1<k_2<\dots<k_r$.

I.e., Either every entry in R is 0, or $\exists\ r\in\mathbb{Z}^+$, $1\le r\le m$, and $k_1,k_2,\dots,k_r\in\mathbb{Z}^+$ with $1\le k_i\le n$ and

(a) $R_{ij}=0$ for $i>r$, $R_{ij}=0$ if $j<k_i$.

(b) $R_{ik_j}=\delta_{ij}$, $1\le i\le r$, $1\le j\le r$.

(c) $k_1<\dots<k_r$.

Examples: $I_{m\times n}$, $O_{n\times n}$, $\begin{bmatrix}0 & 1 & -3 & 0 & \frac{1}{2}\\ 0 & 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$.

Thm. Every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix.

Now consider a homogenous system $RX = 0$, wher R is a row-reduced echelon matrix. let $1, \ldots, r$ be non-zero rows of R, and let the leading non-zero entry of row i occurs in column k_i. The system $RX = 0$ then consists of r non-trivial eq^{n}s. Also, the unknown x_{k_i} will occur (with non-zero coefficient) only in the i^{th} eq^{n}. Let $u_1, \ldots u_{n-r}$ denote the $(n-r)$ unknowns which are different from $x_{k_1}, \ldots, x_{k_r}$, then the r non-trivial eq^{n}s in $RX = 0$ are of the form

$$x_{k_1} + \sum_{j=1}^{n-r} c_{1j} u_j = 0$$

$$\vdots \qquad\qquad \vdots$$

$$x_{k_r} + \sum_{j=1}^{n-r} c_{rj} u_j = 0$$

1.3

Assign any values whatsoever to $u_1, \ldots, u_{n-r}$, to get corresponding values of $x_{k_1}, \ldots, x_{k_r} \longrightarrow$ sol^{n}s to the system.

For examples if $R = \begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

in $RX = 0$, then $r = 2$, $k_1 = 2$, $k_2 = 4$ and two non-trivial eq^{n}s are

$x_2 - 3x_3 + \frac{1}{2}x_5 = 0$ or $x_2 = 3x_3 - \frac{1}{2}x_5$

$x_4 + 2x_5 = 0$ or $x_4 = -2x_5$.

Assign $x_1 = a$, $x_3 = b$, $x_5 = c$, then the sol^n is $(a, 3b - \frac{c}{2}, b, -2c, c)$.

Note: If the no. of non-zero rows, i.e., r, in R is less than n $(r < n)$ then $RX = 0$ has a non-trivial sol^n. That is, a sol^n $(x_1, \ldots, x_n)$ in which not all x_j is 0. For, since, $r < n$, we may choose some x_j which is not among the r unknowns $x_{k_1}, \ldots, x_{k_r}$, & we can then construct a sol^n as above in which x_j is 1.

Thm: If A is an $m \times n$ matrix & $m < n$, then the homogenous system $AX = 0$ has a non-trivial sol^n.

Thm. If A is an $n \times n$ matrix, then A is row-equiv. to $I_{n \times n}$ iff the system $AX = O$ has only the trivial solⁿ.