Lecture 4 Vector spaces

Def^n. A vector space (or linear space) consists of the following: (Vector space over the field)

1. a field F of scalars;
2. a set V of objects, called vectors;
3. a rule (or operation), called vector addition, which associates with each pair of vectors $\vec{\alpha}, \vec{\beta} \in V$ a vector $\vec{\alpha}+\vec{\beta} \in V$, called the sum of $\vec{\alpha}$ & $\vec{\beta}$, in such a way that
   (a) addition is commutative, $\vec{\alpha}+\vec{\beta} = \vec{\beta}+\vec{\alpha}$;
   (b) addition is associative, $\vec{\alpha}+(\vec{\beta}+\vec{\gamma}) = (\vec{\alpha}+\vec{\beta})+\vec{\gamma}$;
   (c) $\exists$ a unique vector $\vec{0} \in V$, called the zero vector, s.t. $\vec{\alpha}+\vec{0} = \vec{\alpha} \ \forall \ \vec{\alpha} \in V$;
   (d) for each $\vec{\alpha} \in V$ $\exists$ a unique vector $-\vec{\alpha} \in V$ s.t. $\vec{\alpha}+(-\vec{\alpha}) = 0$.
4. a rule, called scalar multiplication, which associates with each scalar $c \in F$ & $\vec{\alpha} \in V$ a vector $c\vec{\alpha} \in V$, called the product of c & $\vec{\alpha}$ s.t.
   (a) $1\vec{\alpha} = \vec{\alpha} \ \forall \ \vec{\alpha} \in V$;
   (b) $(c_1 c_2)\vec{\alpha} = c_1(c_2\vec{\alpha})$;
   (c) $c(\vec{\alpha}+\vec{\beta}) = c\vec{\alpha} + c\vec{\beta}$;
   (d) $(c_1+c_2)\vec{\alpha} = c_1\vec{\alpha} + c_2\vec{\alpha}$.

A vector space is a composite object consisting of a field, a set of 'vectors', & two operations w/ certain properties

Examples

① The n-tupple space, F^n. Let F be any field. Let V be the set of all n-tupples $\overline{\alpha} = (x_1, \ldots, x_n)$ of scalars $x_i \in F$. If $\overline{\beta} = (y_1, y_2, \ldots, y_n)$ w/ $y_i \in F$, the sum of $\overline{\alpha}$ & $\overline{\beta}$ is defined by

$$\overline{\alpha} + \overline{\beta} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad \text{(2·1)}$$

The product of a scalar c and vector $\overline{\alpha}$ is defined by

$$c\overline{\alpha} = (cx_1, \ldots, cx_n) \quad \text{(2·2)}$$

② The space of $m \times n$ matrices, $F^{m \times n}$. Let F be any field and let m & n be the integers. Let $F^{m \times n}$ be the set of all $m \times n$ matrices over the field F.

$\overline{A}, \overline{B} \in F^{m \times n}$ then

$$(\overline{A} + \overline{B})_{ij} = A_{ij} + B_{ij}$$

$c \in F$, $\overline{A} \in F^{m \times n}$ then

$$(cA)_{ij} = cA_{ij}$$

③ The space of functions from a set to a field. F be field, S be any non-empty set. V be the set of all f^ns from the set S into F.

For $\bar{f}, \bar{g} \in V$,

$$(f+g)(s) = f(s) + g(s).$$

For $c \in F$, $f \in V$,

$$(cf)(s) = cf(s).$$

(4) The space of polynomial f^ns over a field F.

Let F be a field and let V be the set of all f^ns f from F into F which have the rule of the form

$$f(x) = c_0 + c_1 x + \dots + c_n x^n,$$

where $c_0, c_1, \dots, c_n \in F$ are independent of x.

(5) The field $\mathbb{C}$ of complex nos. $\longrightarrow$ a vector space over the field $\mathbb{R}$ of real nos.

From the def^n of a vector space, we observe: ~~that~~ If for a scalar $c \in F$ and vector $\bar{\alpha} \in V$ we have $c\bar{\alpha} = 0$ then either $c = 0$ or $\bar{\alpha} = \bar{0}$.

For any $\bar{\alpha} \in V$, $-\bar{\alpha} \in V$ since $\bar{0} \in V$ and

$$\bar{0} = 0\bar{\alpha} = (1-1)\alpha = 1 \cdot \bar{\alpha} + (-1) \cdot \bar{\alpha} = \bar{\alpha} + (-1)\bar{\alpha},$$

$$(-1)\bar{\alpha} = -\bar{\alpha}.$$

For any $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4 \in V$, $(\bar{\alpha}_1 + \bar{\alpha}_2) + (\bar{\alpha}_3 + \bar{\alpha}_4)$
$= [\bar{\alpha}_2 + (\bar{\alpha}_1 + \bar{\alpha}_3)] + \bar{\alpha}_4.$

Def^n: A vector $\bar{\beta} \in V$ is said to be a <u>linear combination</u> of the vectors $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \in V$ provided $\exists$ scalars $c_1, \ldots, c_n \in F$ s.t.

$$\bar{\beta} = c_1\bar{\alpha}_1 + \ldots + c_n\bar{\alpha}_n$$
$$= \sum_{i=1}^{n} c_i\bar{\alpha}_i.$$

Other extensions of the associative property of vector add^n & the distributive properties 4 ⓒ and 4 ⓓ of scalar multiplication apply to linear combinations:

$$\sum_{i=1}^{n} c_i\alpha_i + \sum_{i=1}^{n} d_i\alpha_i = \sum_{i=1}^{n} (c_i + d_i)\alpha_i$$

$$c\sum_{i=1}^{n} c_i\alpha_i = \sum_{i=1}^{n} (cc_i)\alpha_i.$$

Def^n: Let V be a vector space over the field F. A <u>subspace</u> of V is a subset W of V which is itself a vector space over F w/ the operations of vector addition and scalar multiplication on V.

Thm. A non-empty subset W of V is a subspace of V if and only if for each pair of vectors $\bar{\alpha}, \bar{\beta}$ in W and each scalar c in F the vector $c\bar{\alpha}+\bar{\beta}$ is again in W.

Proof. Let W be non-empty subset of V s.t. $c\bar{\alpha}+\bar{\beta} \in W$ $\forall\ \bar{\alpha}, \bar{\beta} \in W$ and $\forall\ c \in F$. $\because W$ is non-empty, $\exists\ \vec{\gamma} \in W$, and hence $(-1)\bar{\gamma}+\bar{\gamma} = \bar{0} \in W$ ~~$(-1)\cdot\bar{\alpha} = -\bar{\alpha} \in W$~~. $\Rightarrow$ for any $\bar{\alpha} \in W, c \in F$, we have $c\bar{\alpha} = c\bar{\alpha}+\bar{0} \in W$. I.e., we also have, $(-1)\cdot\bar{\alpha} = -\bar{\alpha} \in W$. At last, $\bar{\alpha}, \bar{\beta} \in W$, then $\bar{\alpha}+\bar{\beta} = 1\bar{\alpha}+\bar{\beta} \in W$. Thus, W is subspace of V.

Conversely (easy part or trivial part) is obvious.

Examples

(a) If V is any vector space, V is a subspace of V; the subset ~~of~~ of V consisting of the zero vector $\overline{0}$ alone is a subspace of V, called the zero subspace.

(Note: Field is non-empty set and has distinct additive identity 0 and multiplicative identity 1. Therefore, any Field at least always has 0 and 1 unlike def^n of vector space).

(b) An $n \times m$ matrix A over the field $\mathbb{C}$ of complex nos. is Hermitian (or self-adjoint) if

$$A_{jk} = \overline{A}_{kj}, \text{ (where } \bar{x} \text{ denotes complex conjugate of } x \in \mathbb{C})$$

for each j,k. A 2×2 matrix is Hermitian if and only if it has the form

$$\begin{bmatrix} z & x+iy \\ x-iy & w \end{bmatrix}, \text{ where } x, y, z, w \in \mathbb{R}.$$

The set of all Hermitian matrices is not a subspace of the space of all $n \times n$ matrices over $\mathbb{C}$. (Why? How?)

What if the given vector space was to be defined over the field $\mathbb{R}$ of real nos.

Q. On $\mathbb{R}^n$, define two operations
$$\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$$
$$c \cdot \bar{\alpha} = -c\bar{\alpha}.$$
The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?

Q. Let V be the set of pairs (x, y) of reals nos. & let F be the field of real nos. Define
$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c\,(x, y) = (cx, 0).$$
Is V, with these operations, a vector space?

Q. Let V be the set of ~~pairs (x, y)~~ all complex valued f^ns f on the real line such that (for all $t \in \mathbb{R}$) $f(-t) = f^*(t)$.
* denotes complex conjugation. Show that V, w/ operations
$$(f+g)(t) = f(t) + g(t)$$
$$(cf)(t) = cf(t)$$
is a vector space over the field of real nos. Give an example of a f^n in V which is not real-valued.