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LA - Assignment - 1

1. RTP every subfield of $(\mathbb{C}, +, \cdot)$ must contain all rational numbers.

Proof:
We start off by considering a ~~field~~ set F over $\mathbb{C}$. We want to reach a field from the set F, while ~~starting~~ minimising no. of elements in set F.

Since 0 is the identity element of $+$, and 1 is the identity element of $\cdot$, ~~$F = \{0,$~~ $0, 1 \in F$

$+, \cdot$, by definition, are always associative & commutative.

~~let us~~
let us assume $F = \{0, 1\}$
Checking ~~for~~ if ~~F is a~~ $(F, +, \cdot)$ is a field.
But $1 + 1 = 2 \notin F$, $\therefore$ F is not ~~closed~~ closed.
Assuming 2 ~~is~~ also $\in F$
Also, similarly, $1 + 2 = 3 \notin F$.
So, for $+$ to be binary operator on F [and F to be closed under $+$]

$F = \{0, 1, 2, 3, \ldots\}$
$= W$

For additive inverse of 1 to exist, $-1 \in F$ $[1 + (-1) = \underset{\text{Identity}}{0}]$
Similarly $-2 \in F$, $-3 \in F$, ...

$\therefore F = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$= \mathbb{Z}$

$\therefore$ F ~~now~~ $= \mathbb{Z}$ has properties of additive closure, identity, inverse present here.

If $2 \in F$, $1/2 \in F$ for multiplicative inverse to exist $\left[2(1/2) = 1\right]$
similarly, $\forall x_i \in F \{x_i \in \mathbb{Z}\}$, $1/x_i \in F$ for multiplicative inverse to exits

$\therefore F = \left\{ \begin{matrix} \{i, i \in \mathbb{Z}\} \\ \{1/i\}, i \in \mathbb{Z}\} \end{matrix} \right\}$

But for closure on $\cdot$, ~~$x_i, 1/x_j, i, j \in \mathbb{Z}$~~
$i, 1/j \in F, i, j \in \mathbb{Z}$ exist.
$\therefore$ $i\left(\frac{1}{j}\right) \in F$ (for closure on $\cdot$)

Taking this as a general form that includes $\{i, i \in \mathbb{Z}\}$ when $j = 1$ & $\{1/j, j \in \mathbb{Z}\}$ when $i = 1$,
the set F can be represented as
$F = \{p/q,\ p, q \in \mathbb{Z}\}$.

But for F to have no duplicates $\{\because$ it is a set$\}$, p, q must be co-prime to each.

$\therefore F = \{p/q;\ p, q \in \mathbb{Z};\ p, q \text{ coprime}\}$

$= \mathbb{Q}$

The set ~~$\mathbb{Q}$~~ F now thus ~~necessarily~~ satisfies multiplicative inverse, identity, closure.

Checking for all properties for a field, addition and multiplication both ~~has~~ formed an abelian group, with the additional property of being distributive over F.

$\therefore$ We have proved that the smallest subfield possible of $\mathbb{C}$ is the set of all rational nos: $\mathbb{Q}$.
We have also proved that ~~every~~ if any field contains $\{0, 1\}$ it must contain $\mathbb{Q}$.
$\because$ ~~$\{0,1\}$~~ $0, 1 \in F\ \forall\ F \subseteq \mathbb{C}$, if $(F, +, \cdot)$ is a field, every subfield of $(\mathbb{C}, +, \cdot)$ must contain all rational nos.

[Hence proved]

2. RTP if $F = \{x + y\sqrt{2} : x, y \in \mathbb{Q}\}$ is a subfield of $(\mathbb{C}, +, \cdot)$

Sol^n: Let $F = \{x + y\sqrt{2} : x, y \in \mathbb{Q}\}$

Firstly, we note that $F \subsetneq \mathbb{R} \subsetneq \mathbb{C}$

because ~~$\sqrt{3} \in F$~~ $\forall f \in F, f \in \mathbb{R}$, but $\sqrt{3} \in \mathbb{R}$, $\sqrt{3} \notin F$

$(F, +, \cdot)$

$\therefore$ ~~We have to~~ proving is a field will be sufficient to prove F is a subfield of $(\mathbb{C}, +, \cdot)$

For F to be a field:

Implied Properties:

① F is closed under $+$

For $f_1, f_2 \in F$

$f_1 = x_1 + y_1\sqrt{2}$

$f_2 = x_2 + y_2\sqrt{2}$

$(f_1 + f_2) = (x_1 + x_2) + (y_1 + y_2)\sqrt{2}$

$\because (x_1 + x_2), (y_1 + y_2) \in \mathbb{Q}$ if $x_1, x_2, y_1, y_2 \in \mathbb{Q}$

Let us call them x_3, y_3

$\therefore f_1 + f_2 = x_3 + y_3\sqrt{2}$; $x_3, y_3 \in \mathbb{Q}$

~~$\therefore$ closed~~ $\therefore (f_1 + f_2) \in F$, $\therefore$ closure holds

② Closure under $\cdot$

For $f_1, f_2 \in F$

$f_1 = x_1 + \sqrt{2}y_1$, $x_1, y_1 \in \mathbb{Q}$

$f_2 = x_2 + \sqrt{2}y_2$, $x_2, y_2 \in \mathbb{Q}$

$f_1 f_2 = (x_1 + \sqrt{2}y_1)(x_2 + \sqrt{2}y_2)$

$= (x_1x_2 + 2y_1y_2) + \sqrt{2}(x_1y_1 + x_2y_1)$

Let $x_3 = x_1x_2 + 2y_1y_2 \in \mathbb{Q}$

$y_3 = x_1y_1 + x_2y_1 \in \mathbb{Q}$

$\therefore f_1 \cdot f_2 \in F$, closure holds.

Properties:

① + is commutative

~~$(x_1+y\sqrt{2})$~~

For $f_1, f_2, f_3 \in F$

~~$f_1+(f_2+f_3) = (f_1+f_2)+f_3$~~

check $f_1+f_2 = f_2+f_1$

$f_1 = x_1+\sqrt{2}y_1$

$f_2 = x_2+\sqrt{2}y_2$

$\therefore f_1+f_2 = (x_1+x_2)+\sqrt{2}(y_1+y_2) = f_2+f_1$

$\therefore$ commutativity holds.

② + is associative

For $f_1, f_2, f_3 \in F$

check

$f_1+(f_2+f_3) = (f_1+f_2)+f_3$

LHS $= (x_1+x_2+x_3)+\sqrt{2}(y_1+y_2+y_3)$

~~$\therefore$ both expressions result in~~

RHS $(x_1+x_2+x_3)+\sqrt{2}(y_1+y_2+y_3)$

$\therefore$ Associativity holds [$\because$ LHS = RHS]

③ Additive Identity exists

identity is 0 $\therefore$ for $x=0, y=0$, identity element $0 \in F$

$\{f_1+0 = f_1 = 0+f_1\}$

④ Additive Inverse exists

Let $f = x+\sqrt{2}y$, $x, y \in \mathbb{Q}$

let p be additive inverse

$p+f = 0$

$\Rightarrow p = -f$

$= (-x)+\sqrt{2}(-y)$

$\because (-x), (-y) \in \mathbb{Q}$

$p \in F$

$\therefore$ it exists.

classmate

Date

Page

⑤ . is commutative

~~p.q =~~ $f_1 \cdot f_2 = f_2 \cdot f_1$

~~Both exp. result in~~ LHS: $(x_1x_2 + 2y_1y_2) + (x_1y_2 + x_2y_1)\sqrt{2}$

RHS: $(x_1x_2 + 2y_1y_2) + (x_1y_2 + x_2y_1)\sqrt{2}$

$\therefore$ LHS = RHS, commutativity exists.

⑥ . is associative

$f_1 \cdot (f_2 \cdot f_3) = (f_1 \cdot f_2) \cdot f_3$

~~both exp. result in~~

~~___~~ by the inherent property of $\cdot$ to be associative, LHS = RHS

$\therefore$ Associativity exists

⑦ Multiplicative Identity exists

Identity is 1. for $(x = 1, y = 0)$, $1 \in F$

$\{f \cdot 1 = f = 1 \cdot f\}$

⑧ Inverse exists :

Let p be inverse for $f = x_1 + \sqrt{2}y_1$

$p \cdot f = 1$

$p \cdot (x + \sqrt{2}y) = 1$

Taking $p = \dfrac{(x - \sqrt{2}y)}{x^2 + 2y^2}$

$1 \cdot (x + \sqrt{2}y) = 1$ is true

$\therefore p = \left(\dfrac{x}{x^2 + 2y^2}\right) - \sqrt{2}\left(\dfrac{y}{x^2 + 2y^2}\right)$ exists

where $x_2 = \dfrac{x}{x^2 + 2y^2}$, $y_2 = \dfrac{y}{x^2 + 2y^2}$ $\in \mathbb{Q}$

$\therefore p \in F$

⑨ . is distributive over add^n.

$f_1 \cdot (f_2 + f_3) = f_1 f_2 + f_1 f_3$ [By inherent props. of + , $\cdot$ we can 'say' this is true

LHS:

$(x_1 + \sqrt{2}y_1) \cdot ((x_2 + x_3) + \sqrt{2}(y_2 + y_3))$

$(x_1(x_2 + x_3) + 2(y_1)(y_2 + y_3))$

$+ (\sqrt{2})(x_1(y_2 + y_3) + y_1(x_2 + x_3))$

$= (x_1x_2 + x_1x_3 + 2y_1y_2 + 2y_1y_3)$
$+ \sqrt{2}(x_1y_2 + x_1y_3 + x_2y_1 + x_3y_1)$

Similarly,

$RHS = (x_1x_2 + x_1x_3 + 2y_1y_2 + 2y_1y_3)$
$+ \sqrt{2}(x_1y_2 + x_1y_3 + x_2y_1 + x_3y_1)$

$\because$ LHS = RHS

$\therefore$ Property holds.

Using properties ① - ⑨, we can say that $(F, +, \cdot)$ is a field

Using our initial argument

$\Rightarrow (F, +, \cdot)$ is a subfield of $(\mathbb{R}, +, \cdot)$