CHAPTER 12

Coupled Oscillations

12.1 Introduction

In Chapter 3, we examined the motion of an oscillator subjected to an external driving force. The discussion was limited to the case in which the driving force is periodic; that is, the driver is itself a harmonic oscillator. We considered the action of the driver on the oscillator, but we did not include the feedback effect of the oscillator on the driver. In many instances, ignoring this effect is unimportant, but if two (or many) oscillators are connected in such a way that energy can be transferred back and forth between (or among) them, the situation becomes the more complicated case of **coupled oscillations.*** Motion of this type can be quite complex (the motion may not even be periodic), but we can always describe the motion of any oscillatory system in terms of **normal coordinates,** which have the property that each oscillates with a single, well-defined frequency; that is, the normal coordinates are constructed in such a way that no coupling occurs among them, even though there is coupling among the ordinary (rectangular) coordinates describing the positions of particles. Initial conditions can always be prescribed for the system so that in the subsequent motion only one normal coordinate varies with time. In this circumstance, we say that one of the **normal modes** of the system has been excited. If the system has n degrees of freedom (e.g., n-coupled one-dimensional oscillators or $n/3$-coupled three-dimensional oscillators), there are in general n normal modes, some of which may be identical. The general motion of the system is a complicated superposition of all the normal modes of oscillation, but we can always find initial conditions such that any given one of the normal modes is independently excited. Identifying each of a system's normal

*The general theory of the oscillatory motion of a system of particles with a finite number of degrees of freedom was formulated by Lagrange during the period 1762–1765, but the pioneering work had been done in 1753 by Daniel Bernoulli (1700–1782).

modes allows us to construct a revealing picture of the motion, even though the system's *general* motion is a complicated combination of all the normal modes.

It is relatively easy to demonstrate some of the coupled oscillator phenomena described in this chapter. For example, two pendula coupled by a spring between their mass bobs, two pendula hung from a rope, and masses connected by springs can all be experimentally examined in the classroom. Similarly, the triatomic molecule discussed here is a reasonable description of CO_2. Similar models can approximate other molecules.

In the following chapter, we shall continue the development begun here and discuss the motion of vibrating strings. This example by no means exhausts the usefulness of the normal-mode approach to the description of oscillatory systems; indeed, applications can be found in many areas of mathematical physics, such as the microscopic motions in crystalline solids and the oscillations of the electromagnetic field.

12.2 Two Coupled Harmonic Oscillators

A physical example of a coupled system is a solid in which the atoms interact by elastic forces between each other and oscillate about their equilibrium positions. Springs between the atoms represent the elastic forces. A molecule composed of a few such interacting atoms would be an even simpler model. We begin by considering a similar system of coupled motion in one dimension: two masses connected by a spring to each other and by springs to fixed positions (Figure 12-1). We return to this example throughout the chapter as we describe various instances of coupled motion.

We let each of the oscillator springs have a force constant* κ: the force constant of the coupling spring is κ_{12}. We restrict the motion to the line connecting the masses, so the system has only two degrees of freedom, represented by the coordinates x_1 and x_2. Each coordinate is measured from the position of equilibrium.

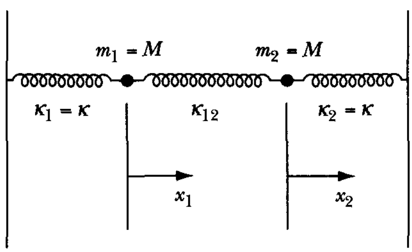

FIGURE 12-1 Two masses are connected by a spring to each other and by springs to fixed positions. This is a system of coupled motion in one dimension.

*Henceforth, we denote force constants by κ rather than by k as heretofore. The symbol k is reserved for (beginning in Chapter 13) an entirely different context.

If m_1 and m_2 are displaced from their equilibrium position by amounts x_1 and x_2, respectively, the force on m_1 is $-\kappa x_1 - \kappa_{12}(x_1 - x_2)$, and the force on m_2 is $-\kappa x_2 - \kappa_{12}(x_2 - x_1)$. Therefore the equations of motion are

$$\left.\begin{aligned} M\ddot{x}_1 + (\kappa + \kappa_{12})x_1 - \kappa_{12}x_2 = 0 \\ M\ddot{x}_2 + (\kappa + \kappa_{12})x_2 - \kappa_{12}x_1 = 0 \end{aligned}\right\} \tag{12.1}$$

Because we expect the motion to be oscillatory, we attempt a solution of the form

$$\left.\begin{aligned} x_1(t) &= B_1 e^{i\omega t} \\ x_2(t) &= B_2 e^{i\omega t} \end{aligned}\right\} \tag{12.2}$$

where the frequency ω is to be determined and where the amplitudes B_1 and B_2 may be complex.* These trial solutions are complex functions. Thus, in the final step of the solution, the real parts of $x_1(t)$ and $x_2(t)$ will be taken, because the real part is all that is physically significant. We use this method of solution because of its great efficiency, and we use it again later, leaving out most of the details. Substituting these expressions for the displacements into the equations of motion, we find

$$\left.\begin{aligned} -M\omega^2 B_1 e^{i\omega t} + (\kappa + \kappa_{12})B_1 e^{i\omega t} - \kappa_{12}B_2 e^{i\omega t} = 0 \\ -M\omega^2 B_2 e^{i\omega t} + (\kappa + \kappa_{12})B_2 e^{i\omega t} - \kappa_{12}B_1 e^{i\omega t} = 0 \end{aligned}\right\} \tag{12.3}$$

Collecting terms and canceling the common exponential factor, we obtain

$$\left.\begin{aligned} (\kappa + \kappa_{12} - M\omega^2)B_1 - \kappa_{12}B_2 = 0 \\ -\kappa_{12}B_1 + (\kappa + \kappa_{12} - M\omega^2)B_2 = 0 \end{aligned}\right\} \tag{12.4}$$

For a nontrivial solution to exist for this pair of simultaneous equations, the determinant of the coefficients of B_1 and B_2 must vanish:

$$\begin{vmatrix} \kappa + \kappa_{12} - M\omega^2 & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} - M\omega^2 \end{vmatrix} = 0 \tag{12.5}$$

The expansion of this secular determinant yields

$$(\kappa + \kappa_{12} - M\omega^2)^2 - \kappa_{12}^2 = 0 \tag{12.6}$$

Hence,

$$\kappa + \kappa_{12} - M\omega^2 = \pm\,\kappa_{12}$$

Solving for ω, we obtain

$$\omega = \sqrt{\frac{\kappa + \kappa_{12} \pm \kappa_{12}}{M}} \tag{12.7}$$

*Because a complex amplitude has a *magnitude* and a *phase*, we have the two arbitrary constants necessary in the solution of a second-order differential equation; that is, we could equally well write $x(t) = |B| \exp[i(\omega t - \delta)]$ or $x(t) = |B| \cos(\omega t - \delta)$, as in Equation 3.6b. Later (see Equation 12.9), we shall find it more convenient to use two distinct *real* amplitudes and the time-varying factors $\exp(i\omega t)$ and $\exp(-i\omega t)$. These various forms of solution are all entirely equivalent.

We therefore have two **characteristic frequencies** (or **eigenfrequencies**) for the system:

$$\omega_1 = \sqrt{\frac{\kappa + 2\kappa_{12}}{M}}, \qquad \omega_2 = \sqrt{\frac{\kappa}{M}} \tag{12.8}$$

Thus, the general solution to the problem is

$$\left.\begin{aligned} x_1(t) &= B_{11}^+ e^{i\omega_1 t} + B_{11}^- e^{-i\omega_1 t} + B_{12}^+ e^{i\omega_2 t} + B_{12}^- e^{-i\omega_2 t} \\ x_2(t) &= B_{21}^+ e^{i\omega_1 t} + B_{21}^- e^{-i\omega_1 t} + B_{22}^+ e^{i\omega_2 t} + B_{22}^- e^{-i\omega_2 t} \end{aligned}\right\} \tag{12.9}$$

where we have explicitly written both positive and negative frequencies, because the radicals in Equations 12.7 and 12.8 can carry either sign.

In Equation 12.9, the amplitudes are not all independent, as we may verify by substituting ω_1 and ω_2 into Equation 12.4. We find

$$\begin{aligned} &\text{for} \quad w = \omega_1\text{:} \quad B_{11} = -B_{21} \\ &\text{for} \quad w = \omega_2\text{:} \quad B_{12} = B_{22} \end{aligned}$$

The only subscripts necessary on the *B*s are those indicating the particular eigenfrequency (i.e., the *second* subscripts). We can therefore write the general solution as

$$\left.\begin{aligned} x_1(t) &= B_1^+ e^{i\omega_1 t} + B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} \\ x_2(t) &= -B_1^+ e^{i\omega_1 t} - B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} \end{aligned}\right\} \tag{12.10}$$

Thus, we have *four* arbitrary constants in the general solution—just as we expect—because we have *two* equations of motion that are of *second* order.

We mentioned earlier that we could always define a set of coordinates that have a simple time dependence and that correspond to the excitation of the various oscillation modes of the system. Let us examine the pair of coordinates defined by

$$\left.\begin{aligned} \eta_1 &\equiv x_1 - x_2 \\ \eta_2 &\equiv x_1 + x_2 \end{aligned}\right\} \tag{12.11}$$

or

$$\left.\begin{aligned} x_1 &= \frac{1}{2}(\eta_2 + \eta_1) \\ x_2 &= \frac{1}{2}(\eta_2 - \eta_1) \end{aligned}\right\} \tag{12.12}$$

Substituting these expressions for x_1 and x_2 into Equation 12.1, we find

$$\left.\begin{aligned} M(\ddot{\eta}_1 + \ddot{\eta}_2) + (\kappa + 2\kappa_{12})\eta_1 + \kappa\eta_2 &= 0 \\ M(\ddot{\eta}_1 - \ddot{\eta}_2) + (\kappa + 2\kappa_{12})\eta_1 - \kappa\eta_2 &= 0 \end{aligned}\right\} \tag{12.13}$$

which can be solved (by adding and subtracting) to yield

$$\left.\begin{aligned} M\ddot{\eta}_1 + (\kappa + 2\kappa_{12})\eta_1 &= 0 \\ M\ddot{\eta}_2 + \kappa\eta_2 &= 0 \end{aligned}\right\} \tag{12.14}$$

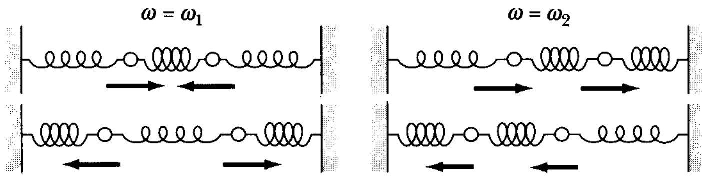

FIGURE 12-2 The two characteristic frequencies are indicated schematically. One is the antisymmetrical mode (masses are out of phase) and the other is the symmetrical mode (masses are in phase).

The coordinates η_1 and η_2 are now *uncoupled* and are therefore *independent*. The solutions are

$$\left.\begin{aligned}\eta_1(t) &= C_1^+ e^{i\omega_1 t} + C_1^- e^{-i\omega_1 t}\\ \eta_2(t) &= C_2^+ e^{i\omega_2 t} + C_2^- e^{-i\omega_2 t}\end{aligned}\right\} \tag{12.15}$$

where the frequencies ω_1 and ω_2 are given by Equations 12.8. Thus, η_1 and η_2 are the *normal coordinates* of the problem. In a later section, we establish a general method for obtaining the normal coordinates.

If we impose the special initial conditions $x_1(0) = -x_2(0)$ and $\dot{x}_1(0) = -\dot{x}_2(0)$, we find $\eta_2(0) = 0$ and $\dot{\eta}_2(0) = 0$, which leads to $C_2^+ = C_2^- = 0$; that is, $\eta_2(t) \equiv 0$ for all values of t. Thus, the particles oscillate always *out of phase* and with frequency ω_1; this is the **antisymmetrical** mode of oscillation. However, if we begin with $x_1(0) = x_2(0)$ and $\dot{x}_1(0) = \dot{x}_2(0)$, we find $\eta_1(t) \equiv 0$, and the particles oscillate *in phase* and with frequency ω_2; this is the **symmetrical** mode of oscillation. These results are illustrated schematically in Figure 12-2. The general motion of the system is a linear combination of the symmetrical and antisymmetrical modes.

The fact that the antisymmetrical mode has the higher frequency and the symmetrical mode has the lower frequency is actually a general result. In a complex system of linearly coupled oscillators, the mode possessing the highest degree of symmetry has the lowest frequency. If the symmetry is destroyed, then the springs must "work harder" in the antisymmetrical modes, and the frequency is raised.

Notice that if we were to hold m_2 fixed and allow m_1 to oscillate, the frequency would be $\sqrt{(\kappa + \kappa_{12})/M}$. We would obtain the same result for the frequency of oscillation of m_2 if m_1 were held fixed. The oscillators are identical and in the absence of coupling have the same oscillation frequency. The effect of coupling is to separate the common frequency, with one characteristic frequency becoming larger and one becoming smaller than the frequency for uncoupled motion. If we denote by ω_0 the frequency for uncoupled motion, then $\omega_1 > \omega_0 > \omega_2$, and we may schematically indicate the effect of the coupling as in Figure 12-3a. The solution for the characteristic frequencies in the problem of three coupled identical masses is illustrated in Figure 12-3b. Again,

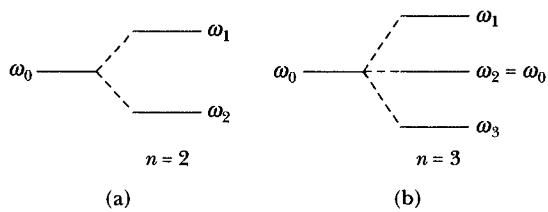

FIGURE 12-3 (a) Coupling separates the common frequency for two identical masses, with one characteristic frequency being higher and one being lower than the frequency ω_0 for uncoupled motion. (b) For three coupled identical masses, one characteristic frequency is smaller than ω_0 and one is larger. For n (number of oscillators) odd, one characteristic frequency is equal to ω_0. The separations are only schematic.

we have a splitting of the characteristic frequencies, with one greater and one smaller than ω_0. This is a general result: For an even number n of identical nearest neighbor coupled oscillators, $n/2$ characteristic frequencies are greater than ω_0, and $n/2$ characteristic frequencies are smaller than ω_0. If n is odd, one characteristic frequency is equal to ω_0, and the remaining $n - 1$ characteristic frequencies are symmetrically distributed above and below ω_0. The reader familiar with the phenomenon of the Zeeman effect in atomic spectra will appreciate the similarity with this result: In each case, there is a symmetrical splitting of the frequency caused by the introduction of an interaction (in one case by the application of a magnetic field and in the other by the coupling of particles through the intermediary of the springs).

12.3 Weak Coupling

Some of the more interesting cases of coupled oscillations occur when the coupling is *weak*—that is, when the force constant of the coupling spring is small compared with that of the oscillator springs: $\kappa_{12} \ll \kappa$. According to Equations 12.8, the frequencies ω_1 and ω_2 are

$$\omega_1 = \sqrt{\frac{\kappa + 2\kappa_{12}}{M}}, \qquad \omega_2 = \sqrt{\frac{\kappa}{M}} \tag{12.16}$$

If the coupling is weak, we may expand the expression for ω_1:

$$\omega_1 = \sqrt{\frac{\kappa}{M}}\sqrt{1 + \frac{2\kappa_{12}}{\kappa}} = \sqrt{\frac{\kappa}{M}}\sqrt{1 + 4\varepsilon}$$

where

$$\varepsilon \equiv \frac{\kappa_{12}}{2\kappa} \ll 1 \tag{12.17}$$