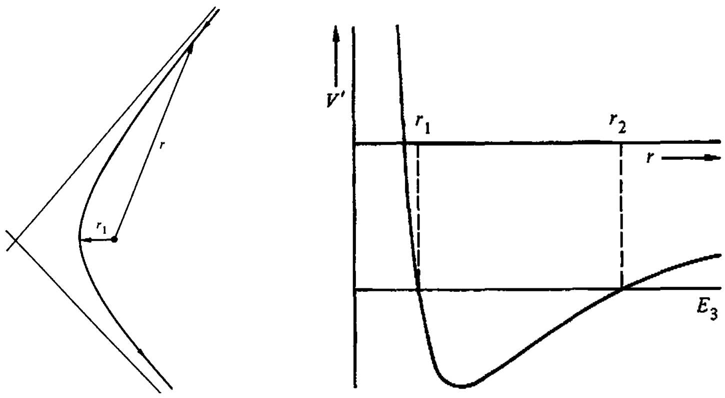

Figure 4: Bound and unbound motion.

3.5 Equations of motion

As mentioned in the previous section, our interest is to obtain the equation for the orbits *i,e,.* a functional dependance bewtween θ and r. We note that

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr = \frac{\ell}{\mu r^2 \dot{r}}dr\,.$$

So from (33) we get

$$\theta(r) = \int \frac{\pm(\ell/r^2)dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu^2 r^2}\right)}}\,. \tag{35}$$

This is the solution for θ in terms of r. To obtain the solution for r in terms of time we use the El EoM

$$\frac{d}{dt}\left(\frac{\partial L}{\partial r}\right) - \frac{\partial L}{\partial r} = 0\,,$$

which gives

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r)\,. \tag{36}$$

In the above equation, if we replace $\dot{\theta}$ in terms of ℓ we get the EoM for r in terms of time. However, in central force problems, the equation of orbit *i.e.,* $r \equiv r(\theta)$ is of interests. Therefore we convert the above equation as a differential equation for r in terms of the θ. To this end, we cast this in a suitable form by change of variable

$$u = \frac{1}{r}\,.$$

We compute

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} = -\frac{\mu}{\ell}\dot{r}\,.$$

In the last line angular $\ell = \mu r^2\dot{\theta}$ has been used. The second derivative of the above is

$$\frac{du^2}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{\ell}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{\ell}\dot{r}\right) = -\frac{\mu}{\ell\dot{\theta}}\ddot{r} = -\frac{\mu^2}{\ell^2}r^2\ddot{r}\,.$$

which can be written as

$$\ddot{r} = -\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}\,.$$

Substituting this in (36) and replacing $\dot{\theta}$ by ℓ we get the EoM for u

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F(u)\,, \tag{37}$$

or in terms of r

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)\,. \tag{38}$$

Solution of this equation gives the orbit.

3.6 Kepler's Problem

Kepler's problem concerns motion of planets around the sun under inverse squared force

$$F(r) = -\frac{GMm}{r^2} = -\frac{k}{r^2}\,,$$

which correspond to a potential of the type

$$U(r) = -\frac{k}{r}\,.$$

The equation of orbit (37) for this system gives

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}(-ku^2) = \frac{\mu k}{\ell^2}\,.$$

The solution of this equation gives

$$u = \frac{\mu k}{\ell^2} + A\cos\theta\,, \Longrightarrow \boxed{r = \frac{\alpha}{1+\epsilon\cos\theta}}\,, \tag{39}$$

where

$$\alpha = \frac{\ell^2}{\mu k}\,, \quad \epsilon = A\frac{\ell^2}{\mu k}\,.$$

The (39) is the orbital equation. Before we compute the positive constant ϵ that determines the behavior of the orbits, we study some general characteristics of the orbits. The behavior of orbits is different for $\epsilon < 1$ and $\epsilon \geq 1$. For $\epsilon < 1$ the denominator of (39) never vanishes, and the radius vector r remains bounded for all values of θ. For $\epsilon \geq 1$ the denominator can vanish for some value of θ so that the radius vector becomes unbounded and approaches infinity. Evidently the value $\epsilon = 1$ is the boundary between the bounded and unbounded solutions. We will see shortly that this boundary scenario is related to the total energy E of the system.

For $\epsilon < 1$, the denominator of (39) oscillates between $1 \pm \epsilon$. Hence the radius vector oscillates between a minimum and a maximum

$$r_{\text{min}} = \frac{\alpha}{1+\epsilon}\,, \quad r_{\text{max}} = \frac{\alpha}{1-\epsilon}\,.$$

The $r = r_{\text{min}}$ for $\theta = 0$ is called the *perihelion*, and $r = r_{\text{max}}$ at $\theta = \pi$ is the *aphelion* of the orbit. The orbit also has a period of 2π so that $r(2\pi) = r(0)$ and the orbit closes on itself after one revolution. We can show that the orbit is nothing but an ellipse. To do so we consider the plane of the orbit as the $X-Y$ plane. We introduce

$$a = \frac{\alpha}{1-\epsilon^2}\,, \quad b = \frac{\alpha}{\sqrt{1-\epsilon^2}}\,, \quad d = a\epsilon\,,$$

and write (39) as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

which is the standard equation for an ellipse where the center C and the origin $\mathcal{O}$ are separated by a distance d as shown in figure 5. The distances a, b are the semi-minor and the semi-major axes and it is related to the ϵ as

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}}\,,$$

which means that ϵ is the eccentricity of the ellipse. So *the orbits of planets are elliptical with the sun at one of the two focuses* – this is *Kepler's first law* of planetary motion.

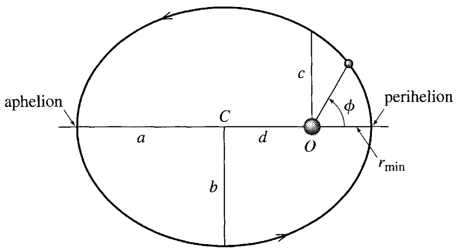

Figure 5: Elliptical orbit of a planet given by equation (39). The sun is at the origin $\mathcal{O}$ which is also one of the focus of the ellipse. The center is at C. The distances a, b are called the semi-major and the semi-minor axes. The parameter $\alpha = \mu k/\ell^2$ the value of the radius vector when $\theta = 90^o$. The closest and the farthest from the sun is called perihelion and the aphelion.

To determine the eccentricity ϵ in terms of total energy E, we note that at $r = r_{\min}$ the radial velocity $\dot{r} = 0$ and the (34) gives

$$E = V'(r_{\min}) = \frac{\ell^2}{2\mu r_{\min}^2} - \frac{k}{r_{\min}} = \frac{1}{2r_{\min}}\left(\frac{\ell^2}{\mu r_{\min}} - 2k\right).$$

We have just found that $r_{\min} = \alpha/(1+\epsilon)$. Substituting this in the above equation and solving for ϵ we get

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\,. \tag{40}$$

The orbits can be classified according to the total energy as

- Hyperbola: $\epsilon > 1 \Longrightarrow E > 0$
- Parabola: $\epsilon = 1 \Longrightarrow E = 0$
- Circle: $\epsilon = 0 \Longrightarrow E = -\frac{\mu k^2}{2\ell^2}$
- Ellipse: $0 < \epsilon < 1 \Longrightarrow -\frac{\mu k^2}{2\ell^2} < E < 0$

We finally discuss the time period of elliptical orbits. From (31)

$$\tau = \int dt = \int \frac{2\mu}{\ell} dA = \frac{2\mu}{\ell} A\,.$$

For an ellipse $A = \pi ab$. From previous expressions of a, b in terms of eccentricity we have $b = \sqrt{\alpha a}$. Substituting in the expressions of τ and squaring we get

$$\tau^2 = \frac{4\pi^2\mu}{k} a^3\,. \tag{41}$$

This is Kepler's third law which states that *squared of the time period is proportional to the cube of the semimajor axis of the ellipse.*