Quiz - 1.

Q. F_1, F_2 - CDFs.

Conti. & strictly inc.

$F_1(x) < F_2(x)$

$\forall x \in \mathbb{R}$

Show that $\exists$ RV X_1, X_2 with CDF F_1 & F_2

$(F_1 < F_2)$

s.t $X_1 > X_2$.

We need to show that $\exists\, X_1, X_2$. We need not show that

$F_1(x) < F_2(x) \Rightarrow X_1 > X_2$.

(In fact we can't show this because X_1 can be less than/equal to X_2).

Wrong attempt

Contrapositive of $X_1 > X_2$: $X_1 \leq X_2$.

$\{X_2 \leq x\} \subseteq \{X_1 \leq x\}$

Say F_1 is CDF of X_1, F_2 is CDF of X_2.

$F_2(x) = P(X_2 \leq x) \leq P(X_1 \leq x) = F_1(x)$

$\lightning$

Contradiction.

This method won't work.

Because.

$X_1 > X_2 \Rightarrow X_1(\omega) > X_2(\omega)\ \forall \omega$.

Negation of $X_1(\omega) > X_2(\omega)$: $X_1(\omega) \leq X_2(\omega)$

Now we can't proceed by the above method.

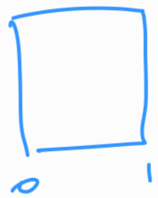
Correct solution

CDF : Monotonic func.

CDF → 1 as $x \to \infty$
CDF → 0 as $x \to -\infty$.

CDF :

$F_X(x) = P(X \le x)$



$P(A) = \text{area}(A)$

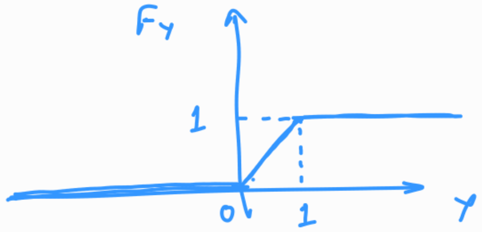
In 1-D, we say

$\mathcal{F}$

$\Omega = [0,1]$
$Y(\omega) = \omega$

$P([a,b]) = \underbrace{b-a}_{\text{Length of the line segment}}$

Consider a RV 'Y' with CDF



$$F_Y(y) = \begin{cases} 0, & y < 0 \\ y, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

Construct X_1, X_2 as

$X_1 = F_1^{-1}(Y)$, $X_2 = F_2^{-1}(Y)$

Inverse is defined because CDF is $\underbrace{\text{strictly inc.}}_{\Rightarrow \text{One one}}$ and $\underbrace{\text{continuous}}_{\Rightarrow \text{Onto}}$

Claim: X_1 is RV whose CDF is F_1, X_2 is RV whose CDF is F_2 and $X_1 > X_2$.

Proof: $F_{X_1}(x) = P(X_1 \le x)$

$= P(F_1^{-1}(y) \le x)$

$= P(Y \le F_1(x))$ ($\because$ F_1 is strictly inc. we can write $F_1^{-1}(y) \le x$ as $Y \le F_1(x)$).

$= F_Y(F_1(x))$

$= F_1(x)$ when $0 \le y \le 1$, then $F_Y(y) = y$ (See above)

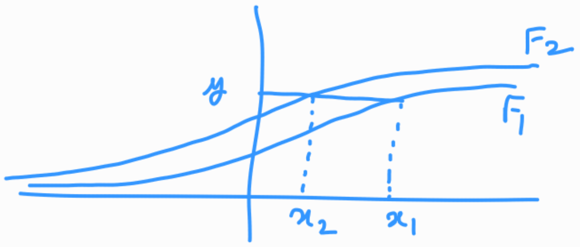
$F_1(x)$ is CDF, so $0 \le F_1(x) \le 1$.

So now we've shown CDF of X_1 is F_1.

|||ly we can show CDF of X_2 is F_2.

Now, $F_1(x) < F_2(x) \; \forall x$.

We need to show $F_1^{-1}(y) > F_2^{-1}(y)$, $y \in [0,1]$



Clearly $x_1 > x_2$

$\Rightarrow F_1^{-1}(y) > F_2^{-1}(y)$

↳ True $\forall y \in [0,1]$

$\Rightarrow F_1^{-1}(Y) > F_2^{-1}(Y)$

$\Rightarrow X_1 > X_2$

→ $E[X \mid Y=y] = \sum_x x \, P_{X|Y}(x \mid y)$ (Conditional expectation of X given $Y=y$).

↳ By def^n.

We know that $E[X] = \sum_x x \, P_X(x)$

Here PMF of X is P_X.

Now if PMF is $P_{X|Y=y}$, then $E[X \mid Y=y] = \sum_x x \, P_{X|Y}(x \mid y)$

→ $\mathrm{var}(X) = E[(X - E[X])^2]$ Conveys deviation from mean.

$\mathrm{var}(X \mid Y=y) = E[(X - E[X \mid Y=y])^2 \mid Y=y]$

When this conditioning is not there, then

$E[(X - E[X \mid Y=y])^2]$ ⟶ This a valid expectation but this is not $\mathrm{var}(X \mid Y=y)$.

Here X is not conditioned on Y.

$\mathrm{var}(X \mid Y=y) = E[X^2 \mid Y=y] - (E[X \mid Y=y])^2$.

↳ This is satisfied only when

$\mathrm{var}(X \mid Y=y) = E[(X - E[X \mid Y=y])^2 \mid Y=y]$.

NOT satisfied when $\mathrm{var}(X \mid Y=y)$ is taken as)

$E[(X - E[X|Y=y])^2]$

Intuition behind law of total variance

$$\boxed{var(X) = E[\underbrace{var(X|Y)}_{RV}] + var(\underbrace{E[X|Y]}_{RV})}$$

We know that $E[X|Y=y] = \sum_x x P_{X|Y}(x|y)$

So $\phi(y) = E[X|Y=y]$

Here Y is RV. $\phi(Y)$ is also RV.

$$\underbrace{\phi(Y)}_{RV} = \begin{cases} E[X|Y=y_1] \text{ with prob. } P_Y(y_1) \\ E[X|Y=y_2] \text{ with prob. } P_Y(y_2) \\ \vdots \end{cases}$$

$\times$

$$g(X) = \begin{cases} g(x_1) \text{ with } P_{x_1} \\ g(x_2) \text{ with } P_{x_2} \\ \vdots \end{cases}$$

$\psi(y) = var(X|Y=y)$

$$\psi(Y) = \begin{cases} var(X|Y=y_1) \text{ with prob. } P_Y(y_1) \\ \vdots \end{cases}$$

$var(X) = E[var(X|Y)] + var(E[X|Y])$

We say $var(X)$ has 2 components.

Say we have 3 sets of students.

S_1	S_2	S_3
st1	st1	st1
st2	st2	st2
$\vdots$	$\vdots$	$\vdots$
stn_1	stn_2	stn_3

X : Quiz score of a random student. | Range (X): takes $n_1+n_2+n_3$ values

To compute var (X), we can compute treating all the students of S_1, S_2, S_3 in one big set containing all students.

Y: Section no. of random student. | Range (Y) = {1, 2, 3}.

$E[X|Y=1] \rightarrow$ Avg. of scores in section 1.

$E[X|Y=2] \rightarrow$ " " " 2

$E[X|Y=3] \rightarrow$ " " " 3

Now $\text{var}(X|Y=1) \rightarrow$ By how much the quiz scores in S_1 deviate from the mean of quiz scores in S_1.

So $\underbrace{P_Y(y_1)\,\text{var}(X|Y=1) + P_Y(y_2)\,\text{var}(X|Y=2) + P_Y(y_3)\,\text{var}(X|Y=3)}_{= E[\text{var}(X|Y)]}$

Expectation of variances within individual sections.

This does not capture the correlation across sections, i.e., this doesn't capture how quiz scores of S_1 compares with scores of S_2.

This is captured by

$\rightarrow \text{var}(E[X|Y])$

$\hookrightarrow$ Variance of averages between sections

$$E[g(x) \mid A] = \sum_{x} g(x)\, P_{X|A}(x)$$

$$E[g(X,Y) \mid A] = \sum_{x,y} g(x,y)\, P_{XY|A}(x,y)$$