Lecture 7

(22 August 2024)

Recap. $X : \Omega \to \mathbb{R}$ is RV is

$\{\omega : X(\omega) \leq x\} \in \mathcal{F} \;\; \forall\, x \in \mathbb{R}$.

Distribution function $F_X(x) = P(X \leq x)$.

$$\lim_{x \to -\infty} F_X(x) = 0$$

$$\lim_{x \to \infty} F_X(x) = 1$$

$$\lim_{\varepsilon \to 0^+} F_X(x+\varepsilon) = F_X$$

$$x < y \implies F_X(x) \leq F_X(y)$$

Discrete Random Variable.

A random variable X is called discrete if it takes values in some countable subset $\{x_1, x_2, \ldots\}$ of $\mathbb{R}$.

A discrete random variable has an associated probability mass function (PMF), $p_X : \mathbb{R} \to [0,1]$ given by

$$p_X(x) = P(X = x)$$

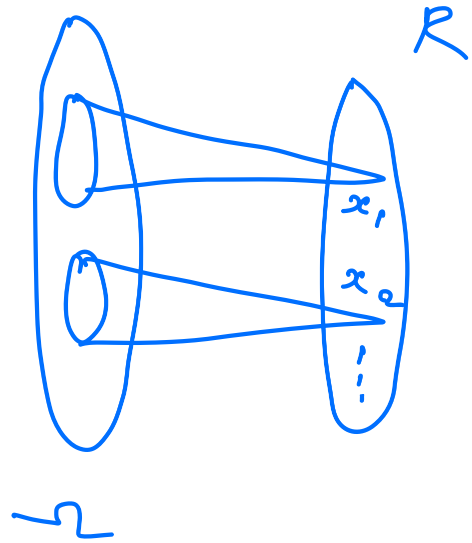
$$= P(\omega \in \Omega : X(\omega) = x).$$

$$F_X(x) = \sum_{i : x_i \leq x} p_X(x_i)$$

<u>Lemma</u>: Let X be a random variable and it takes values $x_1, x_2, \ldots$. Then

$$\sum_{i \in \mathbb{N}} p_X(x_i) = 1.$$

Proof.



$$\sum_{i=1}^{\infty} p_X(x_i)$$

$$= \sum_{i=1}^{\infty} P(X=x_i)$$

$$= \sum_{i=1}^{\infty} P(\omega : X(\omega)=x_i)$$

$$= P\left(\bigcup_{i=1}^{\infty} \{\omega : X(\omega)=x_i\}\right) \quad [\text{additivity}]$$

(because $\{\omega : X(\omega)=x_i\}$ and $\{\omega : X(\omega)=x_j\}$ are disjoint events for $i \neq j$)

$$= P(\Omega)$$

(because $\bigcup_{i=1}^{\infty} \{X=x_i\} = \Omega$)

$$= 1.$$

Functions of Random Variable

Let $X : \Omega \to R$ be a random variable,
consider a real function

$$g : R \to R.$$

$Y = g(X)$ i.e., $Y(\omega) = g(X(\omega))$.

Is Y a random variable?
what are the conditions on g?

we need $\{\omega : Y(\omega) \le y\} \in \mathcal{F} \ \forall y \in R$.

That is

$$Y^{-1}((-\infty, y])$$

$$= X^{-1}(g^{-1}((-\infty, y]))$$

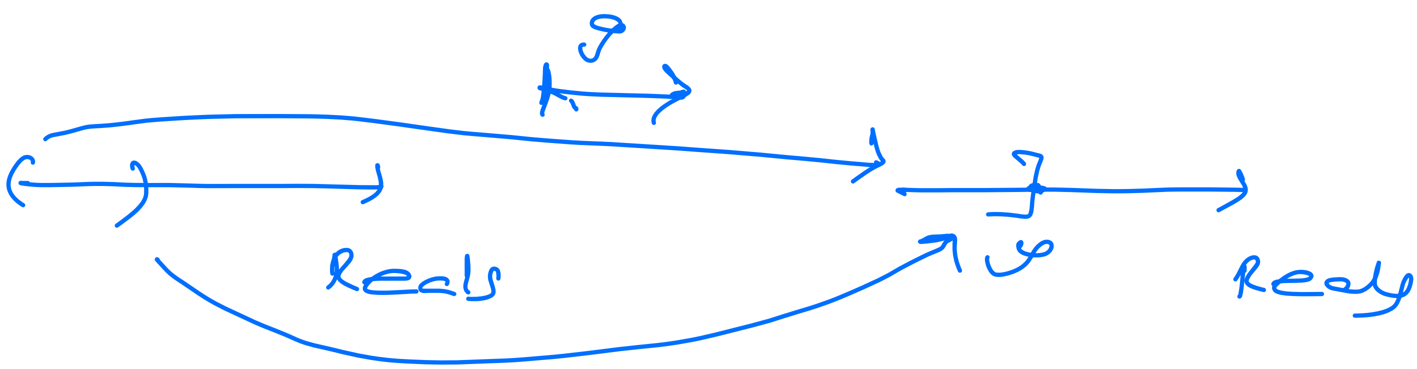
Recall the Borel σ-algebra —
the smallest σ-field that contains
the sets $(-\infty, x]$ $x \in R$.

$\mathcal{B} = \{ (-\infty, x], (-\infty, x), (x_1, x_2), [x_1, x_2],$
$(x_1, x_2], [x_1, x_2), \ldots \}$

we have $X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{B}$.

This is because B can be expressed as a countable union of sets of the form $(-\infty, x]$ and their complements.

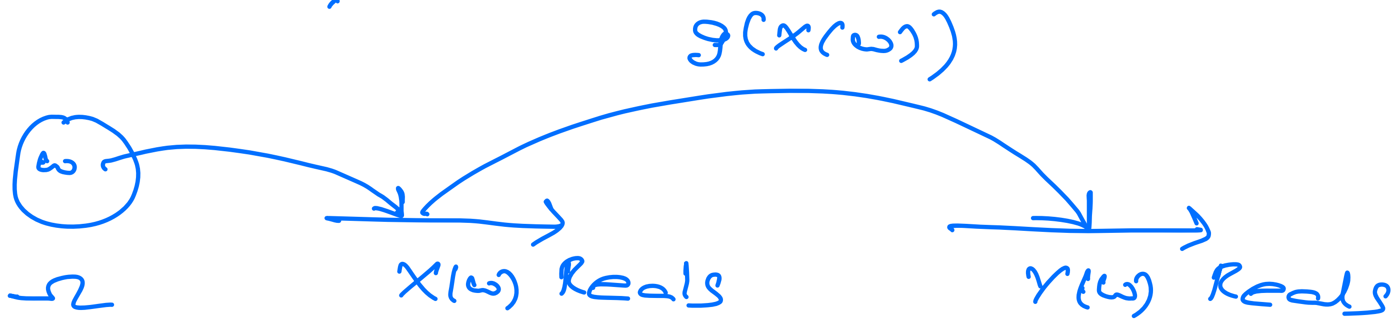
so for $g(x)$ to be a RV it suffices to have $g^{-1}((-\infty, y]) \in \mathcal{B} \quad \forall y \in \mathbb{R}$.



Such a g is called a Borel-measurable function.

In general, <u>all</u> the functions we come across are Borel measurable functions.

Thus, we can assume that $g(X)$ is a random variable whenever X is a random variable in the further discussion.



<u>Lemma</u>. Let X be a discrete random variable with pmf P_X and $Y = g(X)$. Then

$$P_Y(y) = \sum_{x \in \mathcal{X} : g(x) = y} P_X(x),$$

where X takes values in $\mathcal{X}$.

<u>Proof</u>. $P_Y(y) = P(\{\omega : Y(\omega) = y\})$

$= P(\{\omega : g(X(\omega)) = y\})$

$$= P\Big(\bigcup_{x \in \mathcal{X}} \{\omega : X(\omega) = x,\ g(X(\omega)) = y\} \Big)$$

$$= P\Big(\bigcup_{x \in \mathcal{X}} \{\omega : X(\omega) = x,\ g(x) = y\} \Big)$$

$$= P\Big(\bigcup_{\substack{x \in \mathcal{X} : \\ g(x) = y}} \{\omega : X(\omega) = x\} \Big)$$

$$= \sum_{\substack{x \in \mathcal{X} : \\ g(x) = y}} P(\{\omega : X(\omega) = x\})$$

$$= \sum_{\substack{x \in \mathcal{X} : \\ g(x) = y}} P_X(x).$$

Example. Let $Y = |X|$ and

$$P_X(x) = \begin{cases} 1/9, & \text{if } x \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\} \\ 0, & \text{o.w.} \end{cases}$$

$$P_Y(y) = P_X(y) + P_X(-y) = 2/9, \quad \forall y \in [1:4],$$

$$P_Y(0) = 1/9.$$

Expectation

Suppose we have a collection of numbers $a_1, a_2, \ldots, a_n$ their average is a single number that describes the whole collection. Now consider a random variable X, we would like to define a similar notion.

Let X be a discrete random variable that takes values in $\mathcal{X}$. The expectation, or expected value, or mean of X is defined as

$$E[X] = \sum_{x \in \mathcal{X}} x \, p_X(x).$$

Interpretation: Consider a discrete random variable that takes values $x_1, x_2, \ldots, x_m$. This random variable is a result of a random experiment.

Suppose that we repeat this experiment a very large number of times N and that the trials are independent. Let x_i occurs N_i number of times for $i \in [1:m]$,

We consider the average of all the observed values:

$$\frac{\sum_{i=1}^{m} N_i x_i}{N} = \sum_{i=1}^{m} \left(\frac{N_i}{N}\right) x_i$$

$$\approx \sum_{i=1}^{m} x_i \, p_X(x_i)$$

$$\left(\text{as } \frac{N_i}{N} \approx p_X(x_i)\right)$$

roughly

Example. $X \sim Be(p)$, $p_X(1) = p = 1 - p_X(0)$.

$E[X] = 1 \cdot p + 0(1-p) = p$.

Expected value of a function of RV

Let X be a discrete random variable and $g: \mathbb{R} \to \mathbb{R}$ then $Y = g(X)$ is a random variable. To calculate its expectation it may appear we first need to find its PMF P_Y and compute $\sum_y y P_Y(y)$. There is an easier way to do this without computing P_Y.

Law of the Unconscious Statistician:

$$E[g(X)] = \sum_{x \in \mathcal{X}} g(x) P_X(x),$$

$$
\begin{aligned}
E[g(X)] &= E[Y] \\
&= \sum_{y \in \mathcal{Y}} y P_Y(y) \\
&= \sum_{y \in \mathcal{Y}} y \sum_{x \in \mathcal{X} : g(x) = y} P_X(x)
\end{aligned}
$$

$$= \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X} : g(x) = y} y \, P_X(x)$$

$$= \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X} : g(x) = y} g(x) P_X(x)$$

$$= \sum_{x \in \mathcal{X}} g(x) \, P_X(x).$$

<u>Variance</u>,

$$\mathrm{Var}(X) = E\left[(X - E[X])^2\right],$$

measures the amount by which X tends to deviate from mean.

Let $\mu = E[X]$.

$$E[(X-\mu)^2] = \sum_x (x-\mu)^2 P_X(x)$$

$$= \sum_x \left(x^2 + \mu^2 - 2\mu x\right) P_X(x)$$

$$= \sum_x x^2 P_X(x) + \mu^2 - 2\mu\mu$$

$$= \sum_x x^2 p_X(x) - \mu^2$$

$$= E[X^2] - (E[X])^2.$$

Examples of Discrete RVs

Bernoulli Random Variable

Consider the toss of a coin which comes up a head with probability p and a tail with probability $1-p$.

$$X(H) = 1 \quad X(T) = 0$$

$$p_X(1) = p = 1 - p_X(0),$$

Exercise. $X \sim Be(p)$. Show that

(i) $E[X] = p$ (ii) $\mathrm{Var}(X) = p(1-p)$,

Binomial Random Variable

A coin is tossed n times (independently),

$$P(\{H\}) = p = 1 - P(\{T\}).$$

Let X be the total no. of heads in the n-toss sequence.

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$k \in [0:n],$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1,$$

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \frac{n!}{(n-k)!\, k!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

[Take $n' = n-1$ $k' = k-1$]

$$= np \sum_{k'=0}^{n'} \binom{n'}{k'} p^{k'} (1-p)^{n'-k'}$$

$$= np\,(p+1-p)^{n'} = np.$$

$$E[X^2] = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

[$n' = n-1$ $k' = k-1$]

$$= np \sum_{k'=0}^{n'} (k'+1) \binom{n'}{k'} p^{k'} (1-p)^{n'-k'}$$

$$= np\left[\sum_{k'=0}^{n'} k'\binom{n'}{k'} p^{k'}(1-p)^{n'-k'}\right]$$

$$+ np$$

$$= np\left((n-1)p+1\right).$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= np((n-1)p+1) - n^2p^2$$

$$= np - np^2 = np(1-p).$$

Geometric Random Variable

Toss a coin independently until we get a heads.

$$P(\{H\}) = p \quad P(\{T\}) = 1-p$$

X = No. of coin tosses required to get a heads

$$P_X(k) = (1-p)^{k-1}p, \quad k = 1, 2, 3, \ldots$$

__Exercise__. Compute the mean and the variance of a geometric random variable.