# Question 1

1. Find the PDF, the mean, and the variance of the random variable ( X ) with the CDF:

$$F_X(x) = \begin{cases} 1 - \frac{a^3}{x^3}, & \text{if } x \geq a, \\ 0, & \text{if } x < a, \end{cases}$$

   where $a$ is a positive constant.

## Answer

For $x < a$,

$$f_x(x) = 0$$

$$f_x(x) = \frac{dF}{dx} = \frac{3a^3}{x^4}$$

$$E[X] = \int_a^{\infty} \frac{3a^3}{x^4} x dx = \left[ -\frac{3a^3}{2x^2} \right]_a^{\infty} = \frac{3a}{2}$$

$$E[X^2] = \int_a^{\infty} \frac{3a^3}{x^2} dx = \left[ -\frac{3a^3}{x} \right]_a^{\infty} = 3a^2$$

$Var(X) = \frac{3a^2}{4}$

$\Rightarrow$ RVs $A, B$.

$f_{x|A}(x) = 1 \quad 0 \le x \le 1$

~~CDF~~ $= \int$

$f_{x|B}(x) = 3, \quad 0 \le x \le \frac{1}{3}$.

$f_{x|B}(x) = 3$

$\int_0^{1/4} 3 \cdot dx$

$P(A | X \le 1/4)$

$P(A) = P(B) = \frac{1}{2}$

$= \dfrac{P(A \cap (x \le 1/4))}{P(x \le 1/4)}$

$= \dfrac{P(x \le 1/4 | A) P(A)}{P(x \le 1/4)}$

$P(x \le 1/4) \rightarrow P(x \le 1/4 | A) P(A) + P(x \le 1/4 | B) P(B)$

$= \dfrac{\frac{1}{2} \times \int_0^{1/4} f_{x|A}(x) \cdot dx}{\frac{1}{2} \int_0^{1/4} f_{x|A}(x) \cdot dx + \frac{1}{2} \int_0^{1/4} f_{x|A}(x)\, dx}$

$= \frac{1}{2} \left( \dfrac{\frac{1}{4}}{\frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{4}} \right) \frac{1}{2} \times \left( \dfrac{\frac{1}{4}}{\frac{4}{8} \frac{1}{2}} \right)$

$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

# 1 Question 3

We are given the joint probability density function $f_{XY}(x, y)$ as:

$$f_{XY}(x, y) = \begin{cases} k, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the value of $k$

The joint PDF must satisfy the condition that the total probability integrates to 1:

$$\int_0^1 \int_0^y f_{XY}(x, y)\, dx\, dy = 1.$$

Since $f_{XY}(x, y) = k$ for $0 < x < y < 1$, we perform the integration:

$$\int_0^1 \int_0^y k\, dx\, dy = 1.$$

The inner integral with respect to $x$ is:

$$\int_0^y k\, dx = ky.$$

Now, integrate with respect to $y$:

$$\int_0^1 ky\, dy = \frac{k}{2}.$$

Equating this to 1:

$$\frac{k}{2} = 1 \implies k = 2.$$

So, the joint PDF becomes:

$$f_{XY}(x, y) = \begin{cases} 2, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

2. Find the conditional PDF $f_{X|Y}(x|y)$

The conditional PDF $f_{X|Y}(x|y)$ is given by:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}.$$

a. Find $f_Y(y)$ (the marginal PDF of $Y$):

To find $f_Y(y)$, we integrate $f_{XY}(x, y)$ with respect to $x$:

$$f_Y(y) = \int_0^y f_{XY}(x, y)\, dx = \int_0^y 2\, dx = 2y.$$

Thus, the marginal PDF of $Y$ is:

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

b. Find $f_{X|Y}(x|y)$:
Now, the conditional PDF is:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y}.$$

Thus, for $0 < x < y < 1$, the conditional PDF is:

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

3. Find the conditional PDF $f_{Y|X}(y|x)$
Similarly, the conditional PDF $f_{Y|X}(y|x)$ is:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}.$$

a. Find $f_X(x)$ (the marginal PDF of $X$):
To find $f_X(x)$, we integrate $f_{XY}(x,y)$ with respect to $y$:

$$f_X(x) = \int_x^1 f_{XY}(x,y)\,dy = \int_x^1 2\,dy = 2(1-x).$$

Thus, the marginal PDF of $X$ is:

$$f_X(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

b. Find $f_{Y|X}(y|x)$:
Now, the conditional PDF is:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}.$$

Thus, for $0 < x < y < 1$, the conditional PDF is:

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{1-x}, & x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Final Answer:
- The conditional PDF $f_{X|Y}(x|y)$ is:

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- The conditional PDF $f_{Y|X}(y|x)$ is:

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{1-x}, & x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

# 2 Question 4

We are given the joint PDF of two jointly continuous random variables $X$ and $Y$:

$$f_{XY}(x, y) = \begin{cases} 6xy, & 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{x}, \\ 0, & \text{otherwise.} \end{cases}$$

We are tasked with computing the conditional variance $\mathrm{Var}(X|Y = y)$ for $0 \leq y \leq 1$.

## Step 1: Conditional PDF of $X$ given $Y = y$

The conditional PDF of $X$ given $Y = y$, denoted by $f_{X|Y}(x|y)$, is defined as:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}.$$

### a. Find $f_Y(y)$ (the marginal PDF of $Y$)

To find the marginal PDF $f_Y(y)$, we integrate the joint PDF $f_{XY}(x, y)$ over $x$, noting that the bounds on $x$ are determined by $y \leq \sqrt{x}$, which means $x \geq y^2$:

$$f_Y(y) = \int_{y^2}^{1} f_{XY}(x, y)\, dx = \int_{y^2}^{1} 6xy\, dx.$$

Now, perform the integration:

$$\int_{y^2}^{1} 6xy\, dx = 6y \int_{y^2}^{1} x\, dx = 6y \left[\frac{x^2}{2}\right]_{y^2}^{1} = 6y\left(\frac{1}{2} - \frac{y^4}{2}\right).$$

Thus, the marginal PDF of $Y$ is:

$$f_Y(y) = 3y(1 - y^4).$$

### b. Find $f_{X|Y}(x|y)$

Now, the conditional PDF $f_{X|Y}(x|y)$ is:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{6xy}{3y(1 - y^4)} = \frac{2x}{1 - y^4}.$$

Thus, for $y^2 \leq x \leq 1$, the conditional PDF is:

$$f_{X|Y}(x|y) = \begin{cases} \frac{2x}{1-y^4}, & y^2 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

## Step 2: Compute $E[X|Y = y]$

The conditional expectation $E[X|Y = y]$ is given by:

$$E[X|Y = y] = \int_{y^2}^{1} x f_{X|Y}(x|y)\, dx = \int_{y^2}^{1} x \cdot \frac{2x}{1-y^4}\, dx = \frac{2}{1-y^4} \int_{y^2}^{1} x^2\, dx.$$

Perform the integration:

$$\int_{y^2}^{1} x^2\, dx = \left[\frac{x^3}{3}\right]_{y^2}^{1} = \frac{1}{3} - \frac{y^6}{3}.$$

Thus, the conditional expectation is:

$$E[X|Y = y] = \frac{2}{1-y^4}\left(\frac{1}{3} - \frac{y^6}{3}\right) = \frac{2}{3} \cdot \frac{1-y^6}{1-y^4}.$$

## Step 3: Compute $E[X^2|Y = y]$

The second conditional moment $E[X^2|Y = y]$ is given by:

$$E[X^2|Y = y] = \int_{y^2}^{1} x^2 f_{X|Y}(x|y)\, dx = \int_{y^2}^{1} x^2 \cdot \frac{2x}{1-y^4}\, dx = \frac{2}{1-y^4} \int_{y^2}^{1} x^3\, dx.$$

Perform the integration:

$$\int_{y^2}^{1} x^3\, dx = \left[\frac{x^4}{4}\right]_{y^2}^{1} = \frac{1}{4} - \frac{(y^2)^4}{4} = \frac{1}{4} - \frac{y^8}{4}.$$

Thus, the second conditional moment is:

$$E[X^2|Y = y] = \frac{2}{1-y^4}\left(\frac{1}{4} - \frac{y^8}{4}\right) = \frac{1}{2} \cdot \frac{1-y^8}{1-y^4}.$$

## Step 4: Compute Var$(X|Y = y)$

The conditional variance $\text{Var}(X|Y = y)$ is given by:

$$\text{Var}(X|Y = y) = E[X^2|Y = y] - \left(E[X|Y = y]\right)^2.$$

Substitute the expressions for $E[X^2|Y = y]$ and $E[X|Y = y]$:

$$\mathrm{Var}(X|Y=y) = \frac{1}{2} \cdot \frac{1-y^8}{1-y^4} - \left(\frac{2}{3} \cdot \frac{1-y^6}{1-y^4}\right)^2.$$

Simplifying the expressions:

$$\mathrm{Var}(X|Y=y) = \frac{1}{2} \cdot \frac{1-y^8}{1-y^4} - \frac{4}{9} \cdot \frac{(1-y^6)^2}{(1-y^4)^2}.$$

This is the final expression for the conditional variance $\mathrm{Var}(X|Y=y)$.

A4

Q5 Given: RV U & V are Jointly continous

(a) To prove: $P(U=V)=0$

Proof: $F_{UV}(u,v) = \int_{x=-\infty}^{u} \int_{y=-\infty}^{v} f_{UV}(x,y)\,dx\,dy$ where $f_{UV} = \frac{\partial^2 F_{UV}(u,v)}{\partial u \partial v}$

$P(U=V) = \iint_{(u,v \mid u=v)} f_{UV}(u,v)\,du\,dv = \int_{u} f_{UV}(u,u)\,du^2$ $\left[\begin{matrix} u=v \\ du=dv \end{matrix}\right]$

$= 0$ [∵ area of line is zero]

$\Rightarrow P(U=V)=0$. Hence, Proved.

(b) $X = U(0,1)$, $Y=X$.

$P(X=Y) = 1$ [All events map to same value by RV X & Y].

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{o.w} \end{cases}$$

Assume X & Y are jointly continous.

$\Rightarrow \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1 \Rightarrow \int_{x=-\infty}^{\infty} \int_{x}^{x} f_{XY}(x,y)\,dx\,dy = 1$

[∵ for $x \neq y$ $f_{XY}=0$]

$\Rightarrow$ $0=1$ [Contradiction], Thus the RVs X & Y are not jointly continous. $\rightarrow$ No contradiction with Part(a).

$\underline{\underline{Q6}}$ $X$ & $Y \in RV$ with CDF $F_X, F_Y = F_X$ &

PDF $f_X, f_Y = f_X$

To find PDF of $Z = \max(X, Y)$ & $W = \min(X, Y)$

$$f_Z(z) = \frac{\partial}{\partial z} F_Z(z) = \frac{\partial}{\partial z} P(Z < z) = \frac{\partial}{\partial z} P(X < z, Y < z)$$

$$= \frac{\partial}{\partial z} P(X < z) \cdot P(Y < z) = \frac{\partial}{\partial z} F_X(z) F_Y(z)$$

$$= \frac{\partial}{\partial z} [F_X(z)]^2 = 2F_X(z) f_X(z)$$

$\square$

$$f_W(w) = \frac{\partial}{\partial w} P(W < w) = \frac{\partial}{\partial w} [1 - P(W > w)]$$

$$= \frac{\partial}{\partial w} [1 - P(X > w) P(Y > w)] = \frac{\partial}{\partial w} [1 - (1 - P(X < w))(1 - P(Y < w))]$$

$$= \frac{\partial}{\partial w} [- P(X < w) P(Y < w) + P(X < w) + P(Y < w)]$$

$$= f_X(w) + f_Y(w) - f_X(w) f_Y(w) = 2f_X(w) - 2(f_X(w)) F_X(w)$$

$$= 2f_X(w) (1 - F_X(w))$$

$\square$

Q7] We need to compute.

$$\mathbb{E}\left[\frac{\sum_{i=1}^{m} X_i}{\sum_{j=1}^{n} X_j}\right] \quad \text{where } X_i \sim (f_X) \text{ iids.}$$

$$\mathbb{E}\left[\frac{\sum_{i=1}^{m} X_i}{\sum_{j=1}^{n} X_j}\right] = \sum_{i=1}^{m} \mathbb{E}\left[\frac{X_i}{\sum_{j=1}^{n} X_j}\right]$$

Let take a simplified example
where $n = 2$ $m = 1$

$$\mathbb{E}\left[\frac{X_1}{X_1+X_2}\right]$$

we know that

$$\mathbb{E}\left[\frac{X_1}{X_1+X_2}\right] + \mathbb{E}\left[\frac{X_2}{X_1+X_2}\right] = 1$$

also, since $X_1$ and $X_2$ are iids they are interchangable

So let's try to find

$$\mathbb{E}\left[\frac{X_1}{X_1+X_2}\right] = \theta$$

Let $g_1(x_1, x_2) = \dfrac{x_1}{x_1+x_2}$

and $g_2(x_1, x_2) = \dfrac{x_2}{x_1+x_2}$

$$\mathbb{E}\left[\left(g_1(x_1, x_2)\right]\right. = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x_1 x_2)\, f_{X_1 X_2}(x_1 x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x_1}{x_1 + x_2} f_{x_1}(x_1) f_{x_2}(x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x_1}{x_1 + x_1} f_X(x_1) f_X(x_2)\, dx_1\, dx_2$$

Let $x_1 = a$ $\quad x_2 = b$ (~~dv~~ replacing variables)

$$\mathbb{E}\left[g(x_1 x_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{a}{a+b} f_X(a) f_X(b)\, da\, db.$$

Similarly

$$\mathbb{E}\left[g_2(X_1 X_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x_2}{x_1 + x_2} f_X(x_1) f_X(x_2)\, dx_1\, dx_2$$

Let $x_1 = b$ , $x_2 = a$

$$\mathbb{E}\left[g_2(X_1 X_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{a}{a+b} f_X(a) f_X(b)\, da\, db.$$

We can see the value of $\mathbb{E}[g_1(X_1 X_2)]$ & $\mathbb{E}[g_2(X_1 X_2)]$ is same

$$\therefore \quad \mathbb{E}\left[g_1(x_1, x_2)\right] = \mathbb{E}\left[g_2(x_1 x_2)\right] = \frac{1}{2}$$

( their summation $= 1$)

therefore we can extend this idea of m variable

$$\mathbb{E}\left[\frac{x_i}{\sum_{j=1}^{n} x_j}\right] = \frac{1}{n} \quad \forall i$$

$$\therefore \quad \mathbb{E}\left[\frac{\sum_{i=1}^{m} x_i}{\sum_{j=1}^{n} x_j}\right]$$

$$= \sum_{i=1}^{m} \mathbb{E}\left[\frac{x_i}{\sum_{j=1}^{n} x_j}\right]$$

$$= \sum_{i=1}^{m} \frac{1}{n}$$

$$= \frac{m}{n}.$$

# Q8. Proof of Independence for Continuous Random Variables

## Statement

We want to show that continuous random variables $X$ and $Y$ are independent if and only if their joint probability density function $f_{XY}(x, y)$ factorizes as the product $f_{XY}(x, y) = g(x)h(y)$, where $g(x)$ and $h(y)$ are functions of $x$ and $y$, respectively.

## Proof

We will prove both directions of the statement.

### If $f_{XY}(x, y) = g(x)h(y)$, then $X$ and $Y$ are independent

Assume the joint probability density function of $X$ and $Y$ factorizes as:

$$f_{XY}(x, y) = g(x)h(y)$$

We want to show that this implies $X$ and $Y$ are independent, i.e., $f_{XY}(x, y) = f_X(x)f_Y(y)$.

**Step 1: Calculate the marginal distributions.** - The marginal density of $X$ is given by:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_{-\infty}^{\infty} g(x)h(y)\, dy$$

Since $g(x)$ is independent of $y$, it can be factored out:

$$f_X(x) = g(x) \int_{-\infty}^{\infty} h(y)\, dy = g(x) \cdot c_1$$

where $c_1 = \int_{-\infty}^{\infty} h(y)\, dy$.

- Similarly, the marginal density of $Y$ is:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx = \int_{-\infty}^{\infty} g(x)h(y)\, dx$$

Factoring out $h(y)$, we get:

$$f_Y(y) = h(y) \int_{-\infty}^{\infty} g(x)\, dx = h(y) \cdot c_2$$

where $c_2 = \int_{-\infty}^{\infty} g(x)\, dx$.

**Step 2: Normalize the joint distribution.** The joint probability density function must integrate to 1:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy = 1$$

Substituting $f_{XY}(x, y) = g(x)h(y)$, we get:

$$\left( \int_{-\infty}^{\infty} g(x)\, dx \right) \left( \int_{-\infty}^{\infty} h(y)\, dy \right) = c_2 \cdot c_1 = 1$$

Thus, $c_1 \cdot c_2 = 1$.

**Step 3: Verify independence.** Now, multiplying the marginal densities:

$$f_X(x)f_Y(y) = (g(x) \cdot c_1)(h(y) \cdot c_2) = g(x)h(y) \cdot (c_1c_2)$$

Since $c_1 \cdot c_2 = 1$, we have:

$$f_X(x)f_Y(y) = g(x)h(y) = f_{XY}(x, y)$$

Thus, the joint pdf factorizes as the product of the marginal pdfs, which implies that $X$ and $Y$ are independent.

## Only if $X$ and $Y$ are independent, then $f_{XY}(x, y) = g(x)h(y)$

Now, assume that $X$ and $Y$ are independent. By the definition of independence, the joint pdf must satisfy:

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

where $f_X(x)$ and $f_Y(y)$ are the marginal pdfs of $X$ and $Y$, respectively.

**Step 1: Factorization of $f_{XY}(x, y)$.** Since $f_{XY}(x, y) = f_X(x)f_Y(y)$, we can let:

$$g(x) = f_X(x) \quad \text{and} \quad h(y) = f_Y(y)$$

Thus, the joint pdf $f_{XY}(x, y)$ can be written as:

$$f_{XY}(x, y) = g(x)h(y)$$

where $g(x)$ and $h(y)$ are functions of $x$ and $y$ alone.

# Question 9

$$f_{XY}(x,y) = 2e^{-x-y}, 0 < x < y < \infty$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_x^{\infty} 2e^{-x-y}dy$$

$$= 2e^{-x}\int_x^{\infty} e^{-y}dy = -2e^{-x}[e^{-y}]_x^{\infty}$$

$$= -2e^{-x}[0 - e^{-x}] = 2e^{-x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_0^{y} 2e^{-x-y}dx$$

$$= 2e^{-y}\int_0^{y} e^{-x}dx = -2^{-y}[e^{-x}]_0^{y}$$

$$= -2e^{-y}[e^{-y} - 1] = 2e^{-y}(1 - e^{-y})$$

Clearly, $f_{XY}(x,y) \neq f_X(x)f_Y(y)$. Thus, X and Y are not independent.