Quantum Mechanics 2023

SC1.203 :: Quiz-I

Time 40 Mins

6×5

1) Would you get the same outcome if you measured the energies of an infinite ensemble of identical particle-in-a-box systems at time t? Explain your answer.

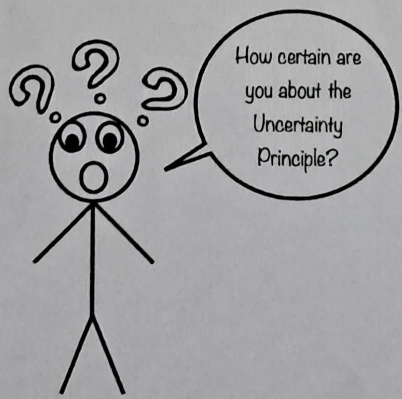
2) Is the operator for p hermitian? Show by explicit integral.

3) Evaluate $[\hat{x}, \hat{p}^2]$.

4) In the harmonic oscillator problem write the Schrödinger equation in terms of the raising and lowering operators.

5) For a free particle, $v_{\text{quantum}} = v_{\text{classical}}/2$. Explain.

6) Prove that eigenvectors/eigenfunctions of a hermitian operator with different eigenvalues are orthogonal.



INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY HYDERABAD

QUANTUM MECHANICS

Monsoon 2023 – CND Core – Credit 4

Instructor: Subhadip Mitra | **Date:** SEPTEMBER 21, 2023

Time: 1 H 30 M (08:30 - 10:00)

Mid-Semester Examination

Total Marks: 50

Instructions:

- **Class notes or books are not permitted. But you may bring one A4 sheet of handwritten material (not photo-copy/printed).**
- **Calculators are allowed.**
- **Do not write anything** (except roll number, seat no. etc.) **on the first page of the answer book.**
- You may skip 'trivial' steps. However, unless the logic is clear, you will **not** get any credit for a problem.
- Illegible answers will **not** be graded.
- **No** 'benefit of doubt' because of bad notation/illegible hand-writing etc.

Q 1. Consider a particle of mass m trapped in a 1D box between $x = -L/2$ to $x = L/2$.

(a) Obtain the general position-space wave function for this particle.

(b) If the particle is in the n^{th} energy state, calculate $\langle x \rangle$, σ_x, $\langle p \rangle$, and σ_p. Verify that the uncertainty principle is not violated.

(c) Is it justified to claim that the ground state energy is nonzero because of the uncertainty principle? Explain your answer.

(d) Even if we don't know the energy of the particle, we know that the wave function of the particle returns to its original form after every $T = 4mL^2/\pi\hbar$, i.e., $\Psi(x,T) = \Psi(x,0)$. How?

[$4+5+3+3 = 15$ **CO:** 1,2,3,5]

Q 2. For a simple harmonic oscillator about the point $x = 0$,

(a) Obtain the wave function for the first excited state upto an overall multiplicative constant. What is its energy? Plot $|\psi_1|^2$ as a function of x.

(b) Let $\psi_n(x)$ be for the normalized (steady state) wavefunction of the n^{th} energy state. Find how $\psi_n(x)$ is related to $\psi_0(x)$.

(c) Find the allowed energies of the *half* harmonic oscillator

$$V(x) = \begin{cases} 1/2m\omega^2x^2, & \text{for } x > 0, \\ \infty, & \text{for } x < 0. \end{cases}$$

[$(5+1+1)+4+4 = 15$ **CO:** 2,3,4,5]

Q 3. (a) Obtain the bound state solution for $V(x) = -\alpha\delta(x)$.

(b) Prove that $[f(x), p] = i\hbar\frac{df}{dx}$

(c) Argue that the eigenvalues of the operator $\hat{L}^2 - \hat{L}_x^2$ are always positive.

[$10+5+5 = 20$ **CO:** 1,2,3,4,5]

$\hat{a}_+^2 = a_+a_- + a_-a_+ + a_-^2$

Quantum Mechanics 2023

SC1.203

Quiz-II

Time 40 Mins

6×5

1) Let an electron be in a spin state

$$\chi = \frac{1}{3}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}.$$

What would be the expectation value if we measure S_x?

2) Assume three noninteracting neutrinos (chargeless fermions) are trapped in a one-dimensional infinite square well of size a in the (one-particle) states ψ_2, ψ_5, ψ_7. Write the three-particle wave function. What is the total energy in the unit of $\pi^2\hbar^2/2m_\nu a^2$? Would the three-particle wave function change if any one of the particles were a boson instead? How?

3) Electrons are fermions, yet they can form covalent bonds. Explain how.

4) Defining $u(r) = rR(r)$, the radial wave function satisfies the following equation:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

Obtain ψ_{100} for an infinite spherical well:

$$\text{V(r)=}\begin{cases} 0, & \text{for } r \le a, \\ \infty, & \text{for } r > a. \end{cases}$$

5) In the Stern-Gerlach experiment, the electron beam (moving $\perp$ to the z axis) undergoes a Hamiltonian of the following type:

$$\text{H(t)=}\begin{cases} 0, & \text{for } t \le 0, \\ -\gamma(B_0 + \alpha Z)S_z, & \text{for } 0 \le t \le T, \\ 0, & \text{for } t \ge T. \end{cases}$$

Obtain the z component of the momentum of the spin down component at a time $t \ge T$.

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY HYDERABAD

QUANTUM MECHANICS

Monsoon 2023 – CND Core – Credit 4

Instructor: Subhadip Mitra

Date: DECEMBER 1, 2023

Time: 03 H 00 M

End Examination

Total Marks: 100

Instructions:

- Attempt all the questions.
- Class notes or books are not permitted. But you may bring one A4 sheet of handwritten material (not photocopied/printed) which you have to submit with your answer script.
- Keep your answers to the point. Unless the logic is clear, you will not get any credit for a problem. Illegible answers will not be graded. No 'benefit of doubt' because of bad notation/illegible hand-writing etc.

Q 1. Consider a finite square well,

$$V(x) = \begin{cases} -V_0 & \text{for } -a < x < a \quad (V_0 > 0) \\ 0 & \text{otherwise,} \end{cases}$$

with a particle of energy $E > 0$ (scattering state).

(a) Show that the probability of the particle reflecting back is nonzero in general.

(b) What happens if $E \gg V_0$ or $E \to 0$? Show that there are some energies for perfect transmission (transmission resonance, this is why you get a very large transmission when you scatter low-energy electrons through noble-gas atoms).

(c) We say that *the absolute value of potential does not matter, only the difference matters. Hence, if we add a constant to the overall potential, nothing changes.* Is this true in Quantum Mechanics? If so, how do we see that? If not, why not?

(d) Suppose, now, the particle has $E < 0$ (i.e., it is in a bound state). What will happen if we add a purely imaginary constant (say, $-i\Gamma$) to the potential instead of a real constant? Will it affect the probability of the particle? If so, how?

[3+3+4+5=15 **CO:** 1,2,4,5]

Q 2. Let $|x\rangle$ denote the state (wave-function) at x. We can define an infinitesimal translation operator $\hat{T}(dx)$ such that

$$\hat{T}(dx')\,|x\rangle = |x+dx'\rangle\,.$$

(a) What properties should such an operator satisfy? In particular, argue for

(i) $\hat{T}^{\dagger}(dx')$,

(ii) $\hat{T}^{-1}(dx')$,

(iii) $\hat{T}(dx')\cdot\hat{T}(dx'')$ and

(iv) $\lim_{dx'\to 0}\hat{T}(dx')$.

(b) Under what condition $\hat{T}(dx') = 1 - i\,\dfrac{\hat{K}}{\hbar}\,d\hat{x}'$ satisfies all the above properties if we ignore terms of second order or higher in dx'? Is $\hat{T}$ hermitian?

(c) Show that

$$[\hat{x},\hat{T}(dx')]\,|x'\rangle = d\hat{x}'\,|x'+d\hat{x}'\rangle \approx d\hat{x}'\,|x'\rangle$$

and obtain $[\hat{x},\hat{K}]$.

(d) If $\langle x|\psi\rangle = \psi(x)$ (where $|\psi\rangle$ is an arbitrary state), what is $\langle x|T(dx)|\psi\rangle$? Expand it to establish the relation between $\hat{K}$ and the momentum operator.

$[4+(2+1)+(3+1)+(2+2)=15$ **CO:** 2,4]

Q 3. (a) Show with the momentum-space wave function $\Phi(p,t)$ that

$$\langle x\rangle = \int \Phi^{*}\left(-\frac{h}{i}\frac{\partial}{\partial p}\right)\Phi\, dp.$$

(b) Prove the Virial theorem:

$$\frac{d}{dt}\langle xp\rangle = 2\langle T\rangle - \left\langle x\frac{dV}{dx}\right\rangle,$$

where T is the kinetic energy.

(c) Consider a periodic potential, i.e., $V(x+\lambda) = V(x)$. Show that the wave function at $(x_0+\lambda)$ is proportional to $\psi(x_0)$ up to a constant (i.e., x-independent) phase.

(d) Explain how one gets dynamic solutions out of the stationary states for the time-independent potential.

[2+3+3+2=10 **CO:** 1,3,4,5]

Q 4. (a) Let, for a system of interest $\{|a_i\rangle\}$ be the set of eigenstates of an Hermitian operator A. Show that

(i) the matrix $A_{ij} = \langle a_i|A|a_j\rangle$ is diagonal,

(ii) the matrix $B_{ij} = \langle a_i|B|a_j\rangle$ is also diagonal where A and B are compatible observables.

(iii) the transformation from the basis $\{|a_i\rangle\}$ to another basis $\{|c_i\rangle\}$ is unitary, where $\{|c_i\rangle\}$ are the eigenstates of another Hermitian operator C incompatible with A or B.

(b) In the case of perturbation theory with degenerate states, why does one first look for some operator that commutes with the perturbed Hamiltonian?

(c) Consider a two-dimensional sqare box

$$V(x,y) = \begin{cases} 0 & \text{if } 0 \le x \le a,\ 0 \le y \le a, \\ \infty & \text{otherwise.} \end{cases}$$

Obtain the first-order corrections to the ground-state energy and the energies of the first and second excited states if we introduce a time-independent perturbation

$$H'(x,y) = \begin{cases} \Delta V_0 & \text{if } 0 \le x \le a/3, \\ 0 & \text{otherwise.} \end{cases}$$

(d) If the lowest-order relativistic correction to the Hamiltonian is given as

$$H' = -\frac{p^4}{8m^3c^2},$$

find the lowest-order relativistic correction to the energy levels of the one-dimensional harmonic oscillator.

[(1+2+2)+3+5+7=20 **CO:** 1,2,3,4,5]

Q 5. A spinning electron constitutes a magnetic dipole. Its dipole moment is proportional to the spin,

$$\vec{\mu} = \gamma\vec{S}$$

where γ is the gyromagnetic ratio. If you put it in a magnetic field $\vec{B}$, it feels a torque. The energy associated with the torque is $-\vec{\mu}\cdot\vec{B}$.

(a) If the magnetic field is constant $\vec{B} = B_0\hat{z}$, then show that $\langle\vec{S}\rangle$ gets titled and it precesses about the field with a constant frequency.

(b) If $\vec{B} = B_0\cos(\omega t)\hat{z}$ (where ω is a constant) and the electron starts out in the spin-down state in the x direction, i.e.,

$$\chi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

then obtain $\chi(t)$ by solving the time dependent Schrödinger equation

$$i\hbar\frac{\partial\chi}{\partial t} = \mathbf{H}\chi,$$

where $\mathbf{H}$ is the Hamiltonian matrix.

(c) Use a Gaussian trial function, $\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$ to obtain the lowest upper bound on the ground state energy of the quartic potential: $V(x) = \alpha x^4$.

[7+8+5=20 **CO:** 2,3,4]

Q 6. Consider a box of volume V containing free electron gas (assume the total number of atoms to be N with each one contributing q electrons). The normalized wave functions are given as

$$\psi_{n_x,n_y,n_z} = \sqrt{\frac{8}{V}}\sin\left(\frac{n_x\pi}{l_x}x\right)\sin\left(\frac{n_y\pi}{l_y}y\right)\sin\left(\frac{n_z\pi}{l_z}z\right)$$

where $V = l_xl_yl_z$. The allowed energies are

$$E_{n_x,n_y,n_z} = \frac{\hbar^2k^2}{2m} = \frac{\hbar^2}{2m}(k_x^2+k_y^2+k_z^2)$$

where the wave vector $\vec{k} = (k_x,k_y,k_z)$ with $k_i = n_i^2/l_i^2$.

(a) Show that the Fermi energy is $E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$ where ρ is the free electron density. How is it related to the chemical potential?

(b) The total energy $E_{tot} \propto V^{-2/3}$. Find the proportionality constant and the degeneracy pressure.

(c) Covalent bonding between two electrons requires the two to be in the singlet state. Explain.

(d) Derive the most probable occupation number for N identical fermions with total energy E (assume N_i particles are in d_i degenerate states of energy E_i, etc.).

[4+6+5+5=20 **CO:** 1,2,3,4,5]