Lecture 14

(26 September 2024)

Geometric and Exponential RVs

CDF defined for any type of RV provides a convenient means of exploring the relations between continuous and discrete random variables, we explore the relation between geometric and exponential RVs.

Let X be a geometric RV with success probability p i.e., X is the no. of trials until the first success in a sequence of independent Bernoulli trials where the probability of success in each trial is p.

$$F_X^{g}(n) = \sum_{k=1}^{n} (1-p)^{k-1} p = \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{p}}$$

$$= 1 - (1-p)^n,$$

for $n = 1, 2, 3, \ldots$

If X is exponential RV

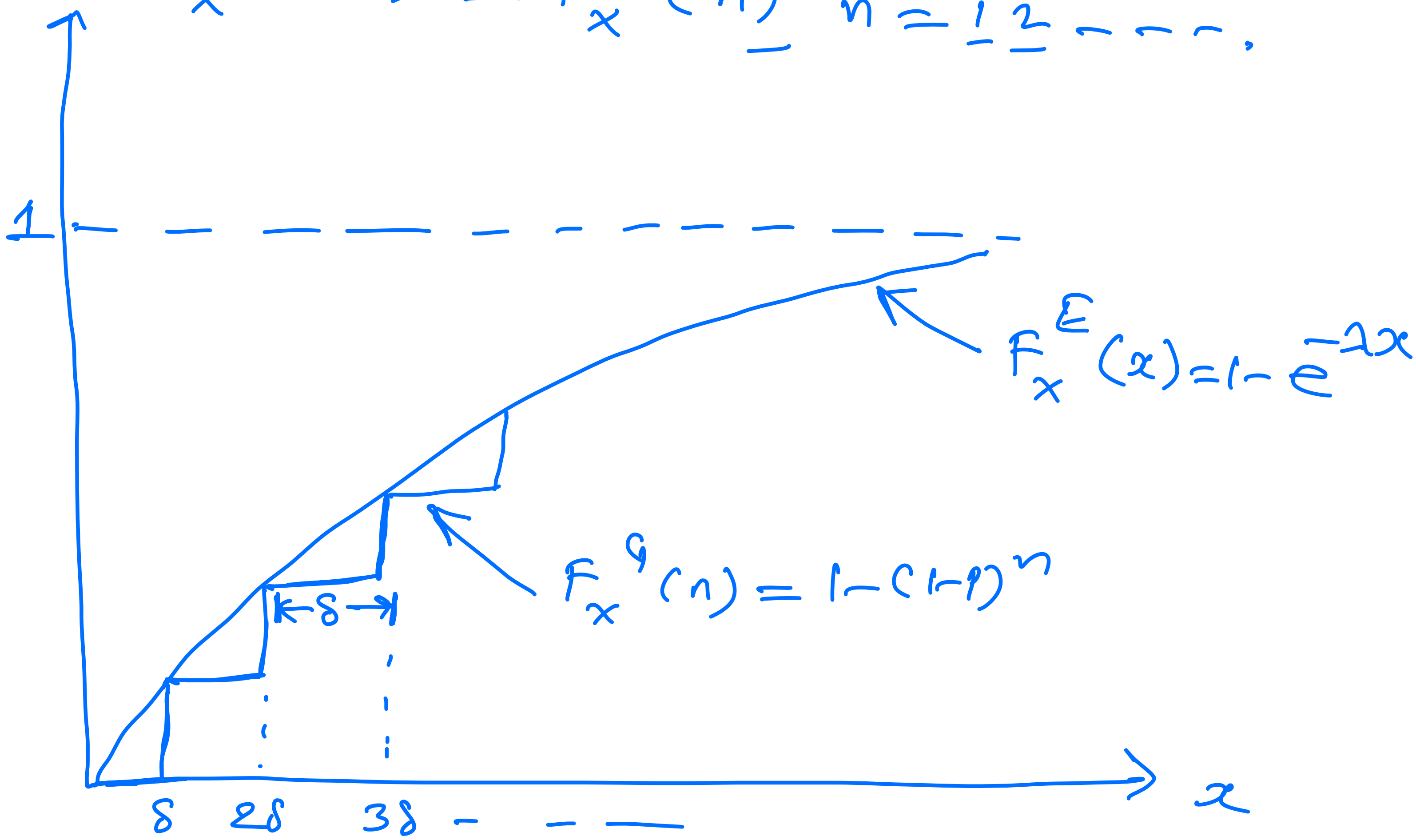
$$F_X^{E}(x) = \begin{cases} \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

For the purpose of comparison choose δ s.t. $e^{-\lambda \delta} = 1-p$, i.e.,

$$\delta = \frac{-\ln(1-p)}{\lambda}.$$

Then, we see that the values of the exponential and the geometric CDFs are equal whenever $x = n\delta$ with $n = 1, 2, \ldots$, i.e.,

$$F_X^E(n\delta) = F_X^g(n) \quad n = 1, 2, \ldots$$



Suppose now we toss a biased coin very quickly (every δ seconds where $\delta \ll 1$) with a small probability of heads (equal to $p = 1 - e^{-\lambda\delta}$). Then the first time to obtain a head (a geometric RV with parameter p) is a close approximation to an exponential RV with parameter λ in the sense that the corresponding CDFs are very close to each other as shown in the above figure.

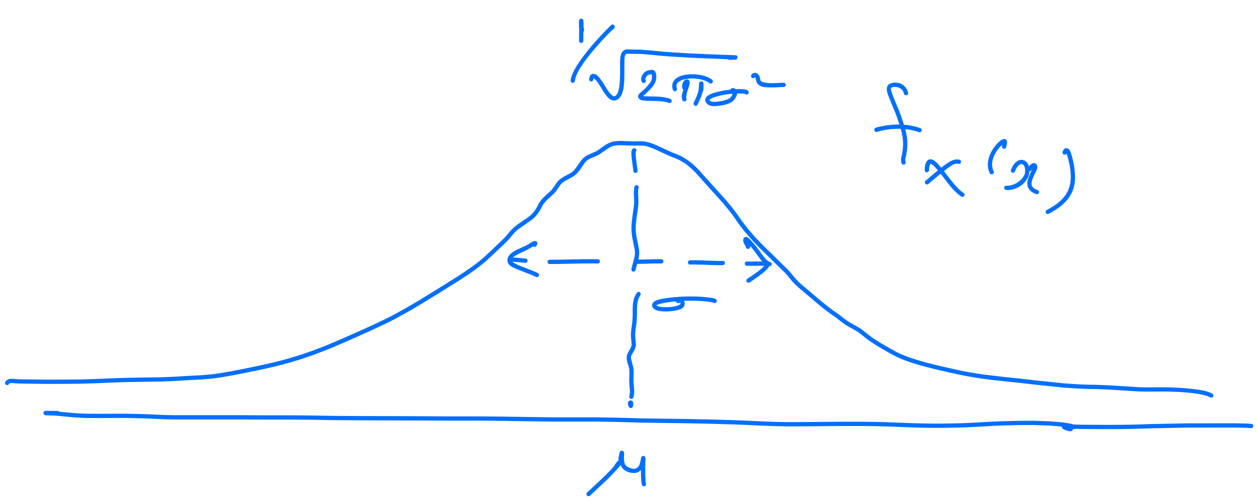
Formally, $\displaystyle \lim_{\delta \to 0} 1 - e^{-\lambda \lfloor \frac{x}{\delta} \rfloor \delta} = 1 - e^{-\lambda x}, \quad x \geq 0.$

Gaussian Random Variable

A continuous RV x is said to be Gaussian or normal if it has a PDF of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $\mu \in \mathbb{R}$, $\sigma \in [0, \infty)$.



We first show that $f_X(x)$ is a valid PDF i.e. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Consider the change of variable $t = \frac{x-\mu}{\sigma}$, we get

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt.$$

Let $I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$.

$$I^2 = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \frac{1}{2\pi} e^{-x^2/2} \cdot e^{-y^2/2}\,dx\,dy$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} \, dx\, dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$, we get

$$\int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{2\pi} e^{-\frac{r^2}{2}} \, r\, dr\, d\theta$$

$$= \int_{r=0}^{\infty} \cancel{2\pi} \cdot \frac{1}{\cancel{2\pi}} e^{-r^2/2} \, r\, dr$$

$$= \left[-e^{-r^2/2} \right]_0^{\infty} = \underline{1}.$$

We compute the mean and variance.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \, dx$$

$$= \int_{-\infty}^{\infty} (t\sigma + \mu) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$= \mu + \int_{-\infty}^{\infty} \sigma t \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$= \mu + \frac{\sigma}{\sqrt{2\pi}} \left[e^{-t^2/2} \right]_{-\infty}^{\infty}$$

$$= \mu + 0 = \mu .$$

$$\therefore E[x] = \mu .$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \, dx$$

$$= \int_{-\infty}^{\infty} (\sigma t + \mu)^2 \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$= \int_{-\infty}^{\infty} \sigma^2 t^2 \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

$$+ \mu^2 + \underbrace{2\mu\sigma \int_{-\infty}^{\infty} t e^{-t^2/2} dt}_{=0}$$

$$= \sigma^2 \int_{-\infty}^{\infty} t^2 \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt + \mu^2$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \left[t \cdot \int t e^{-t^2/2} dt + \int e^{-t^2/2} dt \right]_{-\infty}^{\infty} + \mu^2$$

$$= \sigma^2 \left[\underbrace{\left. \frac{t e^{-t^2/2}}{\sqrt{2\pi}} \right|_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \right] + \mu^2$$

$$= \sigma^2 + \mu^2.$$

$$\mathrm{Var}(x) = E[x^2] - E[x]^2 = \sigma^2.$$

A Gaussian RV with mean 0 & variance 1 is referred to as 'standard normal random variable'.

The CDF of a standard normal RV is denoted by

$$\Phi(x) = F_X(x)$$

$$= P(X \leq x)$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Lemma. $\Phi(-x) = 1 - \Phi(x)$ $x \in \mathbb{R}$.

$$\Phi(-x) = \int_{-\infty}^{-t} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

$$= \int_{-x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds$$

$$= 1 - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \, ds$$

$$= 1 - \Phi(x).$$

In other words if Z is standard normal RV

$$P(Z \leq -x) = P(Z \geq x) \quad x \in \mathbb{R}.$$

A normal RV has several special properties.

Theorem. If X is a normal RV with mean μ and variance σ^2 and if $a \neq 0$, b are scalars then $Y = aX + b$ is also normal with $E[Y] = a\mu + b$, $Var(Y) = \sigma^2 a^2$.

Before proving this theorem let us first find the PDF of a linear function of any RV.

Let X be a continuous RV with PDF f_X and let $Y = aX + b$.

$$F_Y(y) = P(Y \le y)$$

$$= P(aX + b \le y)$$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & \text{if } a > 0 \\ P\left(X \ge \frac{y-b}{a}\right) & \text{if } a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & \text{if } a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & \text{if } a < 0. \end{cases}$$

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a > 0 \\ \frac{-1}{a} f_X\left(\frac{y-b}{a}\right) & a < 0. \end{cases}$$

$$= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

$$\therefore X \sim f_X \implies aX+b \sim \frac{1}{|a|} f_X\left(\frac{y-b}{|a|}\right).$$

Proof of Theorem

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{|a|} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi a^2 \sigma^2}} e^{-\left(\frac{y-(a\mu+b)}{2\sigma^2 a^2}\right)}$$

This is the PDF of a normal RV with mean $\tilde{\mu} = a\mu + b$ and variance $\tilde{\sigma}^2 = a^2\sigma^2$.

$X \sim N(\mu, \sigma^2)$ means X is a normal RV with mean μ and variance σ^2.

Example. $X \sim N(\mu = 60, \sigma^2 = 20^2)$.

Find $P(X \geq 80)$ in terms of $\Phi(\cdot)$.

$$P(X \geq 80) = P\left(\frac{X-60}{20} \geq \frac{80-60}{20}\right)$$

$$= P\left(\frac{X-60}{20} \geq 1\right)$$

Let $Y = \frac{X-60}{20}$, $Y \sim N(0, 1)$

$$P(Y \geq 1) = 1 - P(Y \leq 1)$$

$$= 1 - \Phi(1).$$

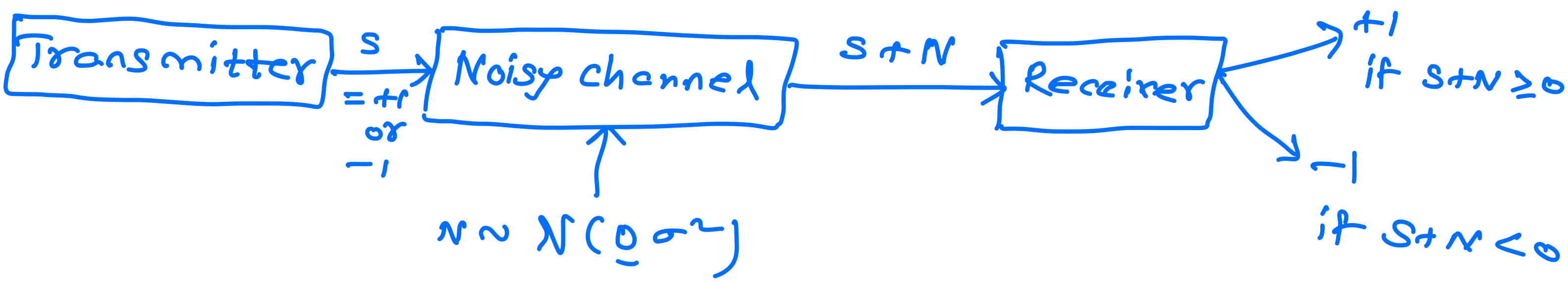
In general

$$P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Normal RVs are often used in signal processing and communications to model noise.

<u>Example</u> (Signal Detection).

A binary message is transmitted as a signal $\underline{s}$ which is either +1 or −1.



What is the probability of error is $s=-1$ is transmitted?

When −1 is transmitted an error occurs if and only if the noise N is at least 1 so that $S+N=-1+N\geq 0$.

So the probability of error when

$s = -1$ is transmitted

$$= P(N \geq 1)$$

$$= 1 - P(N \leq 1)$$

$$= 1 - \Phi\left(\frac{1}{\sigma}\right).$$