Poisson RV

$$P_x(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

The random exp. associated with Poisson RV will be discussed later

↳ A random var. with $P_x(k)$ is called Poisson RV.

→ For a valid RV, $\sum_{k=0}^{\infty} P_x(k) = 1$

Th^m: Consider Bin(n,p). As $n \to \infty$, while keeping $np = \lambda$ constant, we have

$$\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{e^{-\lambda}\lambda^k}{k!} \quad \forall k = 0, 1, 2, \dots$$

Bin(n,p) Keeping np : const. but $n \to \infty$ then Bin(n,p) $\downarrow$ Poisson(λ)

Proof: $\frac{n!}{(n-k)!\,k!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$

$$= \frac{\lambda^k}{k!} \underbrace{\frac{(n-k+1)(n-k+2)\dots(n)}{n^k}}_{\text{As } n\to\infty,\ \to 1} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\to 1 \text{ as } n\to\infty} \underbrace{\left(1-\frac{\lambda}{n}\right)^{n}}_{e^{-\lambda}}$$

(See formula)

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

→ $E[X] = \lambda$

$Var(X) = \lambda$ } Exercise

Eg: Coin toss twice $\Omega, \mathcal{F}, P$; $\mathcal{F} = 2^{\Omega} = 2^4$

$\Omega = \{HH, HT, TH, TT\}$.

$X(\omega)$ = no. of heads, $Y(\omega)$ = no. of tails

($X(\omega) + Y(\omega) = 2$ → True for all ω.)

→ Two RV X, Y on same sample space are said to be jointly discrete if (X, Y) takes countable values in $\mathbb{R}^2$.

$X_1 > X_2$

$\Rightarrow X_1(\omega) > X_2(\omega) \quad \forall \omega$

Negation of the above statement

$\exists \omega$ s.t. $X_1(\omega) \leq X_2(\omega)$

- Joint PMF

$$P_{X,Y}(x, y) = P(X = x, Y = y)$$

$$= P(\underbrace{\{\omega : X(\omega) = x, Y(\omega) = y\}}_{\in \mathcal{F}})$$

→ $\sum_{y} P_{XY}(x, y) = P_X(x)$, $\sum_{x} P_{XY}(x, y) = P_Y(y)$

How?

$$P_X(x) = P(X = x)$$

$$= P(\{X = x\} \cap \Omega)$$

Substitute $\Omega = \bigcup_{y} \{Y = y\}$.

$$= P\left(\{X = x\} \cap \left(\bigcup_{y} \{Y = y\}\right)\right)$$

$$= P\left(\bigcup_{y} \left(\{X = x\} \cap \{Y = y\}\right)\right)$$

Here $\{X = x\} \cap \{Y = y\}$ are disjoint sets

Complete the proof!

→ X, Y are RVs, then $Z = g(X, Y)$ is also a RV (Random vector rather)

$$Z(\omega) = g(X(\omega), Y(\omega)), \quad Z : \Omega \to \mathbb{R}$$
$$g : \mathbb{R}^2 \to \mathbb{R}$$

Eg: $g(x, y) = x + y$

X, Y is also "RV" (technically)
But we have to change the mapping
$\because$ RV: $\Omega \to \mathbb{R}$.
But here $X, Y : \Omega \to \mathbb{R}^2$
So we call
X, Y as RANDOM VECTOR

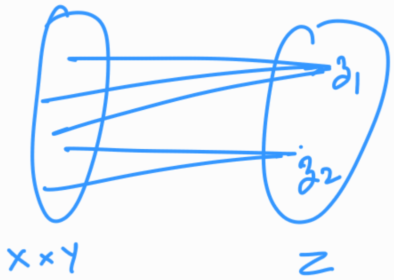
→ $Z = g(X, Y)$ Given $(X, Y) \sim P_{XY}$

$P_Z(z) = ?$

$Y = g(X)$, $P_Y(y) = \sum_{x : g(x) = y} P_X(x)$

IIIly

$$P_Z(z) = \sum_{x, y : g(x, y) = z} P_{XY}(x, y)$$



→ $Z = g(X, Y)$, $E[Z] = \sum_{x, y} g(x, y) P_{XY}(x, y)$

(Proof similar to $E[g(X)] = \sum_x P_X(x) g(x)$

• Independence (of RV)

X & Y (discrete RV) are said to be independent if

$$P_{XY}(x, y) = P_X(x) P_Y(y) \quad \forall x, y.$$

Exercise

$\exists g, h$ s.t

$P_X(x,y) = g(x)\,h(y) \quad \forall\, x, y \implies X \,\&\, Y$ are independent

Here $g(x)$, $h(y)$ are just some func of x & y resp.

(need not be PMF).

Eg: $P_X(x,y) = \underbrace{2P(X)}_{g(x)} \underbrace{\left(\frac{P(Y)}{2}\right)}_{h(y)}$

Here $g(x)$ & $h(y)$ are not PMF.

Eg: Consider $(X, Y) \in \{0, 1\}$.

Given:

$P_{XY}(1,1) = P_X(1)\,P_Y(1)$

Are X & Y independent?

Proof: $P_{X,Y}(1,0) = P_X(1) - P_{XY}(1,1)$

$\left(\because \sum_y P_{XY}(x,y) = P_X(x). \text{ Here } x = 1,\ y = \{0,1\} \right).$

$$P_{X,Y}(1,0) = P_X(1) - P_X(1)\,P_Y(1)$$
$$= P_X(1)\left(1 - P_Y(1)\right)$$
$$= P_X(1)\,P_Y(0)$$

$|||^{ly}$ it decomposes $\forall (x,y) \in (X,Y)$

$\rightarrow$ Given a binary RV, and that P_{XY} decomposes for one of the pairs, then the (X,Y) is independent. $(\forall (x,y))$