# Lecture 2

(5 August 2024)

## Recap

Different Approaches to Probability

A. Classical approach

$$P(E) = \frac{\text{no. of outcomes favourable for } E}{\text{total possible no. of outcomes}}$$

B. Relative frequency approach

$$P(E) = \lim_{n\to\infty} \frac{n_E}{n}, \quad \text{where}$$

$n_E$ = no. of times $E$ occurs and
$n$ = total no. of trials.

C. Axiomatic approach

- Review of set theory

C. Axiomatic approach to Probability

$(\Omega, \mathcal{F}, P)$

Sample space | Event space | Probability law

<u>Sample space $\Omega$</u>

Sample space is the set of all possible outcomes of a random experiment.

The elements of a sample space are outcomes of the experiment.

The random experiment should produce exactly one out of all the possible outcomes.

The outcomes of the sample space should be mutually exclusive and collectively exhaustive.

<u>Example</u>. Roll a die. which of the following are potential sample spaces?

(i) $\{\underline{1}, \underline{2}, \underline{2 \text{ or } 3}, \underline{3 \text{ or } 4}, \underline{5}, 6\}$

(ii) $\{\underline{1}, \underline{2}, \underline{3}, \underline{4}, 6\}$

(i) is not a sample space because it is not mutually exclusive.

(ii) is not a sample space because it is not collectively exhaustive.

$\Omega = \{\underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, 6\}$ is a sample space

<u>Examples</u>.

(i) Finite sample space

Roll a die

$\Omega = \{\underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, 6\}$

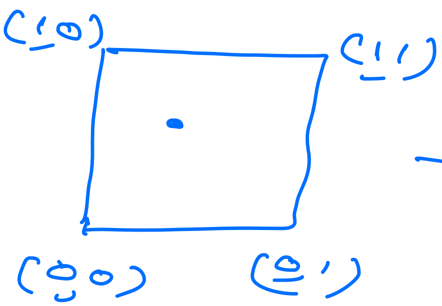
(ii) Countably infinite sample space

Toss a coin until you see a heads

$\Omega = \{H, TH, TTH, \ldots\}$

(iii) Uncountably infinite sample space

Consider throwing a dart on a $1 \times 1$ square target.




$$\Omega = \{(x,y) : 0 \le x \le 1, \; 0 \le y \le 1\}.$$

## Event space $\mathcal{F}$

An event is a subset of the sample space. The collection of all events is called an event space. An event space should be a $\sigma$-field.

A collection of sets $\mathcal{F}$ is said to be a $\sigma$-field if it satisfies the following.

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

<u>Proposition</u>.

(i) $A_1, A_2, \ldots, A_n \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}$

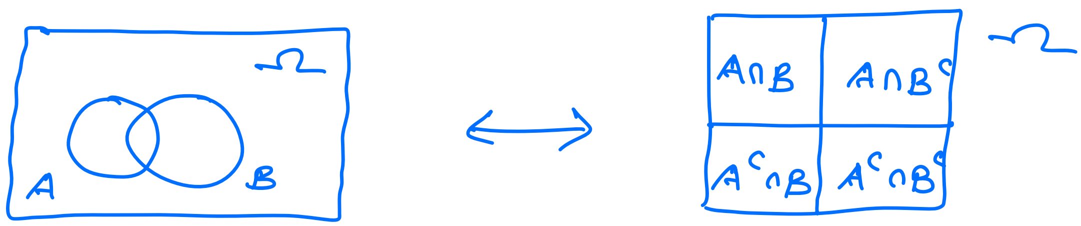
(ii) $A, B \in \mathcal{F} \Rightarrow A \setminus B \in \mathcal{F}$

<u>Examples</u>

(i) $\mathcal{F} = \{\Omega, \phi\}$

(ii) $\mathcal{F} = \{\Omega, \phi, E, E^c\}$

(iii) Smallest $\sigma$-field containing two sets $A$ and $B$.




Let $A \triangle B = (A \setminus B) \cup (B \setminus A)$, symmetric difference between $A$ and $B$

$\mathcal{F} = \{ A, B, A^c, B^c, A^c \cap B, (A^c \cap B)^c,$

$A \cap B^c, (A \cap B^c)^c, A \cup B, (A \cup B)^c, A \cap B, (A \cap B)^c,$

$A \triangle B, (A \triangle B)^c, \Omega, \phi \}$

Thus $\mathcal{F}$ can have at most 16 elements

(no. of possible unions of sets taken from $A \cap B$, $A^c \cap B$, $A \cap B^c$, $A^c \cap B^c$)

At most 16 because not all of them are always distinct, e.g., take $A, B$ s.t. $A \cap B = \phi$.

Example. $\Omega = \{1, 2, 3, 4\}$, $A = \{1\}$, $B = \{1, 2\}$.

## Probability Law $P$.

A probability law or a probability measure $P$ is a set function

$$P : \mathcal{F} \longrightarrow [0,1]$$

that satisfies the following axioms.

1) (Non-negativity), $P(E) \geq 0$ for all $E \in \mathcal{F}$

2) (Normalization). $P(\Omega) = 1$.

3) (Additivity). If $A_1, A_2, \ldots$ are disjoint (i.e., mutually exclusive events) then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i),$$

## Examples.

(i) $\Omega = \{1, 2, 3, 4, 5, 6\}$

$P(\{1\}) = 0.1$ $\quad$ $P(\{4\}) = 0.3$

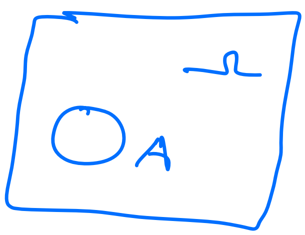
$P(\{2\}) = 0.05$ $\quad$ $P(\{5\}) = 0.25$

$P(\{3\}) = 0.05$ $\quad$ $P(\{6\}) = 0.25$

This defines a probability law (i.e., one can verify that this defn. satisfies the 3 axioms).

(ii) $\Omega = \{(x, y) : 0 \le x \le 1 \quad 0 \le y \le 1\}$.

$\mathcal{F}$ contains subsets of $\Omega$.




$P(A) = \text{area of } A$