CS 302.1 - Automata Theory

Lecture 11

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H Y D E R A B A D

Quick Recap

The standard TM model is quite robust. It can simulate other seemingly "powerful" variants such as

- Lazy TM
- Multi-tape TM
- Two-way infinite tape TM
- Enumerator
- Non-deterministic TM

The set of problems that are decided by a standard TM = the set of problems decided by any of these variants

Turing Machines: some definitions

Total Turing Machines: A TM M is total if for all input strings $w \in \Sigma^*$, $M(w)$ **accepts or rejects but never runs infinitely.**

On every input, M halts

An **Algorithm** is nothing but a Total Turing Machine.

Recursive Language/Turing Decidable/Decidable: A language L is called Recursive or Turing decidable or Decidable if there exists a Total Turing Machine M for L, i.e.

$\forall \omega \in L, M(\omega)$ accepts
$\forall \omega \notin L, M(\omega)$ rejects $\Big\}$ Halts on all inputs

Total TM $M = $ On input w,
If $M(w)$ reaches an accept state, ACCEPT
If $M(w)$ reaches a reject state, REJECT

Recursively Enumerable Language/Turing Recognizable (RE): A language L is called Recursively Enumerable (RE) or Turing Recognizable if

$\forall \omega \in L, M(\omega)$ **accepts**
$\forall \omega \notin L, M(\omega)$ **doesn't accept**   (rejects or runs infinitely)

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$M = $ On input w,
If $M(w)$ reaches an accept state, ACCEPT
If $M(w)$ reaches a reject state, REJECT
If $M(w)$ loops, ……….

L is in RE if **L is recognized by some Turing Machine M**, i.e. **$L(M) = L$**. It halts for ALL the YES instances.

All Recursive Languages are Recursively Enumerable but not vice versa.

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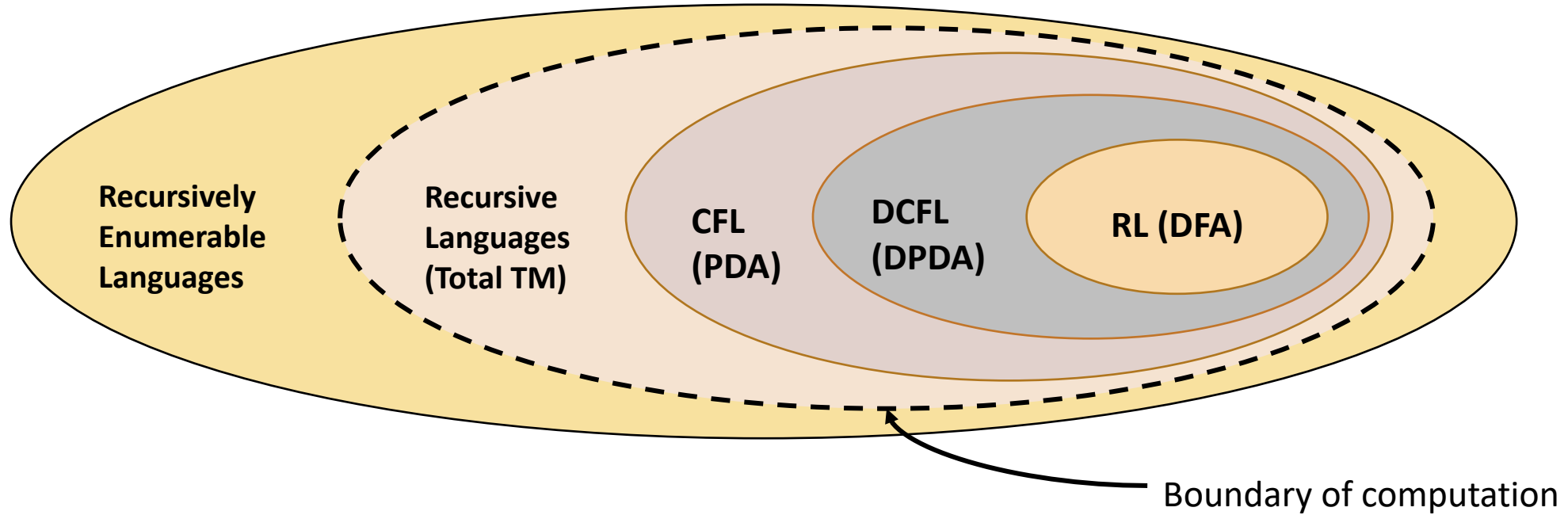**Recursively Enumerable Language/Turing Recognizable:** A language L is called Recursively Enumerable (RE) or Turing Recognizable if

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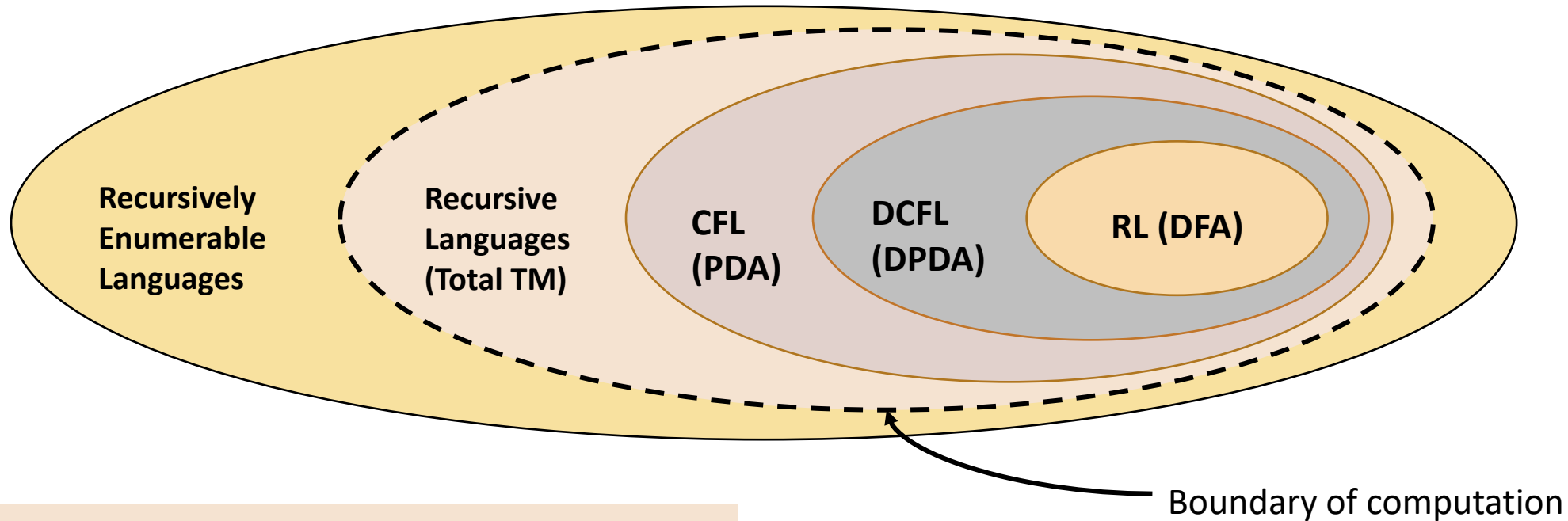$$\forall \omega \notin L, M(\omega) \textbf{ doesn't accept} \qquad \text{(rejects or runs infinitely)}$$

Co-Recursively Enumerable Language/co-Turing Recognizable (Co-RE/$\overline{RE}$/nRE): A language L is **Co-Recursively Enumerable (co-RE/$\overline{RE}$) or Co-Turing Recognizable** if

$$\forall \omega \in L, M(\omega) \textbf{ doesn't reject} \qquad \text{(accepts or loops)}$$
$$\forall \omega \notin L, M(\omega) \textbf{ rejects}$$

Hierarchy of Languages

Recursively Enumerable Languages

Recursive Languages (Total TM)

CFL (PDA)

DCFL (DPDA)

RL (DFA)

Boundary of computation

Hierarchy of Languages

Boundary of computation
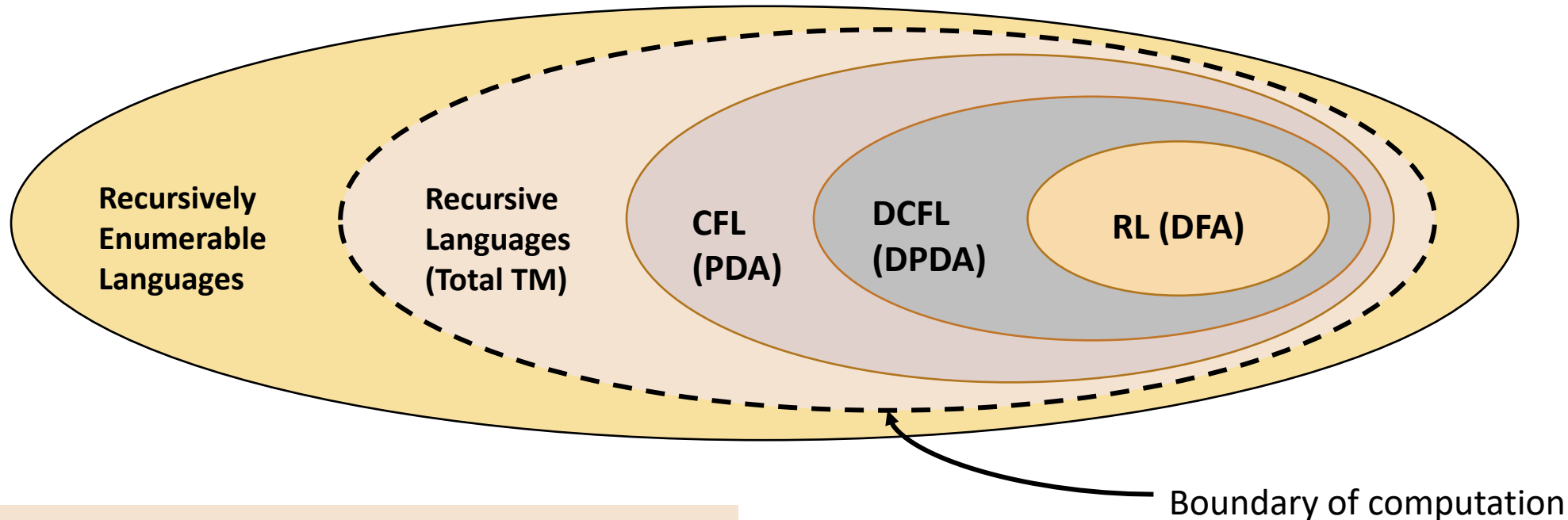
Any **problem that is not Recursive (not decidable) is called Undecidable**. There exists some input w for which the Turing Machine loops forever and hence, cannot **decide** whether or not w belongs to the Language.

We **cannot write Algorithms to decide the membership** of undecidable problems

Hierarchy of Languages

Boundary of computation

Any **problem that is not Recursive (not decidable) is called Undecidable**. There exists some input w for which the Turing Machine loops forever and hence, cannot **decide** whether or not w belongs to the Language.

We **cannot write Algorithms to decide the membership** of undecidable problems

There are **problems in RE which are not Recursive**. For such problems there exists some $\omega \notin L$, the TM **never halts but rather loops forever**. So, such problems are **undecidable**.

However, they can recognize any $\omega \in L$, so these undecidable problems are also called partially decidable.

Turing Machines: some definitions

Undecidable language: A language L is undecidable if it is not decidable/recursive. Any TM for L **will loop infinitely for some input** ω. You cannot write an algorithm to decide the membership of L.

Turing Machines: some definitions

Undecidable language: A language L is undecidable if it is not decidable/recursive. Any TM for L **will loop infinitely for some input ω**. You cannot write an algorithm to decide the membership of L.

Undecidable languages can be of two kinds:

- **Partially decidable Language:** A language L is partially decidable if L is **Recursively Enumerable as well as Undecidable (not recursive)** (TM accepts all the YES instances and loops infinitely for at least one NO instance), i.e.

$$\forall \omega \in L, M(\omega) \text{ \textbf{accepts}}$$
$$\forall \omega \notin L, M(\omega) \text{ \textbf{doesn't accept} but } \exists \text{ at least one instance where the program will loop forever.}$$

Turing Machines: some definitions

Undecidable language: A language L is undecidable if it is not decidable/recursive. Any TM for L **will loop infinitely for some input ω.** You cannot write an algorithm to decide the membership of L.

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- **Partially decidable Language:** A language L is partially decidable if L is **Recursively Enumerable as well as Undecidable (not recursive)** (TM accepts all the YES instances and loops infinitely for at least one NO instance), i.e.

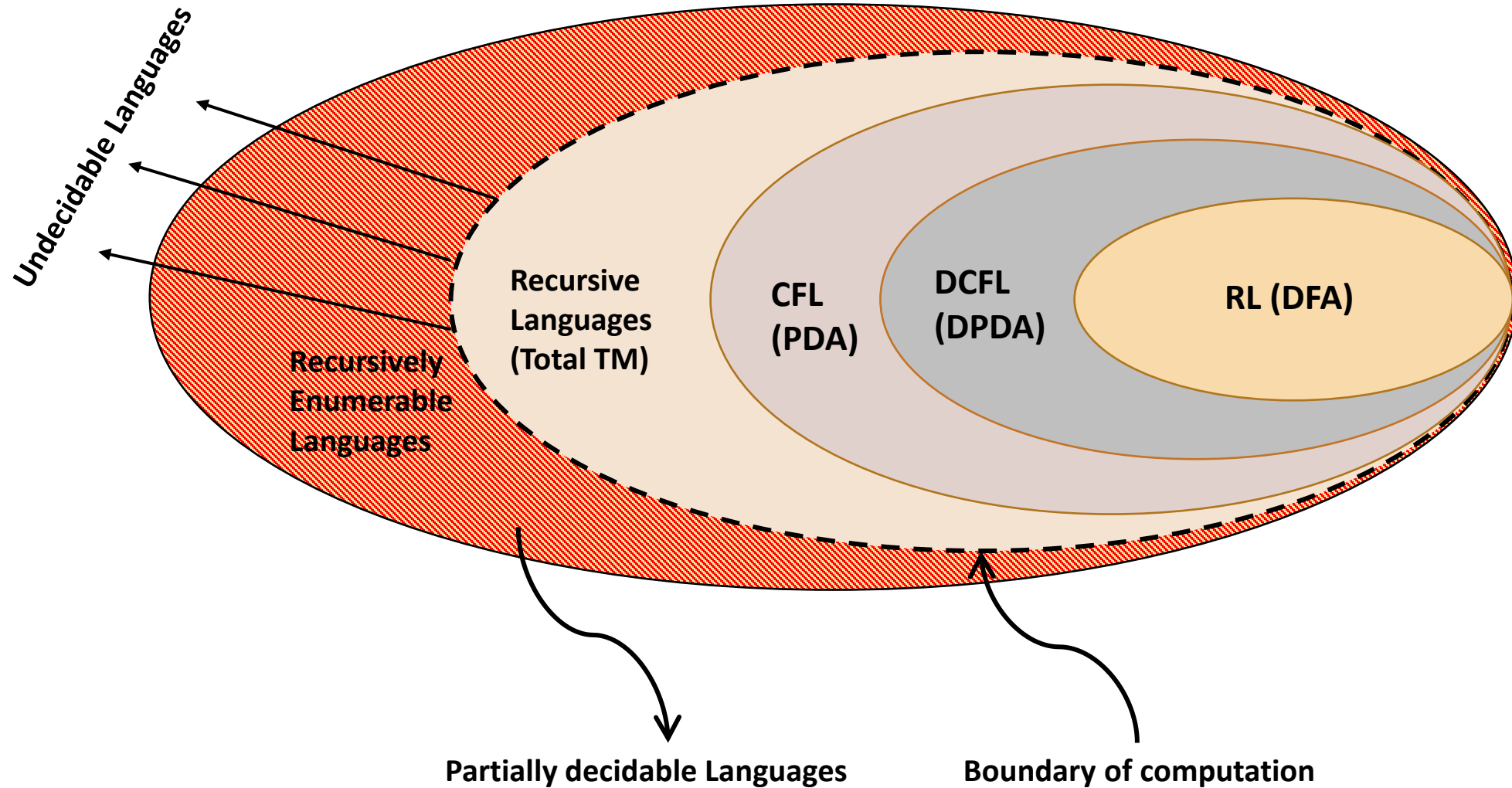 $\forall \omega \in L, M(\omega)$ **accepts**
    $\forall \omega \notin L, M(\omega)$ **doesn't accept** but $\exists$ at least one instance where the program will loop forever.

- **Completely undecidable language:** A language L is completely undecidable if $\boldsymbol{L}$ **is undecidable but not partially decidable** (TM loops infinitely for at least one YES instance), i.e.
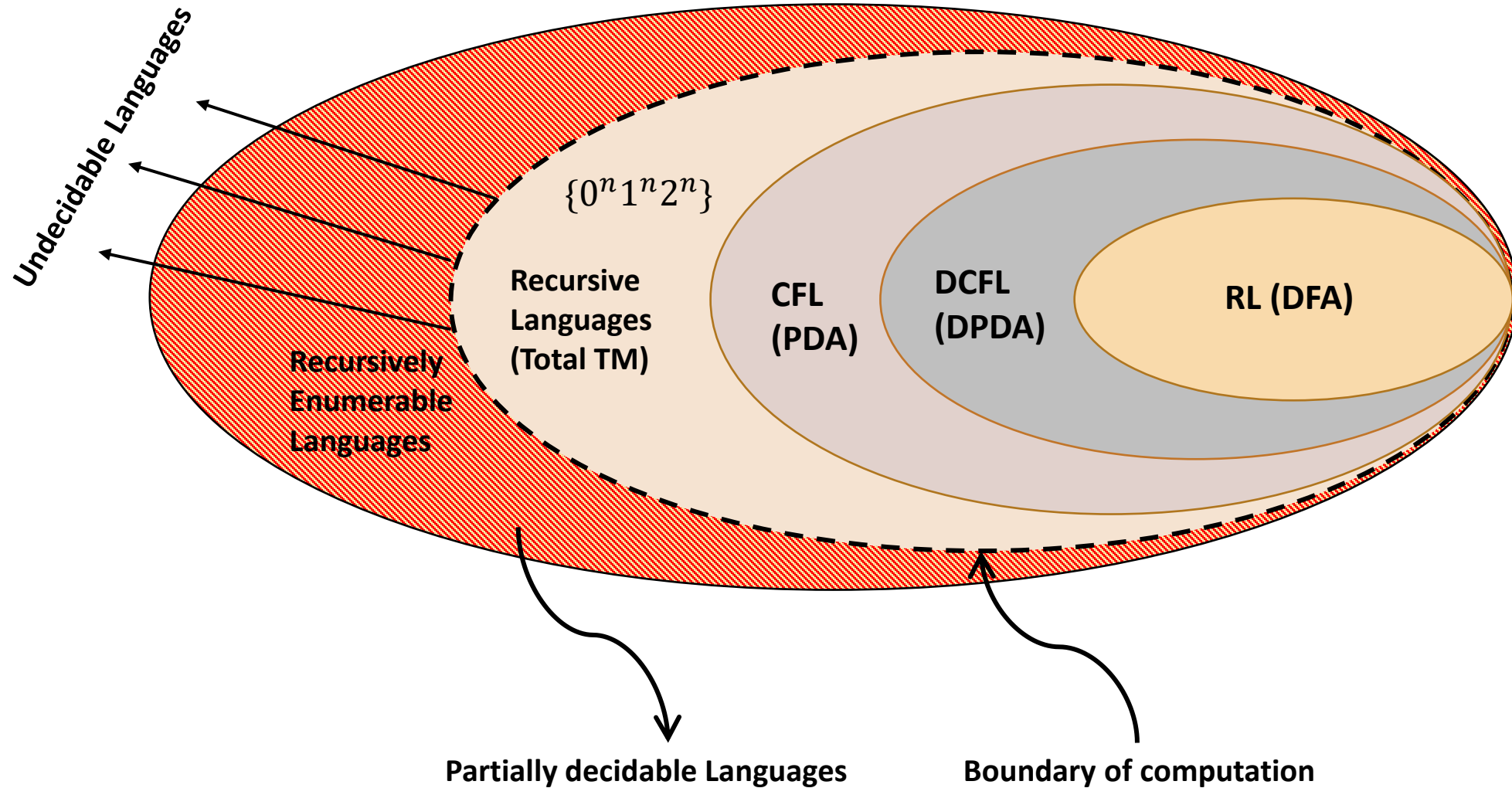
    $\forall \omega \in L, M(\omega)$ **doesn't accept** and $\exists$ at least one instance where the program will loop forever
    $\forall \omega \notin L, M(\omega)$ **rejects/loops forever**

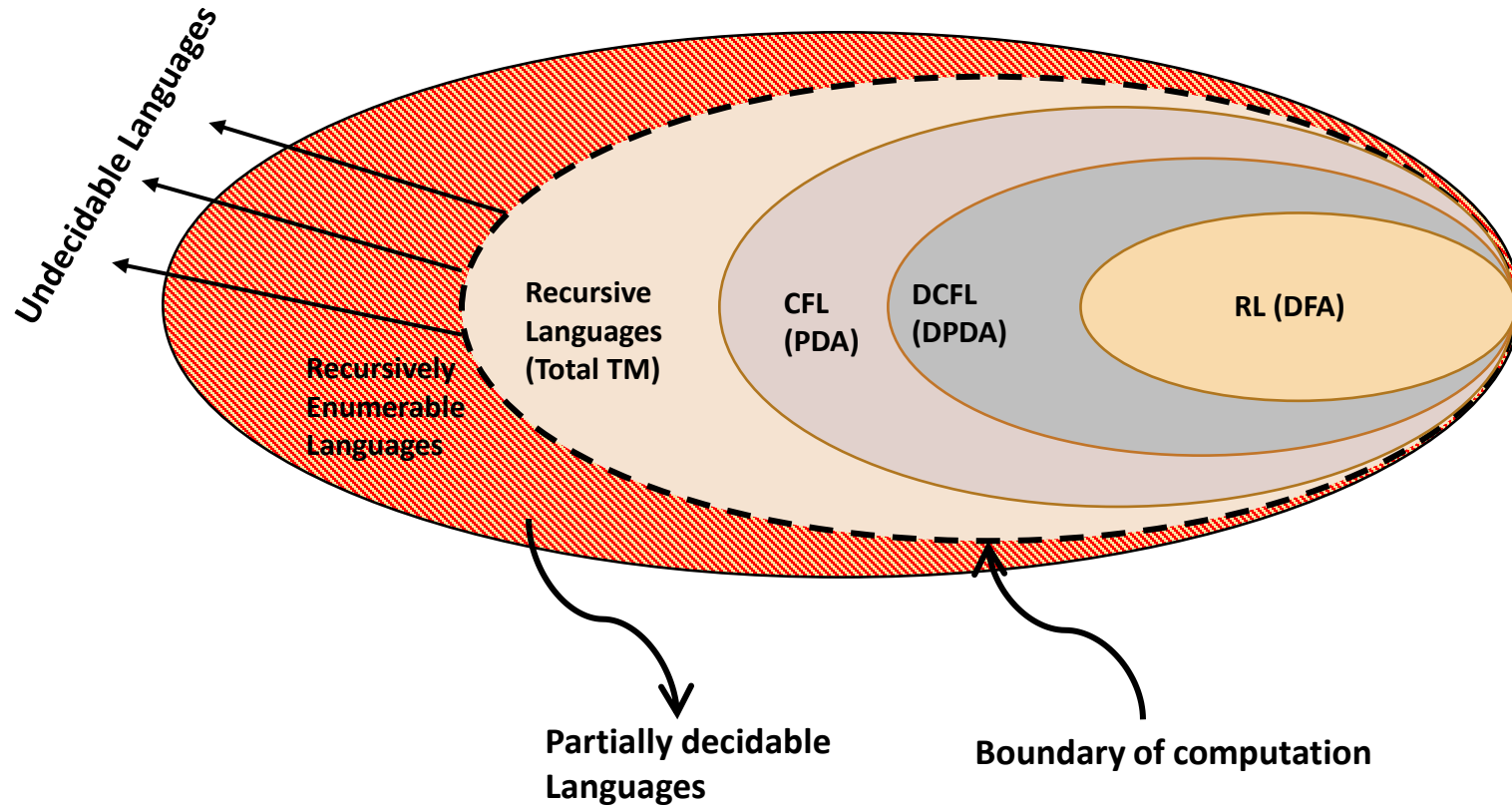# Hierarchy of Languages

Hierarchy of Languages

Undecidable Languages

$\{0^n 1^n 2^n\}$

Recursively Enumerable Languages

Recursive Languages (Total TM)

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Partially decidable Languages

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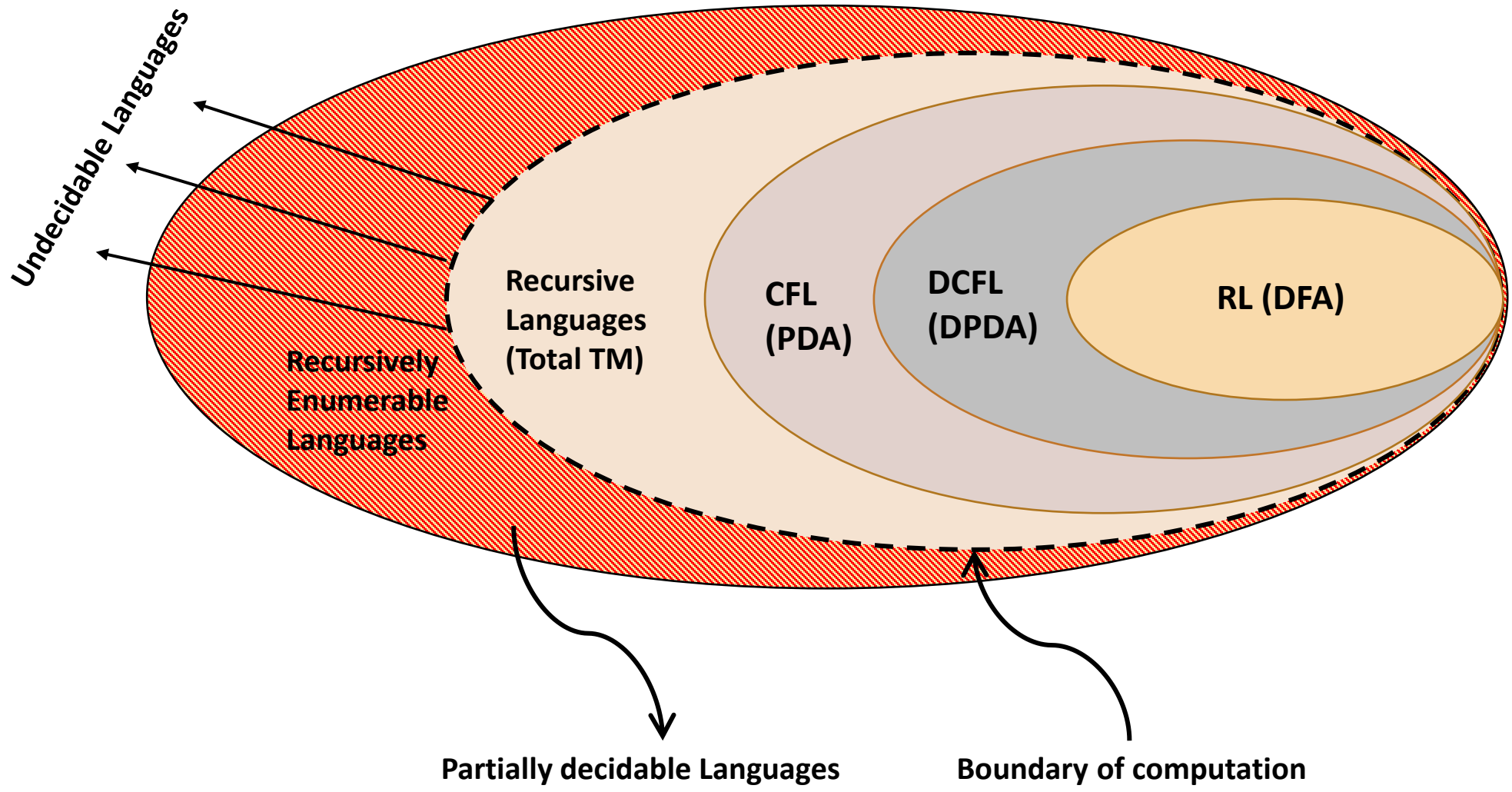
Boundary of computation

Any Language here is computable/decidable (you can write algorithms)

$(RL \equiv DFA) \subseteq (DCFL \equiv DPDA) \subseteq (CFL \equiv PDA) \subseteq (Recursive\ Lang \equiv Total\ TM)$ $\subseteq RE$

Undecidable Languages – Problems here are NOT computable

Boundary of computation

Encoding

The input to a TM are often strings/sequences of strings.

$M(w_1, w_2) =$ If w_1 is a substring of w_2, ACCEPT
Otherwise, REJECT.

Not just numbers, seemingly complicated objects such as a **graph, a DFA, a CFG and even a Turing Machine** itself can be encoded as a string – and hence can be an input to a TM.

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Consider this example:

$M(\langle M_1, w \rangle) =$ Run M_1 on input w.
If $M_1(w)$ accepts, ACCEPT
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- $\langle M_1 \rangle$ is the encoding of TM M_1 as a string.
- M simulates the run of M_1 on input w.
- Observe that M can accept a description of itself as input.

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- Encoding objects such as TMs as strings will help define a Universal Turing Machine U_{TM} which is a DTM that accepts as input the encoding of a DTM M and an input string w, and simulates $M(w)$.

- To prove that problems related to regular languages, CFLs are decidable/undecidable, we need to provide encodings DFAs/CFGs as inputs to a TM.

- How can we encode objects as strings? We will show a simple encoding of a DTM into a binary string.

Encoding a Turing Machine

- We will provide a simple mapping from a DTM to a **binary string**.

- Of course, this is not the only encoding.

- You can come up with your own encoding.

Encoding a Turing Machine

Recall that a DTM M is a 7-tuple $(Q, \ \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$.

- Let $Q = \{q_0, \cdots, q_{m-1}\}, \Sigma = \{0,1, \cdots, k - 1\}, \Gamma = \{0,1, \cdots, n - 1\}$. As $\Sigma \subseteq \Gamma, \text{k} < \text{n}$ and without loss of generality B corresponds to the last symbol $n - 1$ in Γ.

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- Any state $q_i \in Q$ can be encoded as a binary string, where

$$\langle q_0 \rangle = 0, \langle q_1 \rangle = 1, \langle q_2 \rangle = 10, \cdots$$

- Any symbol in Γ (or Σ) can be encoded as

$$\langle 0 \rangle = 0, \langle 1 \rangle = 1, \langle 2 \rangle = 10, \cdots$$

- The directions $\langle L \rangle = 0$ and $\langle R \rangle = 1$. So the transition function $\delta(q_i, a) = (q_j, b, L/R)$ is just the sequence

$$\langle \langle q_i \rangle, \langle a \rangle, \langle q_j \rangle, \langle b \rangle, \langle L/R \rangle \rangle$$

- All such transitions are listed in lexicographic order into

$$\langle \delta \rangle = \langle \langle \delta_0 \rangle, \langle \delta_1 \rangle, \langle \delta_2 \rangle, \cdots \rangle$$

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Following these encodings we can simply encode the DTM M as

$$\langle M \rangle = (\ \langle m \rangle, \langle k \rangle, \langle n \rangle, \langle \delta \rangle, 0/1, \langle q_{accept} \rangle, \langle q_{reject} \rangle\ )$$

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We are almost there but not quite. We have to find a way to combine this tuple of binary strings into one bigger binary string. Note that $\langle \delta \rangle$ itself is a tuple of binary strings.

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We can combine multiple sequences of binary strings into one as follows. Consider the sequence

$$\langle \langle a_1 \rangle, \langle a_2 \rangle, \cdots, \langle a_n \rangle \rangle = \langle \langle a_1 \rangle \# \langle a_2 \rangle \# \cdots \# \langle a_n \rangle \rangle,$$

where a_i are binary strings of finite length.

We claim that using the following map suffices

$$0 \mapsto 00$$
$$1 \mapsto 01$$
$$\# \mapsto 1$$

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Why does this work?

- For a 0 in an odd position, the symbol immediately following it corresponds to the symbol that was encoded

- We can identify the delimiter as the 1 that appears in an odd position.

Encoding a Turing Machine

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To recover a_1 and a_2 from the encoding:
- For any 0 in odd positions, the symbol that follow in the even positions, belong to a_1.
- If a 1 is obtained in an odd position, it corresponds to the delimiter/partition $\Rightarrow a_1$ has been recovered, now a_2 will be obtained similarly.
- This can be generalized to multiple tuples of binary strings which is what we need to encode M.

Encoding a Turing Machine

So for any DTM M, we obtain an encoding

$$\langle M \rangle = (\langle m \rangle, \langle k \rangle, \langle n \rangle, \langle \delta \rangle, 0/1, \langle q_{accept} \rangle, \langle q_{reject} \rangle)$$

such that $\langle M \rangle \in \{0,1\}^*$.

Every DTM corresponds to a binary string but the reverse is not necessarily true. Some binary strings are not valid descriptions of DTMs.

Can we make this a bijection?

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Can we make this a bijection?

- Lexicographically generate finite-length binary strings.
- For any length k, there are 2^k binary strings of length k
- So any TM that can be described by a k-length binary string will be within this finite set.
- Some of these will not correspond to a valid DTM. Ignore them.
- Relabel the first binary string that corresponds to a valid TM as 0.
- Relabel the second binary string that corresponds to a valid TM as 1.

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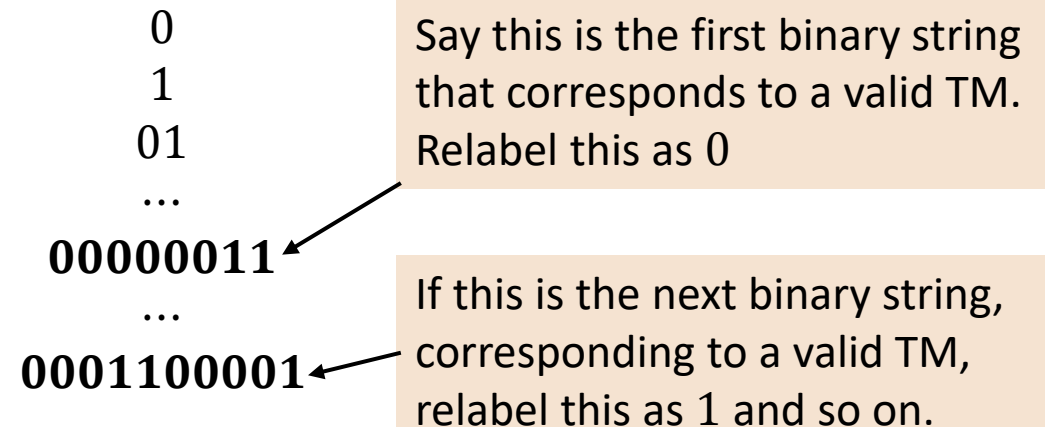# Encoding a Turing Machine

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Now we have a one-one mapping (bijective relationship) between the set of finite-length binary strings and DTMs.

Universal Turing Machines

Now that we have shown how to encode objects including Turing Machines as binary strings, we can now define **Universal Turing Machines** – or Turing Machines that simulate other Turing Machines.

Universal Turing Machine: A Universal Turing Machine, denoted as U_{TM} accepts as input (i) the encoding of a Turing Machine M, (ii) an input string w and **simulates M running on w**, i.e.

$$U_{TM}(\langle M, w \rangle) = \begin{cases} \textbf{ACCEPTS, if } M(w) \textbf{ accepts} \\ \textbf{REJECTS, if } M(w) \textbf{ rejects} \\ \textbf{LOOPS INFINITELY, if } M(w) \textbf{ loops infinitely} \end{cases}$$
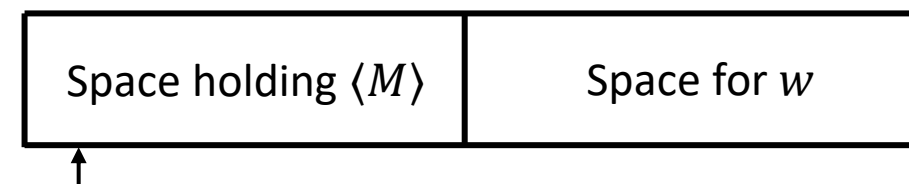
By the **Church-Turing thesis**, a U_{TM} can perform any computation on any feasible computational device.

So, in principle using U_{TM}, Turing Machines can answer questions about Turing Machines!

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such that $\langle M \rangle \in \{0,1\}^*$.

Space holding $\langle M \rangle$	Space for w

U_{TM} checks

- the space for w to determine the symbol currently being read

- And the space containing $\langle M \rangle$ for determining the transition function to be implemented

Some Decidable Languages

Much like Turing Machines, DFAs, NFAs, CFGs can also be encoded as binary strings. In fact, a bijection can be established between binary strings and these objects.

This is useful as it helps answer the decidability of Languages related to them.

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Examples: The following languages are decidable

- $A_{DFA} = \{\langle DFA, w \rangle | w \in L(DFA)\}$

$M =$ On input $\langle DFA, w \rangle$:

- Simulate the run of $\langle DFA \rangle$ on w.
- If w is accepted, output $ACCEPT$
- If w is rejected, output $REJECT$

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$M =$ On input $\langle DFA \rangle$:

- Mark the start state of $\langle DFA \rangle$
- Repeat until no new states are marked
  - Mark any state that has an incoming transition from a marked state
- If the final state is unmarked, $ACCEPT$, else $REJECT$

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- $A_{CFG} = \{\langle CFG, w \rangle | w \in L(CFG)\}$

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So, for any DTM M, we obtain an encoding

$$\langle M \rangle = (\langle m \rangle, \langle k \rangle, \langle n \rangle, \langle \delta \rangle, 0/1, \langle q_{accept} \rangle, \langle q_{reject} \rangle)$$

such that $\langle M \rangle \in \{0,1\}^*$.

Examples: The following languages are decidable

- $A_{DFA} = \{\langle DFA, w \rangle | w \in L(DFA)\}$
- $E_{DFA} = \{\langle DFA \rangle | L(DFA) = \Phi\}$
- $A_{CFG} = \{\langle CFG, w \rangle | w \in L(CFG)\}$

$M =$ On input $\langle CFG, w \rangle$:

- Convert $\langle CFG \rangle$ into CNF
- List all derivations of $2|w| - 1$ steps
- If any of these derivations yield w, $ACCEPT$, else $REJECT$

Some Decidable Languages

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Or, run the CYK algorithm

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Idea similar to DFAs: Check if the Start Variable leads to any terminal

Some Decidable Languages

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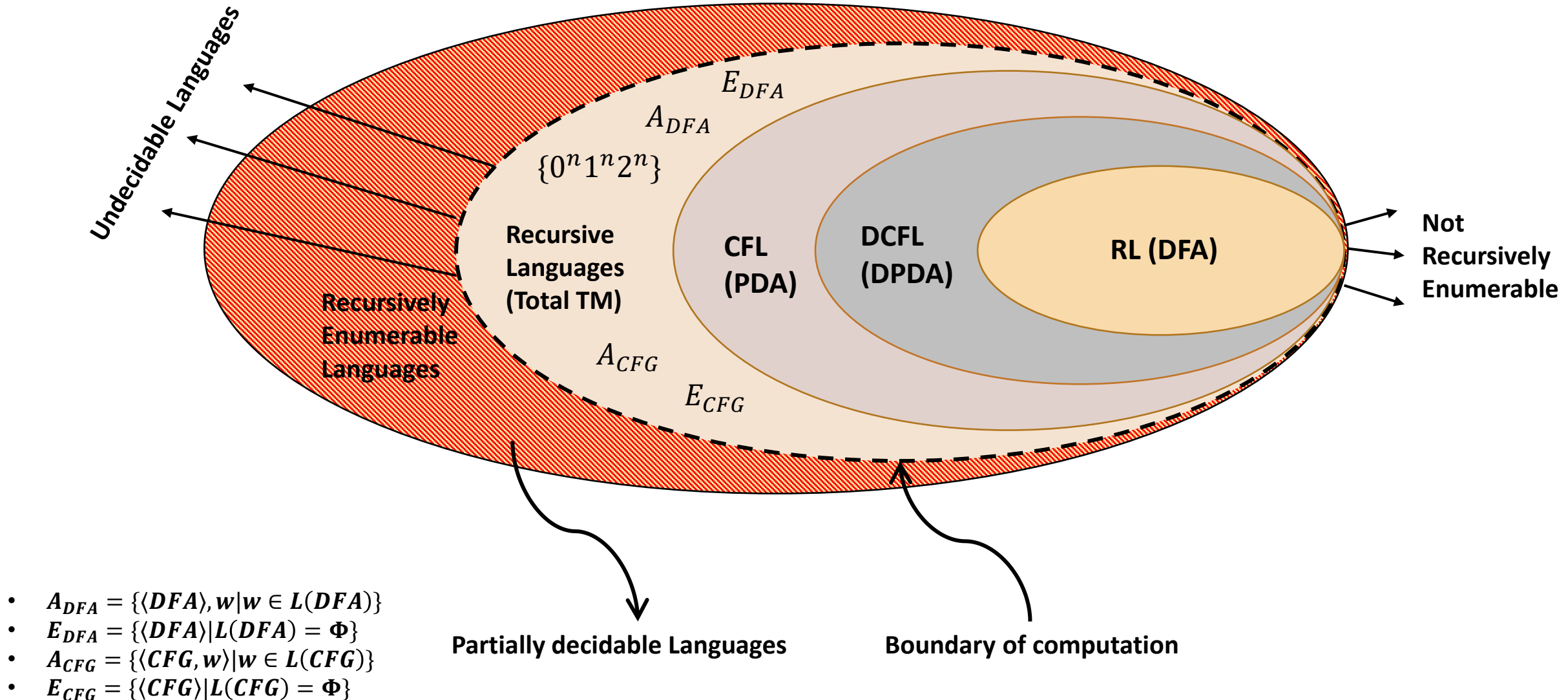such that $\langle M \rangle \in \{0,1\}^*$.

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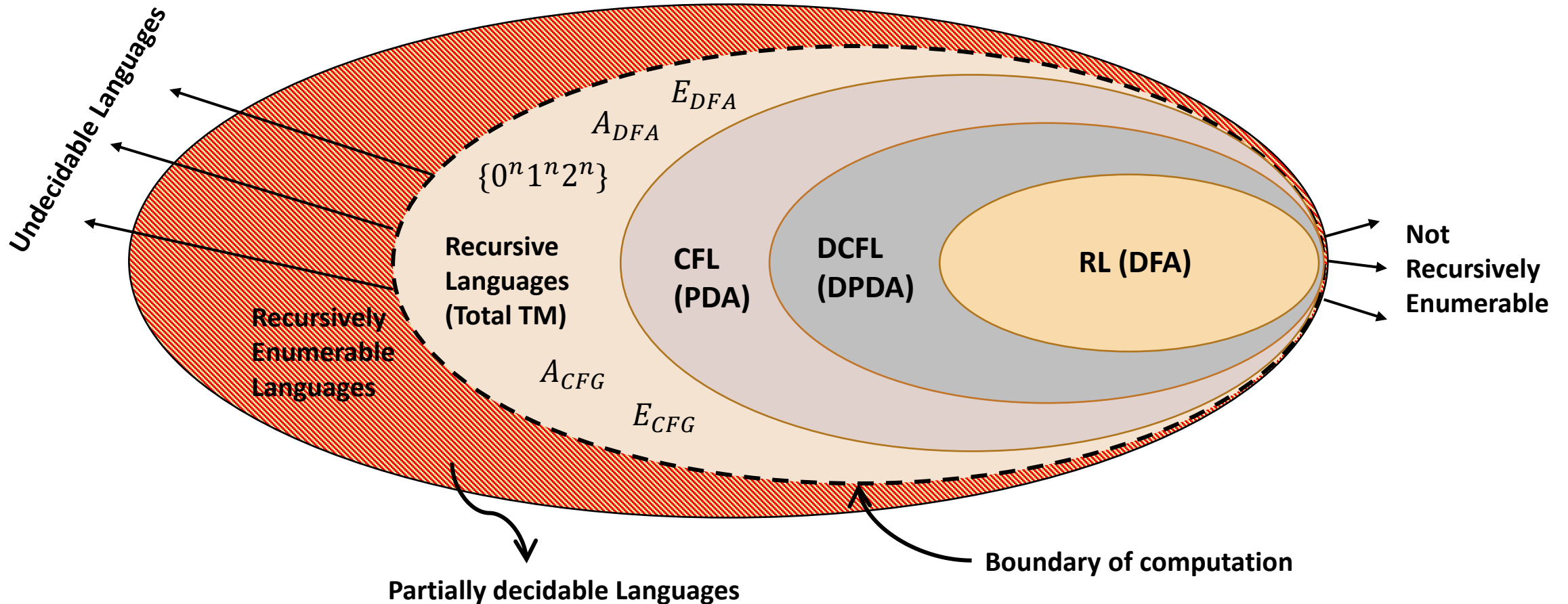- $\boldsymbol{E_{CFG} = \{\langle CFG \rangle | L(CFG) = \Phi\}}$

$M =$ On input $\langle CFG \rangle$:

- Mark all terminal symbols
- Repeat until no new variables are marked
  - Mark any V, s.t. $V \rightarrow X_1 X_2 \cdots X_l$.
- If S is unmarked, $ACCEPT$. Else $REJECT$

Some Decidable Languages

$A_{DFA} = \{\langle DFA \rangle, w | w \in L(DFA)\}$

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What about undecidable languages?

An undecidable problem

$A_{TM} = \{\langle M, w\rangle | M$ **accepts input** $w\}$. **Is** A_{TM} **decidable?**

A_{TM}: Does there exist a Total Turing Machine A that accepts as input a Turing Machine M and an input string w and outputs ACCEPT, if $M(w)$ accepts w and REJECT, if $M(w)$ does not accept w (rejects or loops forever)?

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Every (finite length) binary string is a TM and vice versa. So, the **input may have two copies of the same string (say w):**

- The first copy corresponds to the encoding of some TM M_w.
- The second copy is the input string $w = \langle M_w \rangle$.

$$A(w, w) = A(M_w, \langle M_w \rangle)$$

In this case, A simulates the run of TM M_w on the input string w, which is the binary encoding of M_w itself

An undecidable problem

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There can be inputs such as $A(\langle w, w \rangle)$

Let $w = \langle M_w \rangle$

$$A(\langle w, w \rangle) = \begin{cases} \text{ACCEPTS, if } M_w(\langle M_w \rangle) \text{ accepts} \\ \\ \text{REJECTS, if } M_w(\langle M_w \rangle) \text{ rejects or loops infinitely} \end{cases}$$

An undecidable problem

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- We will show that if such a Total TM A exists, we run into the following contradiction

  Using A, we can build a new Total TM for which there exists an instance for which the machine **both accepts and rejects**!

An undecidable problem

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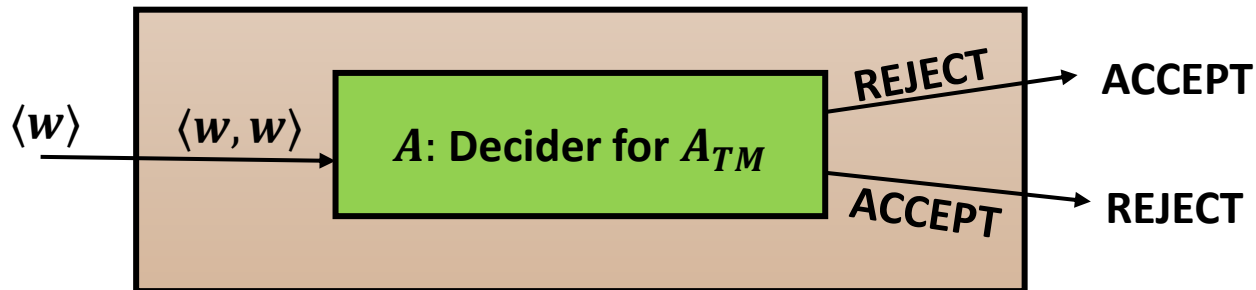
Proof: Let us assume that a Total Turing machine A exists. Then we can construct a special Total Turing Machine D that accepts an input w and uses A as a subroutine to simulate $A(\langle w, w \rangle)$ in the following way:

An undecidable problem

$$A(\langle M, w \rangle) = \begin{cases} \textbf{ACCEPTS, if } M(w) \textbf{ accepts} \\ \\ \textbf{REJECTS, if } M(w) \textbf{ rejects or loops infinitely} \end{cases}$$

Proof: Let us assume that a Total Turing machine A exists. Then we can construct a special Total Turing Machine D that accepts an input w and uses A as a subroutine to simulate $A(\langle w, w \rangle)$ in the following way:

$D(w) = \{$ **Run** $A(\langle w, w \rangle)$

  If $A(\langle w, w \rangle)$ **accepts,** D **outputs** $REJECT$
  If $A(\langle w, w \rangle)$ **rejects,** D **outputs** $ACCEPT$
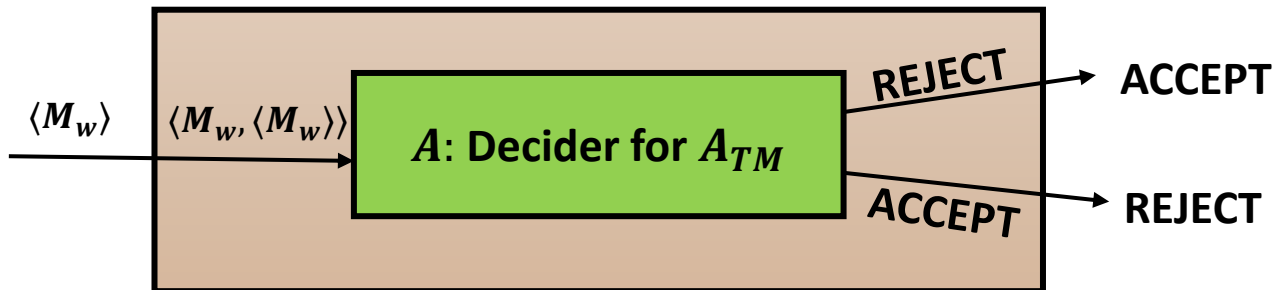
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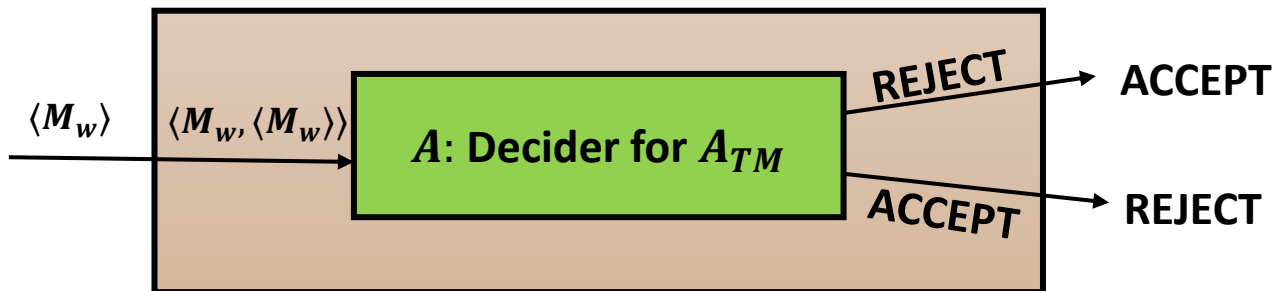

Let $w = \langle M_w \rangle$**, then**

$$D(\langle M_w \rangle) = \begin{cases} \textbf{ACCEPTS, if } M_w(\langle M_w \rangle) \textbf{ doesn't accept} \\ \\ \textbf{REJECTS, if } M_w(\langle M_w \rangle) \textbf{ accepts} \end{cases}$$

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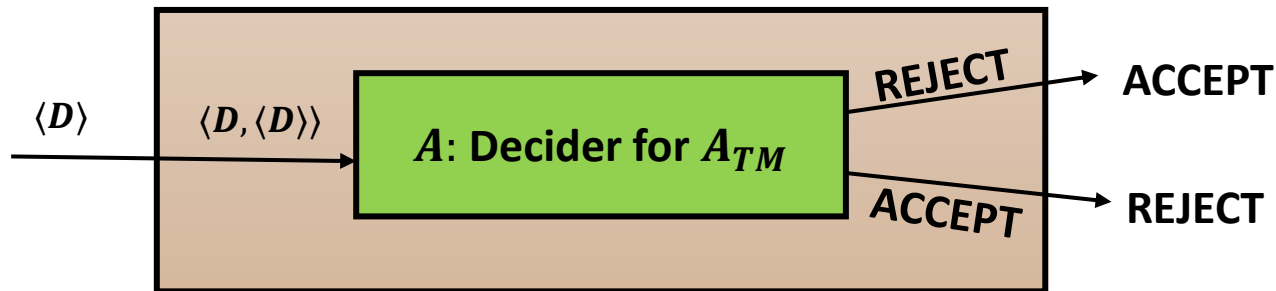
What happens when $w = \langle D\rangle$ i.e., $M_w = D$?

An undecidable problem

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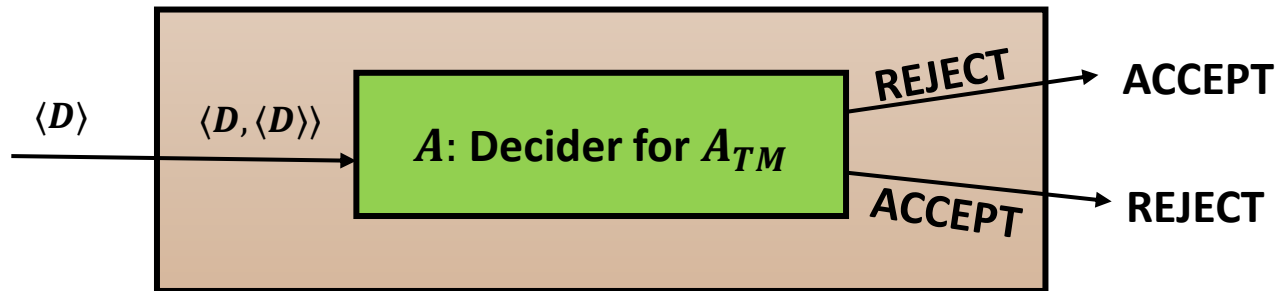

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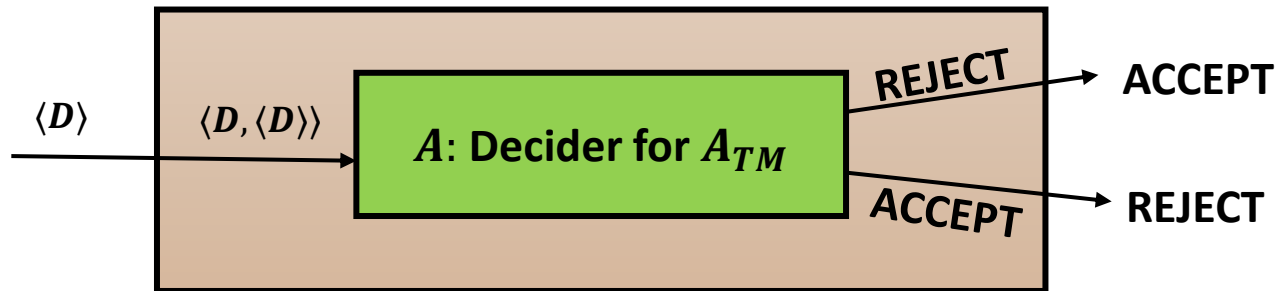

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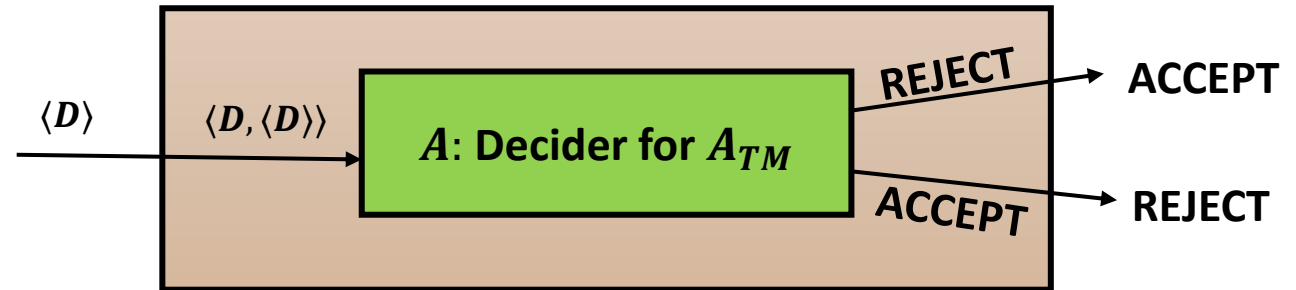
CONTRADICTION!

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CONTRADICTION!

- D cannot be a Total TM as it cannot decide input $\langle D \rangle$.
- If a total TM A existed, we could have constructed a total TM D.
- So, a total TM A cannot exist and hence A_{TM} **is not decidable.**

An undecidable problem

$A_{TM} = \{\langle M, w \rangle | M$ accepts input $w\}$. Is A_{TM} decidable? NO!

$$A(\langle M, w \rangle) = \begin{cases} \text{ACCEPTS, if } M(w) \text{ accepts} \\ \\ \text{REJECTS, if } M(w) \text{ rejects or loops infinitely} \end{cases}$$

Is $A_{TM} \in RE$?

An undecidable problem

$A_{TM} = \{\langle M, w \rangle | M \text{ accepts input } w\}$. **Is A_{TM} decidable? NO!**

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Of course, $A_{TM} \in RE$ as A halts whenever M accepts w and so

$U =$ **On input $\langle M, w \rangle$:**
- **Simulate M on input w**
- **If M accepts w, $ACCEPT$; if M rejects w, $REJECT$**

U recognizes A_{TM}

An undecidable problem

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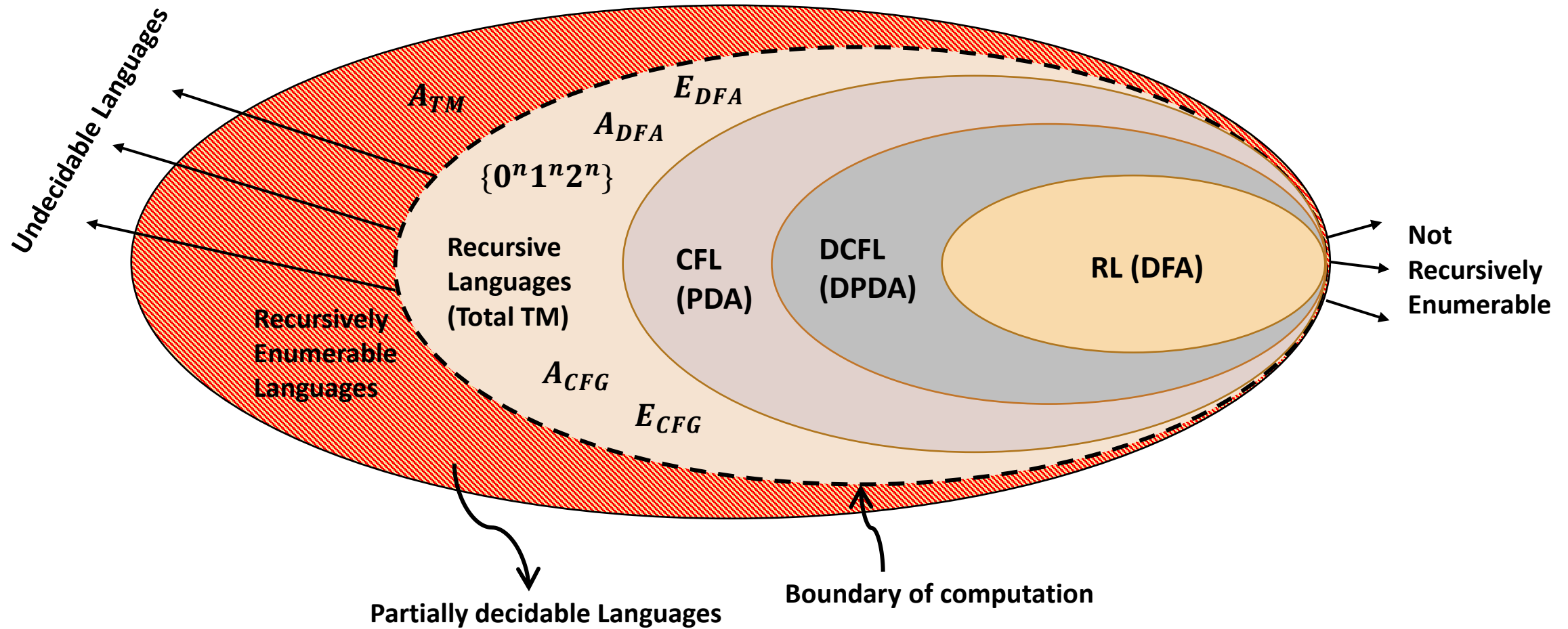
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U recognizes A_{TM}

- A_{TM} is undecidable
- $A_{TM} \in RE$ but not recursive
- A_{TM} is partially decidable

Undecidable Languages

A_{TM}

E_{DFA}

A_{DFA}

$\{0^n 1^n 2^n\}$

Recursively Enumerable Languages

Recursive Languages (Total TM)

A_{CFG}

E_{CFG}

CFL (PDA)

DCFL (DPDA)

RL (DFA)

Not Recursively Enumerable

Partially decidable Languages

Boundary of computation

Next Lecture:
- Halting Problem
- More on Recursive & RE languages
- Co-RE language

Thank You!