Assignment 1

(MA6.102) Probability and Random Processes, Monsoon 2024

Release date: 9 August 2024, Due date: 17 August 2024

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. Construct the smallest σ-Field that contains three events A, B, and C.

Problem 2. Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Show that $\frac{1}{6} \leq P(A \cap B) \leq \frac{1}{2}$. Give examples to demonstrate that the extreme values can be attained with equalities.

Problem 3. Consider the events $A_1, A_2, \ldots, A_n$ such that at least one of the events is certain to occur, but certainly no more than two occur (with respect to a probability law P). If $P(A_i) = p$, and $P(A_i \cap A_j) = q$, $i \neq j$, show that $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n}$.

Problem 4. (a) For a sequence of real number $(a_n)_{n\in\mathbb{N}}$, show that $|a_n - a| \to 0$ as $n \to \infty$ if and only if $a_n \to a$ as $n \to \infty$.
(b) If $P(A_n \triangle A) \to 0$ as $n \to \infty$, then $P(A_n) \to P(A)$ as $n \to \infty$ (note that $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Problem 5. Consider a biased coin with the probability of heads being p. Let p_n denote the probability that an even number of heads occurs in n independent tosses.
(a) Let E_n be the event that an even number of heads occurs in n tosses. Using the fundamental definition of conditional probability (i.e., $P(A|B) = \frac{P(A\cap B)}{P(B)}$), prove that

$$P(E_n \mid n^{\text{th}} \text{ toss results in tails}) = p_{n-1}.$$

(Note: This is not obvious.)
(b) Argue that $p_0 = 1$, and using the total probability theorem, derive the recurrence equation:

$$p_n = p(1 - p_{n-1}) + (1 - p)p_{n-1}.$$

(c) Solve the recurrence equation above.

Problem 6. Prove the following inequalities.

$$\sum_{i=1}^{n} P(A_i) - \sum_{1\leq i<j\leq n} P(A_i \cap A_j) \leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i\in[1:n]:i\neq r} P(A_i \cap A_r),$$

for a fixed $r \in [1 : n]$.
[Hint: Use mathematical induction.]

Problem 7. You are handed two envelopes, and you know that each contains a positive integer dollar amount and that the two amounts are different. The values of these two amounts are modeled as constants that are unknown. Without knowing what the amounts are, you select at random one of the two envelopes, and after looking at the amount inside, you may switch envelopes if you wish. A friend claims that the following strategy will increase above $\frac{1}{2}$ your probability of ending up with the envelope with the larger amount:

- Toss a fair coin repeatedly
- let X be equal to $\frac{1}{2}$ plus the number of tosses required to obtain heads for the first time
- switch if the amount in the envelope you selected is less than the value of X

Is your friend correct? Justify.

Problem 8. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type I), 0.4 against a quarter of the players (call them type II), and 0.5 against the remaining quarter of the players (call them type III). You play a game against a randomly chosen opponent. What is the probability that you played the game with an opponent of type II given that you win the game?

Problem 9. (a) Prove, or disprove via counterexample: If A is independent of B, and A is independent of C, then A is independent of $B \cup C$.

(b) For any two events A and B, show that $P(A|A \cup B) \geq P(A|B)$.

Problem 10. Let $\mathcal{C} = \{A_1, A_2, \ldots, A_n\}$ be a collection of mutually independent events. Prove that, for any $i \in [1:n]$, $(\mathcal{C} \setminus A_i) \cup \{A_i^c\}$ is also a collection of mutually independent events.