Lecture 4

(12 August 2024)

Recap.

Continuity of Probability

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)$$

Corollaries

(i) $A_1 \subseteq A_2 \subseteq \cdots \Rightarrow$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n)$$

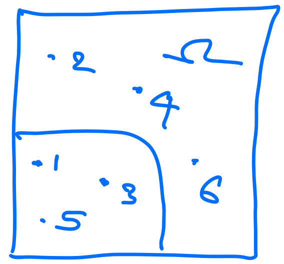
(ii) $B_1 \supseteq B_2 \supseteq \cdots \Rightarrow$

$$P\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} P(B_n)$$

(iii) $P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$

[An elaborate proof for n sets just to illustrate explicit connections with inclusion-exclusion]

Conditional Probability



Roll a die.

$$P(\text{outcome is } 1 \mid \text{outcome is even})$$

$$= 0$$

$$P(A|B) = 0 \quad \text{if} \quad A \cap B = \phi$$

$$P(\text{outcome} \in \{1, 2\} \mid \text{outcome is even})$$

$$= \frac{1}{3} = \frac{1/6}{3/6} = \frac{P(A \cap B)}{P(B)}$$

$P(A|B)$ is proportional to $P(A \cap B)$.
(for a fixed B)

$$P(A|B) = \alpha P(A \cap B)$$

$P(B|B) = 1$ intuitively

$$\Rightarrow \alpha = 1/P(B).$$

Definition. The conditional probability of an event A given an event B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0,$$

Exercise. Show that

$$P_B(A) \triangleq P(A|B) \quad \text{is}$$

a probability law w.r.t. Ω & $\mathcal{F}$, i.e., it satisfies all the three axioms.

Example. A family has two children. Assume that every birth results in a boy with probability $1/2$, independent of other births.

(i) What is the probability that both are boys, given that at least one is a boy?

(ii) What is the probability that both are boys given that the younger one is a boy?

$$\Omega = \{ B\underline{B} \;\; B\underline{G} \;\; G\underline{B} \;\; GG \}$$

$P(BB \mid$ at one is a box$)$

$$= P(\{BB\} \mid \{B\underline{G} \;\; G\underline{B} \;\; BB\})$$

$$= \frac{P(\{BB\} \cap \{B\underline{G} \;\; G\underline{B} \;\; BB\})}{P(\{B\underline{G} \;\; G\underline{B} \;\; BB\})}$$

$$= \frac{P(\{BB\})}{P(\{B\underline{G} \;\; G\underline{B} \;\; BB\})} = \frac{1/4}{3/4} = 1/3 .$$

$P(\{BB\} \mid$ younger is a box$)$

$$= \frac{P(\{BB\} \cap \{B\underline{B} \;\; BG\})}{P(\{B\underline{B} \;\; BG\})} = 1/2 .$$

Independence

Two events A and B are called independent events if

$$P(A \cap B) = P(A)P(B).$$

Interpretation: $P(A|B) = P(A)$ $P(B) > 0$.

Example. Two fair dice are rolled.

$A = \{\text{Sum is } 7\}$ $B = \{1^{st} \text{ roll is } 1\}$.

A and B are independent

$C = \{\text{sum is } 8\}$

C and B are not independent

Three events A_1, A_2, A_3 are (mutually) independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j) \quad i \neq j$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3).$$

Events $A_1, A_2, \ldots, A_n$ are (mutually) independent if

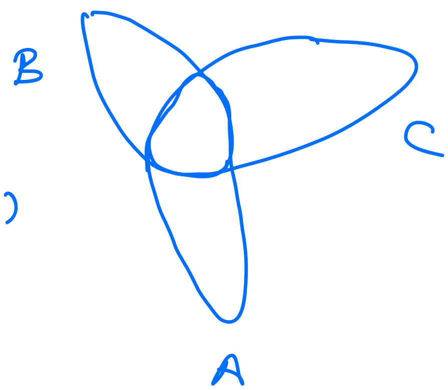
$$P\left(\bigcap_{i\in \mathcal{I}} A_i\right) = \prod_{i\in \mathcal{I}} P(A_i), \quad \mathcal{I} \subseteq [1:n].$$

$A_1, A_2, \ldots, A_n$ are pairwise independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j), \quad i \neq j.$$

Pairwise independence does not imply (mutual) independence.

<u>Example</u>.



$P(A) = P(B) = P(C)$

$= 1/5$

$P(A\cap B) = P(B\cap C) = P(C\cap A) = P(A\cap B\cap C) = p$

$p = 1/25$ pairwise indep but not mutually independent.

A collection of events $A_1, A_2, ---, A_n$ is called a partition of Ω if $\bigcup_{i=1}^{n} A_i = \Omega$ and $A_i \cap A_j = \phi$ $i \neq j$.

Total Probability Theorem

Let $\{A_1, A_2, ---, A_n\}$ be a partition of Ω such that $P(A_i) > 0$ $\forall i \in [1:n]$. Then for any arbitrary event B

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)\, P(A_i).$$

Proof. $P(B) = P(B \cap \Omega)$

$$= P\left(B \cap \bigcup_{i=1}^{n} A_i\right)$$

$$= P\left(\bigcap_{i=1}^{n} (B \cup A_i)\right)$$

$$= \sum_{i=1}^{n} P(B \cap A_i)$$

(by additivity)

$$= \sum_{i=1}^{n} P(B|A_i) P(A_i).$$

Bayes' Theorem

Let $\{A_1, A_2, ---, A_n\}$ be a partition of Ω such that $P(A_i) > 0 \ \forall i \in [1:n]$. Then for any arbitrary event B with $P(B) > 0$

$$P(A_i|B) = \frac{P(B|A_i) P(A_i)}{\sum_{j=1}^{n} P(B|A_j) P(A_j)},$$

Proof. $P(A_i|B) = P(A_i \cap B)/P(B)$

$$= P(B|A_i) P(A_i) \Big/ \sum_{j=1}^{n} P(B|A_j) P(A_j)$$

by total probability theorem.

Example. In answering a multiple-choice question, a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $1-p$ be the probability that the student guesses. Assume that the student who guesses at the answer will be correct with probability $1/m$, where m = no. of multiple-choice options. Find

$$P(\text{student knew the answer} \mid \text{they answered correctly}).$$

Proof. $A_1 = \{\text{student knew the answer}\}$

$A_2 = \{\text{student guesses}\}$

$B = \{\text{student answered question correctly}\}$

$$P(A_1|B) = \frac{P(B|A_2)\,P(A_1)}{P(B)}$$

$$= \frac{P(B|A_1)\,P(A_1)}{P(B|A_1)\,P(A_1) + P(B|A_2)\,P(A_2)}$$

$$= \frac{1 \cdot p}{1 \cdot p + \frac{1}{m} \cdot (1-p)} = \frac{mp}{mp + 1 - p}.$$