Date: / /

PRP - Module 1

~~Lec3~~

- $P(\bigcup_{i=1}^{\infty} A_i) = \lim_{n \to \infty} P(\bigcup_{i=1}^{n} A_i)$ - (Eq 4, lec 3)

- Probability Law
  - → Non negativity → $P(E) \geq 0 \;\; \forall E \in \mathcal{F}$
  - → Normalisation → $P(\Omega) = 1$
  - → Additivity → $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for disjoint A_i

- Total Probability Theorem

for any B $P(B) = \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$ for a partition defined as $\{A_1, A_2 \ldots A_n\}$: $P(A_i) > 0$ ~~the for any B~~

- Bayes' Theorem (Lec 4 - Eq 8)

for $P(B) > 0$ & A_i as defined above,

$$P(A_i|B) = \frac{P(B|A_i)\, P(A_i)}{\sum_{j=1}^{n} P(B|A_j)\, P(A_j)}$$

- Multiplication Rule

$$P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i | \bigcap_{j=1}^{i-1} A_j)$$

- Conditional Independence (Lec 5, Eq 13)

iff $P(A \cap B | C) = P(A|C) \cdot P(B|C)$

$\iff P(A | B \cap C) = P(A|C)$

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Extra

* Continuity

$\lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$

- i) $A_1 \subseteq A_2 \subseteq \cdots A_n$
  - $\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$
- ii) $B_1 \supseteq B_2 \supseteq \cdots B_n$
  - $\lim_{n\to\infty} P(B_n) = P\left(\bigcap_{i=1}^{\infty} B_i\right)$

Common R.Vs

- Bernoulli

  $E[X] = P$, $Var(X) = P(1-P)$

- Geometric $\rightarrow P(1-P)^{k-1}$, $k = 1, 2 \ldots$

  $E[X] = 1/P$ $var(X) = (1-P)/P^2$

- Binomial $\rightarrow P_X(k) = \binom{n}{k} P^k (1-P)^{n-k}$; $k = 0, 1 \ldots n$

  $E[X] = nP$, $Var(X) = np(1-p)$

- Poisson $\rightarrow P_X(k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$; $k = 0, 1 \ldots$

- Gaussian $\rightarrow f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\dfrac{(x-\mu)^2}{2\sigma^2}\right)$

  $E[X] = \mu$, $Var(X) = \sigma^2$

Valid σ field $\rightarrow$ ① $\Omega \in F$

② $A \in F \Rightarrow A^c \in F$

③ $A_1, A_2 \ldots A_n \in F \Rightarrow \bigcup_{i=1}^{n} A_i \in F$

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PRP- Module 2

- Random Variable ($X : \Omega \to \mathbb{R}$)

  $X : \{\omega : X(\omega) \leq x\} \in \mathcal{F}$

- Theorem: Given Ω, $\mathcal{F}$ & $X : \Omega \to \mathbb{R}$, following holds.
  - $X^{-1}((-\infty, x)) \in \mathcal{F}$
  - $X^{-1}([x_1, x_2]) \in \mathcal{F}$
  - $X^{-1}(\{x\}) \in \mathcal{F}$
  - $X^{-1}((x_1, x_2)) \in \mathcal{F}$

- CDF Properities: (Lec 6, Pg 8)
  - $x < y \Rightarrow F_X(x) \leq F_X(y)$
  - $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$
  - F_X is right cont.
  - $P(X > x) = 1 - F_X(x)$
  - $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$
  - $P(X = x) = F_X(x) - \lim_{\varepsilon \to 0^+} F_X(x + \varepsilon)$

- Let $Y = g(X)$. Then

  $P_Y(y) = \sum_{x \in X : g(x) = y} P_X(x)$

- Prove
  - $E[g(X)] = \sum_{x \in X} g(x) P_X(x)$ (Pg 10, L 7)
  - $E[g(x, y)] = \sum_{x, y} g(x, y) P_{X,Y}(x, y)$ (Pg 8, L 8)

Date: / /

- Poisson RV → $P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$ $(\lambda > 0)$

  ↑ limiting case of binomial R.V.

- For binomial $RV(n,p)$, if $n \to \infty$ & $np = \lambda$ (constant) then
  $\lim_{n\to\infty} P_Y(k) = e^{-\lambda}\cdot\lambda^k/k!$
- Poisson → $E[X] = Var(X) = \lambda$

- Independence of R.V:
  X & Y are ind if $P_{XY}(x,y) = P_X(x)\cdot P_Y(y)\ \forall x,y$ =>
  iff $\{X = x\}$ & $\{Y = y\}$ are independent $\forall x, y$.

- If X & Y are INDEPENDENT & DISCRETE,
  then $E[XY] = E[X]E[Y]$
  $= \sum_{xy} x\cdot y\, P_{XY}(x,y)$ ($\boxed{\text{Pg 3 end L-9}}$)
- n RVs are IND. iff $P_{X_1X_2\cdots X_n}(x_1,x_2,\cdots x_n) = \prod_{i=1}^{n} P_{X_i}(x_i)$

- $E[aX + b] = aE[X] + b$
- $Var[aX + b] = a^2 Var[X]$
- $Var[X+Y] = Var[X] + Var[Y]$ if X, Y are IND.
- if $Z = X + Y$ & X & Y are IND, then
  $P_Z(z) = \sum_y P_X(z-y)P_Y(y) = P_X * P_Y$
  → (L9, Pg 10)

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Conditioning of RVS.

- Con. on an event A

$$P_{X|A}(x) = P(\{X = x\}/A) = \frac{P(\{\omega : X(\omega) = x\} \cap A)}{P(A)}$$

$\left(\sum_x \quad " \quad = 1\right)$

- If $\{A_1 \ldots A_n\}$ form a partition with $P(A_i) > 0 \; \forall i$, then $P_X(x) = \sum_{i=1}^{n} P(A_i) \, P_{X|A_i}(x)$

- Con. one RV on another.

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)} \text{ if } P_Y(y) > 0$$

$\sum_{x,y} P_{X,Y}(x,y) = 1$

$$= \frac{P(X = x \wedge Y = y)}{P(Y = y)}$$

Conditional Expectations :

- $E[X|Y=y] = \sum_x x P_{X|Y}(x|y)$
- $E[X|A] = \sum_x x P_{X|A}(x)$
- $E[g(x)|A] = \sum_x g(x) P_{X/A}(x)$ → Prove

- Total Expectation Theorem :

Prove $E[X] = \sum_{i=1} P(A_i) E[X|A_i]$ $\{A_i\}$ form a partition, > 0

(Pg 3, L10)

on putting $\{Y = y\}$, $y \in Y$ as the partition,

$\rightarrow E[X] = \sum_y P_Y(y) E[X/Y=y]$

$\Phi(Y) \equiv E[X|Y]$ <> $\Phi(y) = E[X/Y=y]$

$E[\Phi(Y)] = E[X]$ ← Prove like

R.V => $E[E[X|Y]] = E[X]$ ← LAW OF ITERATED EXPECTATIONS

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Conditional Independence

- X & Y are conditionally IND given A if

$$P_{XY/A}(x,y) = P_{X/A}(x) \cdot P_{Y/A}(y) \quad \forall x,y$$

$$\parallel$$

$$\frac{P(X=x \cap Y=y \cap A)}{P(A)}$$

- Conditional Variance $\leftarrow$ R.V

$$Var(X|Y=y) = \varphi(y) = E[X^2/Y=y] - (E[X/Y=y])^2$$

- Law of Total Variance

$\hookrightarrow$ $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

$$\boxed{\int_0^\infty x P(x) = \int_0^\infty P(X>x)}$$

$$P_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu, \quad Var[X] = \sigma^2$$

$$E[X] = \int_0^\infty P(Y>y)\,dy = 1 - F_Y(y)$$

- Uncorr $\Rightarrow$ IND
- " if $E[(X-E[X])(Y-E[Y])] = 0$

MATRIKAS

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L3 Module 3 - Continuous R.V

- $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$, $x \in R$ & $f(x) = F_X'(x)$
- $f_X(x) = \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}$

Proof → $\int_R f_X(x)dx = 1$ & $P(X=x) = 0 \ \forall \ x \in R$

↳ $E[Y] = \int_{-\infty}^{\infty} P(Y>y)\,dy$

Eg

① uniform R.V → $f_X(x) = \begin{cases} 1/b-a & ; a \le x \le b \\ 0 & o/w \end{cases}$

$E[X] = \frac{a+b}{2}$, $Var(X) = \frac{(a-b)^2}{12}$

L4

② Exponential R.V ⇒ $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \ge 0 \\ o/w \end{cases}$

↳ $E[X] = 1/\lambda$, $Var(X) = 1/\lambda^2$

- Relation b/w geometric (discrete) & exp. (cont):

$\lim_{\delta \to 0} 1 - e^{-\lambda \lfloor \frac{x}{\delta} \rfloor \delta} = 1 - e^{-\lambda x}$; $x \ge 0$

- Gaussian Random Variable:

$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$; $\mu \in R$ & $\sigma \in [0,\infty)$

$E[X] = \mu$, $Var(X) = \sigma^2$

$\Phi(x) = F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

STANDARD NORMAL R.V for $\mu = 0, \sigma = 1$ (ie Mean = 0, Var = 1)

- For any (μ, σ) if $a > 0$ & $b \in R$ then: $Y = aX + b$

⇒ $E[Y] = a\mu + b$, $Var(Y) = \sigma^2 a^2$

MATRIKAS

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L13

$F_{X,Y}(x,y) = P(X \le x, Y \le y)$ [X & Y can be both C & D]

- 2 RV, X & Y on $(\Omega, \mathcal{F}, P)$ are
- Jointly continuous iff $F_{X,Y}(x,y) = \int_{v=-\infty}^{y}\int_{u=-\infty}^{x} f_{XY}(u,v)\,du\,dv$

$\Rightarrow f_{XY}(x,y) = \frac{\partial^2 (F_{XY}(x,y))}{\partial x \partial y}$

$\Downarrow$

Probability per unit area in the vicinity of (x,y)

- Joint cont $\Rightarrow$ Individually cont.

Conditioning RV on event

$F_{X/A}(x) = \frac{P(X \le x \cap X \in B)}{P(A)}$ [let $A = \{X \in B\}$, $B \subseteq \mathbb{R}$]

TRASH

$= \frac{\int_{*} f_X(u)\,du}{P(X \in B)}$ $(* = (-\infty, x) \cap B)$

$f_{X/A} = \begin{cases} f_X(x)/P(X \in B) & \text{if } x \in B \\ 0 & \text{ow} \end{cases}$

$\Rightarrow f_X = \sum_{i=1}^{n} P(A_i) f_{X/A_i}(x)$ (A_i is partition)

↳ Proof (use $F_X(x)$ diff Pg 13)

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L16

$$F_{X/A}(x) = P(X \le x \mid A) = \frac{P(X \le x \cap A)}{P(A)} = \int_{-\infty}^{x} f_{X/A}(u)\,du$$

- Conditioning One RV on another :

($\because P(Y=y)=0$ for cont }, consider :

$$F_{X/y<Y\le y+\Delta y}(x) = P(\underset{A}{X \le x} \mid \underset{B}{y < Y \le y+\Delta y}) = \frac{P(A,B)}{P(B)}$$

$$= \frac{F_{XY}(x, y+\Delta y) - F_{XY}(x,y)}{F_Y(y+\Delta y) - F_Y(y)}$$

but differentiate N & D by Δy

→ Def → $f_{X/Y}(x/y) = \lim_{\Delta y \to 0} f_{X/\{y<Y\le y+\Delta y\}}(x)$

$$\Rightarrow \boxed{f_{X/Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)}}$$

All $E[X/_{A\text{ or }Y}]$ formulae are same

- Independence

↳ if $f_{XY}(x,y) = f_X(x) \cdot f_Y(y) \; \forall x,y$

→ Prove for C & D both : Ind iff $F_{XY}(x,y) = F_X(x) \cdot F_Y(y) \; \forall x, y$

→ (Prove) : $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ for IND X & Y

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L17

- Bayes' Rule

$f_{XY}(x,y) = f_X(x)\, f_{Y/X}(y/x) = f_Y(y)\, f_{X/Y}(x,y)$

$\Rightarrow f_{X/Y}(x/y) = \dfrac{f_X(x) \cdot f_{Y/X}(y/x)}{\int_{t=-\infty}^{\infty} f_X(t)\, f_{Y/X}(y/t)\,dt}$

- $P(A|Y=y) = \dfrac{P(A)\, f_{Y/A}(y)}{f_Y(y)}$ → Prove → Pg 3

$= \dfrac{P(A)\, f_{Y/A}(y)}{P(A) f_{Y/A}(y) + P(A^c) f_{Y/A^c}(y)}$

→ Given X → Discrete RV, Y → cont

- $P(X=x|Y=y) = \dfrac{P_X(x)\, f_{Y/X}(y/x)}{\sum_{x'=x} P_X(x')\, f_{Y/X}(y/x')}$

MAP Rule

$\hat{x}_{MAP}(y) = \underset{x \in X}{\arg\max} \left(P_{X/Y}(x/y) \right) \cancel{P(y)}$

$\hat{x}_{MLE}(y) = \text{''} \; (P_{X/Y}(x/y))$

$= \left(\underset{x}{\arg\max}\; f_{Y/X}(y/x) \cdot P_X(x) \right) \cdot \dfrac{1}{f_Y(y)}$

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L18

PDF formula for a monotonic function of a continuous R.V.

- Function of R.Vs

if $Y = g(X)$ then $\boxed{f_Y(y) = f_X(g^{-1}(y))\Big|_{(Y=y)} \left| \frac{d}{dy}(g^{-1}(y)) \right|_{(Y=y)}}$

- eg $Y = aX + b$ $(= g(X))$ $\left[\Rightarrow X = \left(\frac{Y-b}{a}\right) = g^{-1}(Y) \right]$

Note: Procedure for the above :

→ Find $F_Y(y) = P(Y \le y) = P(g(X) \le y)$ in terms of F_X

→ Differentiate it to find pdf

- Let X & $Y = g(X) \Rightarrow$ cont. $R = \{I_n\}$ such that (partition of R, intervals $I_1, I_2 \ldots$)

$g(X)$ is strictly monotone & differentiable on each $I_i \; \forall \, i \in [1:n]$.

$\hookrightarrow f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|\big|_{x=x_i}}$,

where x_i are roots to $g(x) = y$

- $Z = X + Y$

$f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$

convolution

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L19 $Z = X + Y$ — cont, ind

$F_Z(z) = P(X+Y \le z) = P((X,Y) \in B_z)$

(due to joint cont) $B_z = \{(x,y) : x+y \le z\}$

$= \iint_{(x,y)\in B_z} f_{X,Y}(x,y)\,dx\,dy$

$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_X(x)\, f_Y(y)\,dx\,dy$

$= \int_x f_X(x)\, F_Y(z-x)\,dx$ | $f_Z(z) = \frac{d\,F_Z(z)}{dz}$

$= \int_x f_X(x)\, f_Y(z-x)\,dx$

$= \int_y f_Y(y)\, f_{\cancel{Z}\,X}(z-y)\,dy$

Eg

$X, Y \sim$ uniform $[0,1]$; $X \perp Y$

$Z = X/Y$

A) $f_{XY}(x,y) = \begin{cases} 1 ; & 0 \le x, y \le 1 \\ 0 & o/w \end{cases}$

$P(Z \le t) = P(Y \le Xt) = P((X,Y) \in B_t)$

$B_t = \{(x,y) : y \le xt\}$

Case 1 → $t < 1$

$F_Z(t) = \int_{B_t} 1\,dx\,dy$

$= \frac{t}{2}$

Case 2 → $t \ge 1$

$F_Z \to 1 - \frac{1}{2t}$

MATRIKAS

• Two fn of 2 RVs

$(X, Y) \sim f_{XY}(x, y)$

$Z = g_1(X, Y)$

$W = g_2(X, Y)$

$f_{Z,W} = ?$

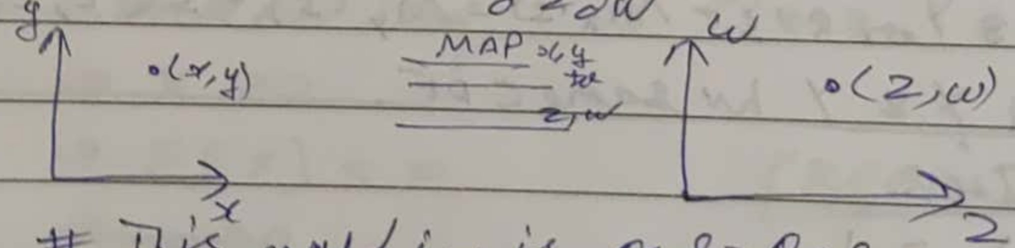
steps ① $F_{ZW}(z, w) = P(Z \le z, W \le w)$

$= P(g_1(X,Y) \le z, g_2(X,Y) \le w)$

then $\cancel{\iint_{B_{zw}} f_{ZW}(x,y)\,dx\,dy} = P((X,Y) \in B_{zw})$

$B_{zw} = \{(x,y) : g_1(x,y) \le z,\ g_2(x,y) \le w\}$

② $f_{ZW}(z, w) = \dfrac{\partial^2 F_{ZW}(z,w)}{\partial z \partial w}$

③ 

This mapping is one-one

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L20

Moment Generating Functions (MGFs)

→ MGF of X is a function $M_X : \mathbb{R} \to [0, \infty)$:

$$M_X(s) = E[e^{sX}]$$

→ Domain/ROC → $D_X = \{ s \in \mathbb{R} : M_X(s) < \infty \}$

→ Discrete → $M_X(s) = \sum_x e^{sx} P_X(x)$

→ Cont → " $= \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$

Theorem (No Proof)

(i) If $M_X(s)$ is finite $\forall s \in [-\varepsilon, \varepsilon]$ Then $M_X(s)$ uniquely determines CDF of X

(ii) If X & Y are RV s.t $M_X(s) = M_Y(s)\ \forall s\ [-\varepsilon, \varepsilon]$, $\varepsilon > 0$ Then X & Y hv same CDF.

Properties

- $M_X(0) = 1$
- $\frac{d^n}{ds^n} M_X(s) \Big|_{s=0} = E[X^n]$ — EZ Proof full math.
- $Y = aX + b \to M_Y(s) = e^{bs} \cdot M_X(as)$
- $Z = X + Y$ & X IND Y $\Rightarrow M_Z(s) = M_X(s) M_Y(s)$
- $Z = \sum_{i=1}^{N} X_i$ ① X_i → iid w/ MGF M_X ② N is IND of X_i with MGF M_N

$\Rightarrow M_Z(s) = M_N(\log M_X(s))$

PROOF

Characteristic Fn

$\phi_x(t) = E[e^{itx}]$

- $|\phi_x(t)| \leq 1$

<u>Properties</u>

- $\phi_x(0) = 1$
- $\left.\frac{d^n}{dt^n} \phi_x(t)\right|_{t=0} = i^n E[x^n]$
- $Y = aX + b \rightarrow \phi_Y(t) = e^{ibt}\phi_x(at)$
- $Z = X + Y$ (X IND Y)

  $\phi_Z(t) = \phi_X(t) \cdot \phi_Y(t)$

L-21
- Markov Ineq $\rightarrow P(X \geq a) \leq \frac{E[X]}{a}$ $\forall a > 0$

  $\rightarrow X > 0$

  $\rightarrow E[X] < \infty$ <u>PROOF</u>

  ~~$\rightarrow a \geq 0$~~

- Chebyshev's Ineq $\rightarrow P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$ $\forall c > 0$

  $(\mu, \sigma < \infty)$

REV
- Chernoff Bounds $\rightarrow P(X \geq a) \leq \inf_{s>0} \frac{E[e^{sX}]}{e^{aS}}$

  $\rightarrow S \in [-\varepsilon, \varepsilon]$

  $\rightarrow P(X \leq a) \leq \inf_{s<0} \frac{E[e^{sX}]}{e^{as}}$

Strength $\rightarrow$ Chernoff > Cheby > Mark

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Convergence

↳ $(x_n)_{n \in N}$ conv to x if for every $\epsilon > 0$, $\exists\, n_\epsilon \in N$ such that $\forall\, n \geq n_\epsilon$, $|x_n - x| < \epsilon$

In Probability

↳ $X_1, X_2 \dots X_n$ be a seq. of RV on $(\Omega, \mathcal{F}, P)$.
$(X_n)_{n \in N}$ converges to RV X if
$P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$ $\forall\, \epsilon > 0$

L22

WLLN → PROOF using Cheby

Let $X_1, X_2 \dots$ be a seq. of iid R.v w/ $(\mu, \sigma^2 < \infty)$
then for every $\epsilon > 0$,

$$\to P\left(\left| \frac{\sum_{i=1}^{n} x_i}{n} - \mu \right| > \epsilon \right) \to 0 \text{ as } n \to \infty$$

OR

→ $\sum_{i=1}^{n} x_i / n$ converges to μ in probability

true for $\sigma^2 = \infty$ as well → Proof not needed

(X_n) converges to X in d

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \;\; \forall\, x \text{ at which } F_X(x)$$

$F_X(x) = P(X \leq x)$ is cont.

SLLN

$$P(\omega : \lim_{n \to \infty} M_n(\omega) = \mu) = 1$$

x_i are iid w/ mean μ
& $\underbrace{\frac{\sum_{i=1}^{n} x_i}{n}}_{= \mu_N} \to \mu$

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~~L22~~

~~L22~~ ~~Central Limit Theorem~~

WLLN assumes $E[X^2] < \infty$

SLLN " $E[X^4] < \infty$

PROVE (L23 end)

$$(X_n \xrightarrow{a.s} X) \Rightarrow (X_n \xrightarrow{P} X) \Longrightarrow (X_n \xrightarrow{D} X)$$

$$(X_n \xrightarrow{m.s}) \Rightarrow (X_n \xrightarrow{P} X)$$

~~L22~~

X_i are iid seq, $\mu, \sigma < \infty$

$$Z_n = \frac{\sum_{i=1}^{n} x_i - n\mu}{\sqrt{n}\,\sigma} \text{ then } \lim_{n\to\infty} P(Z_n \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

ie $N[0,1]$

Collary → X_i → iid w/ $\mu=0, \sigma=1$

CLT says → $\frac{\sum x_i}{\sqrt{n}} \xrightarrow{d} N[0,1]$ → GOOD Proof

~~L23~~ ~~SLLN~~

$$P\left(\left\{w : \lim_{n\to\infty} \frac{\sum x_i(w)}{n} = \mu\right\}\right) = 1$$

→ Prove with $E[X^4]$

#

~~L-23~~
L-24

correlation → $R_x(t_1, t_2) = E[X_1, X_2]$
Covariance → $C_x(t_1, t_2) = Cov(X_{t_1}, X_{t_2})$
$= E[X_{t_1} X_{t_2}] - \mu_x(t_1)\mu_x(t_2)$

Random Processes

Discrete → $X_t \; ; t \in N$
Cont → $X_t \; ; t \in R$

Bernoulli's Process

→ No. of trials to first success → $P_T(t) = p \cdot (1-p)^{t-1}, t \in N$
↳ $E[T] = 1/p, V(T) = \frac{1-p}{p^2}$

→ s success in n trials → Binomial → $\binom{n}{k} p^k (1-p)^{n-k}, E[S] = np$
$Var(S) = np(1-p)$

→ Fresh start property : $Y_i = X_{n+i}$ is IND of past RVs
→ Arrival time : time of kth arrival.
→ Interarrival (T_k) = number of trials following (k-1) success until next success.
↳ All T_i have Geometric dist
↳ $E[Y_k] = k/p, \; Var(Y_k) = \frac{k(1-p)}{p^2}$
↳ $PMF(Y_k) = Y_k \in \{k, k+1 \ldots\}$

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L-23

~~Pf Almost Sure Convergence → $P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$~~

~~P Sure convergence → $f_{X_n} \to f_X$~~

Eg:

$P_{Y_K}(t) = P(Y_K = t) = P((k-1) \text{ in } t-1) \cdot P(t^{th} \text{ trial})$

(Due to independence property of Ber P)

$= P \cdot \binom{t-1}{k-1} (1-P)^{t-k} P^{k-1}$

$= \binom{t-1}{k-1} (1-P)^{t-K} P^{K}$ ← Pascal PMF of order K

Poisson Process $\left(P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}\right)$

→ Count no. of arrivals in $[0, t]$

→ $(N_t, t \in [0,\infty])$ is Poisson with rate λ iff

→ $N(0) = 0$

→ N_t has independent increments i.e. $N_{t_i} - N_{t_j}$ are all IND $\forall\ i > j$

→ $N_{t_i} - N_{t_{i-1}}$: No. of arrivals in $[t_{i-1}, t]$

→ Intervals in $T > 0$ has Poisson (λT)

(next Pg) $\Rightarrow N_{t+T} - N_t \sim \text{Poisson}(\lambda T) \quad \forall t \in [0, \infty]$

→ Poisson → $E[N(t)] = \lambda t$, $Var(N(t)) = \lambda t$

→ $R_N(t_1, t_2) = E[N_{t_1}((N_{t_2} - N_{t_1}) + N_{t_1})]$

$= E[N_{t_1}]E[N_{t_2} - N_{t_1}] + E[N_{t_1}]$

$= \lambda t_1 + \lambda^2 t_1 t_2$

→ $C_N(t_1, t_2) = \lambda t_1$

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$P(X_1 > t) = P(N(t) = 0)$
$= e^{-\lambda t}$ ← using ⊛

$\Rightarrow F_{X_1}(t) = \begin{cases} 1 - e^{-\lambda t} & ; t \geq 0 \\ 0 & ; o/w \end{cases}$

$S_k \rightarrow$ time of k^{th} arrival

$f_{S_1 S_2}(t_1, t_2) = \begin{cases} \lambda^2 e^{-\lambda t_2} & ; 0 < t_1 < t_2 \\ 0 & o/w \end{cases}$

↓ Jacob

$f_{X_1 X_2}(x_1, x_2) = f_{S_1 S_2}(x_1, x_1 + x_2)$ $(|J| = 1)$
$= \lambda^2 e^{-\lambda(x_1 + x_2)}$

X_1 & X_2 are IND, EXP w.r.t λ.

L25/ SSS

$X_t, t \in \mathbb{R}$ is SSS if $\forall\, t_1 \ldots t_n \in \mathbb{R}$, $n \in \mathbb{N}, \tau \in \mathbb{R}$,

cont → $F_{X_{t_1} X_{t_2} X_{t_3} \ldots X_{t_n}}(x_1, x_2 \ldots x_n) = F_{X_{t_1+\tau} X_{t_2+\tau}} \ldots$

discrete $(n_1, n_2 \ldots n_n \in \mathbb{Z}, n \in \mathbb{N}, k \in \mathbb{Z})$

WSS ⇒ I) $\mu_X(t_1) = \mu_X(t_2)$

II) $R_X(t_1, t_2) = R_X(t_1 + \tau, t_2 + \tau)$

↳ ie $E[X_{t_1} X_{t_2}] = E[X_{t_1+\tau} X_{t_2+\tau}]$

→ can be written as $R_X(t_2 - t_1)$

$R_X(\tau) = E[X(t)\, X(t - \tau)]$

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Properties of $R_x(T)$

① $R_x(0) = E[X_t, X_t] = \underbrace{E[X_t^2]}_{\text{expected power in } X_t \text{ at } t} \geq 0$

② $R_x(-T) = R_x(T)$

③ $|R_x(T)| \leq R_x(0)$

$$\leq \sqrt{E[X_t^2]\,E[X_{t-T}^2]} \quad \leftarrow \text{By Cauchy-S}$$

i.e $|E[XY]| \leq \sqrt{E[X^2]\,E[Y^2]}$

$$= \sqrt{R_x(0) \cdot R_x(0)}$$

$$\leq R_x(0)$$

- $E[(X_{t+T} - X_t)^2] = 2(R_x(0) - R_x(T))$

Power Spectral Density $\equiv FT(R_x(T))$

$$S_x(f) = \int_{-\infty}^{\infty} R_x(T)\, e^{-j2\pi f T}\, dT$$

$$R_x(T) = \int_{-\infty}^{\infty} S_x(f)\, e^{j2\pi f T}\, df$$

Properties

① $S_x(f)$ is real & even & ≥ 0

② Expected Power $= E[X_t^2] = R_x(0) = \int_{-\infty}^{\infty} S_x(f)\, df$

Gaussian R.V

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{\sqrt{(2\pi)^2 |K|}} \exp\left(-\frac{1}{2}[x_1 - \mu_1, x_2 - \mu_2] K^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}\right)$$

~~Ed~~: $K_{ij} = Cov(X_i, X_j)$, $i, j = 1, 2$