Lecture 16

(3 October 2024)

Recap : $F_{XY}(x,y) = P(X \le x, Y \le y)$

Jointly continuous RVs

$$F_{XY}(x,y) = \int_{v=-\infty}^{\infty} \int_{u=-\infty}^{\infty} f_{XY}(u,v)\,du\,dv$$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\,\partial y}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$E[g(X_1, X_2, \ldots, X_n)]$$

$$= \int_{x_i=-\infty,\ i\in[1:n]}^{\infty} g(x_1, \ldots x_n)\, f_{X_1 \ldots X_n}(x_1, \ldots x_n)\, dx_1 \ldots dx_n$$

$$E\left[\sum_{i=1}^{n} a_i x_i\right] = \sum_{i=1}^{n} a_i E[x_i]$$

$$F_{X|A}(x) = P(X \leq x | A)$$

$$= \frac{P(X \leq x \cap A)}{P(A)}$$

$$= \int_{-\infty}^{x} f_{X|A}(u)\, du$$

$f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x) \underline{P(A_i)}$ where $A_1, A_2, \ldots, A_n$ form a partition of Ω such that $P(A_i) > 0 \ \forall i$.

Conditioning One RV on another

We would like to define conditional PDF, say, $f_{X|Y=y}(x)$, similar to $f_{X|A}(x)$. However, we cannot consider

$$F_{X|Y=y}(x) = P(X \le x \mid Y=y)$$

as $P(Y=y)=0$ for continuous RV Y.

Instead, we consider

$$F_{X|\{y<Y\le y+\Delta y\}}(x) = P(X \le x \mid y < Y \le y+\Delta y)$$

$$= \frac{P(X \le x,\ y < Y \le y+\Delta y)}{P(y < Y \le y+\Delta y)}$$

$$= \frac{F_{XY}(x,\ y+\Delta y) - F_{XY}(x, y)}{F_Y(y+\Delta y) - F_Y(y)}$$

we know that

$$f_{X|\{y<Y\le y+\Delta y\}}(x) = \frac{\partial}{\partial x} F_{X|\{y<Y\le y+\Delta y\}}(x).$$

we define

$$f_{X|Y}(x|y) = \lim_{\Delta y\to 0} f_{X|\{y<Y\le y+\Delta y\}}(x).$$

so $f_{X|Y}(x|y)$

$$= \lim_{\Delta y\to 0} f_{X|\{y<Y\le y+\Delta y\}}(x)$$

$$= \lim_{\Delta y\to 0} \frac{\partial}{\partial x} F_{X|\{y<Y\le y+\Delta y\}}(x)$$

$$= \frac{\partial}{\partial x} \lim_{\Delta y\to 0} \frac{\left(F_{XY}(x,y+\Delta y) - F_{XY}(x,y)\right)/\Delta y}{\left(F_Y(y+\Delta y) - F_Y(y)\right)/\Delta y}$$

$$= \frac{\partial}{\partial x}\left[\frac{\frac{\partial}{\partial y} F_{XY}(x,y)}{f_Y(y)}\right]$$

$$= \frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y} \Big/ f_Y(y)$$

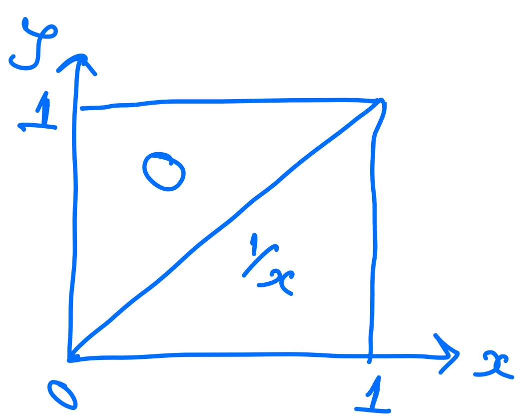
$$= \frac{f_{XY}(x,y)}{f_Y(y)} \,.$$

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \frac{f_Y(y)}{f_Y(y)} = 1 \,.$$

<u>Example</u>. Let X and Y have joint PDF

$$f_{XY}(x,y) = \frac{1}{x}, \quad 0 \le y \le x \le 1 \,.$$

Find f_X, $f_{Y|X}(y|x)$.

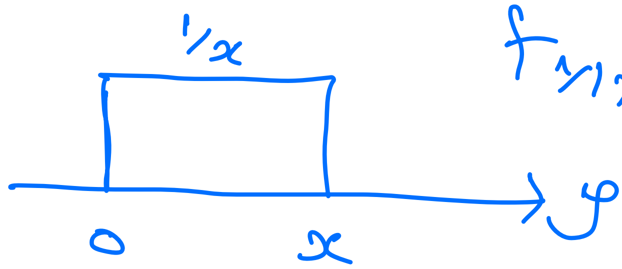


$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_{y=0}^{y=x} \frac{1}{x}\, dy = 1, \quad 0 \le x \le 1.$$

$X \sim$ Uniform $[0, 1]$.

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = 1/x, \quad 0 \le y \le x \le 1.$$



$f_{Y|X}(y|x)$

$Y \mid X = x \sim$ Uniform $[0, x]$.

Conditional Expectation

$$E[X|A] = \int_{-\infty}^{\infty} x\, f_{X|A}(x)\, dx$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

$E[X|Y] = \phi(Y)$, $\phi(y) = E[X|Y=y]$.

Expected Value Rule

$$E[g(x) \mid A] = \int_{-\infty}^{\infty} g(x) f_{x|A}(x)\, dx$$

$$E[g(x) \mid Y=y] = \int_{-\infty}^{\infty} g(x) f_{x|y}(x|y)\, dx$$

Total Expectation Theorems

(1) Let $A_1, A_2, \ldots, A_n$ form a partition of the sample space and let $P(A_i) > 0$ $\forall i$. Then

$$E[x] = \sum_{i=1}^{n} E[x \mid A_i]\, P(A_i).$$

(2) $$E[x] = \int_{-\infty}^{\infty} E[x \mid Y=y] f_y(y)\, dy.$$

$$= E[E[x \mid Y]].$$

<u>Proof</u>. $\sum_{i=1}^{n} E[X|A_i]P(A_i)$

$$= \sum_{i=1}^{n} \int_{-\infty}^{\infty} x f_{X|A_i}(x)\,dx \; P(A_i)$$

$$= \int_{-\infty}^{\infty} x \sum_{i=1}^{n} f_{X|A_i}(x)P(A_i)\,dx$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx$$

(since $f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x)P(A_i)$)

$$= E[X].$$

$$\int_{-\infty}^{\infty} E[X|Y=y]\,f_Y(y)\,dy$$

$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} x f_{X|Y}(x|y)\,f_Y(y)\,dx\,dy$$

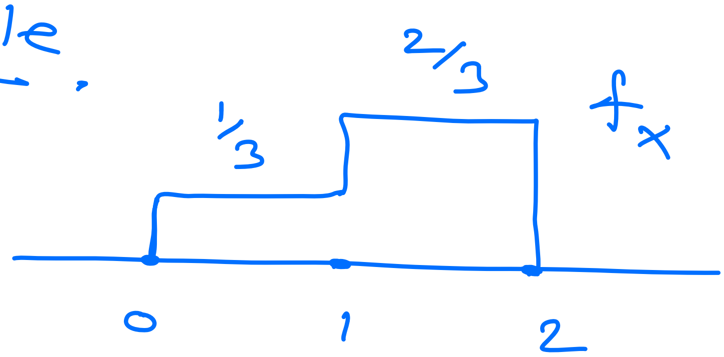
$$= \int_{x=-\infty}^{\infty} x \int_{y=-\infty}^{\infty} f_{XY}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= E[X].$$

The total expectation theorem can often be used to calculate mean and variance.

Example.



$A_1 = \{0 \le x \le 1\}$ $A_2 = \{1 \le x \le 2\}$.

$P(A_1) = \frac{1}{3}$ $P(A_2) = \frac{2}{3}$.

$f_{X|A_1}$, $f_{X|A_2}$ are uniform.

$E[X|A_1] = 1/2$, $E[X|A_2] = 3/2$,

Recall that $U \sim \text{Uniform}[a,b] \Rightarrow$

$$E[U^2] = \frac{a^2+ab+b^2}{3} .$$

$E[X^2|A_1] = \frac{1}{3}$, $E[X^2|A_2] = \frac{7}{3}$.

$E[X] = E[X|A_1]P(A_1) + E[X|A_2]P(A_2)$

$= 7/6$.

$E[X^2] = E[X^2|A_1]P(A_1) + E[X^2|A_2]P(A_2)$

$= \frac{15}{9}$.

$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{11}{36}$.

Conditional Variance

$\text{Var}(X|Y=y) = E[X^2|Y=y] - E[X|Y=y]^2$.

Independence.

Recall that for discrete RV X & Y we defined X & Y are independent if the events $\{X=x\}$ and $\{Y=y\}$ are independent for each pair (x, y), i.e.,

$$p_{XY}(x, y) = p_X(x)\, p_Y(y).$$

Can we have the same definition of independence for continuous RVs as well?

NO!

This is because it is trivially true that

$$0 = P(X=x, Y=y) = P(X=x)\, P(Y=y) = 0.$$

We define that two continuous RVs are independent if

$$f_{XY}(x,y) = f_X(x) f_Y(y), \quad \forall x, y.$$

Example. The joint PDF of two RVs X and Y is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}}.$$

Note that

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$$

$$= f_X(x) f_Y(y).$$

$\therefore$ X and Y are independent Gaussian RVs.

Exercise

(i) Two continuous RVs X and Y are independent if and only if

$F_{XY}(x,y) = F_X(x)\, F_Y(y)$ for all x, y.

(ii) Two discrete RVs X and Y are independent if and only if

$F_{XY}(x,y) = F_X(x)\, F_Y(y)$ for all x, y.

Theorem: If X and Y are jointly continuous independent RVs. For any two functions g & h

$$E[g(X)h(Y)] = E[g(X)]\,E[h(Y)].$$

Proof: We first show that $g(X)$ and $h(Y)$ are independent.

Let $Z = g(X)$ $W = h(Y)$

$$F_{ZW}(z,w) = P(Z \le z, W \le w)$$

$$= P(g(X) \le z, h(Y) \le w)$$

$$= P((X,Y) \in \{(x,y) : g(x) \le z, h(y) \le w\})$$

$$= \int_{y : h(y) \le w} \int_{x : g(x) \le z} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{x : g(x) \le z} f_X(x)\,dx \int_{y : h(y) \le w} f_Y(y)\,dy$$

$$= P(g(X) \le z)\, P(h(Y) \le w)$$

$$= F_Z(z)\, F_W(w).$$

$$\Rightarrow f_{ZW}(z,w) = f_Z(z)\, f_W(w)$$

$$\Rightarrow E[ZW] = E[Z]\, E[W].$$