Review

- FFT

Primitive n^{th} root of unity.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_{4\times 1} \rightarrow 4^{nth} \text{ primitive root of unity. } \{\pm i\}$$

0- indexing

$i,j \rightarrow \omega^{ij}$ ($\omega = \pm i$)

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} i^0 & i^0 & i^0 & i^0 \\ i^0 & i & i^2 & i^3 \\ i^0 & i^2 & i^4 & i^6 \\ i^0 & i^3 & i^6 & i^9 \end{bmatrix} \end{array} = \begin{array}{c|c} & \begin{matrix} & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \end{array}$$

b_0 $\quad b_0 = a_0 + a_1 + a_2 + a_3$

b_1 $\quad b_1 = a_0 + a_1 i - a_2 - a_3 i$

b_2 $\quad b_2 = a_0 - a_1 + a_2 - a_3 = (a_0 + a_2) - (a_1 + a_3)$

b_3 $\quad b_3 = a_0 - a_1 i - a_2 + a_3 i$

$$b_i = \sum_{j=0}^{n-1} \omega^{ij} \cdot a_j = \left(\sum_{j=0}^{\frac{n}{2}-1} \omega^{ij} \cdot a_j\right) + \left(\sum_{j=\frac{n}{2}}^{n-1} \omega^{ij} \cdot a_j\right)$$

$$= \left(\sum_{j=0}^{\frac{n}{2}-1} \omega^{ij} \cdot a_j\right) + \left[\omega^{\frac{n}{2}i} \cdot \left(\sum_{j=0}^{\frac{n}{2}-1} \omega^{ij} \cdot a_{j+\frac{n}{2}}\right)\right]$$

Say $i = 2p$

$$= \left(\sum_{j=0}^{\frac{n}{2}-1} (\omega^2)^{pj} \cdot a_j\right) + \left[\omega^{\frac{n}{2}\cdot 2p} \cdot \left(\sum_{j=0}^{\frac{n}{2}-1} (\omega^2)^{pj} \cdot a_{j+\frac{n}{2}}\right)\right]$$

$i = 0$

$\underbrace{}_{= w^{n} = 1}$ $\Rightarrow p = 0$

$$= \sum_{j=0}^{\frac{n}{2}-1} \left[(w^2)^{pj} \cdot \left(a_j + a_{j+\frac{n}{2}} \right) \right]$$

$$b_0 = \sum_{j=0}^{1} (w^2)^{0 \cdot j} (a_j + a_{j+2})$$

$$= (a_0 + a_2) + (a_1 + a_3)$$

$$b_2 = \sum_{j=0}^{1} (w^2)^{1 \cdot j} \cdot (a_j + a_{j+2})$$

$$= (w^2)^{1 \cdot 0} (a_0 + a_2) + \underline{\underline{(w^2)}}^{1 \cdot 1} (a_1 + a_3)$$

$$= (a_0 + a_2) - (a_1 + a_3)$$

$$\left(w^{\frac{n}{2}}\right)^{2p+1}$$

-1

$i = 2p+1$

$$\left(\sum_{j=0}^{\frac{n}{2}-1} w^{ij} \cdot a_j \right) + \left[\underbrace{w^{\frac{n}{2} i}}_{= -1} \cdot \left(\sum_{j=0}^{\frac{n}{2}-1} w^{ij} \cdot a_{j+\frac{n}{2}} \right) \right]$$

$$= \left(\sum_{j=0}^{\frac{n}{2}-1} (w^2)^{pj} \cdot a_j \cdot w^j \right) - \left[\sum_{j=0}^{n/2-1} (w^2)^{p \cdot j} \cdot a_{j+\frac{n}{2}} \cdot w^j \right]$$

$$= \sum_{j=0}^{\frac{n}{2}-1} (w^2)^{p \cdot j} \cdot \left(\left(a_j - a_{j+\frac{n}{2}} \right) \cdot w^j \right)$$

$$b_1 = \sum_{j=0}^{1} (w^2)^{0 \cdot j} \left(a_j - a_{j+\frac{n}{2}} \right) \cdot w^j$$

$$= (a_0 - a_2) \cdot w^0 + (a_1 - a_3) \cdot i$$

$$= a_0 + a_1 i - a_2 - a_3 i$$

$$b_3 = \sum_{j=0}^{1} (w^2)^{1 \cdot j} \left(a_j - a_{j+\frac{n}{2}} \right) \cdot w^j$$

$$= 1 \cdot (a_0 - a_2) \cdot w^0 + \underline{(w^2)} \cdot (a_1 - a_3) \cdot w$$

$$= (a_0 - a_2) + w^3 (a_1 - a_3)$$

$$= a_0 - a_2 - a_1 i + a_3 i$$

DFT of dim 2.

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & -1 \end{array} \qquad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$v_e = \begin{bmatrix} a_0 + a_{\frac{n}{2}} \\ a_1 + a_{\frac{n}{2}+1} \\ \vdots \\ a_{\frac{n}{2}-1} + a_{n-1} \end{bmatrix}, \quad v_o = \begin{bmatrix} (a_0 - a_{\frac{n}{2}})\, w^0 \\ (a_1 - a_{\frac{n}{2}+1}) \cdot w^1 \\ \vdots \\ (a_{\frac{n}{2}-1} - a_{n-1})\, w^{\frac{n}{2}-1} \end{bmatrix}$$

$$v_e = \begin{bmatrix} a_0 + a_2 \\ a_1 + a_3 \end{bmatrix} \qquad v_o = \begin{bmatrix} a_0 - a_2 \\ (a_1 - a_3)\, i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_0 + a_2 \\ a_1 + a_3 \end{bmatrix} = \begin{bmatrix} a_0 + a_1 + a_2 + a_3 \\ a_0 - a_1 + a_2 - a_3 \end{bmatrix} \begin{matrix} \leftarrow b_0 \\ \leftarrow b_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_0 - a_2 \\ (a_1 - a_3)\, i \end{bmatrix} = \begin{bmatrix} a_0 + a_1 i - a_2 - a_3 i \\ a_0 - a_1 i - a_2 + a_3 i \end{bmatrix} \begin{matrix} \leftarrow b_1 \\ \leftarrow b_3 \end{matrix}$$

$$\mathbb{Z}_m$$

$$\begin{tikzcd}
\text{Coefs} \arrow[rr, bend left, "O(n\log n)"] & & \text{Eval} \\
P \cdot Q \arrow[rr, "FFT"] \arrow[d, "O(n^2)"'] & & \text{Evals of } P \text{ and } Q \text{ at } 2n \text{ points} \arrow[d, "O(n)"] \\
R = P.Q & & \text{Evals of } R \text{ at } 2n \text{ points} \arrow[ll, "\text{InvFFT}"', "O(n\log n)"] \\
\text{Coefs} & & \text{Eval} \arrow[ll, bend left]
\end{tikzcd}$$

Inverse DFT.

$$w^{ij} = \frac{w^{-ij}}{n} = \frac{w^{n-ij}}{n}.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \times \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{InvDFT}_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b_i = \sum_{j=0}^{n-1} \frac{\omega^{-ij}}{n} \cdot a_j$$

$$\omega^{-(i \cdot j + \frac{n}{2})}$$

$$= \left(\omega^{-\frac{n}{2} i}\right) \cdot \omega^{-i \cdot j}$$

$$b_i = \sum_{j=0}^{n-1} \frac{\omega^{-ij}}{n} \cdot a_j$$

$$= \left(\sum_{j=0}^{\frac{n}{2}-1} \left(\frac{\omega^{-i \cdot j}}{n} \right) \cdot a_j \right) + \left(\sum_{j = \cancel{\frac{n}{2}}\, 0}^{\cancel{n-1}\, \frac{n}{2}-1} \left(\frac{\omega^{-i \cdot j}}{n} \right) \cdot a_j \right)$$

$$\frac{n}{2}+j \qquad j+\frac{n}{2}$$

$$= \left(\sum_{j=0}^{\frac{n}{2}-1} \left(\frac{\omega^{-ij}}{n} \right) a_j \right) + \underbrace{\left(\omega^{-\frac{n}{2} \cdot i} \right)} \left(\sum_{j=0}^{\frac{n}{2}-1} \frac{\omega^{-ij}}{n} \cdot a_{j+\frac{n}{2}} \right).$$

$i = 2p$

$$= \left(\sum_{j=0}^{\frac{n}{2}-1} \frac{(\omega^2)^{-p \cdot j}}{n} \cdot a_j \right) + \left(\omega^{-\frac{n}{2} \times 2p} \right) \cdot \left(\sum_{j=0}^{\frac{n}{2}-1} \frac{(\omega^2)^{-i \cdot j}}{n} \cdot a_{j+\frac{n}{2}} \right)$$

$i = 2p+1$

$$\left(\sum_{j=0}^{\frac{n}{2}-1} \frac{(\omega^2)^{-p \cdot j}}{n} \cdot a_j \cdot \omega^{-j} \right) + \omega^{-\frac{n}{2} \times (2p+1)} \left[\sum_{j=0}^{\frac{n}{2}-1} \frac{(\omega^2)^{-p \cdot j}}{n} \cdot a_{j+\frac{n}{2}} \cdot \omega^{-j} \right]$$

Even i

$$\sum_{j=0}^{\frac{n}{2}-1} \frac{(\omega^2)^{-pj}}{n} \cdot \left(a_j + a_{j+\frac{n}{2}} \right)$$

Odd i

$$\left[\sum_{j=0}^{\frac{n}{2}-1} \frac{(\omega^2)^{-p \cdot j}}{n} \cdot \left(a_j - a_{j+\frac{n}{2}} \right) \cdot \omega^{-j} \right]$$

$$= \sum_{j=0}^{\frac{n}{2}-1} \frac{(w^2)^{-p \cdot j}}{(n/2)} \cdot \frac{(a_j + a_{j+\frac{n}{2}})}{2}$$

$$= \sum_{j=0}^{\frac{n}{2}-1} \frac{(w^2)^{-p \cdot j}}{(n/2)} \cdot \frac{(a_j - a_{j+\frac{n}{2}})}{2 \cdot w^j}$$

$$v_e = \begin{bmatrix} (a_0 + a_{\frac{n}{2}})/2 \\ (a_1 + a_{\frac{n}{2}+1})/2 \\ \vdots \\ (a_{\frac{n}{2}-1} + a_{n-1})/2 \end{bmatrix} \qquad v_o = \begin{bmatrix} (a_0 - a_{\frac{n}{2}}) \cdot 1/2w^j \\ (a_1 - a_{\frac{n}{2}+1}) \cdot 1/2w^j \\ \vdots \\ (a_{\frac{n}{2}-1} - a_{n-1}) \cdot \frac{1}{2w^j} \end{bmatrix}$$