Quiz 2 Solutions

1. sol:-

P (A is taller than B | A is shorter than c)

$$= P(X_1 > X_2 \mid X_1 < X_3)$$

$$= \frac{P(X_2 < X_1 < X_3)}{P(X_1 < X_3)}$$

Note that $P(X_1 < X_2 < X_3) + P(X_1 < X_3 < X_2) + P(X_2 < X_3 < X_1) + P(X_2 < X_1 < X_3) + P(X_3 < X_1 < X_2) + P(X_3 < X_2 < X_1) = 1.$

we claim that all these probabilities are equal by symmetry, i.e., by using the independent and identically distributed nature.

$$P(X_1 < X_2 < X_3) = \int_{x_1=-\infty}^{\infty} \int_{x_2=x_1}^{\infty} \int_{x_3=x_2}^{\infty} f_{X_1X_2X_3}(x_1, x_2, x_3)\, dx_1 dx_2 dx_3$$

$$= \int_{x_1=-\infty}^{\infty} \int_{x_2=x_1}^{\infty} \int_{x_3=x_2}^{\infty} f_X(x_1) f_X(x_2) f_X(x_3)\, dx_1\, dx_2\, dx_3$$

$$= \int_{s=-\infty}^{\infty} \int_{t=s}^{\infty} \int_{u=t}^{\infty} f_X(s) f_X(t) f_X(u)\, ds\, dt\, du$$

$$= P(X_i < X_j < X_k)$$

for $(\underline{i}\,\underline{j}\,\underline{k}) = (\underline{1}\,\underline{2}\,\underline{3})\,(\underline{1}\,\underline{3}\,\underline{2})\,(\underline{2}\,\underline{1}\,\underline{3})\,(\underline{2}\,\underline{3}\,\underline{1})$
$(\underline{3}\,\underline{1}\,\underline{2})\,(\underline{3}\,\underline{2}\,\underline{1})$.

$$\therefore P(X_i < X_j < X_k) = 1/6 .$$

$$P(X_1 < X_3) = P(X_2 < X_1 < X_3) + P(X_1 < X_2 < X_3) + P(X_1 < X_3 < X_2)$$

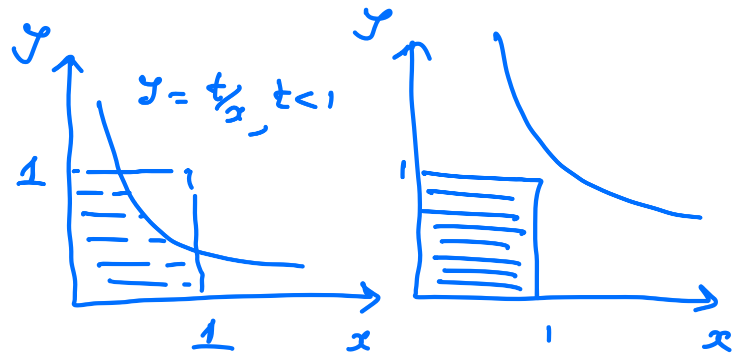
$$= 3/6 = 1/2$$

$\therefore P(A \text{ is taller than } B \mid A \text{ is shorter than } C)$

$$= \frac{1/6}{1/2} = 1/3 .$$

2. Sol:- For $t<0$ $F_Z(t) = P(Z \le t) = 0$.

For $t>0$ we have



$$
\begin{aligned}
F_Z(t) &= P(Z \le t) \\
&= P(XY \le t) \\
&= P\left((X,Y) \in \{(x,y) : xy \le t\}\right) \\
&= \int\limits_{(x,y):\, xy \le t} f_{XY}(x,y)\, dx\, dy
\end{aligned}
$$

$$
= \begin{cases} t + \displaystyle\int_{x=t}^{1} \int_{y=0}^{t/x} dx\, dy & \text{if } t<1 \\[2ex] \displaystyle\int_{x=0}^{1} \int_{y=0}^{1} dx\, dy & \text{if } t>1 \end{cases}
$$

$$
= \begin{cases} t - t\log t & \text{if } t<1 \\ 1 & \text{if } t>1 \end{cases}.
$$

$$
\therefore\ f_Z(t) = \begin{cases} -\log t & \text{if } 0<t<1 \\ 0 & \text{otherwise.} \end{cases}
$$

3. sol:- $y_1 = x_1 + x_2$ $y_2 = \frac{x_2}{x_1+x_2}$

$y_1 = g_1(x_1, x_2) = x_1 + x_2$

$y_2 = g_2(x_1, x_2) = \frac{x_2}{x_1+x_2}$

$y_2 = \frac{x_2}{y_1} \Rightarrow x_2 = y_1 y_2, \quad x_1 = y_1 - y_1 y_2$

$x_1 = h_1(y_1, y_2) = y_1 - y_1 y_2$

$x_2 = h_2(y_1, y_2) = y_1 y_2.$

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1(x_1, x_2)}{\partial x_1} & \frac{\partial g_1(x_1, x_2)}{\partial x_2} \\ \frac{\partial g_2(x_1, x_2)}{\partial x_1} & \frac{\partial g_2(x_1, x_2)}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ \frac{-x_2}{(x_1+x_2)^2} & \frac{x_1}{(x_1+x_2)^2} \end{vmatrix}$$

$$= \frac{1}{x_1+x_2}.$$

$$f_{Y_1 Y_2}(y_1, y_2) = \left. \frac{f_{X_1 X_2}(x_1, x_2)}{|J(x_1, x_2)|} \right|_{\substack{x_1 = h_1(y_1, y_2) \\ x_2 = h_2(y_1, y_2)}}$$

$$= \begin{cases} y_1 f_X(y_1 - y_1 y_2) f_X(y_1 y_2), & \text{if } y_1 > 2, \; \frac{1}{y_1} < y_2 < 1 - \frac{1}{y_1} \\ 0, & \text{o.w.} \end{cases}$$

$$= \begin{cases} \frac{1}{y_1^3 y_2^2 (1 - y_2)^2}, & y_1 > 2, \; \frac{1}{y_1} < y_2 < 1 - 1/y_1 \\ 0, & \text{o.w.} \end{cases}$$