(CS1.301) Algorithm Analysis & Design (Monsoon 2024) Due Date: 10:00 PM, 12^{th} Nov

Problem Set 5

Instructions:

- Discussions amongst the students are not discouraged, but all write-ups must be done individually and include all collaborators' names.
- Referring sources other than the lecture notes is discouraged as solutions to some problems can be found easily via a web search. But if you use an outside source (e.g., textbooks, other lecture notes, or any online material), mention the same in your write-up. This will not affect your grades. However, when caught, dishonesty of any sort shall be heavily penalized.
- Be clear in your arguments. Vague arguments shall not be given full credit.
- Ensure you provide and analyze the run time complexity for questions involving an algorithm.
- Unless specified otherwise, proofs for each solution are required.

Question 1 [2 marks]

Prove or disprove the following:

Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e; and let (A, B) be a minimum s-t cut with respect to these capacities $c_e : e \in E$. Now suppose we add 1 to every capacity; then (A, B) is still a minimum s-t cut with respect to these new capacities $\{1 + c_e : e \in E\}$.

Question 2 [3 marks]

There are n research groups attending a conference. Group i consists of r_i researchers. The conference organizers have prepared m discussion rooms, where $m \geqslant n$. Room j has only s_j seats available. Each group wants to distribute its researchers across different rooms to maximize opportunities for collaboration with researchers from other groups.

Can you determine if it's possible to seat the researchers so that no two members from the same group are seated in the same room?

Question 3 [3 marks]

Let G be an arbitrary flow network such that the capacity c_e of each edge e is a positive even integer.

1. Prove that the maximum flow value in the network is an even integer.

2. Prove that there exists a maximum flow f such that the flow f_e through each edge e is an even integer.

Question 4 [4 marks]

Suppose you are given a directed graph $G = (V, E)$, with a positive integer capacity c_e on each edge e, a source $s \in V$, and a sink $t \in V$. You are also given a maximum s-t flow in G, defined by a flow value f_e on each edge e. The flow f is acyclic: There is no cycle in G on which all edges carry positive flow. The flow f is also integer-valued. Now, suppose we pick a specific edge $e' \in E$ and reduce its capacity by 1 unit.
Show how to find a maximum flow in the resulting capacitated graph in time $O(m + n)$, where m is the number of edges in G and n is the number of nodes.

Question 5 [6 marks]

In a battlefield, depicted as an $n \times m$ grid, certain cells are occupied by mines. The objective is to clear these mines, which incurs a cost. There are two methods for removing the mines:

1. Select an individual cell containing a mine and eliminate it at a cost x.

2. Choose two neighboring cells (having a common edge), each having a mine, and clear both at a cost y.

Provide an algorithm to determine the minimum cost necessary to eliminate all the mines completely.

Question 6 [8 marks]

A packaging company operates within a city, represented by a directed graph with n nodes and m edges. Each edge in the graph has a specified weight limit or capacity, denoted by c_i. The company's task is transporting goods from node 1 to node n along a simple path. The company possesses x trucks, which can carry any possible weight. However, it is necessary that all trucks must carry the same weight. Additionally, all trucks are dispatched from node 1 simultaneously, meaning no truck waits for others to traverse an edge. Therefore, it is crucial to ensure that the total weight transported across any edge does not exceed the weight limit assigned to that edge.

Provide an $O(mx\log(c_{\max}))$ algorithm to determine the maximum total weight of goods that can be delivered by these x trucks. $c_{\max}$ is the maximum weight limit across all the edges. You can assume that the weights are integral.