We are given that $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables, and $X = X_1 + X_2 + \cdots + X_n$. Each X_i is a Bernoulli random variable with parameter p_i, i.e.,

$$P(X_i = 1) = p_i \quad \text{and} \quad P(X_i = 0) = 1 - p_i,$$

where $0 \leq p_i \leq 1$ for each i.

The mean of X is constrained to be μ, i.e.,

$$E[X] = \mu.$$

We want to show that the variance of X, $\text{Var}(X)$, is maximized when all p_i's are equal, specifically $p_1 = p_2 = \cdots = p_n = \frac{\mu}{n}$.

Step 1: Mean of X

The expected value of X is the sum of the expected values of the X_i's:

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = p_1 + p_2 + \cdots + p_n.$$

Since $E[X] = \mu$, we have the constraint:

$$p_1 + p_2 + \cdots + p_n = \mu.$$

Step 2: Variance of X

The variance of X is the sum of the variances of the independent X_i's:

$$\text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n).$$

For a Bernoulli random variable X_i with parameter p_i, the variance is given by:

$$\text{Var}(X_i) = p_i(1 - p_i).$$

Thus, the variance of X is:

$$\text{Var}(X) = p_1(1 - p_1) + p_2(1 - p_2) + \cdots + p_n(1 - p_n).$$

Step 3: Objective

We want to maximize the variance $\text{Var}(X) = \sum_{i=1}^{n} p_i(1 - p_i)$ subject to the constraint $p_1 + p_2 + \cdots + p_n = \mu$.

Step 4: Perfect Square Method

$$
\begin{aligned}
\sum_{i=1}^{n} p_i(1-p_i) &= -\sum_{i=1}^{n}\left(\left(p_i-\frac{\mu}{n}\right)^2+\left(\frac{\mu}{n}\right)^2+\left(1-\frac{2\mu}{n}\right)p_i\right)\\
&= -\sum_{i=1}^{n}\left(p_i-\frac{\mu}{n}\right)^2-n\left(\frac{\mu}{n}\right)^2-\left(1-\frac{2\mu}{n}\right)\mu
\end{aligned}
$$

Hence, it will reach its minimum value when $p_i = \frac{\mu}{n}$

Step 5: Conclusion

The variance of X is maximized when all p_i's are equal, specifically $p_1 = p_2 = \cdots = p_n = \frac{\mu}{n}$.

Q2

$$P(X>m+n \mid X>m) = P(X>n)$$

$$\Rightarrow \frac{P(X>m+n \cap X>m)}{P(X>m)} = P(X>n)$$

$$\Rightarrow P(X>m+n) = P(X>m)\,P(X>n)$$

By mathematical induction it can be proved,

$$P(X>m) = P(X>1)^m$$

Let $P(X>1) = c \Rightarrow 0 \le c \le 1$

$$\Rightarrow P(X>m) = c^m$$

Thus, $P(X=m) = P(X>m-1) - P(X>m)$

$$= c^{m-1} - c^m$$

$$= c^{m-1}(1-c)$$

Let $p = 1-c$

$\Rightarrow P(X=m) = (1-p)^{m-1}\,p$ which is the pdf for Geometric R.V.

Solution-3 Assignment 3

Problem 3:

Solution:

(a)

1. **Define Bernoulli Random Variables:**

   Let U and V be independent Bernoulli random variables with:

   - $P(U = 1) = p$, $P(U = 0) = 1 - p$
   - $P(V = 1) = p$, $P(V = 0) = 1 - p$

2. **Construct X and Y:**

   Define:

   - $X = U + V$
   - $Y = U - V$

3. **Calculate Expected Values:**

   - $E[X] = E[U + V] = E[U] + E[V] = p + p = 2p$
   - $E[Y] = E[U - V] = E[U] - E[V] = p - p = 0$

4. **Calculate E[XY]:**

   $E[XY] = E[(U + V)(U - V)] = E[U^2 - V^2] = E[U] - E[V] = p - p = 0$

5. **Verify Uncorrelatedness:**

   $Cov(X, Y) = E[XY] - E[X]E[Y] = 0$

Hence, X and Y are uncorrelated.

6. **Verify Non-Independence:**

   $X=U-V$ and $Y=U+V$ are not independent because U and V are shared components in both X and Y. Knowing the value of X gives information about the possible values of Y and vice versa.

   To check if X and Y are independent, let's look at the pair ($X = 2$, $Y = 0$) and $p=1/2$

   - For $X = 2$ and $Y = 0$, both U and V must be 1 (since $X = U + V$ and $Y = U - V$).
   - Thus, $P(X = 2, Y = 0) = P(U = 1, V = 1) = p \times p = 1/2 \times 1/2 = 1/4$.

   Now, calculate P_X(2) and P_Y(0):

   - $P\_X(2) = P(U = 1, V = 1) = 1/4$.
   - $P\_Y(0) = P(U = V) = P(U = 0, V = 0) + P(U = 1, V = 1) = 1/4 + 1/4 = 1/2$.

   Therefore:

   - $P\_X(2) \times P\_Y(0) = 1/4 \times 1/2 = 1/8$.

   Since $P(X = 2, Y = 0) = 1/4 \neq 1/8$, X and Y are not independent.

Conclusion:

X and Y are uncorrelated but not independent.

(b)

Setup:

Let X and Y be Bernoulli random variables, each taking values in {0, 1}, with probabilities:

- $P(X = 1) = p\_X$, $P(X = 0) = 1 - p\_X$
- $P(Y = 1) = p\_Y$, $P(Y = 0) = 1 - p\_Y$

We will check that uncorrelatedness leads to independence by verifying the condition $P_{XY}(x, y) = P_X(x) * P_Y(y)$ for all possible pairs (x, y).

Step 1: Uncorrelatedness condition

Uncorrelatedness means that the covariance of X and Y is zero:
$Cov(X, Y) = E[XY] - E[X]E[Y] = 0$

We know that:

- $E[X] = p_X$
- $E[Y] = p_Y$

Now, calculate E[XY]:
$E[XY] = P(X = 1, Y = 1)$

Thus, the uncorrelatedness condition gives us:
$P(X = 1, Y = 1) = E[X] * E[Y] = p_X * p_Y$

Step 2: Verify independence for all (x, y)

We need to check that $P_{XY}(x, y) = P_X(x) * P_Y(y)$ for all four possible pairs (x, y): (0, 0), (0, 1), (1, 0), and (1, 1).

1. **For (X = 1, Y = 1):**
    - From the uncorrelatedness condition, we have: $P(X = 1, Y = 1) = p_X * p_Y$.
    - On the other hand: $P_X(1) = p_X$ and $P_Y(1) = p_Y$.
    - Thus: $P_X(1) * P_Y(1) = p_X * p_Y$.
2. **For (X = 1, Y = 0):**
    - $P(X = 1, Y = 0) = P(X = 1) * P(Y = 0) = p_X * (1 - p_Y)$.
    - On the other hand: $P_X(1) = p_X$ and $P_Y(0) = 1 - p_Y$.
    - Thus: $P_X(1) * P_Y(0) = p_X * (1 - p_Y)$.
3. **For (X = 0, Y = 1):**
    - $P(X = 0, Y = 1) = P(X = 0) * P(Y = 1) = (1 - p_X) * p_Y$.
    - On the other hand: $P_X(0) = 1 - p_X$ and $P_Y(1) = p_Y$.
    - Thus: $P_X(0) * P_Y(1) = (1 - p_X) * p_Y$.

4. **For (X = 0, Y = 0):**
   - P(X = 0, Y = 0) = P(X = 0) * P(Y = 0) = (1 - p_X) * (1 - p_Y).
   - On the other hand: P_X(0) = 1 - p_X and P_Y(0) = 1 - p_Y.
   - Thus: P_X(0) * P_Y(0) = (1 - p_X) * (1 - p_Y).

Conclusion:

Since we have verified that P_XY(x, y) = P_X(x) * P_Y(y) for all possible pairs (x, y) = (0, 0), (0, 1), (1, 0), and (1, 1), and since uncorrelatedness leads directly to P(X = 1, Y = 1) = P_X(1) * P_Y(1), we have shown that uncorrelatedness guarantees independence in this case.

#4 To prove: $|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$ & equality holds for $X = \alpha Y$

let R.V $Z := (X - \alpha Y)^2$

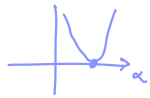
$\mathbb{E}[Z] \geq 0$ [all elements are non negative] with equality when $X = \alpha Y$.

$\mathbb{E}[(X - \alpha Y)^2] \geq 0$

$\mathbb{E}[X^2] + \alpha^2 \mathbb{E}[Y^2] - 2\alpha \mathbb{E}[XY] \geq 0$

$\alpha^2 \mathbb{E}[Y^2] - 2\mathbb{E}[XY](\alpha) + \mathbb{E}[X^2] \geq 0$

↳ Quadratic equation in α which remains non negative.



either no roots or equal roots.

$\Rightarrow b^2 - 4ac \leq 0$

$\Rightarrow 4[(\mathbb{E}[XY])^2 - \mathbb{E}[X^2]\mathbb{E}[Y^2]] \leq 0$

$\Rightarrow |\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$

To prove: $\rho(X, Y) = \dfrac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{var(X)\, var(Y)}} \leq 1 \quad \forall X, Y.$

$$\rho(X,Y) = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{var(X)\, var(Y)}} = \frac{\mathbb{E}[XY] - 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{var(X)\, var(Y)}}$$

$$= \mathbb{E}\left[\frac{X - \mathbb{E}[X]}{\sigma_x} \cdot \frac{Y - \mathbb{E}[Y]}{\sigma_y}\right]$$

$\sigma_x, \sigma_y = \sqrt{var(X)}, \sqrt{var(Y)}$

consider $A = \dfrac{X - \mathbb{E}[X]}{\sigma_x}$ & $B = \dfrac{Y - \mathbb{E}[Y]}{\sigma_y}$

$\mathbb{E}[A] = \mathbb{E}[B] = 0$

$\sigma_A = \sigma_B = 1 \Rightarrow \mathbb{E}[A^2] = \sigma_A^2 + (\mathbb{E}[A])^2 = 1$

$\mathbb{E}[B^2] = \sigma_B^2 + (\mathbb{E}[B])^2 = 1$

$\rho(X,Y) = \mathbb{E}[A \cdot B] \leq \sqrt{\mathbb{E}[A^2]\, \mathbb{E}[B^2]} = \sqrt{1 \cdot 1} = 1$

$\Rightarrow \boxed{\rho(X,Y) \leq 1}$

□

Question 5

Let $\varphi(Y) = E[X|Y]$.

For any function g : R → R, show that $E[\varphi(Y)g(Y)] = E[Xg(Y)]$

Argue that the law of iterated expectations, ie., $E[E[X|Y]] = E[X]$, is a special case of this.

Proof

$E[\varphi(Y)g(Y)] = \Sigma_y \varphi(Y) * g(Y) * p_Y(y)$

$= \Sigma_y (\Sigma_x p(x|y)x) * g(Y) * p_Y(y)$

$= \Sigma_y (\Sigma_x x * p(x|y) * p_Y(y)) * g(y)$

$= \Sigma_y \Sigma_x x p_{xy}(x, y) g(y)$

$= E[Xg(Y)]$

Now to show that law of iterated expectation is a special case of this, we just substitute $g(Y) = 1$

$$E[\varphi(Y)g(Y)] = \sum_y \sum_x x p_{xy}(x, y) * 1$$

$$E[\varphi(Y)g(Y)] = \sum_x x \sum_y p_{xy}(x, y) * 1$$

$$E[\varphi(Y)g(Y)] = \sum_y x p_X(x)$$

Therefore for the case of $g(Y) = 1$

$$E[\varphi(Y)g(Y)] = E[X]$$

Question 6

Part A.

For any discrete random variable X and any event A such that $P(A) > 0$, show that :

$E[1_A X] = E[X|A] \cdot P(A)$

where 1_A is the indicator random variable of event A.

Solution:

RHS:

$$E[X|A] = \sum_x p_x(x|A)x$$

$$E[X|A] = \sum_x \frac{P(x \cap A)}{P(A)} * x$$

$$E[X|A])P(A) = \sum_x P(X = x \cap A) * x$$

LHS:

$$E[1_A X] = \sum_x 1_A(x) * x * p_X(x)$$

$1_A = 0$, in the cases for x where $\omega \cap A = \phi$ st $X(\omega) = x$,

And

$1_A = 0$, in the cases for x where $\omega \cap A = \phi$ st $X(\omega) = x$,

So our expression now becomes :

$E[1_A X] = \sum_x P(X = x \cap A) * x$

And now, we can equate LHS and RHS, and our proof is complete

Part B.

Let (X) denote the sum of outcomes obtained by rolling a die twice, and let A_i be the event that the first die shows i, for $i \in [1:6]$.

Compute $E[X|A_i]$, for $i \in [1:6]$).

We can decompose:

$X = D_1 + D_2$, where D_1, D_2 are dice random variables for the first and second toss respectively.

$E[X|A_i] = E[D_1 + D_2|D_1 = i]$

$\therefore E[X|A_i] = i + E[D_2|D_1 = i]$ (D_1 is constant in that space, with value of i.)

Since D_1 and D_2 are independent

$E[X|A_i] = i + E[D_2] = i + 3.5$, for a fair dice.