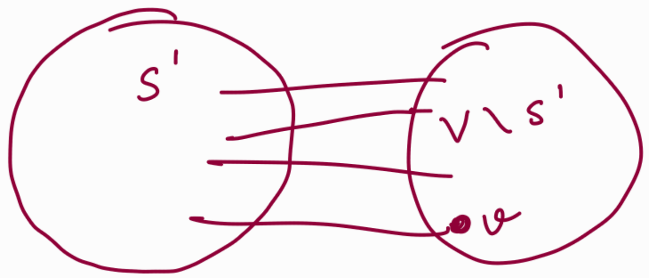
Correctness of Djikstra

Proof by induction : Proof by induction on the size of the set visited (S').

Base case : $|S| = 1$

$S' = \{s\}$, $d[s] = 0$

Inductive hypo : The statement is true $\forall$ S' s.t $|S'| = k$ and $k \geq 1$.
($k \leq n-1$)



First we look at

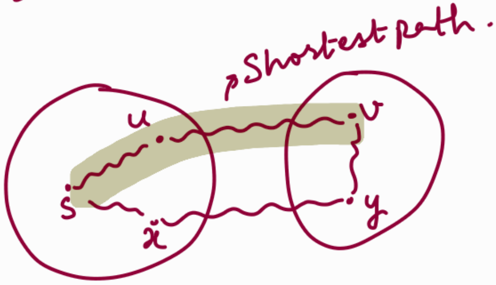
$N(S') \cap (V \setminus S')$

For each $v \in N(S') \cap (V \setminus S')$, compute d' values.

Pick element that attains min. d' value.

Let v attain the min d' value.

Here $d[v]$ is set to $d'[v]$.



How Q is does there exist other path than $s \rightsquigarrow u \rightsquigarrow v$ which gives a shorter path?

Let $x \in S'$ and $(x, y) \in E$.

Claim : $d'[v] \leq d'[y]$. $y \neq v$

$s \rightsquigarrow u \to v$

$= d[u] + wt(u, v)$

$s \rightsquigarrow x \to y \rightsquigarrow v$

$= \underbrace{d[x] + wt(x, y)} + \ell(y, v)$

$\geq d'[y] + \ell(y, v)$

$\left(\because d'[y] = \min\limits_{\substack{(w,y)\in E \\ w \in S'}} \{ d[w] + wt(w,y) \} \right)$

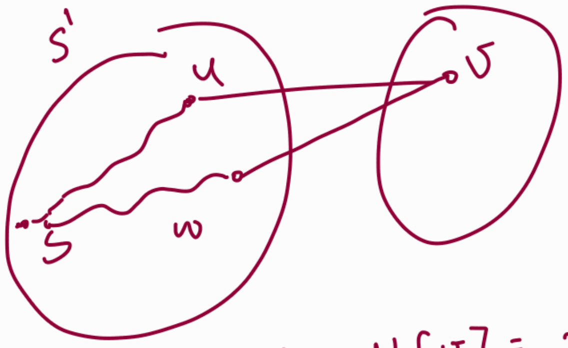
So if x gives min. then equality exists else strictly greater.

$\geq d'[v] + l(y,v) \geq d'[v]$

$(\because l(y,v) > 0)$

Assumption here is edge wt. non-zero positive numbers.

Equality if $l(y,v) \geq 0$



$s \rightsquigarrow w \to v \geq d'[v]$

Say $d'[v]$ attains min via u.

So $d'[v] = \min\limits_{\substack{(a,v)\in E \\ a \in S'}} \{ d'[a] + wt(a,v) \}$.

So $d'[u] + wt(u,v) > d'[w] + wt(w,v)$

→ If there are same dist. values which can be obtained via diff. vertices, then picking any vertex arbitrarily wouldn't change the val. of dist. (doesn't matter). or we could take lexicographically smallest one

Running time analysis

Q1. Should d' be computed every time?

→ In each iteration of the while loop, only the neighbours of the min. vertex could have their d' values updated.

→ Extract min. from the data structure which stores d' values.

→ If v attains min. value then deg(v) many updates are performed.

→ (n-1) iterations of while loop & O(n) book keeping

For visited, d', d

1 extract min. → deg(v) no. of updates.

$\Rightarrow$ $(n-1)$ extract mins + $2m$ updates

Total complexity : $O(n)$ + $(n-1)$ ExtractMin + $2m$ updates.

• Minimum spanning tree

$G = (V, E)$

→ Spanning Tree : Subgraph of G s.t

(i) it's a tree

(ii) contains / covers all the vertices V.

→ If G is connected, then so must the spanning tree.

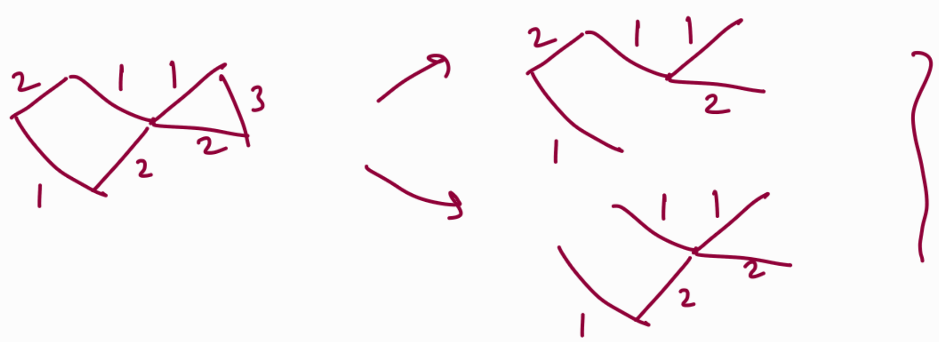
→ Minimum spanning tree given $wt : E \to \mathbb{R}$.

$$\Downarrow$$

$$wt(\text{spanning tree } T) = \sum_{e \in T} wt(e) \text{ is minimum.}$$

$wt = 8$

NOTE : It's possible to have multiple MSTs.



MST.

$wt = 7$

Algorithm to find MST

Shortest path doesn't give MST & MST doesn't give shortest path

Q: If edge wts. are distinct, then we have a unique MST.

Proof:

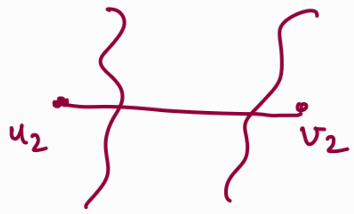
For the sake of contradiction, assume that T_1 & T_2 both are valid distinct trees having wt. C^*.

T_1	T_2
$E(T_1)$	$E(T_2)$
$\exists\ e_1 \in T_1$	$\exists\ e_2 \in T_2$
$e_1 \notin T_2$	$e_2 \notin T_1$
Let $(u_1, v_1) = e_1$	Let $(u_2, v_2) = e_2$

Say $wt(e_1) < wt(e_2)$

Remove e_2 from T_2

Add e_1 to $T_2 \setminus \{e_2\}$



"Swapping arguments"

"Exchange argument".

Incomplete