- Multiplicative rule

→ $P(A \cap B) = P(A)\,P(B|A)$
$\quad\quad\quad\;\; = P(B)\,P(A|B)$ } By defⁿ.

→ Now for 3 sets,

$$P(A \cap B \cap C) = P(A)\,P(B|A)\,P(C|A \cap B)$$

Consider $(A \cap B)$ to be one set.

$$\text{So } P((A \cap B) \cap C) = P(A \cap B)\,P(C|A \cap B)$$
$$= P(A)\,P(B|A)\,P(C|A \cap B)$$

→ For n events,

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_2 \cap A_1) \dots$$
$$\dots P(A_n | A_1 \cap A_2 \cap \dots \cap A_{n-1})$$

(Can be proved by induction).

- Conditional independence

Events A and B are conditionally independent given C if

$$P(A \cap B | C) = P(A|C)\,P(B|C)$$

For indep. events,
$P(A \cap B) = P(A)\,P(B)$
$P(A|B) = P(A)$.

Similarly for $P(A \cap B|C)$
$\downarrow$
$P(A|B \cap C) = P(A|C)$ → Prob. of A given $B \cap C$ is same as prob. of A given C. So prob. of A doesn't depend on B.

How? $$P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)}$$
$$= \frac{P(B \cap C)\,P(A|B \cap C)}{P(C)}$$
$$= P(B|C)\,P(A|B \cap C)$$

Now $P(A|C)\,P(B|C) = P(B|C)\,P(A|B \cap C)$

$\Rightarrow P(A|B\cap C) = P(A|C)$ when $P(B|C) \neq 0$

→ In general, conditional independence & independence are not the same, i.e., $P(A\cap B|C) = P(A|C)\,P(B|C) \not\Leftrightarrow P(A\cap B) = P(A)\,P(B)$.

Eg: $\Omega = \{HT, TH, HH, TT\}$.

$P(\{\omega\}) = \frac{1}{4}, \ \omega \in \Omega$.

H_1 : Event where first toss is head $= \{HT, HH\}$.

H_2 : Event where second toss is head $= \{TH, HH\}$.

E : Both tosses are different $= \{HT, TH\}$

H_1 & H_2 are independent! (Independence is not at all intuitive to me)

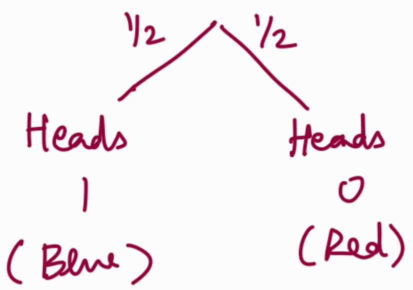
$$\left(P(H_1 \cap H_2) = \frac{1}{4}, \quad P(H_1) = \frac{2}{4} = P(H_2) \right)$$

But H_1 & H_2 are not independent given E.

$$P(H_1 \cap H_2 \mid E) \neq P(H_1 \mid c)\, P(H_2 \mid c)$$

$$\left(P(H_1 \mid c) = \frac{1}{2}, \quad P(H_2 \mid c) = \frac{1}{2}, \quad P(H_1 \cap H_2 \mid E) = 0 \right)$$

Eg: Consider two coins



Coins are chosen each with prob. $1/2$.

Two indep. tosses once a coin is chosen.

H_1 : 1^{st} toss is heads, H_2 : 2^{nd} toss is heads.

B : Blue coin is chosen

$P(B) = \frac{1}{2}$

$P(H_2 | H_1) = 1$. But $P(H_1) \neq 1$ → But this is not 1 becoz only if you choose blue it is 1. Else it is 0

↓

First coin only shows heads. So if you see heads in the first toss, then it is the blue coin and thus prob to see heads in the next toss is 1 as well.

So H_1 & H_2 are not independent ($\because P(H_2|H_1) \neq P(H_1)$)

But $P(H_1 \cap H_2 \mid B) = P(H_1 \mid B)\, P(H_2 \mid B)$ i.e., H_1 & H_2 are conditionally independent.

- Review of counting

Permutation : Given n distinct elements, no. of k-length distinct sequence is ${}^{n}P_{k} = \frac{n!}{(n-k)!}$

$$\overbrace{\boxed{n} \boxed{n-1} \boxed{n-2 \ \ldots}}^{k}$$

Combination : No. of k-size subsets.

$$= {}^{n}C_{k} = \frac{n!}{(n-k)!\,k!}$$

($\because$ Ordering is not imp., ${}^{n}P_{k}$ is divided by $k!$)

(For each subset there are $k!$ ways to arrange

$\longrightarrow$ Given n objects

$n_1 \quad n_2 \quad n_3 \quad \cdots \quad n_r$

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2\cdots-n_{r-1}}{n_r}$$

$$= \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

(EXERCISE)