Lecture 19

(14 October 2024)

Recap

X, $Y = g(X)$ are continuous random variables

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|},$$

where $x_i = a_i(y)$ are the roots of $g(x) = y$

$Z = X + Y$, X & Y are independent

$$f_Z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_X(x)\, f_Y(z-x)\, dx.$$

Functions of Two Random Variables

$$Z = g(X, Y), \text{ i.e.,}$$

$$Z(\omega) = g(X(\omega), Y(\omega)), \quad \forall \omega \in \Omega.$$

Sum of Independent Random Variables:

$Z = X + Y$, X and Y are independent

$$F_Z(t) = P(Z \leq t)$$

$$= P(X + Y \leq t)$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{t-x} f_{XY}(x, y)\, dy\, dx$$

$$= \int_{x=-\infty}^{\infty} f_X(x) \int_{y=-\infty}^{t-x} f_Y(y)\, dy\, dx$$

$$= \int_{x=-\infty}^{\infty} f_X(x) F_Y(t-x)\, dx$$

$$\Rightarrow F_Z'(t) = \frac{d}{dt} \int_{x=-\infty}^{\infty} f_X(x) F_Y(t-x)\, dx$$

$$= \int_{x=-\infty}^{\infty} f_X(x) \frac{d}{dt} F_Y(t-x)\, dx$$

$$= \int_{x=-\infty}^{\infty} f_X(x) f_Y(t-x)\, dx .$$

$$\therefore f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx .$$

X is discrete, Y is continuous, X & Y are independent

$$Z = X + Y$$

$$F_Z(z) = P(X + Y \leq z)$$

$$= \sum_x P(X+Y \leq z \mid X = x)\, P_X(x)$$

$$= \sum_x P(Y \leq z - x \mid X = x)\, P_X(x)$$

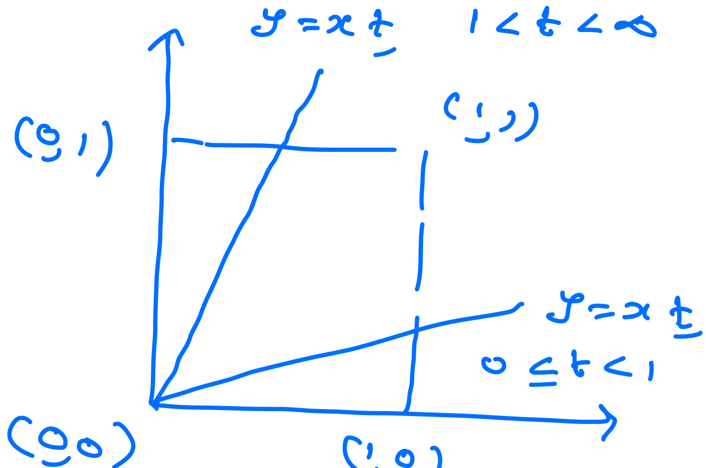
$$= \sum_x P(Y \leq z - x)\, P_X(x)$$

$$= \sum_x F_Y(z-x)\, P_X(x)$$

$$\Rightarrow f_Z(z) = \sum_x f_Y(z-x)\, P_X(x).$$

Exercise. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, X and Y are independent. $Z = X + Y$. Then show that $Z \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Example. If X and Y are independent RVs that are uniformly distributed on $[0,1]$. What is the PDF of $Z = \frac{Y}{X}$?



$$F_Z(t) = P(Z \leq t)$$

$$= P(Y \leq Xt)$$

For $0 < t < 1$

$P(Y \leq Xt)$

$$= P\left((X,Y) \in \{(x,y) \in [0,1]^2 : y \leq xt\}\right)$$

$$= \int_{x=0}^{1} \int_{y=0}^{xt} f_{XY}(x,y)\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{xt} 1\,dy\,dx = \int_{x=0}^{1} xt\,dx = \frac{t}{2}.$$

For $1 < t < \infty$

$$F_Z(t) = P(Y \le X t)$$

$$= P\left((X,Y) \in \{(x,y) \in [0,1]^2 : y \le xt\} \right)$$

$$= \frac{1}{2t} + 1 - \frac{1}{t} = 1 - \frac{1}{2t}.$$

$$\therefore F_Z(t) = \begin{cases} t/2, & \text{if } 0 < t \le 1 \\ 1 - \frac{1}{2t}, & \text{if } 1 < t < \infty. \end{cases}$$

$$\Rightarrow f_Z(t) = \begin{cases} 1/2, & \text{if } 0 < t \le 1 \\ 1/2t^2, & \text{if } t > 1 \\ 0, & \text{otherwise.} \end{cases}$$

Two Functions of Two Random Variables

Let $\underline{x}, \underline{y}$ be jointly continuous RVs, with f_{xy}.

$$Z = g_1(\underline{x}, \underline{y})$$

$$W = g_2(\underline{x}, \underline{y}).$$

The general procedure to find f_{ZW} is:

(1) Compute

$$F_{ZW}(\underline{z}, \underline{w}) = P(Z \le \underline{z}, W \le w)$$

$$= P(g_1(\underline{x}, \underline{y}) \le \underline{z}, g_2(\underline{x}, \underline{y}) \le w)$$

$$= P((\underline{x}, \underline{y}) \in B_{\underline{z}w})$$

$$= \int_{(\underline{x}, \underline{y}) \in B_{\underline{z}w}} f_{xy}(\underline{x}, \underline{y})\, dx\, dy,$$

where $B_{\underline{z}w} = \{(z', w') : g_1(\underline{x}, \underline{y}) \le \underline{z}, g_2(\underline{x}, \underline{y}) \le w\}.$

(2) Take double derivative

$$f_{ZW}(z, w) = \frac{\partial^2 F_{ZW}(z, w)}{\partial z\, \partial w}.$$

If $g_1(x, y)$ and $g_2(x, y)$ are continuous, differentiable, and the mapping

$(g_1, g_2) : (x, y) \mapsto (z, w)$ is one-to-one, then, as in the case of one random variable, it is possible to develop a formula to obtain the joint PDF f_{ZW}.

Let $h_1(z, w)$, $h_2(z, w)$ be the inverse transformations,

$$z = g_1(x, y), \quad w = g_2(x, y)$$

$$\Longrightarrow x = h_1(z, w), \quad y = h_2(z, w).$$

Example. $r = g_1(x, y) = \sqrt{x^2 + y^2}$

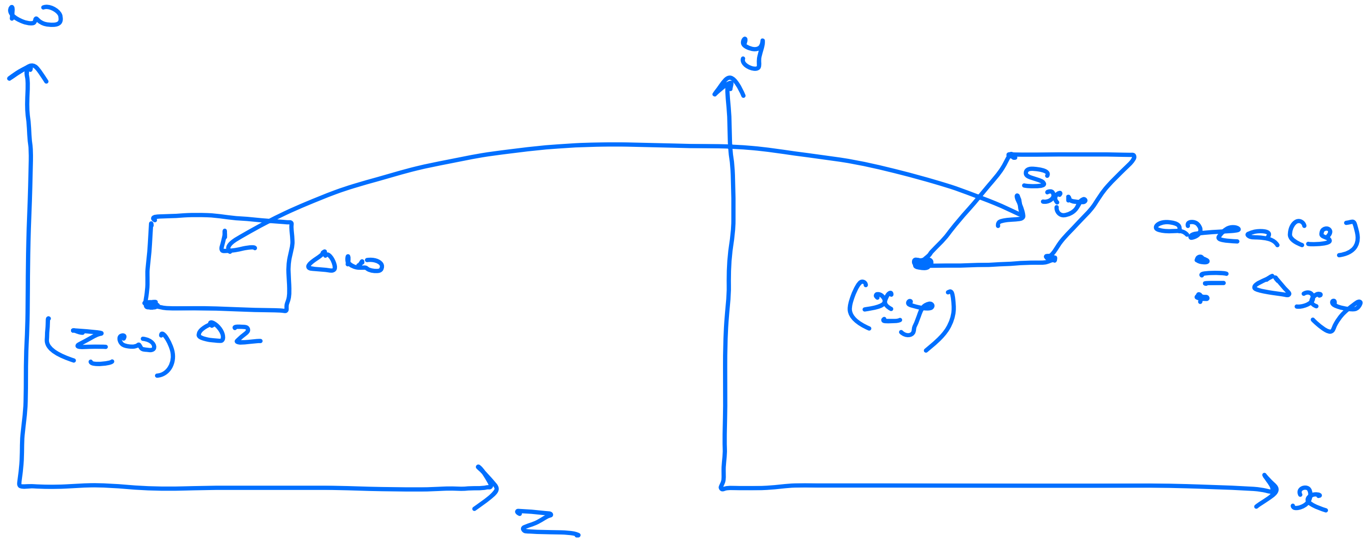
$$\theta = g_2(x, y) = \tan^{-1}(y/x)$$

$x = h_1(r, \theta) = r\cos\theta$,

$y = h_2(r, \theta) = r\sin\theta$.

$(X,Y) \sim f_{XY}$ $\quad Z = g_1(X,Y)$ $\quad W = g_2(X,Y)$

$x = h_1(z,w)$ $\quad y = h_2(z,w)$.



$$P(z < Z \le z+\Delta z,\ w < W \le w+\Delta w)$$

$$= P((X,Y) \in S_{xy})$$

$$\Rightarrow \quad f_{ZW}(z,w)\,\Delta z\,\Delta w = f_{XY}(x,y)\,\Delta_{xy}$$

where $\quad x = h_1(z,w)$ $\quad y = h_2(z,w)$

$$\Rightarrow \quad f_{ZW}(z,w) = f_{XY}(x,y)\left(\frac{\Delta_{xy}}{\Delta z\,\Delta w}\right)$$

$$= f_{XY}(x,y)\,|J(z,w)| \text{ as } \Delta z, \Delta w \to 0,$$

where

$$J(z,w) = \begin{vmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{\partial h_1(z,w)}{\partial z} & \frac{\partial h_1(z,w)}{\partial w} \\ \frac{\partial h_2(z,w)}{\partial z} & \frac{\partial h_2(z,w)}{\partial w} \end{vmatrix}$$

$$= \frac{1}{J(x,y)}, \quad \text{where}$$

$$J(x,y) = \begin{vmatrix} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \end{vmatrix}_{\substack{x=h_1(z,w) \\ y=h_2(z,w)}}.$$

$$\therefore f_{ZW}(z,w) = \frac{f_{XY}(h_1(z,w),\, h_2(z,w))}{|J(x,y)|}, \quad \substack{x=h_1(z,w) \\ y=h_2(z,w)}$$

where

$$J(x,y) = \begin{vmatrix} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \end{vmatrix}.$$

Formally the following statement is true.

Theorem.

If $g_1(x,y)$ and $g_2(x,y)$ are continuous, differentiable, and the mapping

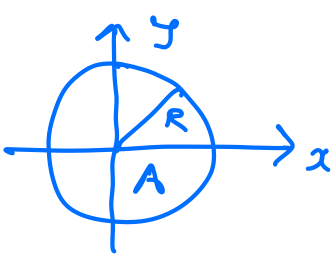
$(g_1, g_2): (x,y) \mapsto (z,w)$ is one-to-one.

Assume (h_1,h_2) is the inverse mapping i.e., $x = h_1(z,w)$, $y = h_2(z,w)$. Let $(g_1,g_2): A \to B$.

Then

$$\int_{(x,y)\in A} f(x,y)\,dx\,dy = \int_{(z,w)\in B} \frac{f(h_1(z,w), h_2(z,w))}{|J(h_1(z,w), h_2(z,w))|}\,dz\,dw.$$

Let us get some intuition about why this has to be true.



$$\int_{(x,y)\in A} 1 \cdot dx\,dy = \int_{x=-R}^{R} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1 \cdot dx\,dy$$

$$= \int_{r=0}^{R} \int_{\theta=0}^{2\pi} 1 \cdot \underline{\underline{r}}\, dr\,d\theta.$$

Here γ is nothing but the Jacobian inverse.

Also from the above discussion on the ratio of infinitesimal areas

$$\int_{(x,y)\in A} f(x,y)\,dx\,dy$$

$$= \int_{(z,w)\in B} \frac{f(h_1(z,w)\,h_2(z,w))}{\left(\frac{dz\,dw}{dx\,dy}\right)} \cdot dz\,dw$$

$$= \int_{(z,w)\in B} \frac{f(h_1(z,w)\,h_2(z,w))}{\left|J(h_1(z,w)\,h_2(z,w))\right|}\,dz\,dw.$$

So $F_{ZW}(u,v) = P(g_1(x,y)\le u,\ g_2(x,y)\le v)$

$$= \int_{\substack{(x,y):\ g_1(x,y)\le u \\ g_2(x,y)\le v}} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{\substack{(z,w):\ z\le u \\ w\le v}} \frac{f_{XY}(h_1(z,w)\,h_2(z,w))}{\left|J(h_1(z,w)\,h_2(z,w))\right|}\,dz\,dw$$

$$= f_{ZW}(z,w).$$

<u>Example</u>. Let X and Y are independent and identically distributed $N(0,1)$.

$$R=\sqrt{X^2+Y^2}, \quad \Theta=\arctan(Y/X).$$

$$P((X,Y)\in\mathbb{R}^2) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$$

$$= P\left(R\in[0,\infty),\ \Theta\in[0,2\pi]\right)$$

$$= \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} f_{R\Theta}(r,\theta)\,dr\,d\theta$$

$$f_{R\Theta}(r,\theta) = \frac{f_{XY}(r\cos\theta,\ r\sin\theta)}{\left|J(x,y)\right|_{\substack{x=r\cos\theta,\\ y=r\sin\theta}}},$$

$$J(x,y) = \begin{vmatrix} \dfrac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial x} & \dfrac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial y} \\[2ex] \dfrac{\partial\left(\tan^{-1}(y/x)\right)}{\partial x} & \dfrac{\partial}{\partial x}\left(\tan^{-1}(y/x)\right) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}}.$$

So, we have

$$f_{ZW}(z,w) = f_{XY}(r\cos\theta, r\sin\theta) \cdot r.$$

Recall that we considered

$$I^2 = \frac{1}{2\pi} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy$$

while showing that Gaussian PDF integrates to 1,

$$I^2 = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{XY}(x,y) \, dx \, dy$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} f_{R\Theta}(r,\theta) \, dr \, d\theta$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} f_{XY}(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{2\pi} e^{-r^2/2}\, r\, dr\, d\theta = 1.$$

Exercise.

Let $(X, Y) \sim f_{XY}$.

$$Z = aX + bY$$

$$W = cX + dY.$$

Assume $ad - bc \neq 0$.

Show that $f_{ZW}(z, w) = \dfrac{f_{XY}\left(\frac{dz - bw}{|ad-bc|}, \frac{-cz + aw}{|ad-bc|}\right)}{|ad-bc|}$.