Assignment 6

(MA6.102) Probability and Random Processes, Monsoon 2024

Release date: 29 October 2024, Due date: 6 November 2024

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. Suppose X and Y are jointly continuous random variables with joint PDF f_{XY}. Let $Z = \min\{X, Y\}$ and $W = \max\{X, Y\}$.

(a) Find the joint PDF f_{ZW} in terms of f_{XY} using Jacobian determinants.

(b) If X and Y are independent and uniformly distributed on $[0, 1]$, compute f_{ZW}.

Problem 2. Let X be a Gaussian random variable with mean μ and variance σ^2. Compute the moment generating function of X, $M_X(s) = \mathbb{E}[\mathrm{e}^{sX}]$.

Problem 3. Recall that a sequence of random variables $X_1, X_2, \ldots$ converges in probability to X if

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0, \text{ for every } \epsilon > 0.$$

If we modify the definition by replacing $P(|X_n - X| > \epsilon)$ with $P(|X_n - X| \geq \epsilon)$, is the resulting definition equivalent to the original one?

Problem 4. In order to estimate f, the true fraction of smokers in a large population, Alvin selects n people at random. His estimator M_n is obtained by dividing S_n, the number of smokers in his sample, by n, i.e., $M_n = \frac{S_n}{n}$. Alvin chooses the sample size n to be the smallest possible number for which the Chebyshev's inequality yields a guarantee that

$$P(|M_n - f| \geq \epsilon) \leq \delta,$$

where ϵ and δ are some prespecified tolerances. Determine how the value of n recommended by the Chebyshev's inequality changes in the following cases.

(a) The value of ϵ is reduced to $\frac{2}{3}$ of its original value.

(b) The probability δ is reduced to $\frac{3}{5}$ of its original value.

Problem 5. Show that if a sequence of random variables $X_1, X_2, \ldots$ converges in distribution to c, then X_n also converges to c in probability.

Problem 6. Suppose a sequence of random variables $X_1, X_2, \ldots$ converges in probability to X and $Y_1, Y_2, \ldots$ converges in probability to Y. Prove that the sequence $X_1 + Y_1, X_2 + Y_2, \ldots$ converges in probability to $X + Y$.