

# Quantum Mechanics

# Mathematical Interlude

in qm, operators can generally be thought of as infinite dimensional matrices

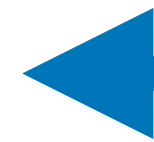
## Fourier transform of a delta function

<https://math.stackexchange.com/a/3814300>

Plancherel's theorem

generalised function, not just a function  
integral is easy to define, the actual fn is tougher

$$\delta(x) = \frac{1}{2\pi} \int e^{ikx} dk$$



$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k) e^{ikx} dk \iff F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

spectrum of a matrix is the set of eigenvalues that can be taken

as in the free particle space

When the **spectrum** of a hermitian operator is continuous, the individual solutions are not-normalisable. Nevertheless, there is a sense of orthonormality and completeness among the eigenvectors.

Let  $f_p(x)$  be the eigenfunction and  $p$  the eigenvalue of the momentum operator.

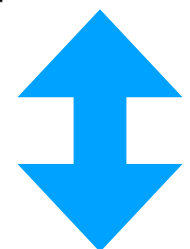
$$\frac{\hbar}{i} \frac{d}{dx} f_p(x) = p f_p(x). \quad f_p(x) = A e^{ipx/\hbar}$$

$$\int_{-\infty}^{\infty} f_{p'}^*(x) f_p(x) dx = |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar} dx = |A|^2 2\pi \hbar \delta(p - p')$$

in the case of simple harmonic oscillator, it was kronecker delta, here its dirac delta

If we pick  $A = 1/\sqrt{2\pi\hbar}$ , so that  $f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ .

$$\langle f_{p'} | f_p \rangle = \delta(p - p'),$$



$$\langle f_m | f_n \rangle = \delta_{mn}$$

Any (square-integrable) function  $f(x)$  can be written in the form

$$f(x) = \int_{-\infty}^{\infty} c(p) f_p(x) dp = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{ipx/\hbar} dp.$$

# Generalised Statistical Interpretation

If you measure an observable  $Q(x, p)$ , you would get one of the eigenvalue of  $\hat{Q}(\hat{x}, \hat{p})$

If the spectrum of  $\hat{Q}$  is discrete, the probability of getting a particular e.value  $q_n$  associated with e.vector  $f_n(x)$  is

$$|c_n|^2, \quad \text{where} \quad c_n = \langle f_n | \Psi \rangle.$$

Ortho-normalised

If the spectrum of  $\hat{Q}$  is continuous with e.values  $q(z)$  associated with e.vectors  $f_z(x)$ , the probability of getting a result in the range  $dz$  is

$$|c(z)|^2 dz \quad \text{where} \quad c(z) = \langle f_z | \Psi \rangle$$

$$\langle Q \rangle = \sum_n q_n |c_n|^2.$$

Upon measurement, the wave function collapses to  $f_n$  or a narrow range about  $f_z$  depending on the precision of the measurement.

$c_n$  is a function of time

Completeness  $\longrightarrow \Psi(x, t) = \sum_n c_n f_n(x)$

$$\begin{aligned} 1 = \langle \Psi | \Psi \rangle &= \left\langle \left( \sum_{n'} c_{n'} f_{n'} \right) \middle| \left( \sum_n c_n f_n \right) \right\rangle = \sum_{n'} \sum_n c_{n'}^* c_n \langle f_{n'} | f_n \rangle \\ &= \sum_{n'} \sum_n c_{n'}^* c_n \delta_{n'n} = \sum_n c_n^* c_n = \sum_n |c_n|^2. \end{aligned}$$

example of the continuous one- the same as the free particle idea

$$\begin{aligned} \Phi(p, t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x, t) dx; \\ \Psi(x, t) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \Phi(p, t) dp. \end{aligned}$$

# Generalised Uncertainty Principle

(value-expectation)<sup>2</sup>: since hermitian, we can write each on one side

For any observable  $A$ , we have

$$\sigma_A^2 = \langle (\hat{A} - \langle A \rangle) \Psi | (\hat{A} - \langle A \rangle) \Psi \rangle = \langle f | f \rangle,$$

where  $f \equiv (\hat{A} - \langle A \rangle) \Psi$ . Likewise, for any *other* observable,  $B$ ,

$$\sigma_B^2 = \langle g | g \rangle, \quad \text{where } g \equiv (\hat{B} - \langle B \rangle) \Psi.$$

Therefore

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2.$$

Now, for any complex number  $z$ ,

$$|z|^2 = [\text{Re}(z)]^2 + [\text{Im}(z)]^2 \geq [\text{Im}(z)]^2 = \left[ \frac{1}{2i} (z - z^*) \right]^2.$$

Therefore, letting  $z = \langle f | g \rangle$ ,

$$\sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} [\langle f | g \rangle - \langle g | f \rangle] \right)^2.$$

if there are two eigenvectors that are commuting means they have common eigenvectors

commuting of 2 operators means  
 $[A, B] = 0 \Rightarrow AB = BA$

But

$$\begin{aligned} \langle f | g \rangle &= \langle (\hat{A} - \langle A \rangle) \Psi | (\hat{B} - \langle B \rangle) \Psi \rangle = \langle \Psi | (\hat{A} - \langle A \rangle) (\hat{B} - \langle B \rangle) \Psi \rangle \\ &= \langle \Psi | (\hat{A} \hat{B} - \hat{A} \langle B \rangle - \hat{B} \langle A \rangle + \langle A \rangle \langle B \rangle) \Psi \rangle \\ &= \langle \Psi | \hat{A} \hat{B} \Psi \rangle - \langle B \rangle \langle \Psi | \hat{A} \Psi \rangle - \langle A \rangle \langle \Psi | \hat{B} \Psi \rangle + \langle A \rangle \langle B \rangle \langle \Psi | \Psi \rangle \\ &= \langle \hat{A} \hat{B} \rangle - \langle B \rangle \langle A \rangle - \langle A \rangle \langle B \rangle + \langle A \rangle \langle B \rangle \\ &= \langle \hat{A} \hat{B} \rangle - \langle A \rangle \langle B \rangle. \end{aligned}$$

Similarly,

$$\langle g | f \rangle = \langle \hat{B} \hat{A} \rangle - \langle A \rangle \langle B \rangle,$$

so

$$\langle f | g \rangle - \langle g | f \rangle = \langle \hat{A} \hat{B} \rangle - \langle \hat{B} \hat{A} \rangle = \langle [\hat{A}, \hat{B}] \rangle,$$

where

$$[\hat{A}, \hat{B}] \equiv \hat{A} \hat{B} - \hat{B} \hat{A}$$

is the commutator of the two operators

$$\sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2.$$

# Energy Time Relation

As a measure of how fast the system is changing, let us compute the time derivative of the expectation value of some observable,  $Q(x, p, t)$ :

$$\frac{d}{dt}\langle Q \rangle = \frac{d}{dt}\langle \Psi | \hat{Q} | \Psi \rangle = \left\langle \frac{\partial \Psi}{\partial t} \left| \hat{Q} \Psi \right. \right\rangle + \left\langle \Psi \left| \frac{\partial \hat{Q}}{\partial t} \Psi \right. \right\rangle + \left\langle \Psi \left| \hat{Q} \frac{\partial \Psi}{\partial t} \right. \right\rangle.$$

Now, the Schrödinger equation says

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$$

(where  $H = p^2/2m + V$  is the Hamiltonian). So

$$\frac{d}{dt}\langle Q \rangle = -\frac{1}{i\hbar}\langle \hat{H} \Psi | \hat{Q} \Psi \rangle + \frac{1}{i\hbar}\langle \Psi | \hat{Q} \hat{H} \Psi \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle.$$

But  $\hat{H}$  is hermitian, so  $\langle \hat{H} \Psi | \hat{Q} \Psi \rangle = \langle \Psi | \hat{H} \hat{Q} \Psi \rangle$ , and hence

$$\boxed{\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{Q}] \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle.}$$

Quaso bracket is similar to Bracket  
Commutator is generalised Quaso bracket

Q does not have explicit time  
dependance  
because closed system is  
considered in qm  
non-dissipative

Now, suppose we pick  $A = H$  and  $B = Q$ , in the generalized uncertainty principle, and assume that  $Q$  does not depend explicitly on  $t$ :

$$\sigma_H^2 \sigma_Q^2 \geq \left( \frac{1}{2i} \langle [\hat{H}, \hat{Q}] \rangle \right)^2 = \left( \frac{1}{2i} \hbar \frac{d\langle Q \rangle}{dt} \right)^2 = \left( \frac{\hbar}{2} \right)^2 \left( \frac{d\langle Q \rangle}{dt} \right)^2.$$

Or, more simply,

$$\sigma_H \sigma_Q \geq \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|.$$

Let's define  $\Delta E \equiv \sigma_H$ , and

$$\Delta t \equiv \frac{\sigma_Q}{|d\langle Q \rangle/dt|}.$$

Then

$$\Delta E \Delta t \geq \frac{\hbar}{2},$$

we cannot in qm, say uncertainty in  
measurement of t, because t is a parameter

and that's the energy-time uncertainty principle. But notice what is meant by  $\Delta t$ , here: Since

$$\sigma_Q = \left| \frac{d\langle Q \rangle}{dt} \right| \Delta t,$$

$\Delta t$  represents the *amount of time it takes the expectation value of  $Q$  to change by one standard deviation*. In particular,  $\Delta t$  depends entirely on what observable ( $Q$ ) you care to look at—the change might be rapid for one observable and slow for another. But if  $\Delta E$  is small, then the rate of change of *all* observables must be very gradual; or, to put it the other way around, if *any* observable changes rapidly, the “uncertainty” in the energy must be large.

energy conservation can be violated for a short time  
violates 1st law of thermodynamics