CS 302.1 - Automata Theory

Lecture 04

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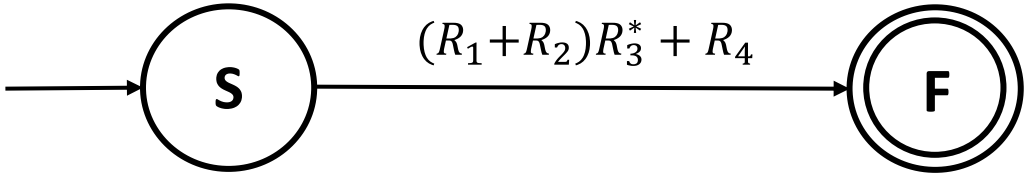
Quick Recap

- RL can also be derived from first principles.
- Regular expressions provide an elegant algebraic framework to represent regular languages.
- We can construct NFAs given a Regular Expression.

A Generalized NFA (GNFA) is similar to an NFA except that transitions contain regular expressions.

Given a DFA M, we obtain the regular expression corresponding to $L(M)$ by constructing a 2-state GNFA via a recursive algorithm.



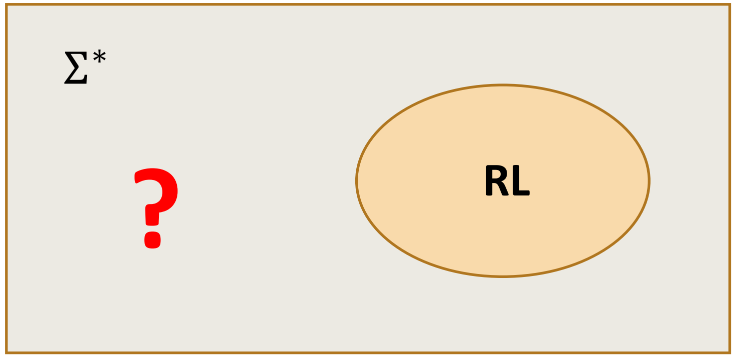
DFA, NFA, Regular Expressions have equal power and all of them correspond to Regular Languages

Pumping Lemma

Recall that so far, we have proven that the following statements are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathcal{L}(D) = L$.
- There is an NFA N such that $\mathcal{L}(N) = L$.
- There is a regular expression R such that $\mathcal{L}(R) = L$.

- Not all languages are regular.



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How do we prove that certain languages are non-regular? We start with an example

Let $\Sigma = \{0,1\}$. Consider the language $L = \{0^n 1^n | n \geq 0\}$ and the following conversation between Karl and Mil.

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Mil: I have a DFA for L.
Karl: How many states are there?
Mil: n-states (say $n = 10$)

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Mil: I have a DFA for L.
Karl: How many states are there?
Mil: n-states (say $n = 10$)
Karl: Then $0^{10}1^{10}$ must be accepted.
By the **pigeonhole principle**, while reading the first $(n = 10)$ symbols, some states need to be revisited. Otherwise $n + 1 = 11$ states would have been present. Hence some loop must be present. How many states are there in the loop?

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Karl: How many states are there?
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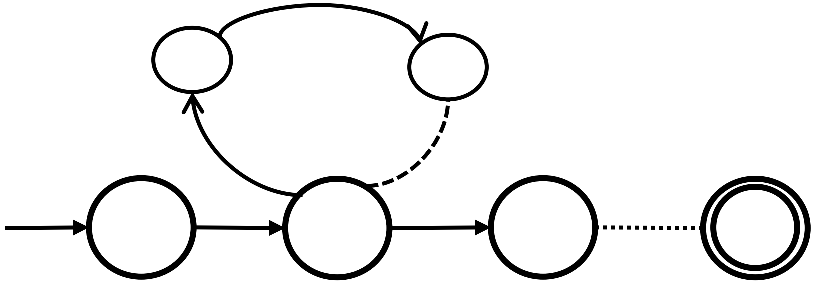
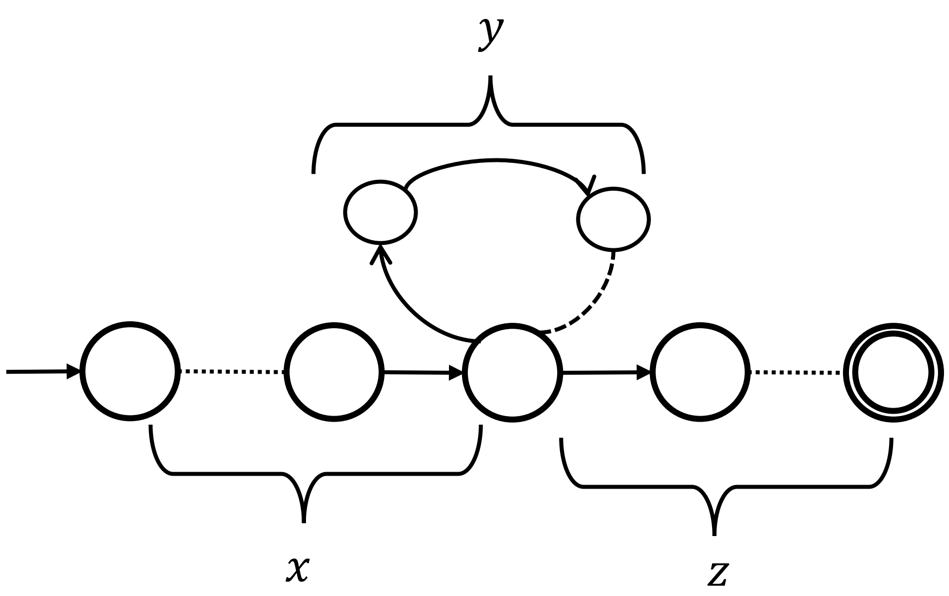
Karl: Then $0^{10}1^{10}$ must be accepted. By the **pigeonhole principle**, while reading the first $(n = 10)$ symbols, some states need to be revisited. Otherwise $n + 1 = 11$ states would have been present. Hence some loop must be present. How many states are there in the loop?
Mil: t-states (say $t = 3$).
Karl: If your DFA accepts $0^n 1^n$, it must also accept $0^{n+t}1^n$. This is because, if we take the loop one extra time, we read t more 0's.



Contradiction as $0^{n+t}1^n \notin L$. So Mil, you never had a DFA for L and in fact, ***L* is not regular.**

Pumping Lemma

If L is a regular language, all strings in the language, larger than a certain length (pumping length) , can be *pumped*: the string contains a certain section that can be repeated *any number of times* and the resulting string still $\in L$.



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(Pumping Lemma) If L is a regular language, then there exists a number p (the pumping length) where for all $s \in L$ of length at least p, there exists x, y, z such that $s = xyz$, such that

1. $|xy| \le p$.
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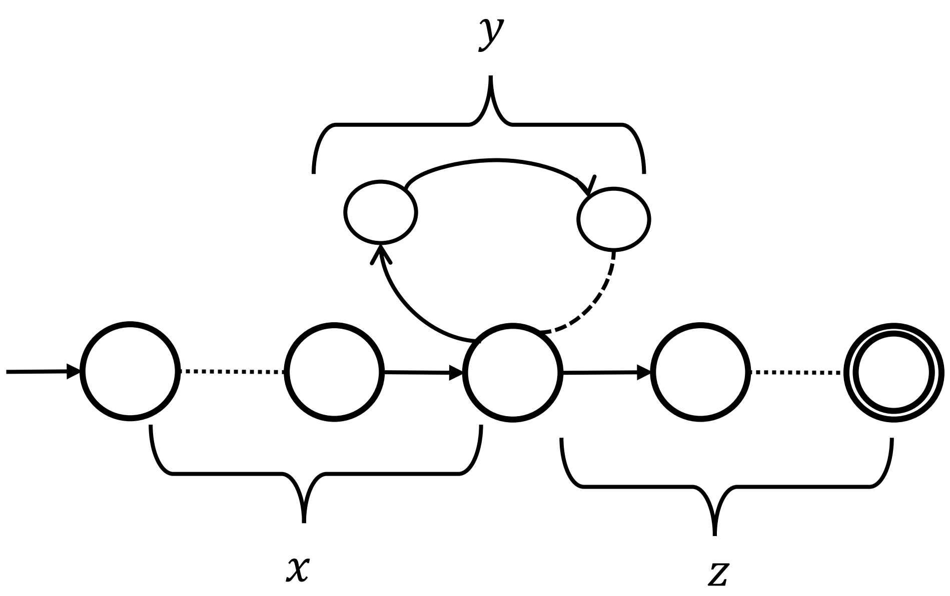
3. $\forall i \ge 0, xy^iz \in L$.



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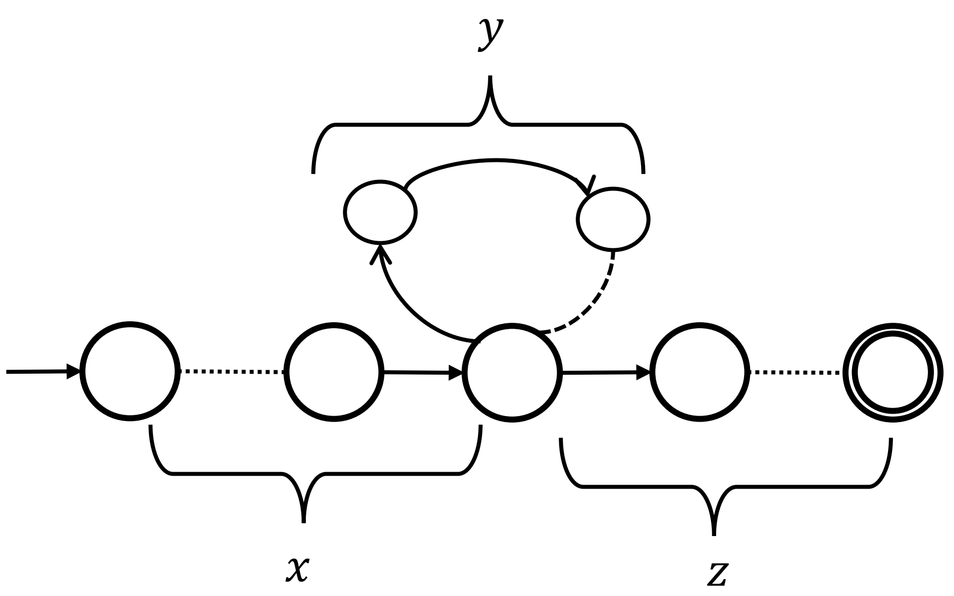
Note: $(A \Rightarrow B) \equiv (\neg B) \Rightarrow (\neg A)$

If L is regular then, pumping property is satisfied

$\equiv$

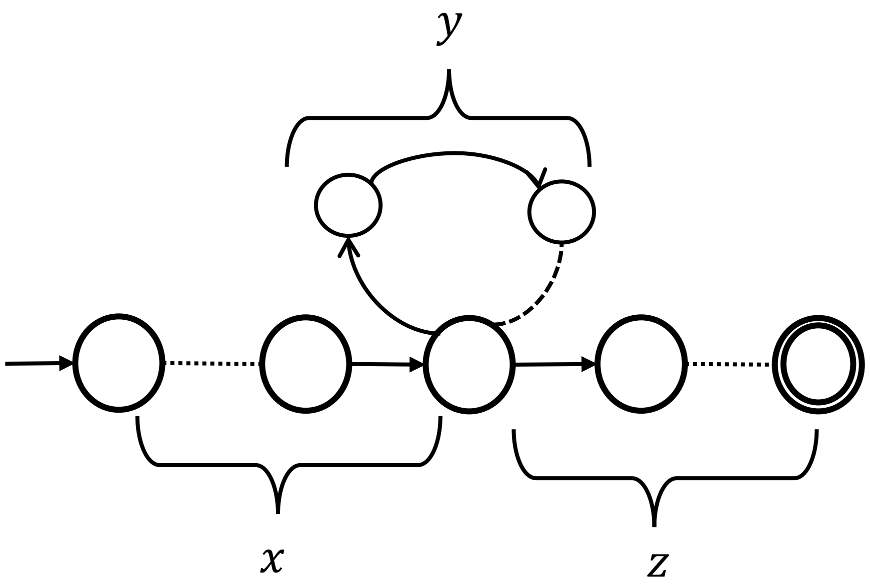
If pumping property is NOT satisfied, then L is NOT regular.



Pumping Lemma

Proof sketch: Suppose that we have a DFA M of p states. Then any run in the DFA corresponding to strings of length at least p, some states are repeated.

This is because of the ***pigeonhole principle***: any such run would encounter $p + 1$ states, but there are p distinct states in the DFA.



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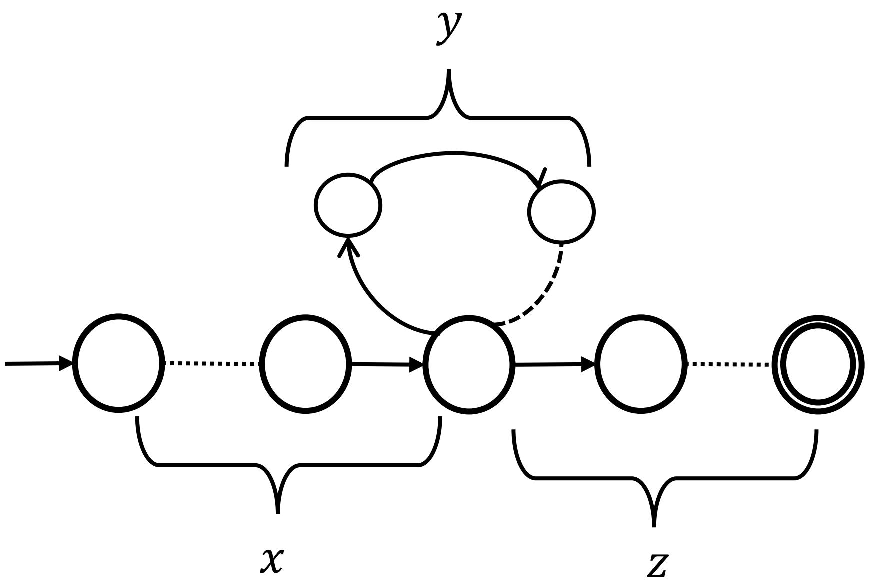
Suppose $s = s_1 s_2 \cdots s_n$ be any such string of length n $(\geq p)$ and suppose $r_1 r_2 \cdots r_{n+1}$ be the sequence of states encountered, while implementing a run of s in M.

As $n+1 \geq p+1$, in the above sequence at least two states must be repeated. Let them be r_j and r_l, i.e., $r_j = r_l$, but $j \neq l$.



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So we can divide the s into three parts, $x = s_1 \ldots s_{j-1}$, $y = s_j \ldots . s_{l-1}$, $z = s_l \ldots s_n$. For a run on M, due to s

- the x part takes us from r_1 to r_j
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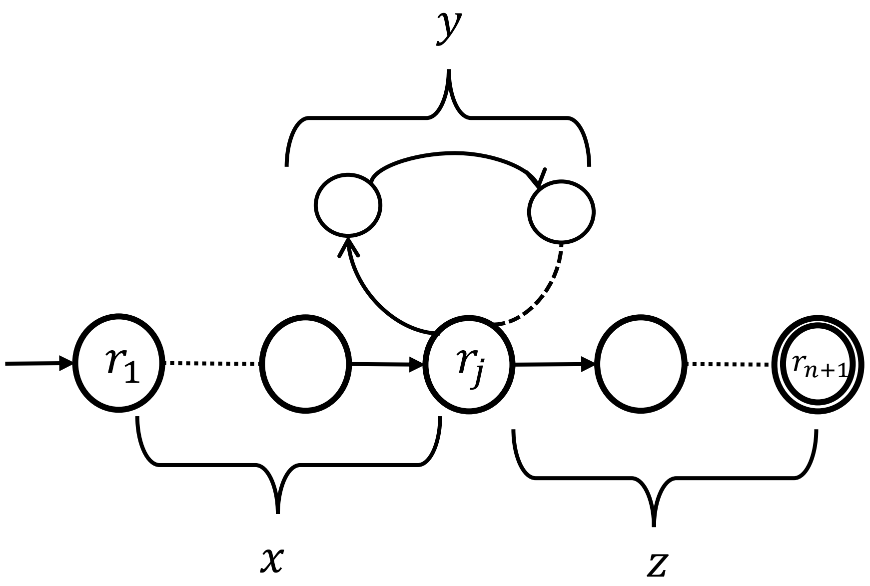

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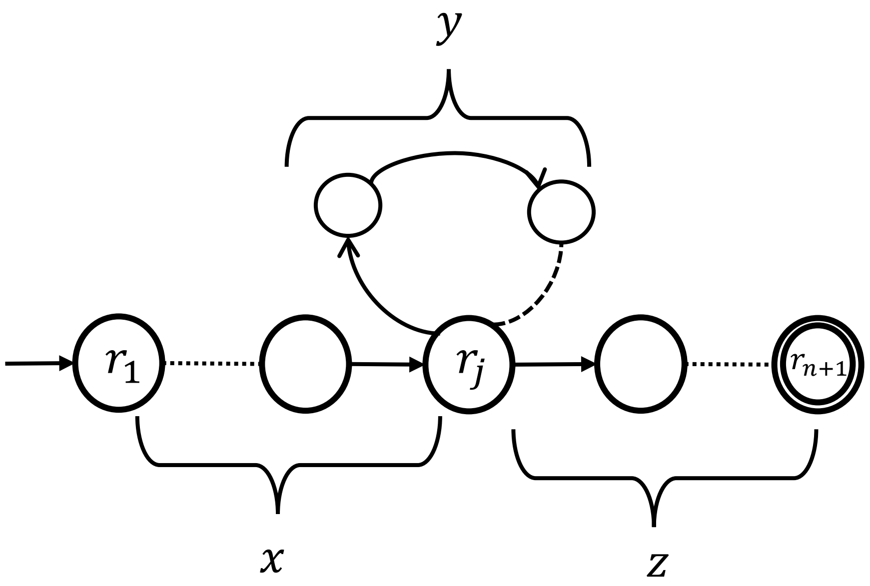

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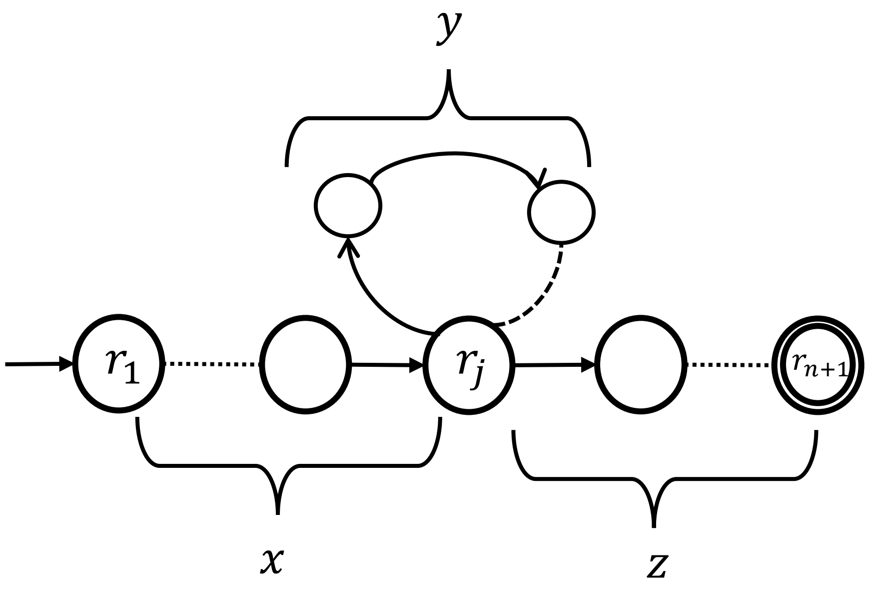

- We can traverse the loop bit any number of times and so $\forall i \geq 0, xy^i z \in L$.
- Also, as $j \neq l, |y| \geq 1$
- While reading the input, within the first p symbols of s, some state must be repeated.

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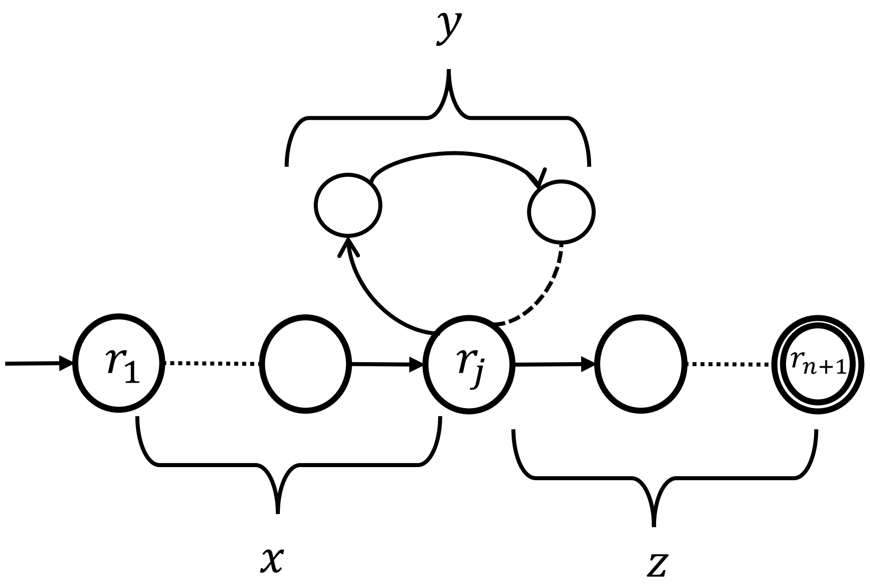

- We can traverse the loop bit any number of times and so $\forall i \geq 0, xy^i z \in L$.
- Also, as $j \neq l$, $|y| \geq 1$, and
- The DFA reads $|xy|$ by then and so $|xy| \leq p$.

Pumping Lemma

In order to prove that a language is non-regular,

- Assume that it is regular and obtain a contradiction.
- Find a string in the language of length $\geq p$ (pumping length) that cannot be pumped.

Examples of languages that are NOT regular:

- $\{0^n 1^n | n \geq 0\}$
- $\{\omega | \omega$ has equal number of 0's and $1's\}$
- $\{\omega | \omega$ is palindrome$\}$

⋮

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(Pumping Lemma) If L is a regular language, then **there exists** a number $\boldsymbol{p}$ (the pumping length) where **for all $\boldsymbol{s} \in \boldsymbol{L}$ of length at least $\boldsymbol{p}$, there exists $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$** such that $s = xyz$, such that

1. $|xy| \leq p$.
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Refer to Sipser (or some other textbook) for proofs using Pumping lemma

The story so far…

- We have built devices (DFAs/NFAs) that decides some languages.
- Regular languages are precisely the ones that are accepted by finite automata.
- For any $L \in RL$, we have DFA/NFA M such that $L(M) = L$.
- Regular expressions describe regular languages algebraically.
- There are languages that are not regular.

DFA $\equiv$ NFA $\equiv$ Regular Expressions

Next up:

- How do we generate the strings in a language?
- **Syntax:** What are the set of legal strings in a language?
- Think of the English language (Rules of **grammar**)

Grammars

- **Grammars** provide a way to generate strings belonging to a language. The set of all strings generated by the grammar is the *language* of the grammar.

- ***Grammars generate languages:*** Grammars consist of a set of ***rules*** that allow you to construct strings of the language.

- For some classes of grammars, one can build automata that recognizes the language generated by the grammar.

- In fact, these concepts have been fundamental in attempts to formalize natural languages.

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- Consider these rules

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\begin{aligned}
&Sentence \rightarrow Subject\ Verb\ Object\\
&Subject \rightarrow Noun.phrase\\
&Object \rightarrow Noun.phrase\\
&Noun.phrase \rightarrow Article\ Noun|Noun\\
&Article \rightarrow \boldsymbol{the}\\
&Noun \rightarrow \boldsymbol{boy}|\boldsymbol{girl}|\boldsymbol{soccer}|\boldsymbol{poetry}\\
&Verb \rightarrow \boldsymbol{loves}|\boldsymbol{plays}
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Terminals consist of strings over the alphabet corresponding to the language that the Grammar generates

Variables: $\{Sentence, Subject, Verb, Object, Noun, Noun.phrase, Article\}$, **Terminals:** $\{the, girl, loves, plays, soccer, poetry\}$
Start Variable: $Sentence$

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\end{aligned}$$

The sentence "**the girl plays soccer**" can be derived from this set of rules.

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&\rightarrow \boldsymbol{the\ girl}\ Verb\ Object\\
&\rightarrow \boldsymbol{the\ girl\ plays}\ Object\\
&\rightarrow \boldsymbol{the\ girl\ plays}\ Noun.phrase\\
&\rightarrow \boldsymbol{the\ girl\ plays}\ Noun\\
&\rightarrow \boldsymbol{the\ girl\ plays\ soccer}
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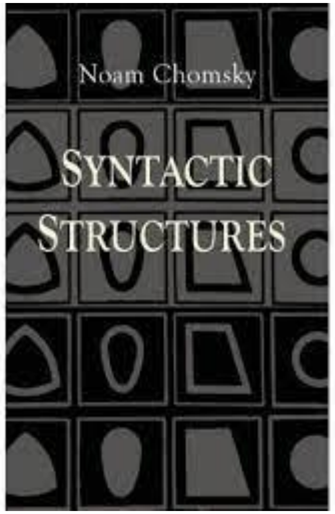
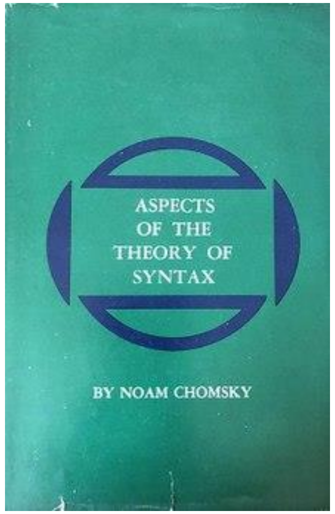
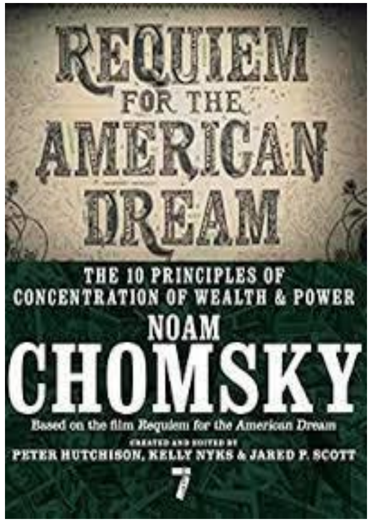
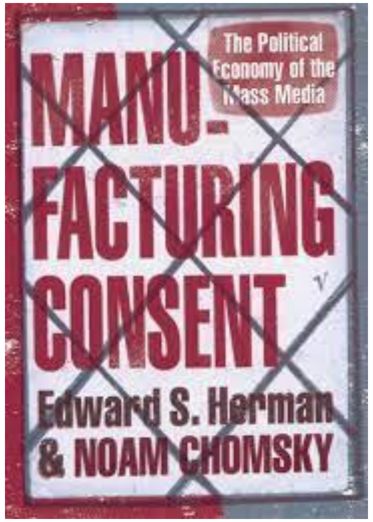
Grammars





Noam Chomsky





- Noam Chomsky did pioneering work on linguistics and formalized many of these concepts.

- Also made great contributions to political economy and has been a champion of anti-imperialist, anti-capitalist, social justice struggles across the globe.

Grammars

(Grammar) Formally, a *Grammar* G is a 4-tuple (V, Σ, P, S) such that

- V is the set of **Variables**
- Σ is the set of **Terminals** (disjoint from V)
- P is the set of production **Rules** [$(V \cup \Sigma)^* V (V \cup \Sigma)^* \to (V \cup \Sigma)^*$]
- S is the **Start Variable** [The variable in the LHS of the first rule is generally the start variable]

Eg: Consider the grammar G

$X \to 1X$
$X \to 0Y$
$Y \to 0X$
$Y \to 1Y$
$Y \to \epsilon$

X is the start variable of the Grammar. Variables: $\{X, Y\}$, Terminals: $\{\epsilon$, 0, 1$\}$

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Grammars can be used to derive strings.

The sequence of **substitutions** (using the rules of G) required to obtain a certain string is called a **derivation**.

- Begin the **derivation** from the **Start variable.**
- Replace any variable according to a rule. Repeat until only terminals remain.
- The generated string is **derived by the grammar.**

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$X \to 1X$
$X \to 0Y$
$Y \to 1Y$
$Y \to 0X$
$Y \to \epsilon$

X: Start Variable
$\{X, Y\}$: Variables
$\{\epsilon, 0, 1\}$: Terminals

The following is a derivation

$$X \to 1X \to 11X \to 110Y \to 1101Y \to \mathbf{1101}$$

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- To show that a string $\boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{G})$, we show that there exists a **derivation ending up in $\boldsymbol{w}$**. The fact that w can be derived using the rules of G, is expressed as $\boldsymbol{S} \overset{*}{\Rightarrow} \boldsymbol{w}$.

- The **language of the grammar, $\boldsymbol{L}(\boldsymbol{G})$** is $\{w \in \Sigma^* | S \overset{*}{\Rightarrow} w\}$

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$Y \rightarrow 0X$
$Y \rightarrow \epsilon$

The string $\mathbf{1101} \in \boldsymbol{L}(\boldsymbol{G})$ because there exists the following derivation

$$X \rightarrow 1X \rightarrow 11X \rightarrow 110Y \rightarrow 1101Y \rightarrow 1101$$

Grammars for Regular Languages

Regular grammar: If the *rules* of the underlying grammar G are of the form

$$Var \to Ter\ \boldsymbol{Var}$$
$$Var \to Ter$$
$$Var \to \boldsymbol{\epsilon}$$

then the language of the grammar is **regular.** Also known as **Right-linear grammar** (all variables are to the right of terminals in the RHS).

Right linear Grammar to DFA

Eg: Consider the grammar G

$X \to 1X$
$X \to 0Y$
$Y \to 1Y$
$Y \to 0X$
$Y \to \epsilon$ (indicates that Y is the final state)

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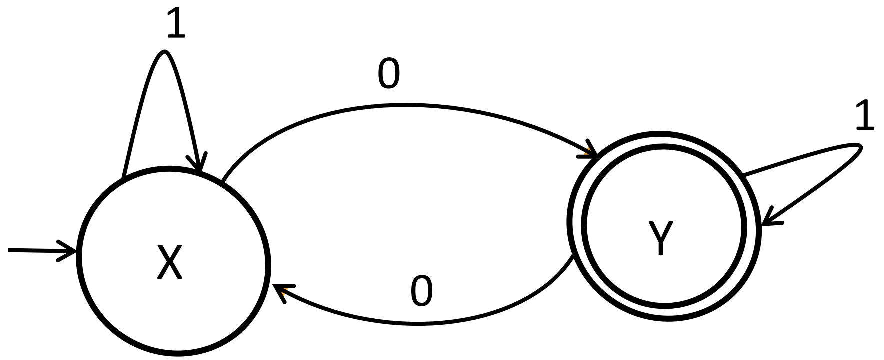
$Y \rightarrow 0X$
$Y \rightarrow \epsilon$ (indicates that Y is the final state)



Grammars for Regular Languages

Regular grammar: If the *rules* of the underlying grammar G are of the form

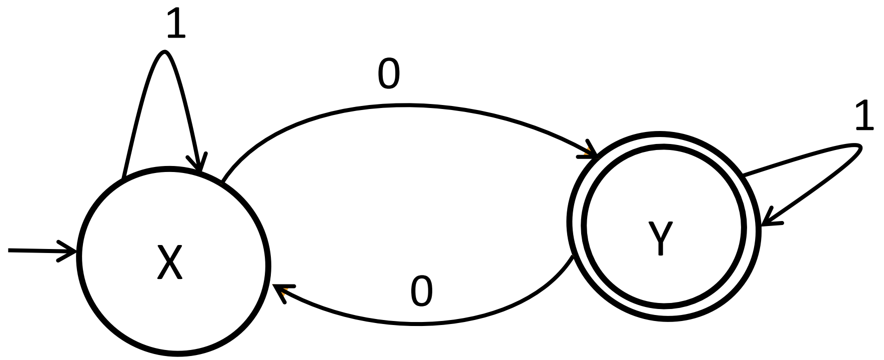
$$Var \rightarrow Ter\ \boldsymbol{Var}$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then the language of the grammar is **regular.** Also known as **Right-linear grammar** (all variables are to the right of terminals in the RHS).

Right linear Grammar to DFA

Eg: Consider the grammar G

$X \rightarrow 1X$
$X \rightarrow 0Y$
$Y \rightarrow 1Y$
$Y \rightarrow 0X$
$Y \rightarrow \epsilon$ (indicates that Y is the final state)



A **run** in a DFA model is analogous to a **derivation** in a linear grammar.

For the string $\mathbf{1101}$:

Derivation: $X \rightarrow 1X \rightarrow 11X \rightarrow 110Y \rightarrow 1101Y \rightarrow 1101$. So $1101 \in L(G)$

Run: $X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{0} Y \xrightarrow{1} Y$ (Accepting Run and so $1101 \in L(M)$).

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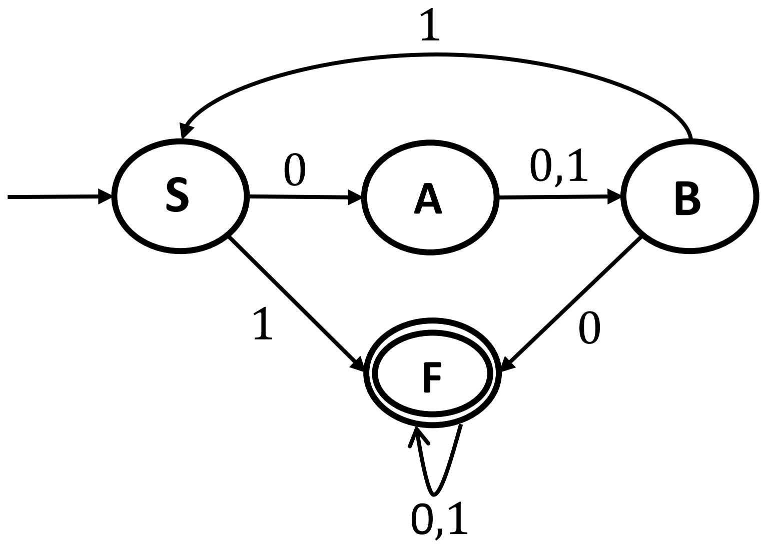
DFA to Right linear Grammar

Consider the following DFA M



The right-linear grammar G for M

$S \to 0A|1F$
$A \to 0B|1B$
$B \to 0F|1S$
$F \to 0F|1F|\epsilon$

Grammars for Regular Languages

Right-linear grammar $\equiv$ DFA $\equiv$ NFA $\equiv$ Regular Expressions

Left linear grammar: If the *rules* of the underlying grammar G are of the form

$$Var \rightarrow \boldsymbol{Var}\, Ter$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then such a grammar is called **Left-linear** (all Variables are to the left of terminals in the RHS).

Right linear grammars are equivalent to Left-linear grammar (We won't be proving it here)

Grammars for Regular Languages

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Right linear grammars are equivalent to Left-linear grammar (We won't be proving it here)

Right-linear grammars and **Left-linear grammars** generate **Regular Languages.**

Note that mixing left-linear grammars and right-linear grammars in the same set of rules **won't generate regular languages**. (e.g: $S \rightarrow aX, X \rightarrow Sb, S \rightarrow \epsilon$)

Left-linear grammar $\equiv$ Right-linear grammar $\equiv$ DFA $\equiv$ NFA $\equiv$ Regular Expressions

Context free Grammars

(Grammar) Formally, a *Grammar* G is a 4-tuple (V, Σ, P, S) such that

- V is the set of **Variables**
- Σ is the set of **Terminals**
- P is the set of production **Rules** $[\,(\boldsymbol{V} \cup \boldsymbol{T})^* \boldsymbol{V} (\boldsymbol{V} \cup \boldsymbol{T})^* \rightarrow (\boldsymbol{V} \cup \boldsymbol{T})^*]$
- S is the **Start Variable** [The variable in the LHS of the first rule is generally the start variable]

Context-Free Grammars: If the *rules* of the underlying grammar G are of the form

$$\boldsymbol{V} \rightarrow (\boldsymbol{V} \cup \boldsymbol{T})^*$$

then such a grammar is called **Context-Free.**

Any language generated by a context-free grammar is called a ***context-free language.***

Immediately we find that the *rules* are less restrictive than left-linear grammars and right-linear grammars. Context free grammars allow

$$Var \rightarrow Anything$$

$$Var \rightarrow String\ of\ Variables |\ String\ of\ Terminals | Strings\ of\ Variables\ and\ Terminals | \epsilon$$

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- So Left linear grammars and Right linear grammars are also context-free grammars.
- **Regular languages** $\subset$ **Context Free Languages**.

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What is the language generated by this grammar?

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Strings that can be derived from G:

$$S \rightarrow \epsilon$$

$$\{\epsilon\}$$

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$$S \rightarrow 0S1 | \epsilon$$

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Strings that can be derived from G:

$$S \rightarrow 0S1 \rightarrow 01$$

$$\{\epsilon, 01\}$$

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Strings that can be derived from G:

$$\{\epsilon, 01, 0011, 000111, 0^4 1^4, \cdots\}$$

What is the language generated by this grammar?

$$L(G) = \{\omega | \omega = 0^n 1^n, n \geq 0\}$$

So although $\boldsymbol{L(G)}$ is not regular, it is context-free.

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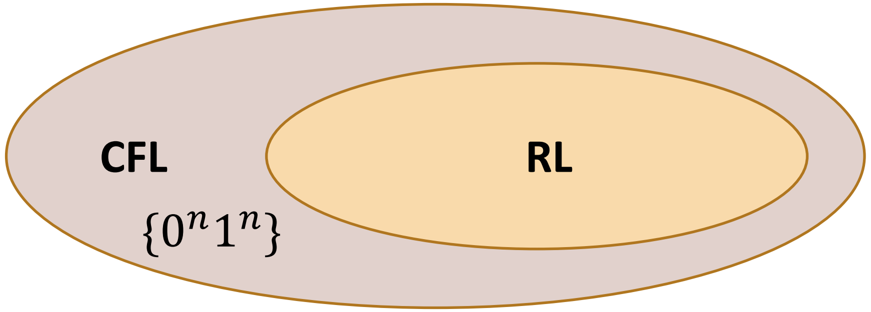

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Thank You!