PRP Assignment - 3

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1. $X = \sum X_i \quad i \in [1:n]$ {followed throughout}

$\because X$ is a Bernoulli RV,

$P(X_i = 1) = p_i$, $P(X_i = 0) = (1-p_i)$

$E[X_i] = 1 \cdot p_i + 0(1-p_i) = p_i$ — ①

$\because X = \sum X_i$

$E[X] = \sum E[X_i]$ ($\because E(\sum_i X_i)$) from linearity of expectation

$\Rightarrow \mu = \sum p_i$ — ②

{μ is mean $\equiv E[X]$}
& $E[X_i] = p_i$ from ①

~~Var(X_i) = ...~~

$Var(X_i) = E[X_i^2] - (E[X_i])^2$
$= 1^2 \cdot p_i - 0^2 \cdot (1-p_i) - p_i^2$
$= p_i - p_i^2$

From linearity of variance

$Var(X) = \sum Var(X_i) \quad (\because X = \sum_i X_i)$
$= \sum (p_i - p_i^2)$
$= \sum p_i - \sum p_i^2$
$= \mu - \sum p_i^2 \quad$ (from ②)

$\max(Var(X)) = \mu - \min(\sum p_i^2)$

We know, AM $\geq$ GM for non-ve real nos

$\Rightarrow \dfrac{p_1^2 + p_2^2 + \dots + p_n^2}{n} \geq (p_1^2 p_2^2 \cdots p_n^2)^{1/n}$

$\Rightarrow \sum_i p_i^2 \geq n(\prod_i p_i^2)^{1/n}$

We know equality holds when every element is same $\Rightarrow p_1^2 = p_2^2 = \dots = p_n^2$ — ③

$\because p_i \geq 0 \ \forall p_i$, ③ $\Rightarrow p_1 = p_2 = \dots = p_n$ ($= p$ (let))

$\therefore \mu = \sum p_i$ (from ②)

$\Rightarrow \mu = np \Rightarrow \boxed{p = \mu/n}$

$\therefore Var(X)$ is maximised for $p = \mu/n$

2. $P(X > m+n \mid X > m) = \dfrac{P((X > m+n) \cap (X > m))}{P(X > m)}$

$\because m, n \in \mathbb{Z}^+$; $(X > m+n) \Rightarrow X > m$

$\therefore P(X > m+n \mid X > m) = \dfrac{P(X > m+n)}{P(X > m)} = P(X > n)$ (given)

$\Rightarrow P(X > m+n) = P(X > m) P(X > n)$

let $P(X > k) = c^k$ (general sol^n for $f(m+n) = f(m) f(n)$)

$\because$ we know from axioms of probability theory,

$c^k \in [0,1] \Rightarrow c \in [0,1]$

$(\because k \in \mathbb{Z}^+$ (given)$)$

$\therefore P(X = k) = P(X > k-1) - P(X > k)$

$= c^{k-1} - c^k$

$= c^{k-1}(1-c)$

let $c = 1-p$ $\quad \because c \in [0,1] \Rightarrow p \in [0,1]$

$\therefore P(X = k) = (1-p)^{k-1} p$, $p \in [0,1]$

Hence with parameter p, this is the PMF of a geometric RV $\Rightarrow X$ is a geometric RV.

3.(i) Let Range of $X = \{-1, 0, 1\}$, Range of $Y = \{-2, 0, 2\}$

let us represent probability laws in such a manner that for $\omega \in \Omega$, $X(\omega) = r_1$ ($r \in R_X$) for many values of ω, & similarly $Y(\omega) = r_2$ for many values of ω, not necessarily same. We represent using constants $\in \mathbb{R}^+$, namely a, b, c, d, the PMFs of various values X & Y can take in the following table

X \ Y	-2	0	2
-1	a	b	a
0	d	c	d
1	a	b	a

$P(X = -1) = a + b + a = P(X = 1) = 2a + b$

$P(X = 0) = d + c + d = 2d + c$

$P_Y(-2) = 2a + d = P_Y(2)$

$P_Y(0) = 2b + c$

$E[X] = \sum_{x \in R_X} x\,P(x) = -1(2a+b) + 0 + 1(2a+b) = 0$

$E[Y] = \sum_{y \in R_y} y\,P(y) = -2(2a+d) + 0 + 2(2a+d) = 0$

$E[XY] = \sum_{\substack{x \in R_x \\ y \in R_y}} xy\,P(x,y) = (-1)(-2)a + (-1)2a + 1(-2)a$
$+ 1.2a + 0$
$= 0$

$\therefore E[X] = E[Y] = E[XY] = 0$
$\Rightarrow E[XY] = E[X]E[Y]$
Hence X, Y are uncorrelated.

$\because$ a, b, c, d are chosen by us arbitrarily, we need to show $\exists\, a, b, c, d \in \mathbb{R}^+$ s.t. that $P(XY) \neq P(X)P(Y)$ for even one x, y; $x \in R_x$, $y \in R_y$

Let $a = 0, b = 1/6, d = 1/7$
From Total probability law, $4a + 2b + 2d + c = 1$
$\Rightarrow c = 1 - \frac{13}{21} = 8/21$

$P(XY) = a$ For $x = 1, y = -2$
$P(X) = 2a + b = 1/6$
$P(Y) = 2a + d = 1/7$
$\therefore P(XY) \neq P(X)P(Y)$ for $X = -1, Y = -2$
$\underset{a}{P(XY)} \neq \underset{1/6}{P(X)}\, \underset{1/7\ (eg)}{P(Y)}$ for $X = 1, Y = 2$

For $X = Y = 0 \Rightarrow P(XY) = c = 8/21$
$P(X) = 2d + c = 2/7 + 8/21 = 14/21$
$P(Y) = 1/3 + 8/21 = 15/21$
$P(XY) \neq P(X)P(Y)$ for $X = Y = 0$.

Thus X & Y are uncorrelated but not independent

b) For uncorrelatedness, $E[XY] = E[X]E[Y]$

$\Rightarrow \sum_{x,y} xy\, P(x,y) = \sum_{x} x P(x) \sum_{y} y P(y)$

$= \sum_{y}\sum_{x} xy\, P(x)P(y) = \sum_{x,y} xy\, P(x)P(y)$

If $x = y \neq 0$, $P(x,y) = P(x)P(y)$ $\forall x, y \Rightarrow P(XY) = P(X)P(Y)$

$\Rightarrow$ If $x = y \neq 0$, uncorrelatedness guarantees independence.

$\forall x, y$

eg: X, Y are Bernoulli RVs

$P(X) = 1/2 = P(Y)$ for 2 diff coins A & B resp.

$P(X)P(Y) = 1/4$ $\quad P(XY) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

$\Rightarrow$ uncorrelatedness guarantees independence

B) Let Z be a RV s.that $Z = X - \alpha Y$, $\alpha \in \mathbb{R}$.
{We know linear combo. of RVs gives another RV}

$\because (X-\alpha Y)^2 \geq 0$

$\Rightarrow E[(X-\alpha Y)^2] \geq 0$

$\Rightarrow E[X^2 + \alpha^2 Y^2 - 2\alpha XY] \geq 0$

$\Rightarrow E[X^2] + \alpha^2 E[Y^2] - 2\alpha E[XY] \geq 0$ — ①

Let $f(\alpha) = \alpha^2 E[Y^2] - 2\alpha E[XY] + E[X^2]$

Discriminant of $f(\alpha)$ {quadratic eq^n in α} $= (-2E[XY])^2 - 4E[X^2]E[Y^2]$
$= 4(E[XY])^2 - 4E[X^2]E[Y^2]$

P ⇒ Q

If $\underbrace{|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}}_{P}$

$\Rightarrow |E[XY]| = E[X^2]E[Y^2]$ for some X, Y

$\Rightarrow$ D of $f(\alpha) = 0$

$\therefore$ For

$f(\alpha) = 0 \Rightarrow E[X^2] + \alpha^2 E[Y^2] - 2\alpha E[XY] = 0$

sol^n exists $\Rightarrow E[(X-\alpha Y)^2] = 0$

$\therefore \overbrace{X = \alpha Y}^{Q}$

$\therefore$ If P then Q proved

Q ⇒ P If $X = \alpha Y$,

RHS of P: $\sqrt{E[X^2]E[Y^2]}$

$= \sqrt{E[\alpha^2 Y^2]E[Y^2]}$

$= |\alpha E[Y^2]|$

LHS of P: $|E[XY]| = |E[(\alpha Y)Y]| = |\alpha E[Y^2]|$

$\because$ LHS = RHS ⇒ P is true

Hence if Q then P proved.

$\therefore$ P ⇔ Q proved ⇒ $|E[XY]| = \sqrt{E[X^2]E[Y^2]}$ iff $X = \alpha Y$

We note that

~~For D ⇒ 0~~ let $X' = X - E[X]$, $Y' = Y - E[Y]$, X', Y' are also RVs

Using Cauchy Schwartz inequality,

$$|E[X'Y']| \leq \sqrt{E[X'^2]\,E[Y'^2]} \quad \text{— (1)}$$

We know that, $E[X+m] = E[X] + m$

& $\quad \text{var}(X+m) = \text{var}(X)$ $\quad m$ is a constant $m \in \mathbb{R}$

by def^n of var ⇒ $\text{var}(X') = \text{var}(X)$, $\text{var}(Y') = \text{var}(Y)$

$E(X') = \text{var}(X)$ also ~~$E[X] = E[X] - E[E[X]]$~~

from (1)

$$|E[(X - E[X])(Y - E[Y])]| \leq \sqrt{E(X')E(Y')}$$

$$\Rightarrow |E[XY - XE(Y) - YE[X] + E[X]E[Y]]| \leq \sqrt{\text{var}(X)\,\text{var}(Y)}$$

$$\Rightarrow |E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y]| \leq \sqrt{\text{var}(X)\,\text{var}(Y)}$$

$$\Rightarrow \left| \frac{E[XY] - E[X]E[Y]}{\sqrt{\text{var}(X)\,\text{var}(Y)}} \right| \leq 1$$

$$\Rightarrow \quad |\rho(X,Y)| \leq 1 \quad \text{(hence proved)}$$

Q5 $\phi(Y) = E[X|Y]$ $\quad$ let $g: \mathbb{R} \to \mathbb{R}$ & let $\phi(Y)g(Y) = h(Y)$

$$E[\phi(Y)\,g(Y)] = E[h(Y)]$$
$$= \sum_{y \in P_Y} h(y)\,P_Y(y)$$

given, $\phi(y) = E[X|Y] = \sum_x x\,P_{X|Y}(x|y)$

$$\therefore E[\phi(Y)\,g(Y)] = \sum_y g(y)\,x\,P_Y(y)\,P_{X|Y}(x|y)$$

We know $P_Y(y)\,P_{X|Y}(x|y) = P_{X,Y}(x,y)$

$$\therefore E[\phi(Y)\,g(Y)] = \sum_y g(y)\,x\,P_{X,Y}(x,y)$$
$$= E[X.g(Y)] \quad \text{(hence proved)}$$

If $g(Y) = 1$

$\Rightarrow E[\phi(Y)] = E[X]$

$\therefore E[E[X|Y]] = E[X]$

$(\because \phi(Y) = E[X|Y]$ (given))

Q6 a) $I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A' \text{ for } \omega \in A' \end{cases}$ logically expe

RTP $E[X|A] = \dfrac{E[I_A X]}{P(A)}$

LHS

$$E[X|A] = \sum_{x \in R_X} x P_{X|A}(x)$$

$$= \sum_{x} x \frac{P(X = X \cap A)}{P(A)}$$

RHS

$$\frac{E[IX]}{P(A)} = \sum_{x,A} \frac{I_A(x) x P_{X|A}(x,A)}{P(A)}$$

We know $I_A(x) = 1$ if $x \in A$ and 0 otherwise

$$= \sum_{\substack{x,A \\ x \in A}} \frac{x P_{X|A}(x,A)}{P(A)} + 0$$

$$= \sum_{x,A} \frac{x P(X = A \cap X)}{P(A)}$$

$\therefore$ LHS = RHS (hence proved)

$$b)\ \text{let } X = X_1 + X_2 \qquad X_1 : \text{outcome of } \overset{1^{st}}{\cancel{\text{first}}} \text{ die's roll}$$
$$X_2 : \text{outcome of } \overset{2^{nd}}{\cancel{\text{second}}} \text{ die's roll}$$

$$E[X|A_i] = E[(X_1+X_2)|A_i] \rightarrow E[(X_1+X_2) | (X_1=i)]$$

$\because$ X_1 & X_2 are independent (subsequent rolls)

$$\Rightarrow E[(X_1+X_2)|(X_1=i)] = E[X_1|X_1=i] + E[X_2|X_1=i]$$

$$E[X_1|X_1=i] = \sum_{x_1 \in R_{X_1}} (x_1)\, P(X_1 = x_1 | x_1 = i)$$

$$P(X_1 = x_1 | x_1 = i) = \begin{cases} 0 & x_1 \neq i \\ 1 & x_1 = i \end{cases}$$

$$\therefore E[X_1|X_1=i] = i(1) + 0$$
$$= i$$

$$E[X_2|X_1=i] = \sum_{x_2 \in R_{X_2}} x_2\, P(X_2 = x_2 | x_1 = i)$$

$\because$ X_2 & X_1 are independent,

$$P(X_2 = x_2 | x_1 = i) = P(X_2 = x_2)$$

$$\therefore E[X_2|X_1=i] = \sum_{x_2} x_2\, P(X_2=x_2)$$
$$= E[X_2]$$
$$= 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + \cdots + 6\left(\frac{1}{6}\right)$$
$$= \left(\frac{6(6+1)}{2}\right)\left(\frac{1}{6}\right) = 7/2$$

$\left\{ P(X_2 = x_2) = 1/6 \text{ for each } x_2 \in R_{X_2} \right\}$

$$\therefore E[X|A_i] = i + 7/2$$

$E[X|A_1] = 9/2$, $E[X|A_2] = 11/2$, $E[X|A_3] = 13/2$

$E[X|A_4] = 15/2$, $E[X|A_5] = 17/2$, $E[X|A_6] = 19/2$