Lecture 9

(2 September 2024)

Recap:

$$\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{e^{-\lambda}\lambda^k}{k!}$$

$np = \lambda$ (constant)

$Z = g(X, Y)$ is also RV

P_{XY} - Joint PMF

$P_X(x) = \sum_y P_{XY}(x, y)$ $\quad P_Y(y) = \sum_x P_{XY}(x, y)$

marginal PMFs

$$P_Z(z) = \sum_{(x,y):\, g(x,y)=z} P_{XY}(x, y)$$

$$E[Z] = E[g(X,Y)] = \sum_{x,y} g(x,y)\, P_{XY}(x,y)$$

X and Y are independent if

$P_{XY}(x, y) = P_X(x)\, P_Y(y) \quad \forall x, y.$

Theorem. If X and Y are independent discrete random variables, then
$E[XY] = E[X]E[Y]$.

Proof.
$$E[XY] = \sum_{x,y} xy\, p_{XY}(x,y)$$
$$= \sum_{x,y} xy\, p_X(x)\, p_Y(y)$$
$$= \sum_{x} x\, p_X(x) \sum_{y} y\, p_Y(y)$$
$$= E[X]E[Y].$$

- RVs X and Y are said to be uncorrelated if $E[XY] = E[X]E[Y]$.

Exercise. X and Y are independent implies $g(X)$ and $h(Y)$ are independent.

Independence of several RVs

Three random variables X_1, X_2 and X_3 are independent if

$$p_{X_1 X_2 X_3}(x_1, x_2, x_3) = p_{X_1}(x_1)\, p_{X_2}(x_2)\, p_{X_3}(x_3),$$

$$\forall\, x_1, x_2, x_3.$$

n random variables are independent if

$$p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p_{X_i}(x_i),$$

$$\forall\, x_1, x_2, \ldots, x_n.$$

<u>Exercise</u>: $X_1, X_2, \ldots, X_n$ are independent implies

$$p_{X_{\mathcal{I}}}(x_{\mathcal{I}}) = \prod_{i \in \mathcal{I}} p_{X_i}(x_i), \quad \forall\, \mathcal{I} \subseteq [1:n].$$

$[x_{\mathcal{I}} := (x_i : i \in \mathcal{I})]$

Some Properties

1) Linearity of Expectation

$$E[X+Y] = E[X] + E[Y]$$

$$Z = X+Y$$

$$E[Z] = E[g(\underline{x}, y)] = \sum_{x, y} (x+y)\, p_{XY}(\underline{x}, y)$$

$$= \sum_{x, y} x\, p_{XY}(\underline{x}, y) + \sum_{x, y} y\, p_{XY}(\underline{x}, y)$$

$$= \sum_{x} x\, p_X(x) + \sum_{y} y\, p_Y(y)$$

$$= E[X] + E[Y].$$

Similarly

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i].$$

Example. Consider a binomial RV Y with parameters n and p.

Let $X_i = \mathbb{1}\{\text{success in } i^{th} \text{ trial}\}$.

Then

$$Y = \sum_{i=1}^{n} X_i .$$

$$E[X_i] = p, \quad \forall i \in [1:n],$$

$$E[Y] = n E[X_i] = np.$$

2) $Y = aX + b$

$$E[Y] = a E[X] + b$$

$$Var(Y) = a^2 Var(X)$$

3) If x and y are independent random variables

$$Var(x+y) = Var(x) + Var(y).$$

$$Var(x+y) = E\left[\left(x+y-E[x+y]\right)^2\right]$$

$$= E\left[\left(x-E[x] + y-E[y]\right)^2\right]$$

$$= E\left[(x-E[x])^2 + (y-E[y])^2 + 2(x-E[x])(y-E[y])\right]$$

$$= Var(x) + Var(y) + E\left[2\left(xy - E[x]E[y]\right)\right] - 2E[\cancel{x}E[y]] + 2E[\cancel{y}E[x]]$$

$$= Var(x) + Var(y)$$

(because x & y are independent)

If $x_1, x_2, \ldots, x_n$ are independent,
then

$$\operatorname{Var}\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \operatorname{var}(x_i).$$

Example. Y is Binomial (n, p).

$$Y = \sum_{i=1}^{n} X_i, \quad X_i = \{\text{success in } i^{th} \text{ trial}\}$$

$$\operatorname{Var}(Y) = n \operatorname{var}(X_i) = np(1-p).$$

4) $Z = X + Y$

$$P_Z(z) = \sum_x P_{XY}(x, z-x)$$

$$P_Z(z) = P(Z=z) = P(X+Y=z)$$

$$= P\left(\bigcup_x \{X=x\} \cap \{Y=z-x\}\right)$$

$$= \sum_x P_{XY}(x, z-x).$$

If X and Y are independent the joint pmf decomposes into product of marginals.

$$p_Z(z) = \sum_x p_X(x)\, p_Y(z-x)$$

$$= \sum_y p_X(z-y)\, p_Y(y)$$

Thus, the pmf p_Z of $X+Y$ is the convolution of pmfs of X & Y,

$$p_Z = p_X * p_Y.$$

Exercise. If X_1 and X_2 are independent geometric random variables with common pmf

$$p_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots$$

$$p_Z(z) = (z-1)\, p^2 (1-p)^{z-2}, \quad z = 2, 3, \ldots$$

Conditioning

Conditioning a RV on an event:

The conditional PMF of a RV X conditioned on a particular event A with $P(A) > 0$ is defined by

$$p_{X|A}(x) = P(X = x \mid A)$$

$$= \frac{P(\{\omega : X(\omega) = x\} \cap A)}{P(A)}.$$

$$\sum_x p_{X|A}(x) = 1.$$

Example: Let X be the roll of a fair six-sided die and let A be the event that the roll is an even number.

$$p_{X|A}(x) = \begin{cases} 1/3, & \text{if } x = 2, 4, 6 \\ 0, & \text{otherwise.} \end{cases}$$

<u>Exercise</u>. If $A_1, A_2, \ldots, A_n$ form a partition of the sample space, with $P(A_i) > 0$ for all i, then

$$p_X(x) = \sum_{i=1}^{n} P(A_i)\, p_{X|A_i}(x).$$

Conditioning one RV on another:

Consider two jointly discrete random variables X and Y. If we know that the value of Y is some particular y with $p_Y(y) > 0$, this provides partial knowledge about the value of X. This knowledge is captured by the conditional PMF $p_{X|Y}$ defined by

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} \quad \text{if } p_Y(y) > 0$$

$$= \frac{P(X=x \cap Y=y)}{P(Y=y)}.$$

$$\sum_x p_{X|Y}(x|y) = 1.$$

$$p_{XY}(x,y) = p_{X|Y}(x|y)\, p_Y(y)$$

$$= p_{Y|X}(y|x)\, p_X(x)$$

$$[\, p_X(x),\ p_Y(y) > 0 \,]$$