Midsem Solutions

1. sol:-

$$P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right)$$

$$= 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right)$$

(by the De Morgan's law)

$$\geq 1 - \sum_{i=1}^{n} P(A_i^c)$$

(by the union bound)

$$= 1 - n + \sum_{i=1}^{n} P(A_i)$$

$$= \sum_{i=1}^{n} P(A_i) - (n-1).$$

2. sol:-

$$P_{ZW}(z, w) = \sum_{\substack{(x, y): \\ g(x)=z,\ h(y)=w}} P_{XY}(x, y)$$

$$= \sum_{\substack{(x, y): \\ g(x)=z,\ h(y)=w}} P_X(x)\, P_Y(y)$$

(since X & Y are independent)

$$= \sum_{x : g(x)=z} P_X(x) \sum_{y : h(y)=w} P_Y(y)$$

$$= P_Z(z) \cdot P_W(w).$$

3. sol:-

$$P_{XY}(x,y) = P_X(x)\, P_{Y|X}(y|x)$$

$$= 2^{-x}\, \mathbb{1}\{y=(-1)^x\}.$$

$$= \begin{cases} 2^{-x}, & \text{if } x \text{ is even, } y=1 \\ 2^{-x}, & \text{if } x \text{ is odd, } y=-1 \\ 0, & \text{o.w.} \end{cases}$$

$$P_Y(1) = \sum_{k=1}^{\infty} 2^{-2k}$$

$$= \frac{1/4}{1-1/4} = 1/3.$$

$$P_Y(-1) = 2/3.$$

$$E[X \mid Y=1] = \sum_{x} x \, p_{X|Y}(x \mid 1)$$

$$= \sum_{k=1}^{\infty} 2k \cdot 2^{-2k} / \left(\tfrac{1}{3}\right)$$

$$= 6 \sum_{k=1}^{\infty} k \left(\frac{1}{4}\right)^{k}$$

$$= 6 \cdot \frac{1/4}{\left(1-\frac{1}{4}\right)^2}$$

$$\left(\text{because } r \frac{d}{dr}\left\{1+r+r^2+\cdots\right\} = \frac{r}{(1-r)^2}\right)$$

$$= \frac{\cancel{6}^{\,3}}{\cancel{4}_{\,2}} \times \frac{\cancel{16}^{\,8}}{\cancel{9}_{\,3}} = \frac{8}{3}.$$

$$E[X|Y=-1] = \sum_x x \, p_{X|Y}(x|-1)$$

$$= \sum_{k=1}^{\infty} (2k-1) \cdot 2^{-2k-1} \Big/ \left(\tfrac{2}{3}\right)$$

$$= 3 \left[\sum_{k=1}^{\infty} 2k \, 2^{2k} - \sum_{k=1}^{\infty} 2^{2k} \right]$$

$$= \frac{8}{3} - 3 \times \frac{1/4}{3/4} = \frac{5}{3} .$$

$$\therefore E[X|Y](\omega) = \begin{cases} \frac{8}{3}, & \text{if } \omega \text{ is even} \\ \frac{5}{3}, & \text{if } \omega \text{ is odd} \end{cases}$$

4. Sol:-

$$F_X(x) = P(X \leq x)$$

$$= P(\{(a,b) : a^2+b^2 \leq x^2\})$$

$$= \begin{cases} 0, & x < 0 \\ \pi x^2/\pi r^2, & 0 \leq x < r \\ 1, & x \geq r \end{cases}$$

$$= \begin{cases} 0, & x < 0 \\ \frac{x^2}{r^2}, & 0 \leq x < r \\ 1, & x \geq r \end{cases}$$

$$f_X(x) = F_X'(x) = \begin{cases} 0, & x < 0 \\ 2x/r^2, & 0 \leq x < r \\ 0, & x \geq r \end{cases}$$

$$E[X] = \int_0^r x f_X(x)\, dx$$

$$= \int_0^r x \cdot \frac{2x}{r^2}\, dx$$

$$= \frac{2x^3}{3}\Big|_0^{\gamma} \cdot \frac{1}{\gamma^2} = \frac{2\gamma}{3}$$

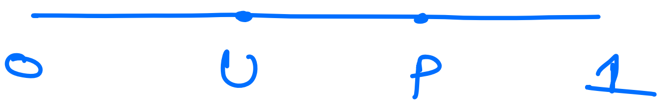
$$E[x^2] = \int_0^{\gamma} x^2 \frac{2x}{\gamma^2}\, dx$$

$$= \frac{2x^4}{4}\Big|_0^{\gamma} \cdot \frac{1}{\gamma^2} = \frac{\gamma^2}{2}$$

$$\mathrm{Var}(x) = E[x^2] - E[x]^2$$

$$= \frac{\gamma^2}{2} - \frac{4\gamma^2}{9} = \frac{9\gamma^2 - 8\gamma^2}{18} = \frac{\gamma^2}{18}.$$

5. Sol^n:-



Let L be the length of the substick that does not contain a given point $p \in [0,1]$.

$$L = g(U)$$

$$= \begin{cases} U & \text{if } U < p \\ 1-U & \text{if } U \geq p \end{cases}$$

$$E[L] = E[g(U)]$$

$$= \int_0^1 g(u)\,du \quad (\text{by LOTUS})$$

$$= \int_0^p u\,du + \int_p^1 (1-u)\,du$$

$$= \frac{p^2}{2} + \left[u - \frac{u^2}{2}\right]_p^1$$

$$= \frac{p^2}{2} + \frac{1}{2} + \frac{p^2}{2} - p$$

$$= p^2 - p + \tfrac{1}{2} = \left(p - \tfrac{1}{2}\right)^2 + \tfrac{1}{4}$$

$$\therefore E[L] = p^2 - p + \tfrac{1}{2} = \left(p - \tfrac{1}{2}\right)^2 + \tfrac{1}{4}.$$

Since $\left(p-\tfrac{1}{2}\right)^2 \geq 0$ the value of p that minimizes the expected length $= \tfrac{1}{2}$.