Lecture 15

(30 September 2024)

Joint CDF

If X and Y are two random variables associated with the same random experiment we define their joint CDF by

$$F_{XY}(x,y) = P(X \le x, Y \le y).$$

Note: Here X, Y can be either continuous or discrete random variables.

Properties of Joint CDF

1) $\lim_{x \to \infty} F_{XY}(x,y) = F_Y(y)$

$\lim_{y \to \infty} F_{XY}(x,y) = F_X(x),$

$$\lim_{\substack{x \to \infty \\ y \to \infty}} F_{XY}(x, y) = 1,$$

2) $$\lim_{x \to -\infty} F_{XY}(x, y) = 0,$$

$$\lim_{y \to -\infty} F_{XY}(x, y) = 0$$

3) If $x_1 \leq x_2$, $y_1 \leq y_2$, then

$$F_{XY}(x_1, y_1) \leq F_{XY}(x_2, y_2).$$

4) $$\lim_{\substack{\varepsilon \to 0^+ \\ \delta \to 0^+}} F_{XY}(x+\varepsilon, y+\delta) = F_{XY}(x, y).$$

5) $$P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$$

$$= F_{XY}(x_2, y_2) + F_{XY}(x_1, y_1) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1),$$

where $x_1 < x_2$, $y_1 < y_2$.

The properties 1) – 4) can be proved exactly along the same lines as that of CDF F_X.

For example

$$\lim_{x\to\infty} F_{XY}(x,y) = \lim_{n\to\infty} P(X\le n, Y\le y)$$

$$= P\left(\bigcup_{n=1}^{\infty} \{X\le n, Y\le y\}\right)$$

(by the continuity of probability)

$$= P\left(\bigcup_{n=1}^{\infty} \{X\le n\} \cap \{Y\le y\}\right)$$

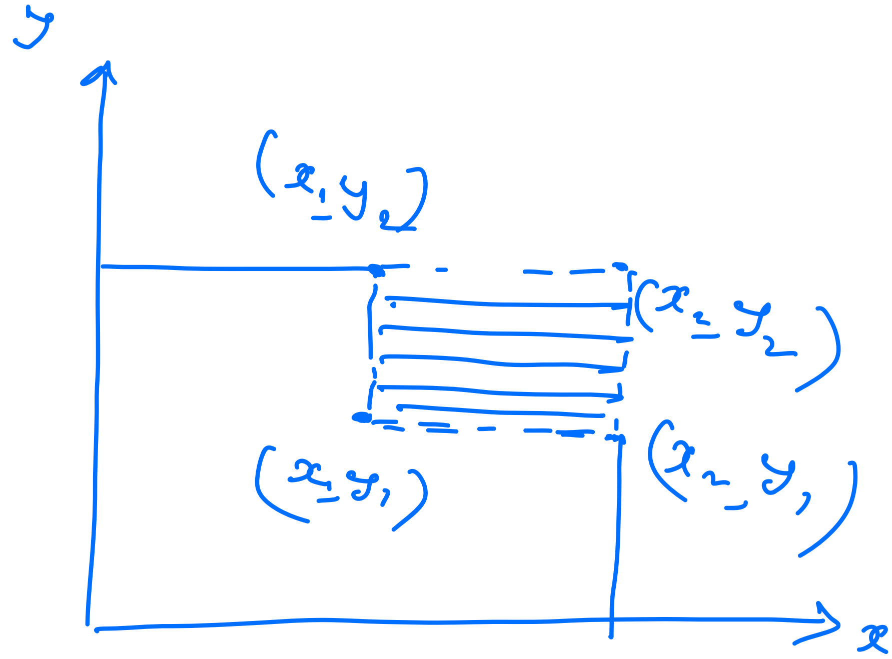
$$= P\left(\Omega \cap \{Y\le y\}\right)$$

(since $\bigcup_{n=1}^{\infty} \{X\le n\} = \Omega$)

$$= P(Y\le y)$$

$$= F_Y(y).$$

5)



$$F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - \left(F_{XY}(x_2, y_1) - F_{XY}(x_1, y_1)\right)$$

$$= F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$

$$= P(x_1 < X \le x_2,\ y_1 < Y \le y_2).$$

Exercise:

If X and Y are discrete RVs taking values in the integers then

(i) $F_{XY}(x, y) = \sum_{l \le x} \sum_{k \le y} p_{XY}(l, k).$

(ii)

$$p_{XY}(x,y) = F_{XY}(x,y) - F_{XY}(x-1,y) - F_{XY}(x,y-1) + F_{XY}(x-1,y-1).$$

Joint CDF of n RVs:

$$F_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n).$$

Definition. Two random variables X and Y on the probability space $(\Omega, \mathcal{F}, P)$ are called jointly continuous if their joint CDF can be expressed as

$$F_{XY}(x,y) = \int_{v=-\infty}^{y} \int_{u=-\infty}^{x} f_{XY}(x,y)\,du\,dv, \quad x,y \in \mathbb{R},$$

for some non-negative integrable function $f : \mathbb{R}^2 \to [0,\infty)$ called the joint probability density function.

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}.$$

we have

$$P(x_1 < X \le x_2, y_1 < Y \le y_2)$$

$$= \int_{y=y_1}^{y_2} \int_{x=x_1}^{x=x_2} f_{XY}(x,y)\,dx\,dy.$$

For any subset $B \subseteq R^2$

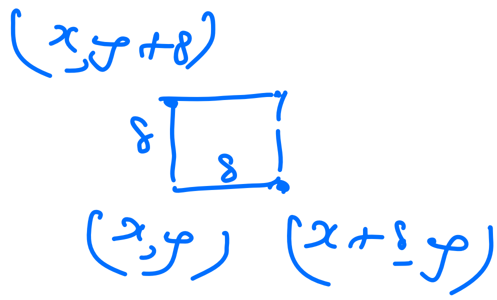
$$P((X,Y) \in B) = \iint_{(x,y) \in B} f_{XY}(x,y)\,dx\,dy.$$

$$B = R^2 \Rightarrow \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1.$$

To interpret the joint PDF we let δ be a small positive number and consider the probability of a small rectangle.

$$P(x < X \leq x+\delta, \; y < Y \leq y+\delta)$$

$$= \int_{y}^{y+\delta} \int_{x}^{x+\delta} f_{XY}(x, y)\, dx\, dy$$



$$\approx f_{XY}(x, y)\, \delta^2$$

So we can view $f_{XY}(x, y)$ as the "probability per unit area" in the vicinity of (x, y).

If X and Y are jointly continuous then they are also individually

Continuous,

$$F_{XY}(x,y) = \int_{v=-\infty}^{y} \int_{u=-\infty}^{x} f_{XY}(x,y)\,du\,dv$$

$$\lim_{y\to\infty} F_{XY}(x,y) = \int_{v=-\infty}^{\infty} \int_{u=-\infty}^{x} f_{XY}(x,y)\,du\,dv$$

$$\Rightarrow F_X(x) = \int_{u=-\infty}^{x} \Bigg(\underbrace{\int_{v=-\infty}^{\infty} f_{XY}(x,y)\,dv}_{f_X(x)} \Bigg) du$$

$\therefore X$ is a continuous RV with PDF

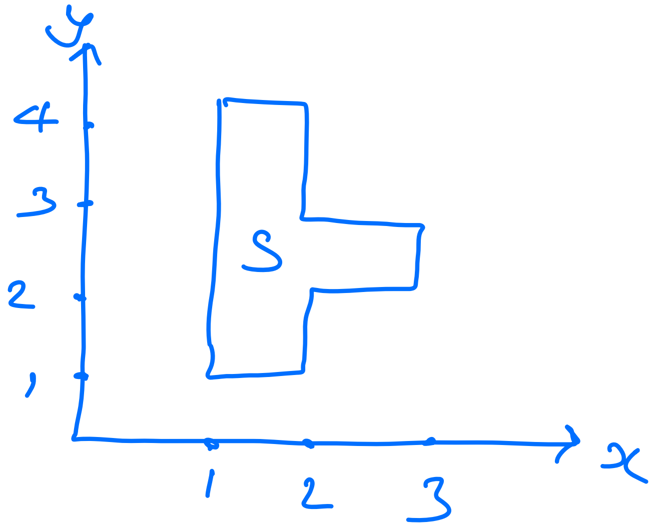
$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy.$$

Similarly Y is a continuous RV with PDF

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx.$$

Example.

Area(S) = 4



$$f_{XY}(x,y) = \begin{cases} 1/4, & \text{if } (x,y) \in S \\ 0 & \text{o.w.} \end{cases}$$

Find f_X, f_Y.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

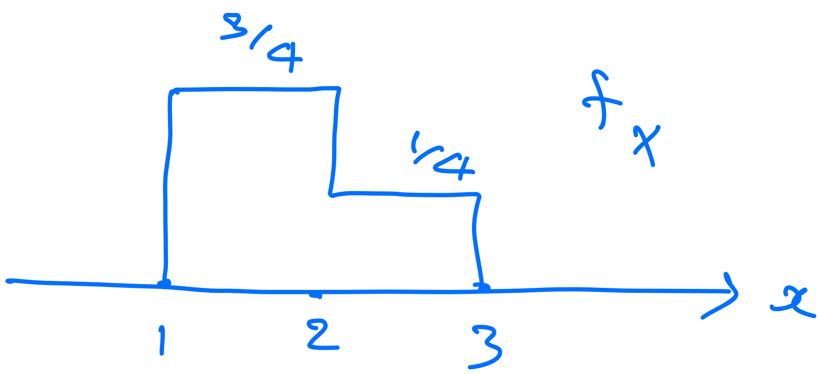
$$= \begin{cases} \int_1^4 1/4\,dy & \text{if } x \in [1;2] \\ \int_2^3 1/4\,dy & \text{if } x \in [2;3] \end{cases}$$

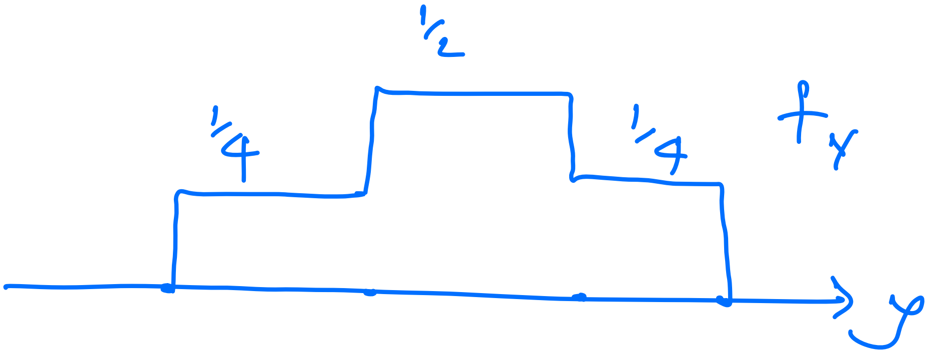
$$= \begin{cases} 3/4 & \text{if } x \in [1:2] \\ 1/4 & \text{if } x \in [2:3] \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \begin{cases} \int_1^2 \frac{1}{4}\,dx & \text{if } y \in [1:2] \cup [3:4] \\ \int_1^3 \frac{1}{4}\,dx & \text{if } y \in [2:3] \end{cases}$$

$$= \begin{cases} \frac{1}{4} & \text{if } y \in [1:2] \cup [3:4] \\ \frac{1}{2} & \text{if } y \in [2:3] \end{cases}$$





Expected value Rule

$$E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

$$E[X+Y] = E[X] + E[Y].$$

<u>More than two</u> <u>Random Variables</u>

RVs X, Y and Z are said to be jointly continuous if

$$F_{XYZ}(x,y,z) = \int_{-\infty}^{z}\int_{-\infty}^{y}\int_{-\infty}^{x} f_{XYZ}(u,v,w)\, du\, dv\, dw,$$

$$f_{XY}(x,y) = \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\, dz$$

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\, dy\, dz$$

$$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i].$$

Conditioning a RV on an Event

Given a RV X and event A s.t. $P(A) > 0$ the conditional CDF is

$$F_{X|A}(x) = P(X \le x \mid A)$$

$$= P(X \le x \cap A) / P(A).$$

$$F_{X|A}(x) = \int_{-\infty}^{x} f_{X|A}(u)\, du$$

Conditional PDF

Let $A = \{X \in B\}$, $B \subseteq \mathbb{R}$,

$$F_{X|A}(x) = \frac{P(X \le x \cap X \in B)}{P(A)}$$

$$= \int_{(-\infty, x) \cap B} f_X(u)\, du \Big/ P(X \in B)$$

$$= \frac{\int_{-\infty}^{x} f_X(u)\, \mathbb{1}\{u \in B\}\, du}{P(X \in B)}.$$

so $f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{P(X \in B)}, & \text{if } x \in B \\ 0 & \text{o.w.} \end{cases}$

Theorem. Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space and assume that $P(A_i) > 0$ for all i. Then

$$f_X(x) = \sum_{i=1}^{n} P(A_i)\, f_{X|A_i}(x).$$

Proof. $F_X(x) = P(X \le x)$

$$= \sum_{i=1}^{n} P(X \le x \mid A_i) P(A_i)$$

$$= \sum_{i=1}^{n} F_{X|A_i}(x) P(A_i)$$

On differentiating w.r.t. x we get

$$f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x) P(A_i).$$