Assignment 2

(MA6.102) Probability and Random Processes, Monsoon 2024

Release date: 22 August 2024, Due date: 31 August 2024

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. Let Ω be a sample space with an associated probability law P and X be a random variable defined on Ω, with PMF P_X. Suppose Y is a function of X, i.e., $Y = f(X)$. Show that

$$P_Y(y) = \sum_{x: f(x)=y} P_X(x),$$

where $P_Y(y) \triangleq P(\{\omega \in \Omega : Y(\omega) = y\})$.

Problem 2. If random variable X takes non-negative integer values, then show that

$$\mathbb{E}[X] = \sum_{n=0}^{\infty} P(X > n).$$

Problem 3. A discrete uniform random variable has a PMF of the form

$$P_X(x) = \begin{cases} \frac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots, b, \\ 0, & \text{otherwise,} \end{cases}$$

where a and b are two integers with $a < b$. Find the variance of X, var(X).

Problem 4. Let F_1 and F_2 be two cumulative distribution functions (CDFs) such that $F_1(x) < F_2(x)$, for all $x \in \mathbb{R}$. Assume that F_1 and F_2 are continuous and strictly increasing. Show that there exists random variables X_1 and X_2, with respective CDFs F_1 and F_2, defined on the same probability space such that $X_1 > X_2$.

Problem 5. Consider a fair coin with probability of heads (and tails) equal to $\frac{1}{2}$. Moreover, consider two dice, first D_1 that has three faces numbered $1, 3, 5$ and second D_2 that has three faces numbered $2, 4, 6$. When rolled, for both D_1 and D_2, each of the three faces are equally likely. A random experiment is conducted as follows. First, the coin is flipped once. If it shows heads, dice D_1 is rolled once, while if the coin shows tails, D_2 is rolled once, and the experiment ends. Let X be the random number seen on the rolled dice in the experiment. Write down the sample space Ω for the whole experiment, an explicit definition of X as a function on Ω, and compute $\mathbb{E}[X]$.

Problem 6. Let X be a discrete random variable that is uniformly distributed over the set of integers in the range $\{a, a+1, \ldots, b\}$, where a and b are integers with $a < 0 < b$. Find the PMFs of the random variables $\max\{0, X\}$ and $\min\{0, X\}$.