Lecture 3

(8 August 2024)

Recap.

$(\Omega, \mathcal{F}, P)$

Ω – sample space

$\mathcal{F}$ – Event space

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2 \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

P – Probability law $P: \mathcal{F} \to \mathbb{R}$ s.t.

(i) (Non-negativity). $P(E) \geq 0$

(ii) (Normalization). $P(\Omega) = 1$

(iii) (Additivity). $A_1, A_2 \dots$ are disjoint

$\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

Properties of Probability Law

(a) $P(A) + P(A^c) = 1$, $P(A) \leq 1$, $P(\phi) = 0$

(b) If $A \subseteq B$, then $P(B) = P(A) + P(B \setminus A) \geq P(A)$.

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof of (a). $A \cup A^c = \Omega$

$$P(A) + P(A^c) = P(\Omega) = 1$$

Proof of (b).

$$B = A \cup (B \setminus A)$$

$$\Rightarrow P(B) = P(A) + P(B \setminus A)$$

$$\geq P(A).$$

Proof of (c).

$$A \cup B = A \cup (B \setminus A)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B \setminus A)$$

$$= P(A) + P(B) - P(A \cap B)$$

More generally if $A_1, A_2, \ldots$ are events, then

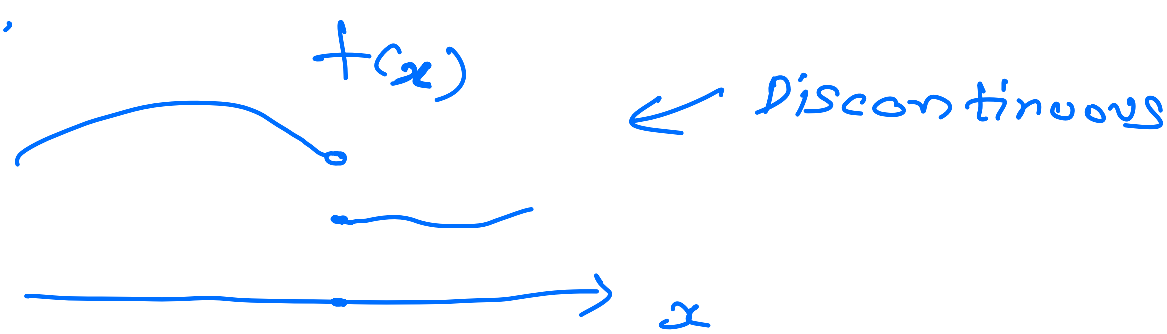
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{1\le i\le n} P(A_i) - \sum_{1\le i<j\le n} P(A_i\cap A_j)$$

$$+ \sum_{1\le i<j<k\le n} P(A_i\cap A_j\cap A_k) - \cdots + (-1)^{n+1} P(A_1\cap A_2\cap \cdots \cap A_n).$$

[Proof by induction]

Continuity of Probability

Recall the continuity of a real function.



For a continuous function f

$$x_n \to x \implies f(x_n) \to f(x).$$

That is $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.

We have a similar notion of continuity for the set function probability law.

<u>Theorem</u> (Continuity of Probability).

For a sequence of events $A_1, A_2, \ldots$ we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right).$$

<u>Proof</u>. Let $B_1 = A_1$

$$B_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j$$

<u>claim 1</u>. $B_i \cap B_{i'} = \phi \quad i \neq i'$.

suppose $i > i'$.

Let $x \in B_i \Rightarrow x \in A_i \setminus \bigcup_{j=1}^{i-1} A_j$.

$$\Rightarrow x \notin A_{i'} \Rightarrow x \notin B_{i'}$$

Let $x \in B_i \Rightarrow x \in A_i \setminus \bigcup_{j=1}^{i-1} A_j$

$$\Rightarrow x \in A_i$$

$$\Rightarrow x \in \bigcup_{j=1}^{i-1} A_j$$

$$\Rightarrow x \notin A_i \setminus \bigcup_{j=1}^{i-1} A_j .$$

<u>Claim 2</u>. $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$, $n \in \mathbb{N}$ and

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i .$$

We prove $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$ using induction.

Assume $\bigcup_{i=1}^{k} A_i = \bigcup_{i=1}^{k} B_i$.

Let $C_k = \bigcup_{i=1}^{k} A_i$.

$$C_{k+1} = C_k \cup A_{k+1}$$

$$= C_k \cup \left(A_{k+1} \setminus C_k\right)$$

$$= C_k \cup \left(A_{k+1} \setminus \bigcup_{i=1}^{k} A_i\right)$$

$$= C_k \cup B_{k+1}$$

$$= \bigcup_{i=1}^{k} B_i \cup B_{k+1}$$

$$= \bigcup_{i=1}^{k+1} B_i .$$

To show $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$, let

$$x \in \bigcup_{i=1}^{\infty} A_i \implies \exists\, n \text{ s.t. } x \in A_i$$

$$\implies x \in \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$$

$$\implies \exists\, n \text{ s.t. } x \in B_i$$

$$\implies x \in \bigcup_{i=1}^{\infty} B_i .$$

Consider

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) \quad \text{[by claim 2]}$$

$$= \sum_{i=1}^{\infty} P(B_i)$$

[by additivity as $B_i \cap B_j = \underline{\phi}$ $i \neq j$]

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right)$$

[by additivity]

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right).$$

[by claim 2]

Corollary.

1) If $A_1, A_2, \ldots$ is a sequence of increasing nested events, i.e., $A_i \subseteq A_{i+1}$, $\forall i \in \mathbb{N}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} P(A_n).$$

2) If $B_1, B_2, \ldots$ is a sequence of decreasing nested events, i.e., $B_i \supseteq B_{i+1}$, $i \in \mathbb{N}$, then

$$P\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n \to \infty} P(B_n).$$

Exercise. Prove the above corollary.

Union Bound for infinite number of events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cup B) \leq P(A) + P(B)$$

we first prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

One can directly prove this via induction using $P(A \cup B) \leq P(A) + P(B)$ as the base case. However we give a bit elaborate proof making explicit connections with the inclusion-exclusion principle (mainly for the illustration purpose).

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$+ \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots A_n)$$

$$= \sum_{i} P(A_i) - \underbrace{\left[\; - \; - \; - \; - \; \right]}_{\geq 0 \;\to\; \text{unclear why.}}$$

$$P(A_1 \cup A_2 \cup A_3) = P\left(A_1 \cup (A_2 \setminus A_1 \cap A_2) \cup \left(A_3 \setminus (A_3 \cap A_1) \cup A_3 \cap A_2\right)\right)$$

$$= P(A_1) + P(A_2) + P(A_3)$$

$$- P(A_1 \cap A_2) - P((A_1 \cap A_3) \cup (A_2 \cap A_3))$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(A_1 \cup (A_2 \setminus A_1 \cap A_2) \cup \cdots \cup \left(A_n \setminus \bigcup_{i<n} (A_n \cap A_i)\right)\right)$$

$$= P(A_1) + P(A_2) - (.) + \cdots P(A_n) - (.)$$

$[A \subseteq B \Rightarrow P(B \setminus A) = P(B) - P(A)]$

$\boxed{\text{This is exactly inclusion-exclusion on simplification}}$

$$\leq \sum_{i=1}^{n} P(A_i)$$

The advantage of this elaborate proof is that it proves the inclusion-exclusion principle and the union bound simultaneously. Moreover this shows how union bound can be obtained from the inclusion-exclusion principle which is otherwise not immediately clear.

Now we prove $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$.

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$\leq \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i)$$

[If $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are convergent sequences and $x_n \leq y_n$ then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$]

$$= \sum_{i=1}^{\infty} P(A_i).$$