PRP A6 : 2023113019

1. $Z = \min\{X,Y\} = g_1(X,Y)$
$W = \max\{X,Y\} = g_2(X,Y)$

a) $f_{ZW} = \left.\dfrac{f_{XY}}{|J(x,y)|}\right|_{x=h_1(z,w),\ y=h_2(z,w)}$

$= \sum_i \dfrac{f_{XY}(x_i,y_i)}{|J(x_i,y_i)|}$ s.t. $(x_i,y_i)$ are soln's of $g_1(x_i,y_i)=z,\ g_2(x_i,y_i)=w$

case① $X \geq Y \Rightarrow z = Y,\ W = X$

$|J(x,y)| = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = |-1| = 1$

~~$J(x,y)$~~

case② $X < Y \Rightarrow W = Y,\ Z = X$

$|J(x,y)| = \begin{vmatrix} \partial g_1/\partial x & \partial g_1/\partial y \\ \partial g_2/\partial x & \partial g_2/\partial y \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$\{X \geq Y\} \cup \{X < Y\} = \Omega$ [Partitions]

$\therefore f_{ZW}(z,w) = \left.\dfrac{f_{XY}(x,y)}{1}\right|_{x=w,\ y=z} + \left.\dfrac{f_{XY}(x,y)}{1}\right|_{x=z,\ y=w}$

$= \begin{cases} f_{XY}(w,z) + f_{XY}(z,w) & w \geq z \\ 0 & \text{o/w} \end{cases}$

b) X, Y independent $\therefore$ $f_{XY}(x,y) = f_X(x)\, f_Y(y)$

$$f_{ZW}(z,w) = f_{XY}(w,z) + f_{XY}(z,w)$$
$$= f_X(w)\, f_Y(z) + f_X(z)\, f_Y(w)$$

We know $w > z$, and $f_X(w) = f_Y(z) = f_Y(w) = f_X(z) = \begin{cases} \frac{1}{1-0} = 1 & \text{for uniform RV} \\ 0 & \text{o/w.} \end{cases}$

$w, z \in [0,1]$

$$\therefore f_{ZW}(z,w) = 1\cdot 1 + 1\cdot 1 = \begin{cases} 2 & 0 \le z \le w \le 1 \\ 0 & \text{o/w.} \end{cases}$$

2. PDF: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$E[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx$$
$$= \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2} + sx\right)}\, dx$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{-\frac{(x-(\mu+s\sigma))^2}{2\sigma^2} + \mu s + \frac{s^2\sigma^2}{2}}\, dx$$
$$= e^{\mu s + (s^2\sigma^2/2)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+s\sigma))^2}{2\sigma^2}}\, dx$$

subs. $x - s\sigma = u \Rightarrow dx = du$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}}\, du = 1$$

(Gaussian norm)

$$\therefore E[e^{sX}] = \boxed{e^{\mu s + s^2\sigma^2/2} = M_X(s)}$$

3. P: $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0 \quad \forall \varepsilon > 0$

Q: $\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$

RTP whether $P \Leftrightarrow Q$

If Q then P

$$P(|X_n-X| \geq \varepsilon) = P(|X_n - X| > \varepsilon) + P(|X_n-X| = \varepsilon)$$

If $\lim_{n\to\infty} P(|X_n-X|\geq\varepsilon) = 0$

$\Rightarrow \lim_{n\to\infty} P(|X_n-X|>\varepsilon) + P(|X_n-X|=\varepsilon) = 0$

$\because P \in [0,1] \Rightarrow \lim_{n\to\infty} P(|X_n-X|>\varepsilon) = 0$ $\left(= \lim_{n\to\infty} P(|X_n-X|=\varepsilon)\right)$

$\therefore Q \to P$

If P then Q

Using $P(\Omega)=1$, $\lim_{n\to\infty} P(|X_n-X|>\varepsilon) + P(|X_n-X|\leq\varepsilon) = 1$

If $\lim_{n\to\infty} P(|X_n-X|>\varepsilon)=0 \Rightarrow \lim_{n\to\infty} P(|X_n-X|\leq\varepsilon)=1 \quad \forall \varepsilon>0$

~~Similarly~~

$\lim_{n\to\infty} P(|X_n-X| \leq \varepsilon)$

Let $\varepsilon' = \lim_{k\to 0^+} \varepsilon + k \Rightarrow \varepsilon' > \varepsilon > 0$ (of P)

$\therefore$ If $|X_n-X| \leq \varepsilon < \varepsilon'$

$\Rightarrow P(|X_n-X|\leq\varepsilon) \leq P(|X_n-X|<\varepsilon')$

but $\lim_{n\to\infty} P(|X_n-X|\leq\varepsilon) = 1$ & $P\in[0,1]$

$\therefore P(|X_n-X|<\varepsilon')=1$

$\because \varepsilon'>0$ & $\inf(\varepsilon) = 0^+ = \inf(\varepsilon')$, $\varepsilon'$ is arbitrary +ve

$\therefore \lim_{n\to\infty} P(|X_n - X| < \varepsilon') = 1 \quad \forall\, \varepsilon' > 0$ ($\varepsilon'$ is arbitrarily small)

(Again)

But we know, $P(\Omega) = 1$

$$\therefore \lim_{n\to\infty} P(|X_n - X| < \varepsilon') = \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon') + P(|X_n - X| < \varepsilon') \quad \forall\, \varepsilon' > 0$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon') = 0 \quad \forall\, \varepsilon' > 0$$

$\varepsilon' \to \varepsilon$ ($\varepsilon > 0$ arbitrary)

$$\Rightarrow \lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0 \quad \forall\, \varepsilon > 0$$

$\therefore P \to Q$

Hence $P \to Q$, & $Q \to P \Rightarrow P \Leftrightarrow Q$

Hence they are equivalent

4. $M_n = \dfrac{S_n}{n}$ $\quad E(M_n) = f$

$S_n = n M_n \Rightarrow E[S_n] = E[nM_n] = nf$

This & the fact that $S_n$ is the no. of smokers in his sample (intuitive reasoning) gives us

$S_n \sim \text{Binomial}(n, f)$ (similar to)

$\text{var}(S_n) = nf(1-f)$ ~~[then only $E[S_n] = nf$ for its derivation]~~

$$\text{var}(M_n) = \frac{\text{var}(S_n)}{n^2} = \frac{f(1-f)}{n}$$

by chebyshev's inequality,

$$P(|M_n - f| \geq k) \leq \frac{\text{var}(M_n)}{k^2} = \frac{f(1-f)}{nk^2} \quad \forall\, k > 0$$

$\because \delta$ is predetermined,

For Chebyshev to always hold,

$$\frac{f(1-f)}{nk^2} < \delta \quad \forall k > 0$$

When $k = \varepsilon$,

$$\frac{f(1-f)}{n\varepsilon^2} \leq \delta$$

$$\Rightarrow n \geq \frac{f(1-f)}{\delta\varepsilon^2}$$

$$n_{min} = \frac{f(1-f)}{\delta\varepsilon^2}$$ [For Chebyshev's recommended $n$]

(a) $\varepsilon' = \frac{2\varepsilon}{3}$

$$\frac{n'_{min}}{n_{min}} = \left(\frac{\varepsilon}{\varepsilon'}\right)^2 = \left(\frac{3}{2}\right)^2 = 9/4$$

$$n'_{min} = \frac{9}{4} n_{min} \quad \Big| \quad \Delta n_{min} = \frac{5}{4} n_{min}$$

(b) $$\frac{n'_{min}}{n_{min}} = \frac{\delta}{\delta'} = \frac{5}{3}$$

$$n'_{min} = \frac{5}{3} n_{min}$$

$$\Delta n_{min} = \frac{2}{3} n_{min}$$

5. $\lim_{n\to\infty} F_{X_n}(t) = F_C(t) \quad (X_n \xrightarrow{D} C)$

where $F_C(t) = P(C \le t) = \begin{cases} 1 & c \le t \\ 0 & c > t \end{cases}$

Required to find

$\lim_{n\to\infty} P(|X_n - c| > \varepsilon) \quad \forall \varepsilon > 0$

$$P(|X_n - c| > \varepsilon) = P(X_n - c > \varepsilon) + P(X_n - c < -\varepsilon)$$
$$= P(X_n > c+\varepsilon) + P(X_n < c-\varepsilon)$$
$$= P(X_n > c+\varepsilon) + F_X(c-\varepsilon)$$

$\because c - \varepsilon < c \quad \forall \varepsilon > 0$

$\Rightarrow F_X(c-\varepsilon) = 0$

$$\therefore \lim_{n\to\infty} P(|X_n - c| > \varepsilon) = \lim_{n\to\infty} P(X_n > c+\varepsilon)$$

$(P(\Omega) = 1)$ $$= 1 - \lim_{n\to\infty} P(X_n \le c+\varepsilon)$$
$$= 1 - F_X(c+\varepsilon)$$
$$= 1 - 1 = 0$$

$\therefore \lim_{n\to\infty} P(|X_n - c| > \varepsilon) = 0 \Rightarrow X_n$ converges to $c$ in prob.

6. $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0 \quad \forall \varepsilon > 0$

Using $\forall \varepsilon > 0$, $\varepsilon' = \varepsilon/2$ can be used ($\because \varepsilon' > 0$)

$\lim_{n\to\infty} P(|X_n - X| > \varepsilon/2) = 0 \quad \forall \varepsilon > 0$ also holds

Similarly

$\lim_{n\to\infty} P(|Y_n - Y| > \varepsilon/2) = 0 \quad \forall \varepsilon > 0$

Let $A$ be $(X_n - X)$, $B$ be $(Y_n - Y)$

$\therefore \lim_{n\to\infty} P(|A| > \varepsilon/2) = 0$, $\lim_{n\to\infty} P(|B| > \varepsilon/2) = 0 \quad \forall \varepsilon > 0$

$\lim_{n\to\infty} P(|A| + |B| > \varepsilon) = 0$

$\cancel{\lim_{n\to\infty} P(|A+B| > \varepsilon) <}$ [illegible]

$\lim_{n\to\infty} P(|A+B| > \varepsilon) \le \lim_{n\to\infty} P(|A| + |B| > \varepsilon)$

(Triangle inequality)

~~But~~ $\because P \in [0,1]$ & $\lim_{n\to\infty} P(|A| + |B| > \varepsilon)$

$\Rightarrow \lim_{n\to\infty} P(|A+B| > \varepsilon) = 0 \quad \forall \varepsilon > 0$

$\Rightarrow \lim_{n\to\infty} P(|X_n + Y_n - (X+Y)| > \varepsilon) = 0 \quad \forall \varepsilon > 0$

$\therefore X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n$ converges to $X + Y$