## Definition

A random variable $x$ is called continuous random variable if its CDF can be expressed as

$$F_x(x) = \int_{-\infty}^{x} f(u)\, du, \quad x \in \mathbb{R},$$

for some integrable function $f: \mathbb{R} \to [0,\infty)$ called the probability density function (PDF),

$$f(x) = F_x'(x)$$

[ by Fundamental Theorem of Calculus ]

$$P(a \leq x \leq b) = \int_{a}^{b} f(x)\, dx.$$

The numerical value of $f(x)$ is not a probability. However we can think of $f(x)\Delta x$ as probability for infinitesimally small $\Delta x$, since

$$P(x < X \leq x + \Delta x) = \int_x^{x+\Delta x} f_X(x)\, dx$$

$$\approx f_X(x)\Delta x.$$

$$f_X(x) = \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}.$$

If $B$ is a (sufficiently nice) subset of $R$ (such as an interval, or a countable union of intervals and so on), then

$$P(X \in B) = \int_B f_X(x)\, dx.$$

Theorem. If a continuous RV X
has a PDF $f_X$, then

(a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

(b) $P(X = x) = 0$ for all $x \in R$

Proof. (a) $\int_{-\infty}^{\infty} f_X(x)\,dx$

$= P(X \in (-\infty, \infty))$

$= P(-\Omega) = 1.$

(b) $P(X = x) = \int_{x}^{x} f(u)\,du = 0,$

# Expectation.

The expectation of a continuous RV $x$ with PDF $f_x$ is given by

$$E[x] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx.$$

Example. $f_x(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{o.w.} \end{cases}$

$$E[x] = \frac{2}{3}.$$

— If $x$ is a continuous RV, $g(x)$ is also a random variable. However $g(x)$ can be either a continuous RV or a discrete RV.

$Y = g(x) = x$ is continuous

If $g(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$, $Y = g(x)$ is discrete

**Theorem.** If $x$ and $g(x)$ are continuous random variables then

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x) \, dx.$$

We first prove the following lemma,

**Lemma.** For a non-negative continuous random variable $y$ (i.e., $f_y$ s.t. $f_y(y) = 0$ when $y < 0$),
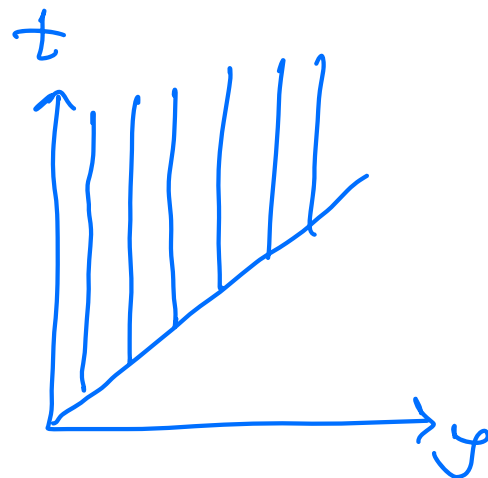
$$E[Y] = \int_{0}^{\infty} P(Y > y) \, dy.$$

**Proof.** $\int_{0}^{\infty} P(Y > y) \, dy$
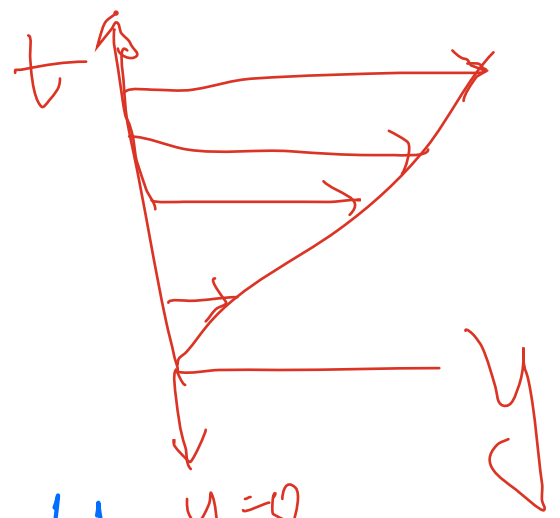
$$= \int_{y=0}^{\infty} \int_{t=y}^{\infty} f_y(t) \, dt \, dy$$



0<y < infinity, y<t < infinity

combining   0<y<t<infinity

equivalent to 0<t<infinity, 0<y<t

$$= \int_{t=0}^{\infty} \int_{y=0}^{t} f_y(t) \, dt \, dy$$

$$= \int_{t=0}^{\infty} f_y(t) \left( \int_{y=0}^{t} dy \right) dt$$

$$= \int_{t=0}^{\infty} t \, f_y(t) \, dt = E[Y].$$

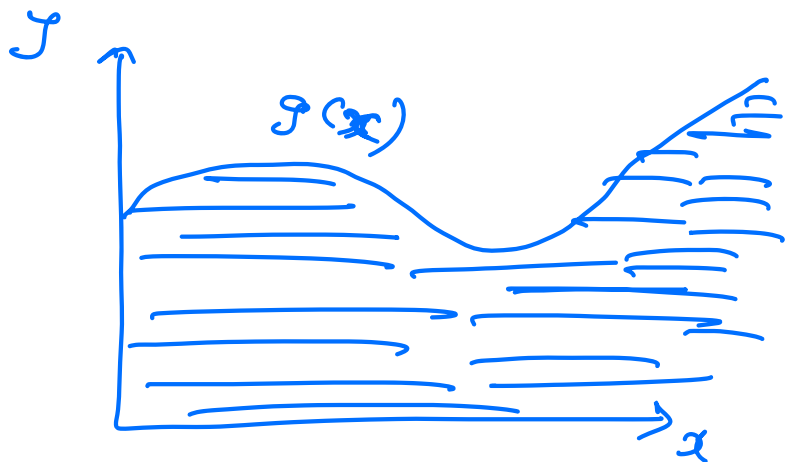We prove the above theorem assuming $g(x)$ is non-negative.

$$E[g(x)] = \int_{0}^{\infty} P(g(x) > y) \, dy$$

$$= \int_{y=0}^{\infty} \int_{x : g(x) > y} f_x(x) \, dx \, dy$$

$$\left[ P(g(x) > y) = P(x \in B) = \int_{B} f_x(x) \, dx \right.$$

$$\text{with } B = \{x : g(x) \geq y\} \quad ]$$

$$= \int_{x=-\infty}^{\infty} \int_{y=0}^{g(x)} f_X(x)\, dx\, dy$$



$$= \int_{x=-\infty}^{\infty} f_X(x) \int_{y=0}^{g(x)} dy\, dx$$

$$= \int_{x=-\infty}^{\infty} g(x)\, f_X(x)\, dx \ ,$$

<u>Exercise</u>. Complete the proof of the theorem for a general real-valued function $g$.
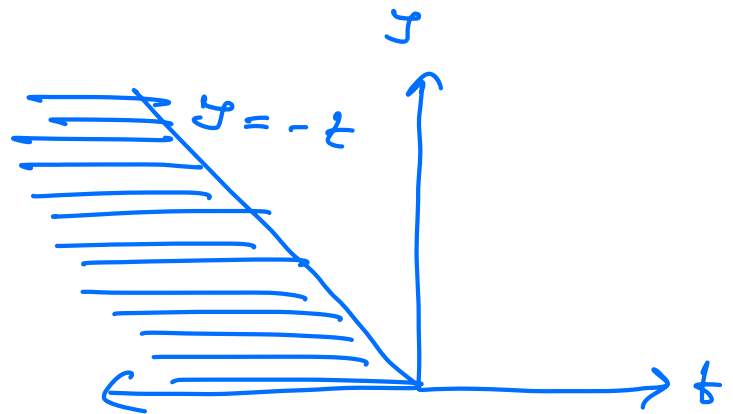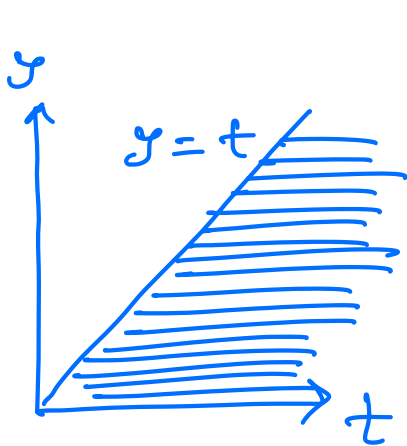
<u>Hint</u>: Show that

$$E[Y] = \int_0^{\infty} P(Y > y)\, dy - \int_0^{\infty} P(Y < -y)\, dy,$$

Proof of the above exercise follows.

$$\int_0^\infty P(Y > y)\, dy - \int_0^\infty P(Y < -y)\, dy$$

$$= \int_{y=0}^\infty \int_{t=y}^\infty f_Y(t)\, dt\, dy - \int_{y=0}^\infty \int_{t=-\infty}^{-y} f_Y(t)\, dt\, dy$$

$$= \int_{t=0}^\infty \int_{y=0}^t f_Y(t)\, dy\, dt - \int_{t=-\infty}^0 \int_{y=0}^{-t} f_Y(t)\, dy\, dt$$



$$= \int_{t=0}^\infty t\, f_Y(t)\, dt + \int_{t=-\infty}^\infty t\, f_Y(t)\, dt$$

$$= E[Y].$$

$$E[g(x)]$$

$$= \int_0^\infty P(g(x) > y)\,dy - \int_0^\infty P(g(x) < -y)\,dy$$

$$= \int_{y=0}^\infty \int_{x:\,g(x) \geq y} f_x(x)\,dx\,dy$$

$$- \int_{y=0}^\infty \int_{x:\,g(x) < -y} f_x(x)\,dx\,dy$$

$$= \int_{x=-\infty}^\infty \int_{y:\,0 \leq y \leq g(x)} f_x(x)\,dy\,dx$$

$$- \int_{x=-\infty}^\infty \int_{y:\,g(x) < -y \leq 0} f_x(x)\,dy\,dx$$

$$= \int_{x=-\infty}^\infty g^+(x) f_x(x)\,dx + \int_{x=-\infty}^\infty g^-(x) f_x(x)\,dx$$

$$= \int_{x=-\infty}^\infty g(x) f_x(x)\,dx. \quad \left[ \begin{array}{l} g^+(x) = \max\{0, g(x)\}, \\ g^-(x) = \max\{0, -g(x)\} \end{array} \right]$$

Variance of x

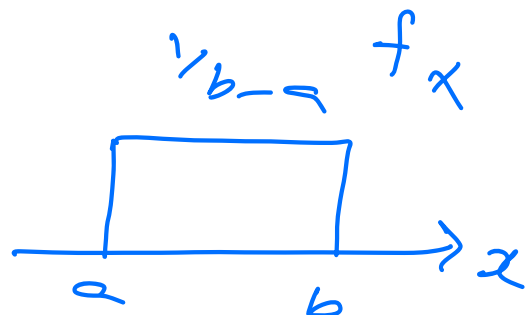$$\text{Var}(x) = E\left[(x - E[x])^2\right]$$

$$= \int_{-\infty}^{\infty} (x - E[x])^2 f_x(x) dx$$

$$= E[x^2] - E[x]^2.$$

Examples of Continuous RVS

Uniform Random Variable

$$f_x(x) = \begin{cases} 1/(b-a), & a \leq x \leq b, \\ 0, & \text{o.w.} \end{cases}$$



$$E[x] = \frac{a+b}{2}$$

$$\text{Var}(x) = E[x^2] - E[x]^2 = \frac{a^2 + b^2 + ab}{3} - \frac{(a+b)^2}{4}$$

$$= (b-a)^2/12$$

# Exponential Random Variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{o.w.}, \end{cases}$$

where $\lambda$ is a positive parameter characterizing the PDF.

An exponential RV can be a good model for the amount of time until an incident of interest takes place, such as

- a message arriving at a computer
- an equipment breaking down, etc.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx$$

$$= \int_{0}^{\infty} x \lambda e^{-\lambda x} \, dx$$

$$= \left[ x \int \lambda e^{-\lambda x} dx - \int 1 \cdot \left( \int \lambda e^{-\lambda x} dx \right) dx \right]_0^\infty$$

$$= \left[ -x e^{-\lambda x} + \int e^{-\lambda x} dx \right]_0^\infty$$

$$= 0 + \left[ \frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty = \frac{1}{\lambda}.$$

$$E[x^2] = \int_{-\infty}^\infty x^2 f_x(x) dx$$

$$= \int_0^\infty \lambda x^2 e^{-\lambda x} dx$$

$$= \left[ x^2 \int \lambda e^{-\lambda x} dx - \int 2x \cdot e^{-\lambda x} dx \right]_0^\infty$$

$$= 0 + 2 \frac{E[x]}{\lambda} = \frac{2}{\lambda^2}$$

$$var(x) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$