$\underline{Th^m}$: If f is any feasible (valid) flow and (S,T) is any cut s.t $s \in S$ and $t \in T$, then total flow $|f|$ is at most the capacity of the cut.

(* Here capacity of the cut (S,T):

$$\sum_{u \in S} \sum_{v \in T} c(u \to v)$$)

(Cut $\Rightarrow$ $S \cup T = V$)

Proof: $$|f| = \sum_{u:\, s \to u \in E} f(s \to u)$$

$$= \underbrace{\sum_{u:\, s \to u \in E} f(s \to u)}_{} + \sum_{v \in V \setminus \{s,t\}} \overbrace{\Big(- \underbrace{\sum_{u:\, u \to v \in E} f(u \to v)}_{} + \underbrace{\sum_{w:\, v \to w \in E} f(v \to w)}_{} \Big)}^{\text{This is } 0 \text{ for each } v \in V \setminus \{s,t\}.}$$

$$= \Big(\sum_{v \in V \setminus \{t\}} \sum_{w} f(v \to w) \Big) - \Big(\sum_{v \in V \setminus \{s,t\}} \sum_{u} f(u \to v) \Big)$$

+ve : Outflow
-ve : Inflow.
w.r.t a vertex.

w, u → all vertices or just neighbours ??

(For rest of the vertices, inflow & outflow cancel each other)

$$= \sum_{v \in S} \sum_{w \in T} f(v \to w) - \sum_{v \in S} \sum_{u \in T} f(u \to v)$$

$$\sum_{v \in S} \cancel{\sum_{w \in S}} f(v \to w) + \sum_{v \in S} \sum_{w \in T} f(v \to w) + \sum_{v \in T/\{t\}} \cancel{\sum_{w}} f(v \to w)$$

$$\sum_{v \in S \setminus \{s\}} \cancel{\sum_{u \in T}} f(u \to v) + \sum_{v \in S \setminus \{s\}} \sum_{u \in T} f(u \to v) + \sum_{v \in T \setminus \{t\}} \cancel{\sum_{u \in T}} f(u \to w)$$

$$\leq \sum_{v \in S} \sum_{w \in T} f(v \to w)$$

$\leq \sum_{v \in S} \sum_{w \in T} c(v \to w)$ = CAPACITY OF THE (S,T) CUT.

$\Rightarrow$ Any feasible flow $\leq$ Cut capacity.

Maximise — Minimise to get best upper bound.

Obs. : A maximum flow can at most be the capacity of a minimum cut.

$\Rightarrow$ If min cut $\leq$ max-flow, then min cut capacity = Max flow.

$\rightarrow$ The algo. terminates if there is no $s \rightsquigarrow t$ path in the residual graph.

$S \leftarrow$ Vertices reachable from source in the residual graph

$T = V \setminus S$

Path with edges of > 0 (res.) capacities.

Edges: $S \to T$

1. $\forall$ edges $u \to v \in E(G_{original})$, $(u \in S)$ $(v \in T)$

the capacity of these edges is saturated.

$f(u \to v) = c(u \to v)$

If not, i.e., if $f(u \to v) < c(u \to v)$ then v could have been reached from source in the residual graph.

$T \to S$

1. $\forall\, u \in T, v \in S$, s.t $u \to v \in E(G_{org.})$,

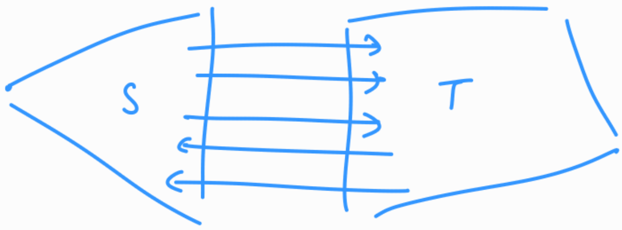
the flow on these edges $= 0$.

If not, i.e., if $f(u \to v) > 0$ then in residual graph

$c(v \to u) = f(u \to v)$ and thus u could have been reachable.

→ Flow is at least $\sum_{u \in S} \sum_{v \in V \setminus S} c(u \to v)$

This Coupled with the fact that feasible flow $\leq$ Capacity of any cut we get max flow and min cut from the algo.



Net Flow across cut is $\sum_{u \in S} \sum_{v \in T} f(u \to v) - \sum_{u \in T} \sum_{v \in S} f(u \to v)$

When algo. terminates, $\sum_{u \in S} \sum_{v \in T} f(u \to v) = \sum_{u \in S} \sum_{v \in T} c(u \to v)$ and $\sum_{u \in T} \sum_{v \in S} f(u \to v) = 0$

(No parallel edges)

Obs. The above statements work as long as we do not have 2-cycle loops in the underlying undir. graph.

* If parallel edges are there, make them a single edge with combined capacity.

Can be resolved.

Introduce 2 new nodes.

Problem solved.

→ Minimum cut may or may not be unique.