Lecture 11

(9 September 2024)

Some Applications

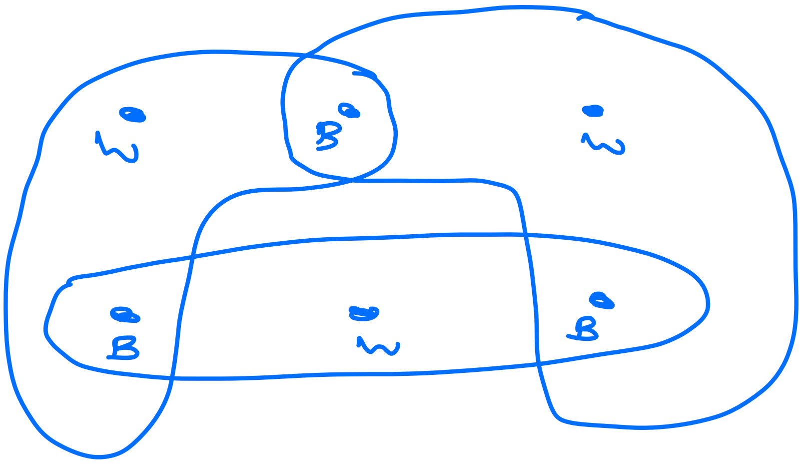
Combinatorics and Graph Theory

2-Coloring: [Existence Proof using Union Bound]

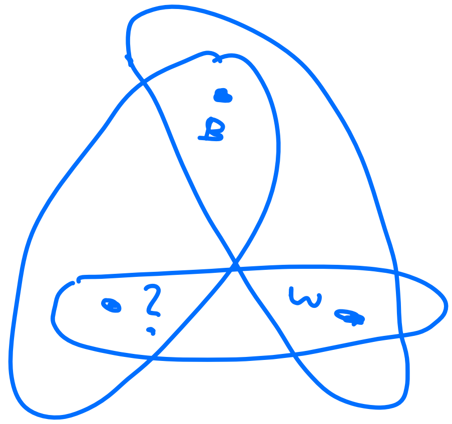
Let S be a set of some elements and $T_1, T_2, \ldots, T_m \subseteq S$ such that $|T_i| = \ell$ for $i \in [1:m]$.

Question.

Can we 2-color S (meaning assign each element of S a color) such that each T_i has elements of both colors (i.e., not monochromatic)?



2-Colorable



Not 2-colorable

$m = 3 < 4 = 2^{l-1}$

$m = 3 \not< 2 = 2^{l-1}$

<u>Theorem</u>. If $m < 2^{l-1}$, then there exists a valid 2-coloring of S such that no T_i is monochromatic.

<u>Proof</u>. Let $S = \{x_1, x_2, ---, x_n\}$.

Randomly color each element of S black or white, independently and identically distributed, each with probability $1/2$. Let E_i be the event that T_i is monochromatic.

$\Omega = \{B, W\}^n$, each ω is a 2-coloring.

$P(\{\omega\}) = 1/2^n$

$$\text{WBBWBBB}\underbrace{\text{BBB}\ldots\text{B}}_{T_i}\text{WBBW}$$

$$P(E_i) = \frac{2^{n-\ell}}{2^n} + \frac{2^{n-\ell}}{2^n} = \frac{2}{2^\ell} = \frac{1}{2^{\ell-1}}.$$

$$P\left(\bigcup_{i=1}^{m} E_i\right) \le \sum_{i=1}^{m} P(E_i)$$

$$= \frac{m}{2^{\ell-1}}$$

$P(\exists \text{ monochromatic } T_i) < 1$ if $m < 2^{\ell-1}$.

$\Rightarrow P(\text{no. monochromatic } T_i) > 0$ if $m < 2^{\ell-1}$.

$\Rightarrow \exists\, \omega$ s.t. the associated coloring has no monochromatic T_i.

This is called the <u>Probabilistic Method</u>.

Estimation

Minimum Mean Square Error Estimation

Let X, Y be jointly discrete random variables with joint pmf p_{XY}. We want to estimate X on observing Y.

$$\underset{\sim p_X}{X} \longrightarrow \boxed{p_{Y|X}} \longrightarrow Y \xrightarrow{f} \hat{X} = f(Y)$$

__Theorem__. The function $f(y) = E[X|Y=y]$ minimizes the expected squared error

$$E[(X-\hat{X})^2] = E[(X-f(Y))^2].$$

__Proof__. $E[(X-f(Y))^2]$

$$= E\left[\, E\left[(X - f(Y))^2 \mid Y\right]\,\right]$$

$$= \sum_{y} p_Y(y)\, E\left[(X - f(y))^2 \mid Y = y\right]$$

$$= \sum_{y} p_Y(y)\, E\left[X^2 + f(y)^2 - 2Xf(y) \mid Y = y\right]$$

$$= \sum_{y} p_Y(y)\left(f(y)^2 - 2E[X \mid Y = y]\, f(y) + E[X^2 \mid Y = y]\right)$$

Each of the terms in the above summation is minimized at

$$f(y) = E[X \mid Y = y].$$

Module 3 (Continuous Random Variables)

- Probability Density Functions
- Joint CDF, Joint PDF, Independence
- Expectation, Variance
- Examples of Continuous RVs
- Conditioning, Bayes' Rule
- Functions of Random Variables

Recall that $X : \Omega \to R$ is a random variable if

$$\{\omega : X(\omega) \leq x\} \in \mathcal{F}, \ \forall x \in R.$$

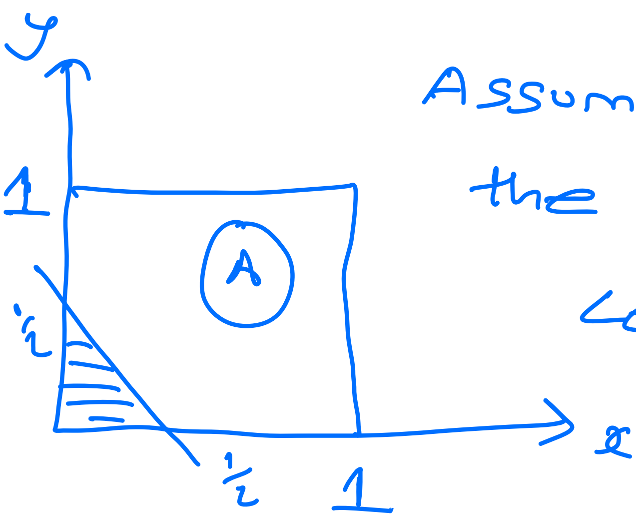
Consider Ω which is uncountable.

Cumulative distribution function

$$F_X(x) = P(X \leq x).$$

We first look at an example of a probability space $(\Omega, \mathcal{F}, P)$ where Ω is uncountable.

$\Omega = [0,1]^2$. Throw a dart on the unit square.

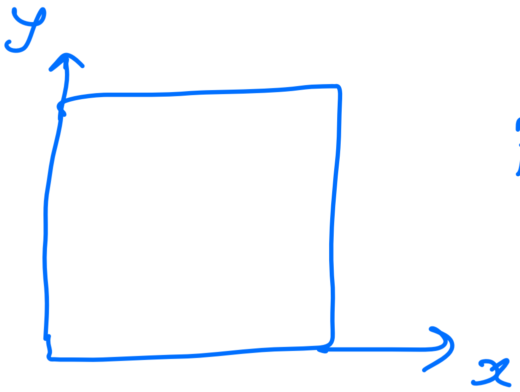


Assume $\mathcal{F}$ to contain all the subsets of Ω.

Let $P(A) = \text{area}(A)$

$$P(\{(x,y): x+y \le 1/2\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1/8.$$

$P((0.4, 0.6)) = 0$



$P(\Omega) = \text{area}([0,1]^2)$

$= 1$

However, $\Omega = \bigcup_{(x,y)\in\Omega} \{(x,y)\}$

Does additivity imply the following?

$$1 = P(\Omega) = \sum_{(x,y)\in\Omega} P(\{(x,y)\}) = 0.$$

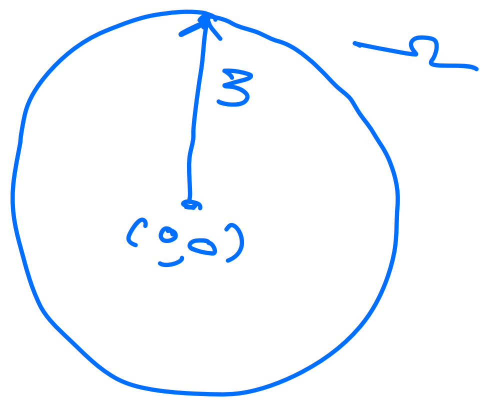
Is there a paradox?

No!

Reason.

Additivity holds only for countable number of disjoint events $A_1, A_2, \ldots,$

Example. A dart is thrown a circular target of radius 3.

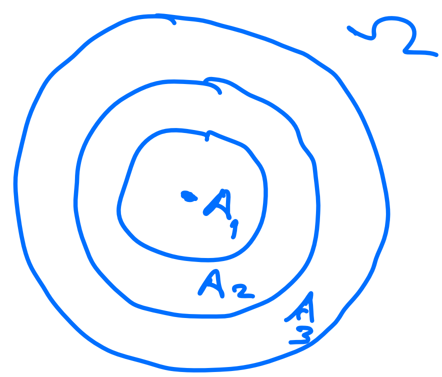


$$\Omega = \{(x,y) : x^2+y^2 < 9\}$$

Consider $P(A) = \dfrac{\text{area}(A)}{9\pi}$, $A \subseteq \Omega$.

A discrete scoring system:

$$A_k = \{(x,y) : k-1 \le \sqrt{x^2+y^2} < k\}.$$



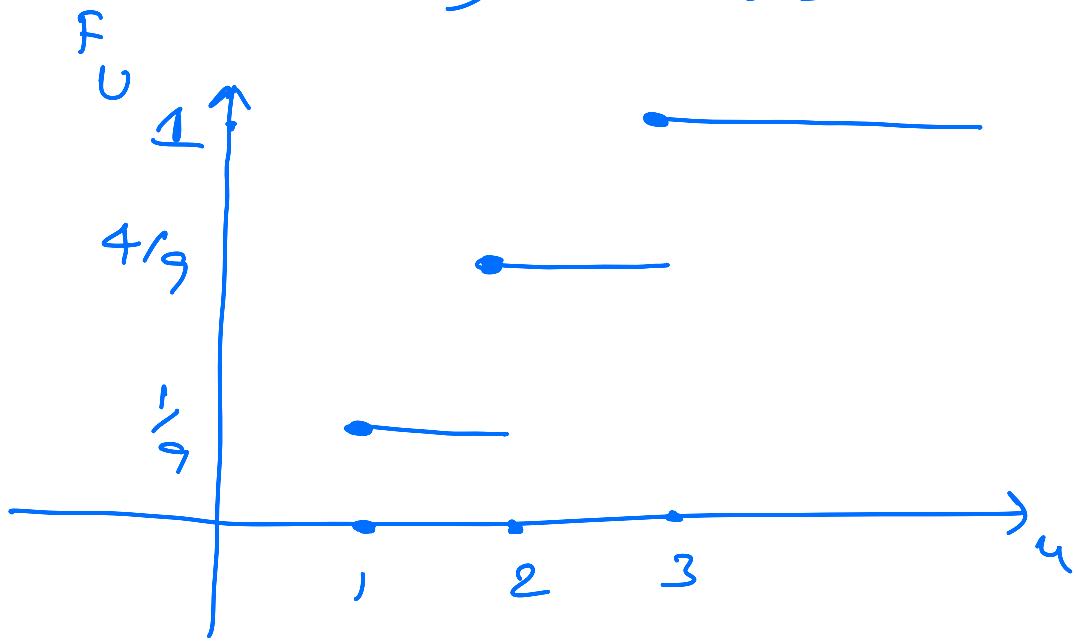
Suppose

$$U(\omega) = k \iff \omega \in A_k.$$

We find the CDF F_X.

$$F_U(x) = \begin{cases} 0, & u < 1 \\ P(A_1), & 1 \le u < 2 \\ P(A_1) + P(A_2), & 2 \le u < 3 \\ 1, & u \ge 3 \end{cases}$$

$$= \begin{cases} 0, & u < 1 \\ 1/9, & 1 \le u < 2 \\ 4/9, & 2 \le u < 3 \\ 1, & u \ge 3 \end{cases} .$$



A continuous scoring system:

$$V(\omega) = V((x,y)) = \sqrt{x^2+y^2} .$$

we find the CDF F_V.

$F_V(v) = 0, \quad v < 0,$

for $0 \le v \le 3$ $F_V(v) = P(V \le v)$

$$= P((x,y) : \sqrt{x^2+y^2} \le v)$$

$$= \frac{\cancel{\pi} v^2}{9\cancel{\pi}} = \frac{v^2}{9}.$$

$$F_V(v) = 1, \quad v \geq 3.$$