- Conditional expectation :

P_X , X , $E[X] = \sum_x x P_X(x)$

$P_{X|Y=y}$ — PMF

$\downarrow$

Q_X

$\rightarrow E[X|Y=y] = \sum_x x P_{X|Y}(x|y)$

$\rightarrow E[X|A] = \sum_x x P_{X|A}(x)$

$\rightarrow E[g(x)|A] = \sum_x g(x) P_{X|A}(x) \longrightarrow$ LOTUS rule for conditional PMF.

- Total expectation theorem

If the events $A_1, A_2, \ldots, A_n$ form a partition of the sample space with $P(A_i) > 0$, then

$$E[X] = \sum_{i=1}^{n} P(A_i)\, E[X|A_i]$$

Proof : $\sum_{i=1}^{n} P(A_i)\, E[X|A_i]$

$$= \sum_{i=1}^{n} P(A_i) \left(\sum_x x P_{X|A_i}(x) \right)$$

$$= \sum_x x \underbrace{\sum_{i=1}^{n} P(A_i) P_{X|A_i}(x)}_{P_X(x)}$$

$$= \sum_x x P_X(x)$$

$$= E[X]$$

Eg:

$\rightarrow$ PMF P_X

$A_1 = \{0 \le x \le 2\}$, $A_2 = \{6 \le x \le 8\}$.

(Here A_1 and A_2 form a partition of sample space)

$$E[X] = \underbrace{P(A_1)E[X|A_1]} + P(A_2)E[X|A_2]$$

Exercise: Compute LHS & RHS and show LHS = RHS

- Conditional expectation as RV

$E[X|Y=y] = \sum_x x \, P_{X|Y}(x|y)$ is not a random var.

Let $\phi(y) = E[X|Y=y]$, $P_Y(y) > 0$.

$\phi : \mathbb{R} \to \mathbb{R}$.

Here $\phi(y)$ changes with changing y because PMF changes.

$\because$ ϕ is a func^c and Y is a RV

So $\phi(Y)$ is a RV.

$$\phi(y) = \sum_x x \, P_{X|Y}(x|y) \Rightarrow \phi(Y) = E[X|Y].$$

This actually doesn't convey anything. It's just a notation.

It makes sense only when we condition it on some particular value $Y=y$.

So if we fix $Y=y$. Then $E[X|Y=y]$ is a constant and thus is not a RV.

But for diff values of y, $E[X|Y=y]$ is RV

$$E[X] = \sum_{i=1}^{n} P(A_i) E[X|A_i]$$, $A_i = \{Y = y_i\}$, $y_1, y_2, \ldots$ are values taken by RV X.

$\{Y=y\}$, $\forall y$ form a partition.

$$E[X] = \sum_{i=1}^{n} P_Y(y) \, E[X|Y=y].$$ (Replacing A_i with $Y=y$ in the above eq^n).

$$= \sum_{i=1}^{n} P_Y(y)\ \phi(y)$$

$$= E[\phi(Y)].$$

$$= E[E[X|Y]] \quad (\text{By definition}).$$

(Here $E[X|Y]$ is a RV) Boom

$E[Y|X] = \psi(X)$. , $\psi(x) = E[Y|X=x]$

$\rightarrow$ $\boxed{E[E[X|Y]] = E[X]}$ $\longrightarrow$ LAW OF ITERATED EXPECTATION

- Conditional independence

X & Y are conditionally independent, given an event A, if

$$\underbrace{P_{XY|A}(x,y)}_{P(X=x,\,Y=y\,|\,A)} = \underbrace{P_{X|A}}_{\frac{P(X=x\,\cap\,A)}{P(A)}}(x)\ P_{Y|A}(y)$$

$$P(X=x, Y=y \mid A)$$

$$= \frac{P((X=x)\cap(Y=y)\cap A)}{P(A)}$$

Exercise :

		y			
		1	2	3	4
x	1	0	$\frac{1}{20}$	0	0
	2	0	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
	3	$\frac{1}{20}$	$\frac{4}{20}$	$\frac{1}{20}$	$\frac{2}{20}$
	4	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{2}{20}$	0

$\swarrow P_{XY}$

Are X & Y are independent?

NO.

$\because P_{XY}(1,1) = 0$

$P(X=1) = \frac{1}{20}$

$P(Y=1) = \frac{2}{20} \quad \left(= \frac{1}{20} + \frac{1}{20}\right).$

$P_{XY}(1,1) \neq P_X(1)P_Y(1)$

$\therefore$ Not indep.

* So that means if the joint PMF has a zero value anywhere & $\because$ the rows or columns can't add up to 0 $\therefore$ not indep.

$A = \{X \geq 3, \ Y \leq 2\}$.

$$P_{XY|A}(x, y) = P_{X|A}(x) \, P_{Y|A}(y) \quad \forall \ x, y.$$

- Conditional variance

$$\text{var}(X) = E[(X - E[X])]^2$$

$\rightarrow$ Gives by how X deviates from its expectation val.

σ_X^2 - variance

σ_X - Standard deviation

$\rightarrow$ $\text{var}(X|Y=y) = E[(X - E[X|Y=y])^2 \mid Y=y]$ $\leftarrow$ Definition

$|||^{\text{lar}}$ to $\text{var}(X) = E[X^2] - (E[X])^2$, we have

$$\text{var}(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$$ $\leftarrow$ Verify

$\psi(y) = \text{var}(X|Y=y)$ $\qquad$ Y RV.

$\text{var}(X|Y) = \psi(Y)$ $\longrightarrow$ Notation.

RV.

$\rightarrow$ $\text{var}(X|Y) = E[X^2|Y] - (E[X|Y])^2$

Exercise : $\text{var}(X) = E[\text{var}(X|Y)] + \text{var}(E[X|Y])$

$\hookrightarrow$ LAW OF TOTAL VARIANCE

Proof : $\text{var}(X) = E[X^2] - (E[X])^2$

$= E[E[X^2|Y]] - (E[E[X|Y]])^2$ ($\because$ Law of iterated expectation)

Complete the proof!