RANDOM VARIABLES

Random variable is a fully deterministic func from Ω to $\mathbb{R}$.

But why is it still called "random" variable? Because the value of domain comes from sample space random experiment.

Eg: $\Omega = \{HH, HT, TH, TT\}$.

$X(HH) = 2$

$X(HT) = 1$

$X(TH) = 1$

$X(TT) = 0$

$X(\omega)$: no. of heads in ω.

When Ω is mapped to 1D real line structure it only makes sense to talk about $X \geq x$ or $X < x$ etc.

Random variable (formal defn) : Given a probability space $(\Omega, \mathcal{F}, P)$, a random variable is a function $X : \Omega \to \mathbb{R}$

s.t $\{\omega : X(\omega) \leq x\} \in \mathcal{F} \ \forall \ x \in \mathbb{R}$.

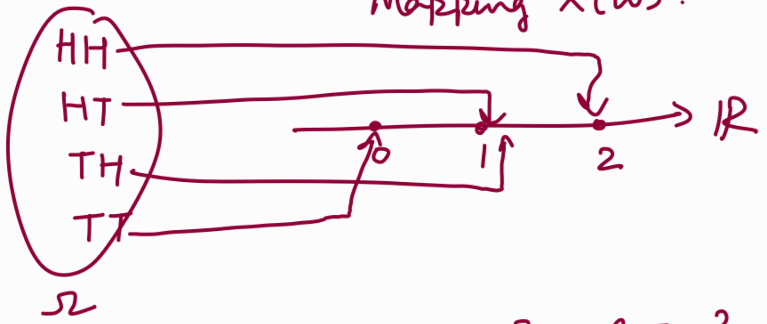
In a way $\underbrace{X^{-1}((-\infty, x])}_{\subseteq \mathbb{R}} = \{\underbrace{\omega}_{\in \Omega} : X(\omega) \leq x\} \in \mathcal{F}$

$X(\omega) \in (-\infty, x] \ \forall \ x \in \mathbb{R}$

$\Big($ $f : A \to B$. $\quad \underbrace{f^{-1}(C)}_{C \subseteq B} = \{x \in A : f(x) \in C\}$

Eg: $X(\omega)$ = no. of heads. $\mathcal{F} = 2^{\Omega}$

Mapping $X(\omega)$.



$\rightarrow \{\omega : X(\omega) \leq 0\} = \{TT\}. \in \mathcal{F}$

$\rightarrow \{\omega : X \leq c\} = \phi \in \mathcal{F}$

where $-\infty < c < 0$

$\rightarrow \{\omega : X \leq c\} = \{TT\} \in \mathcal{F}$

where $0 \leq c < 1$

$\rightarrow \{\omega : X \leq c\} = \{HT, TH\} \in \mathcal{F}$

where $1 \leq c < 2$

$\rightarrow \{\omega : X \leq c\} = \{TT, HT, TH\} \in \mathcal{F}$

where $-\infty < c \leq 1.5$

$\rightarrow \{\omega : X \leq c\} = \Omega \in \mathcal{F}$

where $c \geq 2$

Eg: Consider $\mathcal{F} = \{\phi, \Omega, \{HT, TH\}, \{TT, HH\}\}$

$TT \notin \mathcal{F}$

So $P\{X \leq c\} = \{TT\} \notin \mathcal{F}$

$0 \leq c \leq 1$

So it is NOT a random variable acc. to this given $\mathcal{F}$

$\rightarrow$ So $\mathcal{F}$ is a power set, then the variable is a valid random var.

But need to be careful when $\mathcal{F}$ is restricted because not all subsets would lie in $\mathcal{F}$ then.

- **Theorem**: Given $(\Omega, \mathcal{F}, P)$ and random variable $X : \Omega \to \mathbb{R}$, the following holds:

Notation: $\{X \leq x\} \equiv \{\omega : X(\omega) \leq x\}$

(i) $\{X < x\} \in \mathcal{F} \ \forall x \in \mathbb{R}$

(ii) $\{X = x\} \in \mathcal{F}$

(iii) $\{x_1 \leq X \leq x_2\} \in \mathcal{F}$

(iv) $\{x_1 < X < x_2\} \in \mathcal{F}$

etc.

Proof: (i) $\{X < x\} = \{\omega \in \Omega : X(\omega) < x\}$

$$= \{\omega : X(\omega) \in (-\infty, x)\}$$

$$= X^{-1}((-\infty, x))$$

Let $A_i = (-\infty, x - \frac{1}{i}]$, $i \in \mathbb{N}$

$X^{-1}(A_i) \in \mathcal{F} \ \forall i$ (By defn)

$$\bigcup_{i=1}^{\infty} A_i = (-\infty, \underset{\Downarrow}{x)}$$

Open bracket (Prove it!)

So $\bigcup_{i=1}^{\infty} X^{-1}(A_i) \in \mathcal{F}$ $\because X^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(A_i)$

(ii) $\underbrace{\{X \leq x\} \in \mathcal{F}. \quad \{X < x\} \in \mathcal{F}}$

Intersection of these is $\{X = x\} \in \mathcal{F}$.

$\rightarrow$ Basically $\{X \in S\}$ $S \subseteq \mathbb{R}$.

Discrete random variable

- The concept of random variable
- Distribution func
- Discrete & continuous RVs
- Expectation, variance, functions of RVs — Discrete.
- Multiple RVs, conditioning, independence.