Axiomatic approach.

$(\Omega, \mathcal{F}, P)$

Sample space — Event space — Prob. law / Prob. measure.

• Ω (Sample space) : The set of all possible outcomes of a random exp.

→ In terms of set theory, $\Omega \equiv$ Universal set.

Eg: Ω of coin toss = $\{H, T\}$.

Ω of roll of a dice.

~~$\{1, 2, 3 \text{ or } 4, 4 \text{ or } 5, 6\}$~~

It's not a sample space because elements are NOT mutually exclusive, though it contains all possibilities.

→ Elements should be mutually exclusive & collectively exhaustive

Ω of roll of a dice ~~$\{1, 2, 3, 4, 5\}$~~

$\therefore$ Not collectively exhaustive

(i) Finite sample space.

Eg: Roll a dice : $\{1, 2, 3, 4, 5, 6\}$.

(ii) Countably infinite sample space.

Eg: No. of tosses of a coin until H is observed.

$$\Omega = \{1, 2, 3, \ldots\}$$

$$\Omega = \{H, TH, TTH, \ldots\}$$

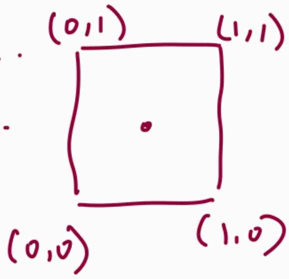
Depends on situation to choose whether the count of tosses or the outcomes of tosses itself.

(iii) Uncountably infinite sample space.

Eg: Throw a needle on the line $[0, 1]$.

Throw a dart on 1×1 square target.

$\Omega = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$.



- Event space $\mathcal{F}$

→ An event is a subset of the sample space.

→ Collection of all events is called event space.

→ There are 2^6 possible subsets i.e., 2^6 possible events possible to consider while rolling a dice.

→ E = even.

$\{E, E^c, \phi, \Omega\}$ — Event space

Def^n: An event space should be σ-field.

- σ-field/σ-algebra: A collection of sets is called σ-field if it satisfies the following:

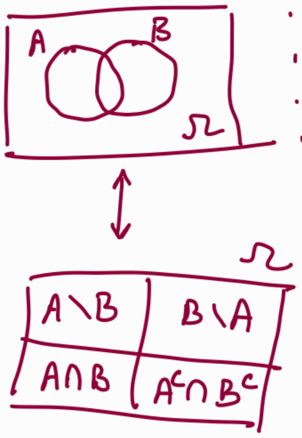
(i) $\Omega \in \mathcal{F}$ (Probability that something occurs is 1. So "something occurs" is an event which is essentially Ω).

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

↳ Though defined for countable union.
It is also true for finite union because
we could consider $A_1 = A,\ A_2 = B,\ A_3 = \phi,$
$A_4 = \phi, \ldots$

Smallest σ-field



Smallest σ-field with A:
$\sigma(A) = \{A, A^c, \phi, \Omega\}$.

Can have at most 16 elements.

Smallest σ-field with A, B is $\sigma(A, B)$.

$\sigma(A,B) = \{A, B, A^c, B^c, A^c \cap B, (A^c \cap B)^c, A \cap B^c,$
$(A \cap B^c)^c, A \cup B, (A \cup B)^c, A \cap B, (A \cap B)^c,$
$A \Delta B, (A \Delta B)^c, \Omega, \phi\}$.

Properties.

→ $A, B \in \mathcal{F} \Rightarrow A \setminus B \in \mathcal{F}$

$\Rightarrow (A \setminus B) \cup (B \setminus A) \in \mathcal{F}$

Exercise : Can union be replaced by intersection in (iii) above? Convince yourself why or why not. Yes.

Unions & intersections are interconvertible using deMorgan

Probability measure / Probability law (P)

Real func $\Rightarrow$ Domain is $\mathbb{R}$.

Set func $\Rightarrow$ Domain is set

Probability law is a set function.

$P : \mathcal{F} \to \mathbb{R}_+$ such that

the following axioms are satisfied :

(i) Non negativity : $P(E) \geq 0$, $E \in \mathcal{F}$

(ii) Normalisation : $P(\Omega) = 1$

(iii) Additivity : If $A_1, A_2, \dots$ are mutually exclusive/ disjoint sets , then $\underbrace{P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)}$

Countably infinite sets.

Eg: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

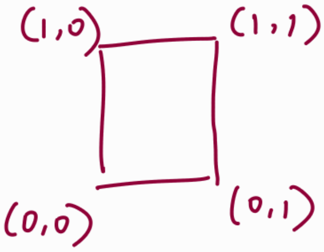
$P(\{i\}) = \frac{1}{6}$, $i \in [1:6]$

↳ Equiprobable.

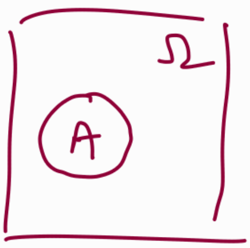
Infact any func where $P(\{i\}) = P_i$ & $\sum_i P_i = 1$ is a probability law.

Exercise: Probability law for a countably infinite sample space & uncountably infinite sample space.

Eg:



$\Omega = \{(i,j) : 0 \leq i \leq 1,\ 0 \leq j \leq 1\}$.



$P(A) =$ Area of A.