① $\Omega \to$ sample space

$P_X : R \to [0,1]$

$X : \Omega \to R$

$f : R \to R$

$Y : \Omega \to R$

$P_Y : R \to [0,1]$

$$P_X(x) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\begin{aligned}
P_Y(y) &= P(\{\omega \in \Omega \mid Y(\omega) = y\}) \\
&= P(\{\omega \in \Omega \mid f(x) = y \text{ for } x = X(\omega)\}) \\
&= \sum_{x : f(x) = y} P(\{\omega \in \Omega \mid X(\omega) = x\}) \\
&= \sum_{x : f(x) = y} P_X(x)
\end{aligned}$$

To prove

$$E[X] = \sum_{n\geq 0}^{\infty} P(X>n)$$

we know by definition of expected value

$$E[X] = \sum_{k\geq 0}^{\infty} k.\, P(X=k)$$

$$\sum_{n\geq 0}^{\infty} P(X>n) = \sum_{n\geq 0}^{\infty} \sum_{k=n+1}^{\infty} P(X=k)$$

on swapping order of summation

$$= \sum_{k=1}^{\infty} \sum_{n\geq 0}^{k-1} P(X=k)$$

$$= \sum_{k\geq 1}^{\infty} P(X=k) \sum_{n\geq 0}^{k-1} 1$$

$$= \sum_{k=1}^{\infty} k.\, P(X=k)$$

$$= E[X]$$

$$\therefore \quad E[X] = \sum_{n\geq 0}^{\infty} P(X>n)$$

Question-3

$$P_X(x) = \begin{cases} \frac{1}{b-a+1}, & k \in [a,b] \cap \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \sum_{x \in \mathcal{X}} x\, p_X(x) = \left(\frac{1}{b-a+1}\right)[a + (a+1) + \dots + (b)]$$

$$= \frac{1}{\cancel{b-a+1}} \frac{\cancel{b-a+1}}{2}[a+b] = \frac{a+b}{2}$$

$$\operatorname{var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X^2] = \sum_{x \in \mathcal{X}} x^2 p_X(x) = \sum_{x \in \mathcal{X}} \frac{1}{(b-a+1)}[a^2 + (a+1)^2 + \dots + b^2]$$

$$= \frac{1}{(b-a+1)}\left[\frac{(b)(b+1)(2b+1)}{6} - \frac{(a-1)(a)(2a-1)}{6}\right]$$

$$= \frac{1}{6(b-a+1)}\left[(2b^3 + 3b^2 + b) - (2a^3 - 3a^2 + a)\right]$$

$$= \frac{2(b-a)(b^2+a^2+ab) + 3(b^2+a^2) + (b-a)}{6(b-a+1)}$$

$$= \frac{(b^2+a^2)}{3} + \frac{b^2+a^2+(2ab+1)(b-a)}{6(b-a+1)}$$

$$= \frac{a^2+b^2}{3} + \frac{a^2+b^2-(2ab+1)}{6(b-a+1)} + \frac{2ab+1}{6}$$

$$= \frac{a^2+b^2}{3} + \frac{2ab+1}{6} + \frac{(b-a-1)}{6}$$

$$= \frac{a^2+b^2}{3} + \frac{b-a+2ab}{6}$$

$$= \frac{2(a^2+b^2)+(b-a)}{6} + 2ab$$

$$= \frac{2(a+b)^2+(b-a)-2ab}{6}$$

$$\mathrm{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2(a+b)^2+(b-a)-2ab}{6} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{(a+b)^2}{12} + \frac{b-a-2ab}{6}$$

3 sol: Let Y be a random variable with CDF

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y & 0 \le y < 1 \\ 1 & y \geq 1 \end{cases}$$

Define RVS

$X_1 = F_1^{-1}(Y)$

$X_2 = F_2^{-1}(Y)$

$P(F_i^{-1}(Y) \le x_i) = P(Y \le F_i(x_i))$

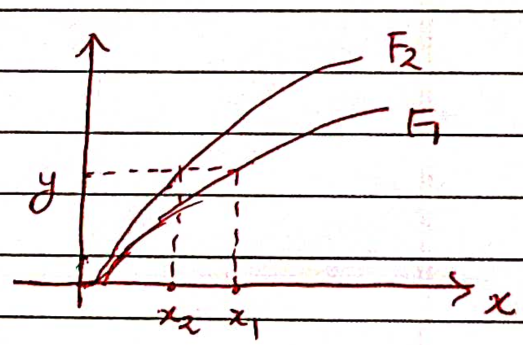
$= F_i(x_i)$

So the CDF of RVS x_1 & x_2 are F_1 & F_2, respectively.

For $y \in [0,1]$,

let $F_1^{-1}(y) = x_1$

$F_2^{-1}(y) = x_2$



suppose $x_1 \le x_2$

$\Rightarrow F_1(x_1) \le F_1(x_2) < F_2(x_2)$

$\Rightarrow y = F_1(x_1) \le F_1(x_2) < F_2(x_2) = y$

This is a contradiction

So, $x_1 > x_2$, i.e. $F_1^{-1}(y) > F_2^{-1}(y)$,

$\forall y \in [0,1]$

$X_1 = F_1^{-1}(Y) > F_2^{-1}(Y) = X_2$

$$\Omega = \left\{ \begin{matrix} (H, 1), (H, 3), \\ (H, 5), (T, 2), \\ (T, 4), (T, 6). \end{matrix} \right\}$$

d

Due to using fair ie and fair coin, all these events are equally likely.

$$X : \Omega : \longrightarrow [1:6]$$

$$X((a, b) \text{ st } (a, b) \in \Omega)$$

$$= b$$

$$P_x(3) = P(\text{Coin} = H \cap D_1 = 3)$$

These 2 events are independent from each other

$$\therefore P_x(3) = P(H)\, P(D_1 = 3 \mid \text{Coin} = H)$$

$$= \frac{1}{2} \times \frac{1}{3}$$

$$\text{lll}^{\text{ly}} \quad P_x(1) = \frac{1}{6} \cdot \left\{ \text{ess} : \frac{1}{2} \times \frac{1}{3} \right\}$$

$$P_x(5) = 1/6$$

$$P_x(2) = P(C = T \cap D_2 = 2)$$

$$= \frac{1}{2} \times \frac{1}{3} = 1/6$$

11'y for 4, 6.

$$E[X] = \sum_x x\, P_x(x)$$

$$= \sum_{i=1}^{6} i\, P_x(i) = \frac{1}{6} \sum_{i=1}^{6} x$$

$$= \frac{1}{6} \times 21 = \underline{\underline{3.5}}.$$

PRP A2 Q6

Pronoy Patra

August 2024

1 Solution 6

Let X be a discrete random variable uniformly distributed over the set of integers in the range $\{a, a+1, \ldots, b\}$, where $a < 0 < b$.

1. PMF of X: Since X is uniformly distributed, the probability of each value is:

$$P(X = k) = \frac{1}{b-a+1}, \quad \text{for } k \in \{a, a+1, \ldots, b\}.$$

2. Random Variable $Y = \max\{0, X\}$:
$Y = 0$ if $X \leq 0$.
$Y = X$ if $X > 0$.
Possible values of Y:
$Y = 0$ when $X \in \{a, a+1, \ldots, 0\}$.
$Y = k$ where k is an integer from 1 to b, when $X = k$.
PMF of Y:

$$P(Y = 0) = P(X \leq 0) = P(X \in \{a, a+1, \ldots, 0\}) = \frac{0-a+1}{b-a+1} = \frac{-a+1}{b-a+1}$$

For $y = k$ where k is an integer from 1 to b:

$$P(Y = k) = P(X = k) = \frac{1}{b-a+1}$$

Thus, the PMF of Y is:

$$P(Y = 0) = \frac{-a+1}{b-a+1}$$

$$P(Y = k) = \frac{1}{b-a+1}, \quad \text{for } k = 1, 2, \ldots, b$$

3. Random Variable $Z = \min\{0, X\}$:
$Z = X$ if $X \leq 0$.
$Z = 0$ if $X > 0$.
Possible values of Z:
$Z = k$ where k is an integer from a to 0, when $X = k$.
$Z = 0$ when $X \in \{1, 2, \ldots, b\}$.

PMF of Z: For $z = k$ where k is an integer from a to 0:

$$P(Z = k) = P(X = k) = \frac{1}{b - a + 1}$$

$$P(Z = 0) = P(X > 0) = P(X \in \{1, 2, \ldots, b\}) = \frac{b}{b - a + 1}$$

Thus, the PMF of Z is:

$$P(Z = k) = \frac{1}{b - a + 1}, \quad \text{for } k = a, a + 1, \ldots, 0$$

$$P(Z = 0) = \frac{b}{b - a + 1}$$

Hence:
PMF of $Y = \max\{0, X\}$:

$$P(Y = 0) = \frac{-a + 1}{b - a + 1}$$

$$P(Y = k) = \frac{1}{b - a + 1}, \quad \text{for } k = 1, 2, \ldots, b$$

PMF of $Z = \min\{0, X\}$:

$$P(Z = k) = \frac{1}{b - a + 1}, \quad \text{for } k = a, a + 1, \ldots, 0$$

$$P(Z = 0) = \frac{b}{b - a + 1}$$