Lecture 25

(14 November 2024)

A continuous-time random process $(X_t, t \in \mathbb{R})$ is strict-sense stationary (or simply stationary) if for all $t_1, t_2, \ldots, t_r \in \mathbb{R}$, $r \in \mathbb{N}$ and $T \in \mathbb{R}$,

$$F_{X_{t_1}, X_{t_2}, \ldots, X_{t_r}}(x_1, x_2, \ldots, x_r) = F_{X_{t_1+T}, \ldots, X_{t_r+T}}(x_1, x_2, \ldots, x_r).$$

A discrete-time random process $(X_n, n \in \mathbb{Z})$ is strict-sense stationary (or simply) stationary if for all $n_1, n_2, \ldots, n_r \in \mathbb{Z}$, $r \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$F_{X_{n_1}, X_{n_2}, \ldots, X_{n_r}}(x_1, \ldots, x_r) = F_{X_{n_1+k}, \ldots, X_{n_r+k}}(x_1, x_2, \ldots, x_r).$$

__Exercise__. Consider a discrete-time random process $(X_n, n \in \mathbb{Z})$ in which the X_n's are i.i.d. with $F_{X_n}(x) = F(x)$ $\forall n$ $\forall x$. Show that X_n is a strict-sense stationary (SSS) process.

In practice, it is desirable if a random process is stationary. For example, suppose that you need to do forecasting about the future of a process X_t. If we know the process is stationary, we can observe the past, which will normally give you a lot of information about how the process will behave in the future.

However, it turns out that not many real-life processes are stationary. Even if a process is stationary, it might be difficult to prove it. Fortunately, it is often enough to show a "weaker" form of stationarity.

A random process is wide-sense stationary (WSS) if, for all $t_1, t_2 \in \mathbb{R}, \tau \in \mathbb{R}$,

(i) $\mu_X(t_1) = \mu_X(t_2)$, i.e., $E[X_{t_1}] = E[X_{t_2}]$,

(ii) $R_X(t_1, t_2) = R_X(t_1+\tau, t_2+\tau)$, i.e.,

$$E[X_{t_1} X_{t_2}] = E[X_{t_1+\tau} X_{t_2+\tau}].$$

The first condition states that $\mu_x(t)$ is constant over time.

$$R_x(t_1, t_2) = E[X_{t_1} X_{t_2}]$$

$$= E[X_{t_1+\tau} X_{t_2+\tau}]$$

$$= E[X_{t_1-t_1} X_{t_2-t_1}]$$

$$= E[X_0 X_{t_2-t_1}]$$

$$= R_x(0, t_2-t_1)$$

$\therefore$ For a WSS process $R_x(t_1, t_2)$ is a function only of $t_2 - t_1$, so we write $R_x(t_2 - t_1)$.

Example. $X_t = \cos(t+U)$ $t \in \mathbb{R}$ where $U \sim \text{Uniform}[0, 2\pi]$. Show that X_t is a WSS process.

$$\mu_x(t) = E[X_t]$$

$$= E[\cos(t+U)]$$

$$= \int_0^{2\pi} \frac{\cos(t+u)\, du}{2\pi}$$

$$= \left[\frac{\sin(t+u)}{2\pi}\right]_0^{2\pi} = \left(\frac{\sin t - \sin t}{2\pi}\right) = 0.$$

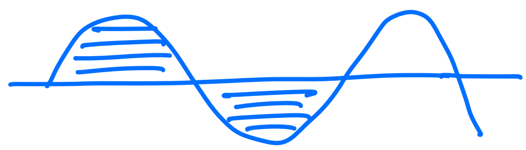
$$R_X(t_1, t_2) = E[X_{t_1} X_{t_2}]$$

$$= E[\cos(t_1+U)\cos(t_2+U)]$$

$$= E\left[\frac{1}{2}\cos(t_1+t_2+2U) + \frac{1}{2}\cos(t_1-t_2)\right]$$

$$= \frac{1}{2}\int_0^{2\pi} \cos(t_1+t_2+2u)\,du + \frac{1}{2}\cos(t_1-t_2)$$

$$= 0 + \frac{1}{2}\cos(t_1-t_2)$$



$= \cos(t_1-t_2)$ for all $t_1 t_2 \in \mathbb{R}$.

Properties of $R_X(\tau)$

(1) $R_X(0) = E[X_t \cdot X_t] = E[X_t^2] \geq 0.$

$E[X_t^2]$ is called expected (or average) Power in X_t at time t. so, for WSS, this is not a function of time.

(2) $R_X(-\tau) = R_X(t, t+\tau)$

$$= E[X_t X_{t+\tau}]$$

$$= E[X_{t+\tau} X_t]$$

$$= R_X(t+\tau, t)$$

$$= R_X(\tau).$$

(3) $|R_X(\tau)| \leq R_X(0).$

$$|R_X(\tau)| = |E[X_t X_{t-\tau}]|$$

$$\leq \sqrt{E[X_t^2] E[X_{t-\tau}^2]}$$

$$= \sqrt{R_X(0) \cdot R_X(0)}$$

$$= R_X(0).$$

Example. suppose that X_t is a WSS process with autocorrelation

$$R_X(\tau) = A e^{-\alpha|\tau|}.$$

Compute $E[(X_8 - X_5)^2]$.

$$E[(X_8 - X_5)^2] = E[X_8^2 + X_5^2 - 2X_8 X_5]$$

$$= R_X(0) + R_X(0) - 2R_X(3)$$

$$= 2A - 2Ae^{-3\alpha}.$$

More generally we have

$$E\left[(X_{t+\tau}-X_t)^2\right]=2\left(R_X(0)-R_X(\tau)\right).$$

Power Spectral Density

The power spectral density of a WSS process X_t is defined as the Fourier transform of $R_X(\tau)$.

$$S_X(f)=\int_{-\infty}^{\infty} R_X(\tau)\, e^{-i2\pi f\tau}\, d\tau .$$

From the inverse Fourier transform we have

$$R_X(\tau)=\int_{-\infty}^{\infty} S_X(f)\, e^{+i2\pi f\tau}\, d\tau .$$

Properties of PSD

- $S_X(f)$ is real and even.

  This follows because $R_X(\tau)$ is real and even and the Fourier transform of a real and even function is real & even.

$$R_x(-\tau) = R_x(\tau)$$

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-i2\pi f\tau} d\tau$$

$$S_x(-f) = \int_{-\infty}^{\infty} R_x(\tau) e^{i2\pi f\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_x(-\tau) e^{-i2\pi f\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_x(\tau) e^{-i2\pi f\tau} d\tau = S_x(f).$$

$S_x(f)$ is even.

$$S_x^*(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{i2\pi f\tau} d\tau$$

$$= S_x(-f) = S_x(f).$$

$\Rightarrow S_x$ is real & even.

so, we have

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\cos(2\pi f\tau)$$

$$= 2\int_{0}^{\infty} R_x(\tau)\cos(2\pi f\tau).$$

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f)\cos(2\pi f\tau)$$

$$= \int_{0}^{\infty} S_x(f)\cos(2\pi f\tau).$$

- The expected power in X_t is

$$E[X_t^2] = R_x(0) = \int_{-\infty}^{\infty} S_x(f)\, e^{-i2\pi f(0)}\, df$$

$$= \int_{-\infty}^{\infty} S_x(f)\, df.$$

This is why $S_x(f)$ is called the power spectral density.

- $S_x(f) \geq 0.$

This follows because $R_x(\tau)$ is a positive semi-definite function and from a property of Fourier transform that the Fourier transform of a positive semi-definite function is non-negative (Bochner's theorem). For $a_1 \ldots a_n \in \mathbb{R}$, $t_1 \ldots t_n \in \mathbb{R}$,

$$\sum_{i,j=1}^{n} a_i a_j R_X(t_i - t_j)$$

$$= \sum_{i,j=1}^{n} a_i a_j E[X_{t_i} X_{t_j}]$$

$$= E\left[\sum_{i,j=1}^{n} a_i a_j X_{t_i} X_{t_j}\right]$$

$$= E\left[\left(\sum_{i=1}^{n} a_i X_{t_i}\right)\left(\sum_{j=1}^{n} a_j X_{t_j}\right)\right]$$

$$= E\left[\left(\sum_{i=1}^{n} a_i X_{t_i}\right)^2\right] \geq 0.$$

$\boxed{\text{$f(t)$ is positive semi-definite function if } \forall a_1 \ldots a_n, t_1 \ldots t_n \text{ we have } \sum_{i,j=1}^{n} a_i a_j f(t_i - t_j) \geq 0.}$

Gaussian Random Vectors

Two random variables $X_1 X_2$ are said to be jointly Gaussian if their joint PDF is

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{\sqrt{(2\pi)^2 |K|}} \exp\left(-\frac{1}{2}\begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} K^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}\right),$$

where $E[X_i] = \mu_i \quad i = 1, 2$

$$k_{ij} = \operatorname{cov}(X_i, X_j), \quad i, j = 1, 2.$$

- K is called covariance matrix. It is a positive semi-definite matrix.

- n RVs $X_1, X_2, \ldots, X_n$ are said to be jointly Gaussian (or multivariate normal) if their joint CDF is

$$f_{X_1, X_2, \ldots, X_n}(x_1, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n |K|}} \exp\left[-\frac{1}{2} (\bar{x} - \bar{\mu}) K^{-1} (\bar{x} - \bar{\mu})^T \right]$$

where $\bar{\mu} = [\mu_1, \mu_2, \ldots, \mu_n]$ with $\mu_i = E[X_i]$

$$k_{ij} = \operatorname{cov}(X_i, X_j), \quad i, j \in [1:n].$$

- If $X_1, X_2, \ldots, X_n$ are jointly Gaussian each X_i is also Gaussian. Moreover any linear combination of X_i's is also Gaussian (can be proved using MGFs similar to the proof of sum of two Gaussians is Gaussian).