# Lecture 22

(28 October 2024)

<u>WLLN</u>. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed RVs with mean $\mu$. Then, for every $\varepsilon > 0$, we have

$$P\left(\left|\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty.$$

<u>Remark</u>. Note that the proof we gave in the last class assumes finite variance. It turns out that this law holds true even if the $X_i$'s have infinite variance. However, a much more elaborate argument is needed (in particular, that involves characteristic functions and complex analysis).

<u>Application of WLLN</u>.

Let $\mathcal{X} = \{a, b, c\}$ and $x \in \mathcal{X}$.

$P_X$ is unknown.

Suppose we are given $n$ i.i.d. samples

from $P_X$. How can we estimate $P_X$?

$$Y_i = \mathbb{1}\{X_i = a\} \quad i \in [1:n].$$

$$\sum_{i=1}^{n} Y_i / n \longrightarrow E[\mathbb{1}\{X_i = a\}] = P_X(a)$$

similarly $\dfrac{\sum_{i=1}^{n} \mathbb{1}\{X_i = b\}}{n}$ and $\dfrac{\sum_{i=1}^{n} \mathbb{1}\{X_i = c\}}{n}$

are the estimates of $P_X(b)$ and $P_X(c)$ respectively.

## Convergence in Distribution

We say a sequence of random variables $X_1, X_2, \ldots$ converges to $X$ in distribution if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x),$$

for all points $x$ at which $F_X(x) = P(X \leq x)$ is continuous.

# Central Limit Theorem

Let $x_1, x_2, \ldots$ be a sequence of i.i.d. random variables with mean $\mu < \infty$ and variance $\sigma^2 < \infty$. Then the random variable

$$Z_n = \frac{\sum_{i=1}^{n} x_i - n\mu}{\sqrt{n}\,\sigma}$$

converges in distribution to the standard Gaussian random variable $N(0,1)$ as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} P(Z_n \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt .$$

## Interpretation

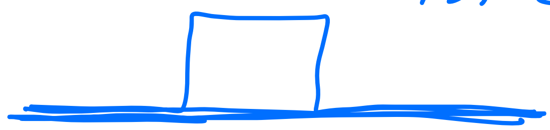
- $X_i$'s are i.i.d. Uniform $[0,1]$.

$$Z_n = \frac{\sum_{i=1}^{n} X_i - \frac{n}{2}}{\left(\sqrt{n/12}\right)}$$

$n=1$

$$Z_1 = \frac{X_1 - \frac{1}{2}}{\sqrt{\frac{1}{12}}}$$

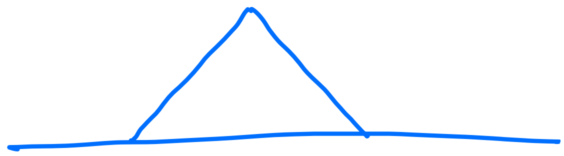
PDF of $Z_1$



$n=2$

$$Z_2 = \frac{X_1 + X_2 - 1}{\sqrt{\frac{2}{12}}}$$

PDF of $Z_2$



$n=3$

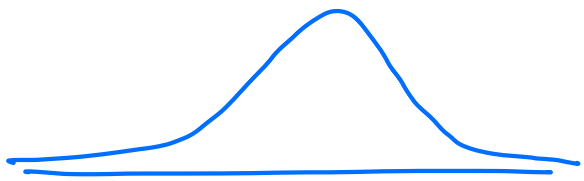
$$Z_3 = \frac{X_1 + X_2 + X_3 - \frac{3}{2}}{\sqrt{\frac{3}{12}}}$$

PDF of $Z_3$

$n=30$

$$Z_{30} = \frac{\sum_{i=1}^{n} X_{20} - 15}{\sqrt{\frac{30}{12}}}$$

PDF of $Z_{30}$



- Let $X_i$'s be i.i.d. with mean 0 & variance 1. CLT says

$$\frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \longrightarrow N(0, 1).$$

Taking $\sqrt{n}$ to the right-hand-side can we say

$$\sum_{i=1}^{n} x_i \text{ is approximately Gaussian}$$

for large enough $n$?

No! This is because $\sum_{i=1}^{n} x_i$ cannot take uncountable real values as $n \to \infty$. so $\sum_{i=1}^{n} x_i / \sqrt{n}$ is necessary.

<u>Proof</u>. $$Z_n = \frac{\sum_{i=1}^{n} y_i}{\sqrt{n}}, \quad \text{where } y_i = \frac{x_i - \mu}{\sigma}.$$

$y_i$ has mean $0$ and variance $1$.

With out loss of generality we will assume that $x_i$'s have mean $0$ and variance $1$, and show that

$$\frac{\sum_{i=1}^{n} x_i}{\sqrt{n}} \xrightarrow{D} N(0, 1).$$

We prove the theorem under the assumption that the MGF $M_x(s) < \infty$ for $s \in [-\varepsilon, \varepsilon]$.

$$M_{Z_n} = E\left[ e^{s \sum_{i=1}^{n} X_i / \sqrt{n}} \right]$$

$$= M_X(s/\sqrt{n})^n$$

we show that

$$\lim_{n\to\infty} n \ln\left( M_X(s/\sqrt{n}) \right) = \frac{s^2}{2}.$$

Let $L(s) = \ln M_X(s)$.

$$\ln M_Z(s) = n \ln M_X(s/\sqrt{n}) = n\, L(s/\sqrt{n}).$$

$$L(0) = 0,$$

$$L'(s) = \frac{M_X'(s)}{M_X(s)} \Rightarrow L'(0) = E[X] = 0,$$

$$L''(s) = \frac{M_X(s) M_X''(s) - M_X'(s)^2}{M_X(s)^2} \Rightarrow L''(0) = 1.$$

$$\lim_{n\to\infty} \ln M_{Z_n}(s) = \lim_{n\to\infty} n \ln\left( M_X(s/\sqrt{n}) \right)$$

$$= \lim_{n\to\infty} \frac{L(s/\sqrt{n})}{(1/n)}$$

$$= \lim_{n\to\infty} \frac{s \, \frac{-1}{2} \, n^{-3/2} \, L'(s/\sqrt{n})}{-1/n^2} \quad (\text{L'Hopital's rule})$$

$$= \lim_{n\to\infty} \frac{s L'(s/\sqrt{n})}{2n^{-1/2}}$$

$$= \lim_{n\to\infty} \frac{s^2 L''(s/\sqrt{n}) \cancel{n^{-3/2}}}{2 \cancel{n^{-3/2}}}$$

$$= s^2 L''(0) = \frac{s^2}{2}.$$

So $\lim_{n\to\infty} M_{Z_n}(s) = e^{s^2/2}$.

Recall that MGF of $N(0,1)$ is $M_N(s) = e^{s^2/2}$.

Since there is a one-to-one correspondence between MGFs and CDFs, the convergence in MGFs is equivalent to the convergence in distribution, i.e., $MGF_{Z_n} \xrightarrow{n\to\infty} MGF_Z$ implies $F_{Z_n} \xrightarrow{n\to\infty} F_Z$. This completes the proof of CLT.

# A Remark on CLT Proof

We would like to remark here that the above proof assumes the existence of MGF over some interval $[-\varepsilon, \varepsilon]$. This may not always be the case. There are some RVs with finite mean and finite variance but MGF does not exist anywhere except for $s=0$, e.g., Pareto distribution. In such cases the above proof will not go through.

However the central limit theorem still holds and a more elaborate argument is needed to prove this via characteristic functions and complex analysis.

# Normal Approximation Based on CLT

Let $S_n = \sum_{i=1}^{n} X_i$, where $X_i$ are i.i.d. with mean $\mu$ and variance $\mu^2$, we are interested in computing $P(S_n \le c)$.

If $n$ is large, it can be approximated in the following way.

$$P(S_n \le c) = P(S_n - n\mu \le c - n\mu)$$

$$= P\left( \frac{S_n - n\mu}{\sqrt{n}\,\sigma} \le \frac{c - n\mu}{\sqrt{n}\,\sigma} \right)$$

$$\approx \Phi\left( \frac{c - n\mu}{\sqrt{n}\,\sigma} \right),$$

where $\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.

Example. Let $X_i \sim$ i.i.d. with mean 2 and variance 1 denote the service times of customers numbered 1 2 - - standing in a queue. Let $Y$ be the total time the bank teller spends serving 50 customers. Find an approximation to $P(90 < Y < 110)$.

$$Y = \sum_{i=1}^{50} X_i$$

$$P(90 < Y < 110) = P\left(\frac{90-100}{\sqrt{50}} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \frac{110-100}{\sqrt{50}}\right)$$

$$= P\left(-\sqrt{2} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \sqrt{2}\right)$$

$$\approx \Phi(\sqrt{2}) - \Phi(-\sqrt{2}).$$

<u>Convergence</u> in <u>mean-square sense</u>

We say a sequence of RVs $X_1, X_2, \ldots$ converges in mean-square sense to $X$ if

$$\lim_{n \to \infty} E\left[(X_n - X)^2\right] = 0.$$

**Theorem.** If $X_n \to X$ in mean-square sense, then $X_n \to X$ in probability also.

**proof.** $X_n \to X$ in mean-square sense

$$\Rightarrow \lim_{n\to\infty} E[(X_n - X)^2] = 0.$$

Consider

$$P(|X_n - X| > \varepsilon) = P(|X_n - X|^2 > \varepsilon^2)$$

$$\leq \frac{E[(X_n - X)^2]}{\varepsilon^2}$$

(by Markov's inequality)

$$\Rightarrow \lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0,$$ i.e.,

$X_n \to X$ in probability.