CS 302.1 - Automata Theory

Lecture 03

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Quick Recap

- DFAs and NFAs are equivalent
- For every NFA we can obtain a “Remembering DFA” that accepts the same language.
- The language accepted by finite automata are called Regular Languages.
- Regular operations: Union, Complement, Concatenation, **Star**.

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  - If $\Sigma = \{a\}$, $\Sigma^* = \{\epsilon, a, aa, aaa, \ldots \ldots\}$

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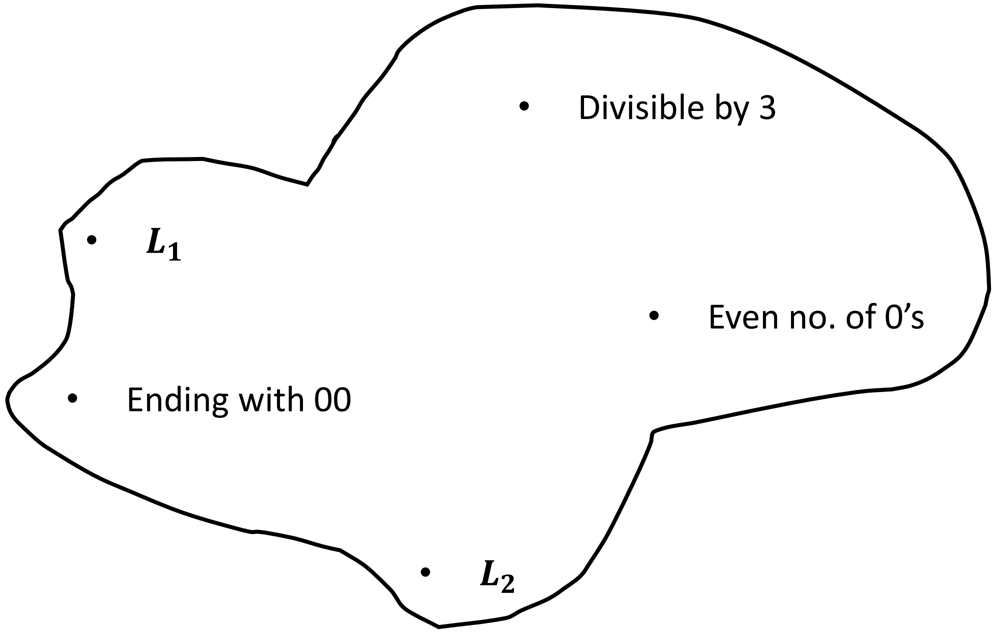
  - If $\Sigma = \{a\}$, $\Sigma^* = \{\epsilon, a, aa, aaa, \dots \dots\}$
  - If $\Sigma = \{\Phi\}, \Sigma^* = \{\epsilon\}$

Closure of Regular Languages

We want to check whether the set of regular languages are **closed** under some operations.

What does this mean?

- We pick up points within the set of all regular languages (say L_1 and L_2)
- Perform *set operations* such as Union, concatenation, Star, intersection, reversal, complement etc on them.
- Observe whether the resulting language still belongs to the set of all regular languages.
- If so, we say, regular languages are **closed** under that operation.



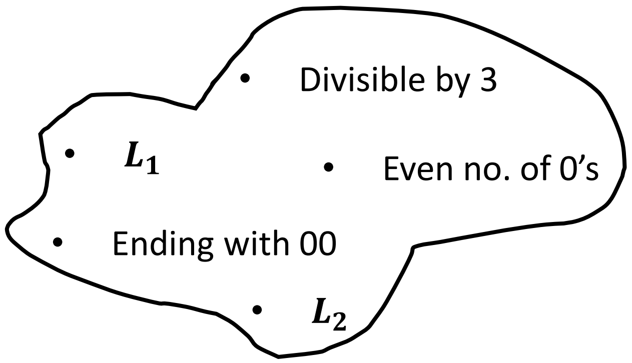
Set of all regular Languages

For example, the **natural numbers are closed under addition/multiplication** and **not under subtraction/division**.

Closure of Regular Languages

Q: Is the set of all regular languages **closed under union**?

Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?



Set of all regular Languages

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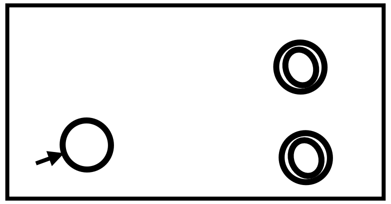
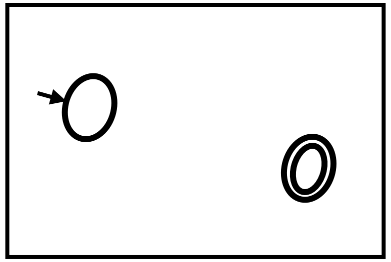
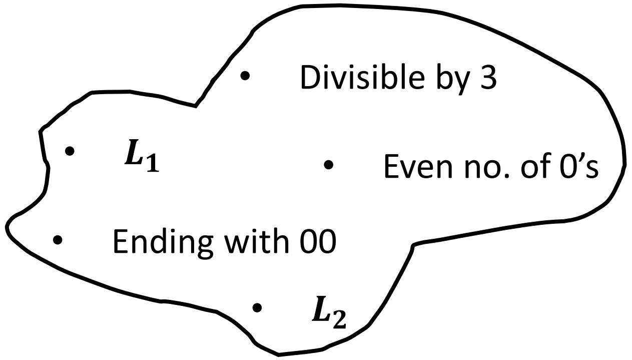
Using M_1 and M_2, we will show how to construct an NFA M that accepts $L = L_1 \cup L_2$, i.e. $L(M) = L_1 \cup L_2$.

Suppose the DFA M_1 is



And the DFA M_2 is





Set of all regular Languages

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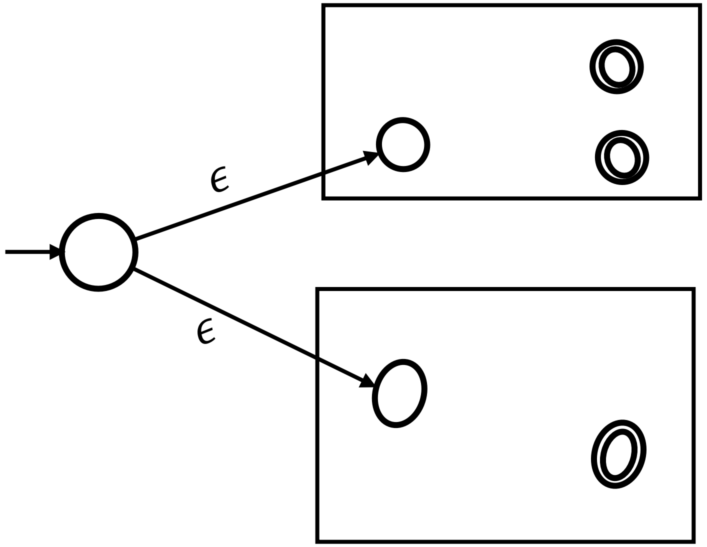
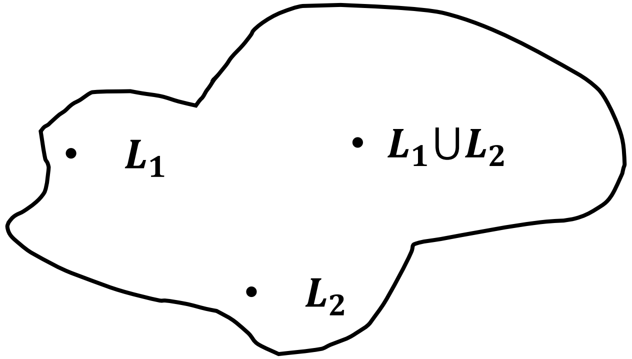
NFA M accepting $\boldsymbol{L = L_1 \cup L_2}$





Set of all regular Languages

Closure of Regular Languages

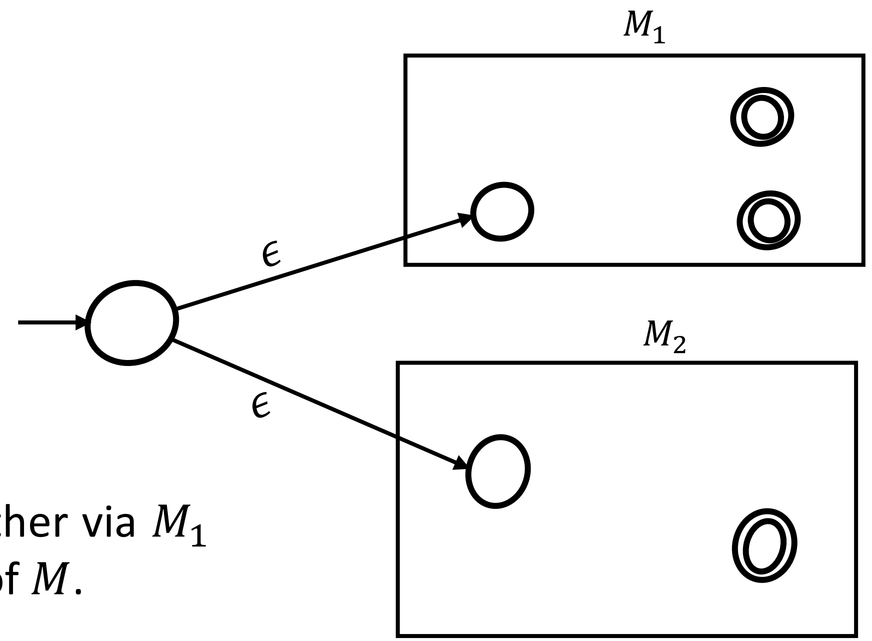
Q: Is the set of all regular languages **closed under union**?

Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?

Proof: In order to prove that $L(M) = L_1 \cup L_2$, we show two things:

(i) $L \subseteq L_1 \cup L_2$

Let $\omega \in L$, i.e. ω is accepted by M. The final state for L can be reached either via M_1 or M_2. Thus ω must be accepted by either of them to reach the final state of M.



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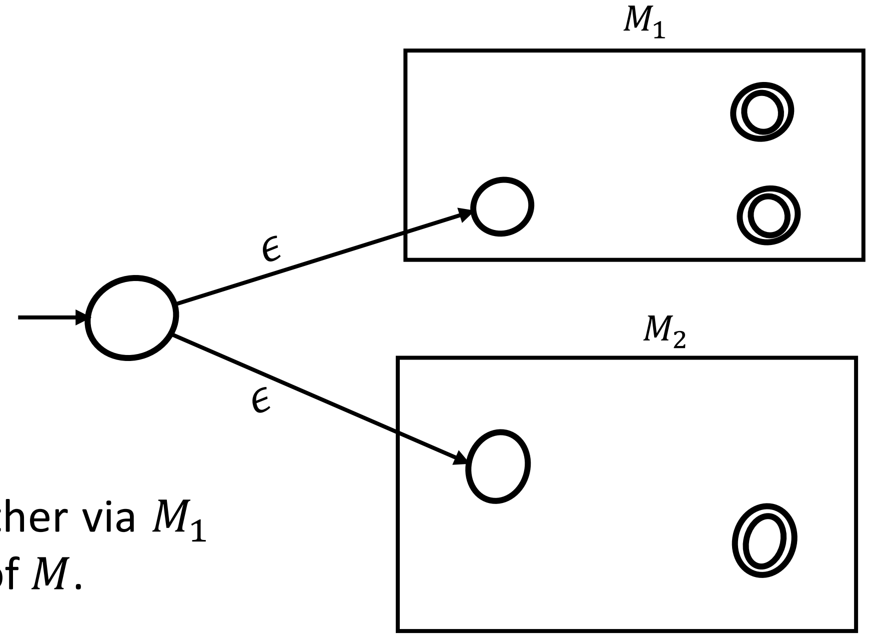

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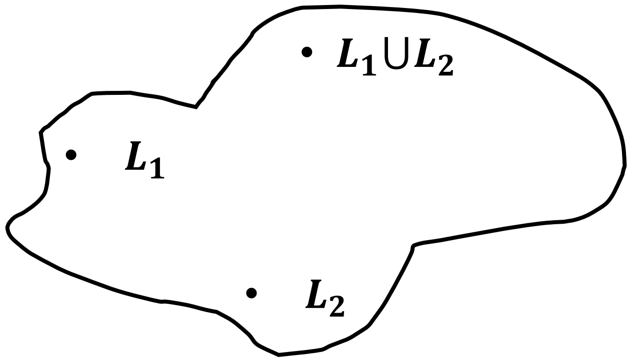
(ii) $L_1 \cup L_2 \subseteq L$

Let $\omega \in L_1 \cup L_2$. Then, $\omega \in L_1$ or $\omega \in L_2$.

Thus, ω must reach the final state of M_1 or M_2. But since the start state of M_1 or M_2 can be reached from the start state of M by taking an ϵ-transition, $\omega \in L$.



Set of all regular Languages

Closure of Regular Languages

Q: Is the set of all regular languages **closed under concatenation**? Suppose L_1 and L_2 are regular languages. Is $L = L_1.L_2$ also regular?

Proof: Since L_1 and L_2 are regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$ and a DFA M_2 that accepts L_2, i.e. $L(M_2) = L_2$.

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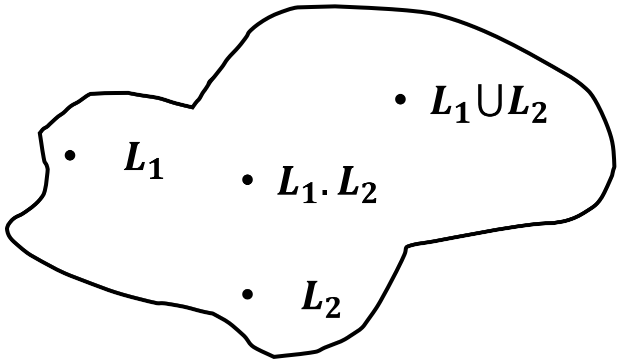
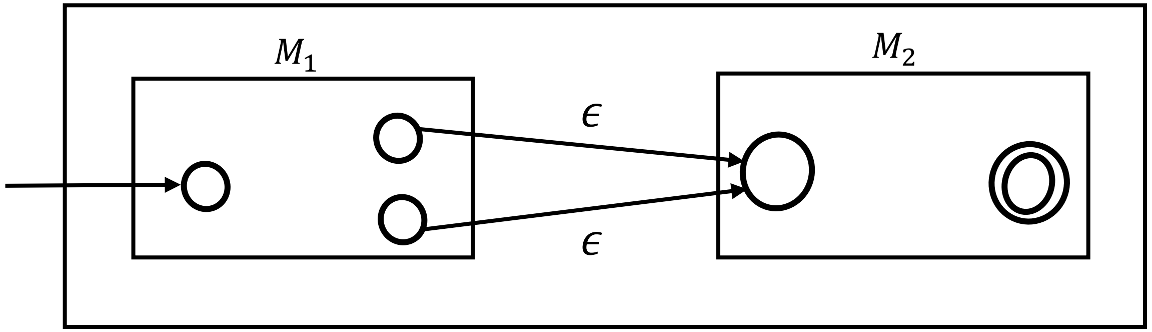


Set of all regular Languages

$L_1.L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$



NFA M accepting $\boldsymbol{L = L_1.L_2}$

Closure of Regular Languages

Q: Is the set of all regular languages **closed under star**? Suppose L_1 is a regular language. Is L_1^* also regular?

Proof: Since L_1 is regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$. Using M_1, we will show how to construct an NFA M that accepts $L = L_1^*$.

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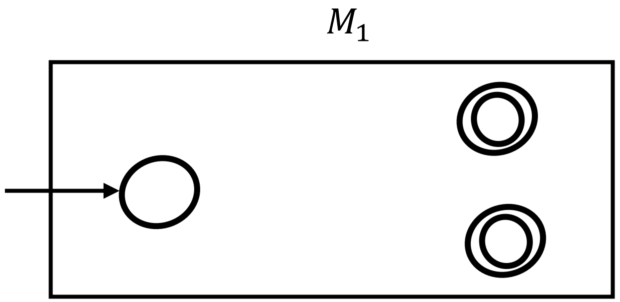

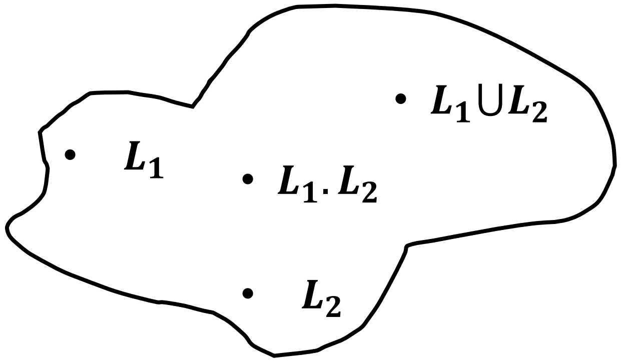

Set of all regular Languages

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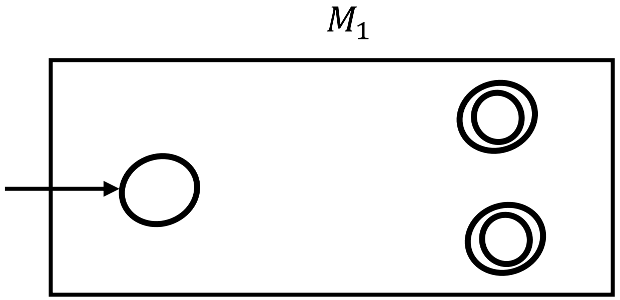

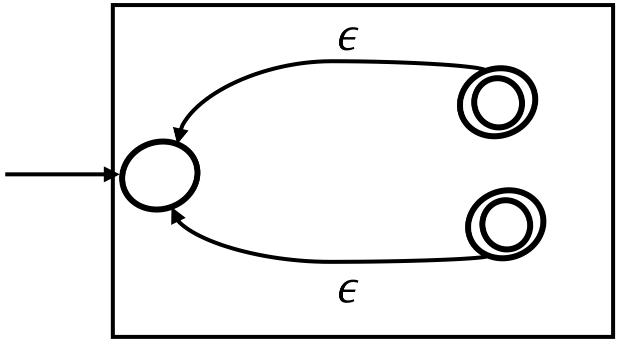

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Steps:

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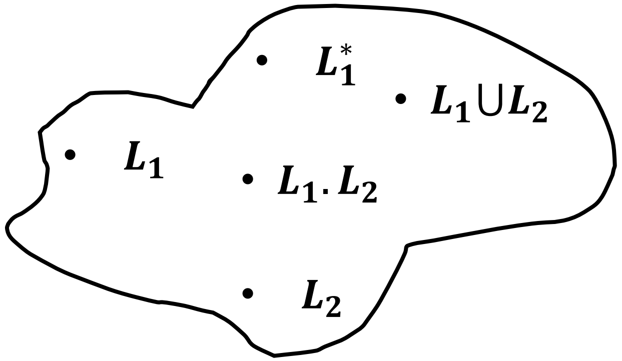

Set of all regular Languages

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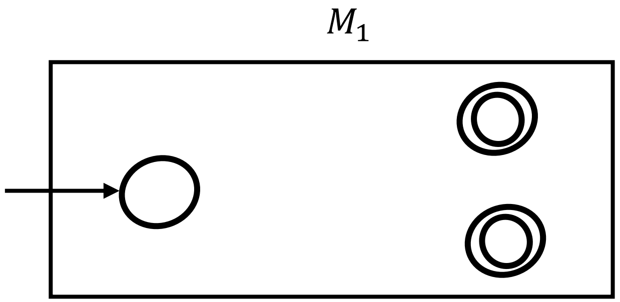

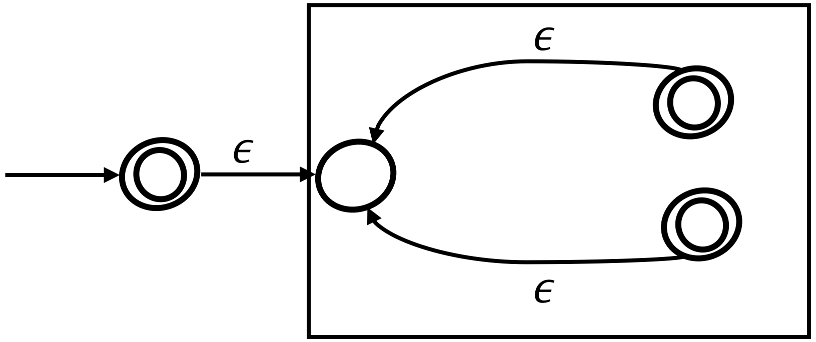

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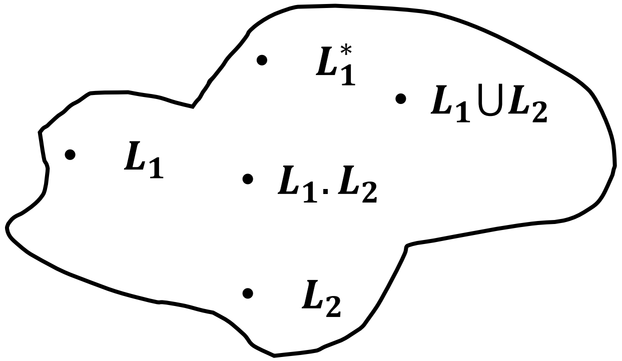


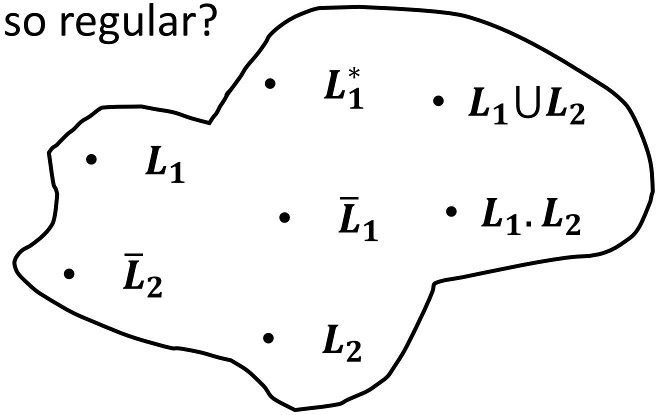
Set of all regular Languages

Steps:

- Make ϵ-transitions from the final states of L_1 to the initial state of L_1.
- Make a new final state as the start state and make an ϵ-transition from this state to the previous start state of L_1.

Closure of Regular Languages

Q: Is the set of all regular languages **closed under complement**? If L is regular, then is $\overline{L}$ also regular?



Set of all regular Languages

Proof: Given a DFA M, such that $L(M) = L$, construct the **toggled DFA** M' from M, by

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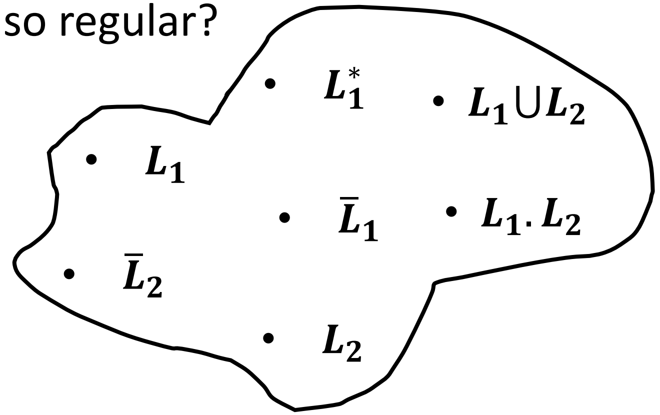

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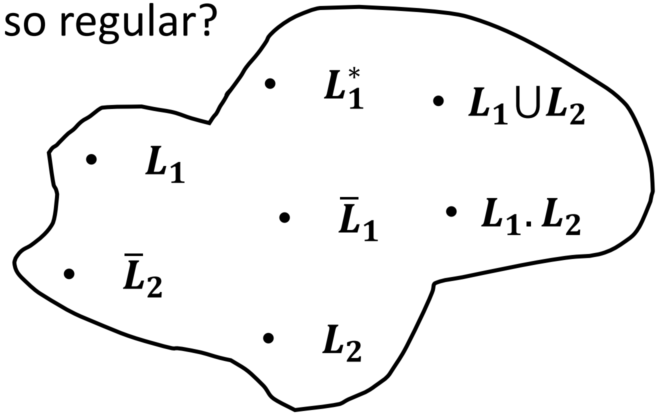

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	NFA N	**Toggled NFA N'**
Run 1	Rejecting	
Run 2	Accepting	

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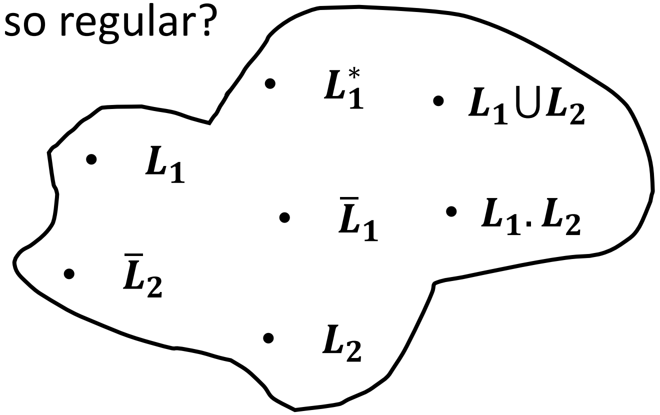

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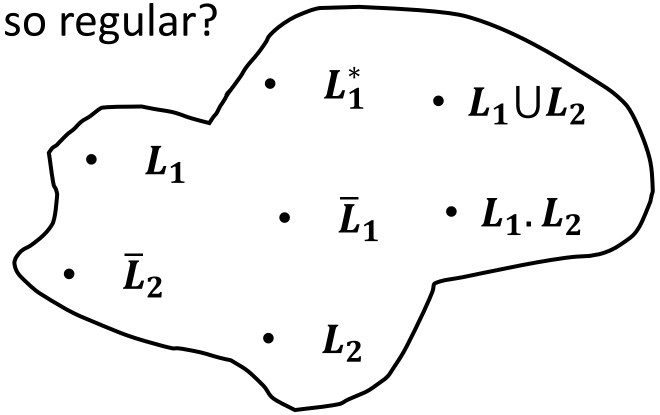

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Contradiction! So No, the **toggled NFA does not accept $\overline{L}$**.

	NFA N	**Toggled NFA N'**
Run 1	Rejecting	Accepting
Run 2	Accepting	Rejecting

Closure of Regular Languages

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Proof: We shall use the fact that regular languages are **closed** under union and complement.

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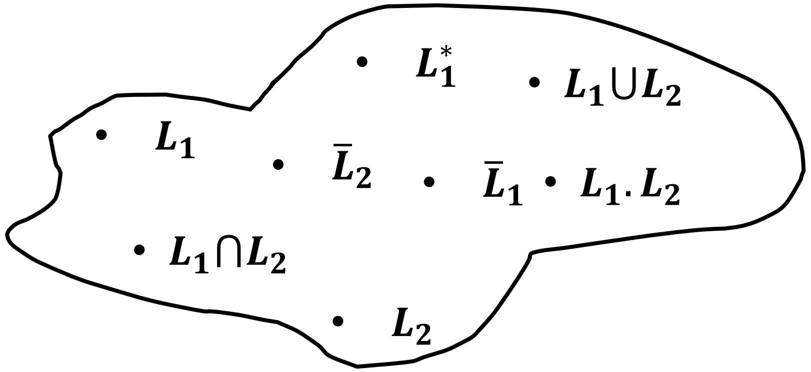
Given a DFA for L_1 and a DFA for L_2, we know how to construct an NFA for $\overline{L_1}, \overline{L_2}$ as well as for $L_1 \cup L_2$. Using these constructions and the aforementioned relationship, we can construct an NFA for $L = L_1 \cap L_2$



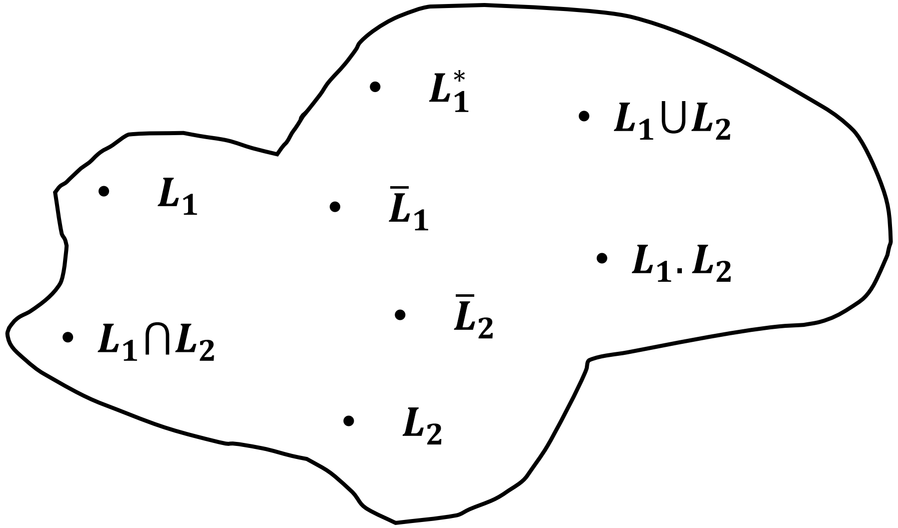
Set of all regular Languages

Closure of Regular Languages

Summary:

Regular Languages are closed under:

- **Union**
- **Intersection**
- **Star**
- **Complement**
- **Concatenation**



Set of all regular Languages

Regular Languages

If Σ is an alphabet, then

- $\Sigma^0 = \{\epsilon\}$
- $\Sigma^2 = \{a_1 a_2 | a_1 \in \Sigma,\ a_2 \in \Sigma\}$
- $\Sigma^k = \{a_1 a_2 \cdots a_k | a_i \in \Sigma\ |1 \leq i \leq k\}$
- $\Sigma^* = \{\bigcup_{i\geq 0} \Sigma^i\} = \{\Sigma^0 \cup \Sigma^1 \cup \Sigma^2\ \cdots\} = \{a_1 a_2 \cdots a_k | k \in \{0,1,\cdots\}\ \&\ a_j \in \Sigma, \forall j \in \{1,2,\cdots,k\}\}$

A Language $L \subset \Sigma^*$ and $L^* = \{\bigcup_{i\geq 0} \mathrm{L}^i\}$

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Regular Language (alternate definition): Let Σ be an alphabet. Then the following are the regular languages over Σ:

- The empty language Φ is regular
- For each $a \in \Sigma, \{a\}$ is regular.
- Let L_1, L_2 be regular languages. Then $L_1 \cup L_2$, $L_1 . L_2$, L_1^* are regular languages.

Regular Expressions

A regular expression describes regular languages algebraically. The algebraic formulation also provides a powerful set of tools which will be leveraged to prove

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- derive properties of regular languages

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Syntax for regular expressions (Recursive definition): R is said to be a regular expression if it has one of the following forms:

- Φ is a regular expression, $L(\Phi) = \Phi$
- ϵ is a regular expression, $L(\epsilon) = \{\epsilon\}$
- Any $a \in \Sigma$ is a regular expression, $L(a) = \{a\}$

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- $R_1 + R_2$ is a regular expression if R_1 and R_2 are regular expressions, $L(R_1 + R_2) = L(R_1) \cup L(R_2)$
- R^* is a regular expression if R is a regular expression, $L(R^*) = \big(L(R)\big)^*$

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- R^* is a regular expression if R is a regular expression, $L(R^*) = \left(L(R)\right)^*$
- $R_1 R_2$ is a regular expression if R_1 and R_2 are regular expressions, $L(R_1 R_2) = L(R_1). L(R_2)$
- (R) is a regular expression if R is a regular expression, $L\left((R)\right) = R$

Regular Expressions

Syntax for regular expressions:

Regular Expression	Regular Language	Comment
Φ	$\{\}$	The empty set
ϵ	$\{\epsilon\}$	The set containing ϵ only
a	$\{a\}$	Any $a \in \Sigma$
$R_1 + R_2$	$L(R_1) \cup L(R_2)$	For regular expressions R_1 and R_2
$R_1 R_2$	$L(R_1).L(R_2)$	For regular expressions R_1 and R_2
R^*	$\left(L(R)\right)^*$	For regular expressions R
(R)	$L(R)$	For regular expressions R

Order of precedence: (), *, . , +

A language $\boldsymbol{L}$ is regular if and only if for some regular expression $\boldsymbol{R}$, $\boldsymbol{L(R) = L}$.

RE’s are equivalent in power to NFAs/DFAs

Regular Expressions

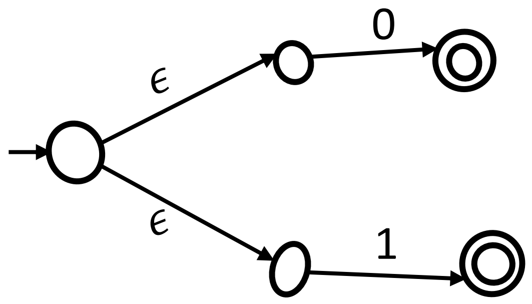
Syntax for regular expressions:

Regular Expression R	$\boldsymbol{L(R)}$
01	$\{01\}$
$01 + 1$	$\{01,1\}$
$(0+1)^*$	$\{\epsilon, 0, 1, 00, 01, \cdots\}$
$(01+\epsilon)1$	$\{011,1\}$
$(0+1)^*01$	$\{01,001,101,0001, \cdots\}$
$(0+10)^*(\epsilon+1)$	$\{\epsilon, 0, 10, 00, 001,010, 0101, \cdots\}$

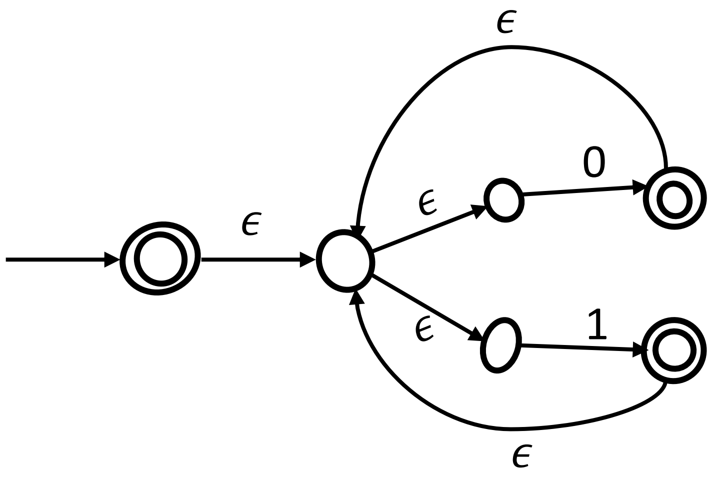
Regular Expressions

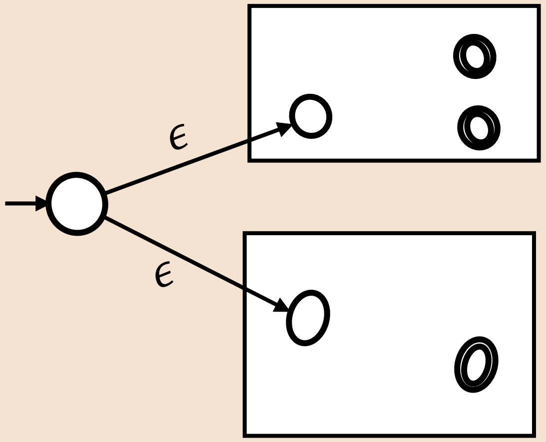
NFA for RE: $(\mathbf{0}+\mathbf{1})^*\mathbf{01}$

(i) NFA for $(0+1)$

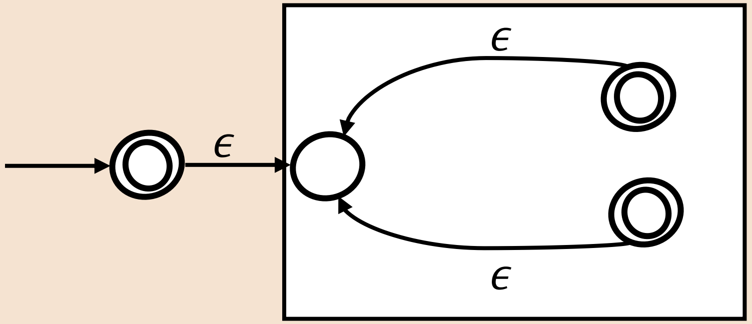


(ii) NFA for $(0+1)^*$





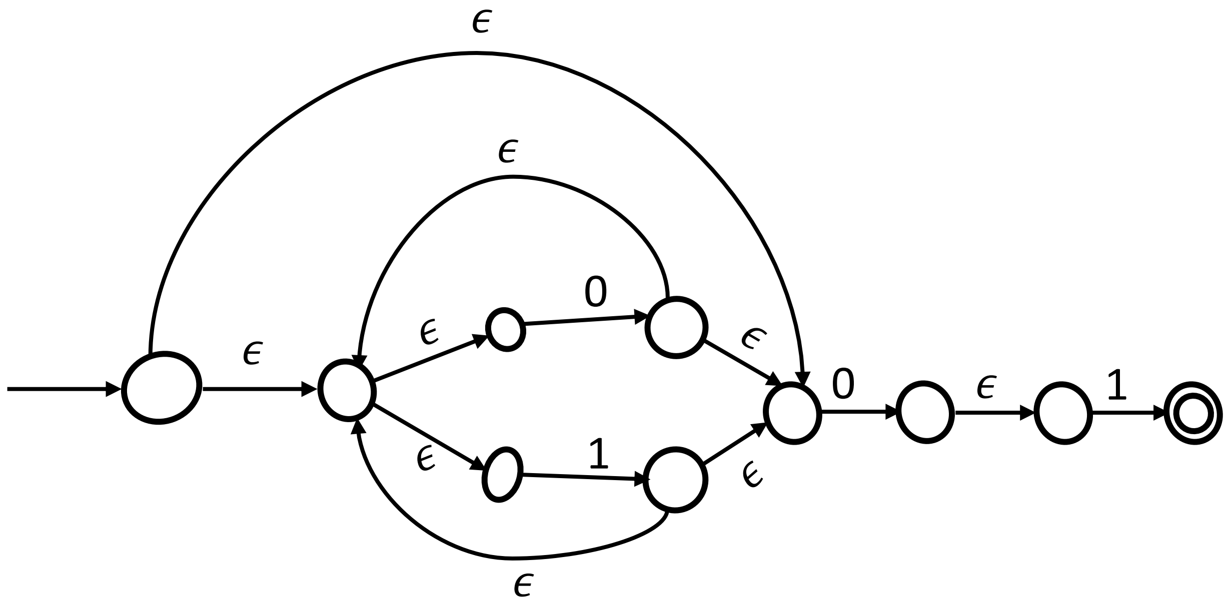
$L_1 \cup L_2$

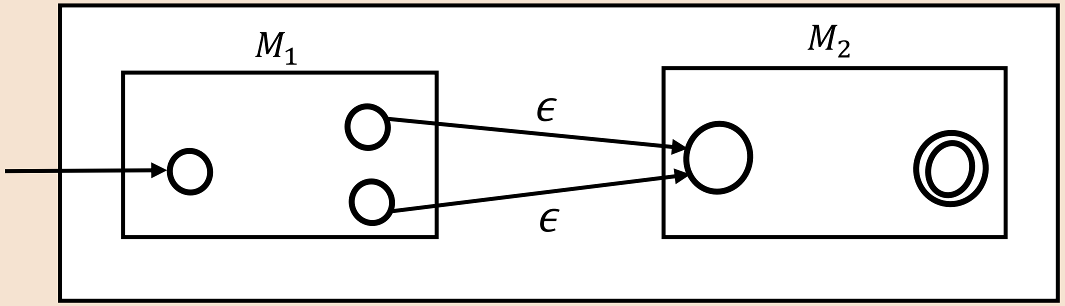


L_1^*

Regular Expressions

NFA for $(\mathbf{0}+\mathbf{1})^*\mathbf{01}$





$\boldsymbol{L_1.L_2}$

Regular Expressions

Let $\Sigma = \{a, b\}$.

Language	**Regular Expression**
$\{\omega \mid \omega$ ends in "ab"$\}$	$(a+b)^*ab$
$\{\omega \mid \omega$ has a single a $\}$	b^*ab^*
$\{\omega \mid \omega$ has at most one $a\}$	$b^* + b^*ab^*$
$\{\omega \mid \, \|\omega\|$ is even$\}$	$((a+b)(a+b))^* = (aa+bb+ab+ba)^*$
$\{\omega \mid \omega$ has "ab" as a substring$\}$	$(a+b)^*ab(a+b)^*$
$\{\omega \mid \|\omega\|$ is a multiple of 3$\}$	$\big((a+b)(a+b)(a+b)\big)^*$

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Some algebraic properties of Regular Expressions:

- $R_1 + (R_2 + R_3) = (R_1 + R_2) + R_3$
- $R_1(R_2R_3) = (R_1R_2)R_3$
- $R_1(R_2 + R_3) = R_1R_2 + R_1R_3$
- $(R_1 + R_2)R_3 = \mathrm{R_1R_3 + R_2R_3}$
- $R_1 + R_2 = R_2 + R_1$
- $R_1^*R_1^* = R_1^*$
- $(R_1^*)^* = R_1^*$
- $R\epsilon = \epsilon R = R$
- $R\Phi = \Phi R = \Phi$
- $R + \Phi = R$
- $\epsilon + RR^* = \epsilon + R^*R = R^*$
- $(R_1 + R_2)^* = (R_1^*R_2^*)^* = (R_1^* + R_2^*)^*$

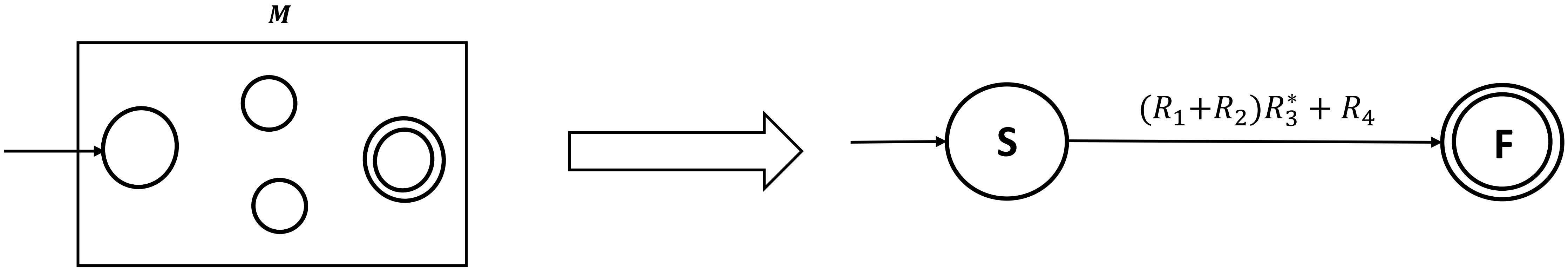
DFA to Regular Expressions

If a language is regular then it accepts a regular expression. We could draw equivalent NFAs for Regular Expressions.

How can we obtain Regular expressions given a DFA?

Given a DFA M, we **recursively** construct a two-state **Generalized NFA** (GNFA) with

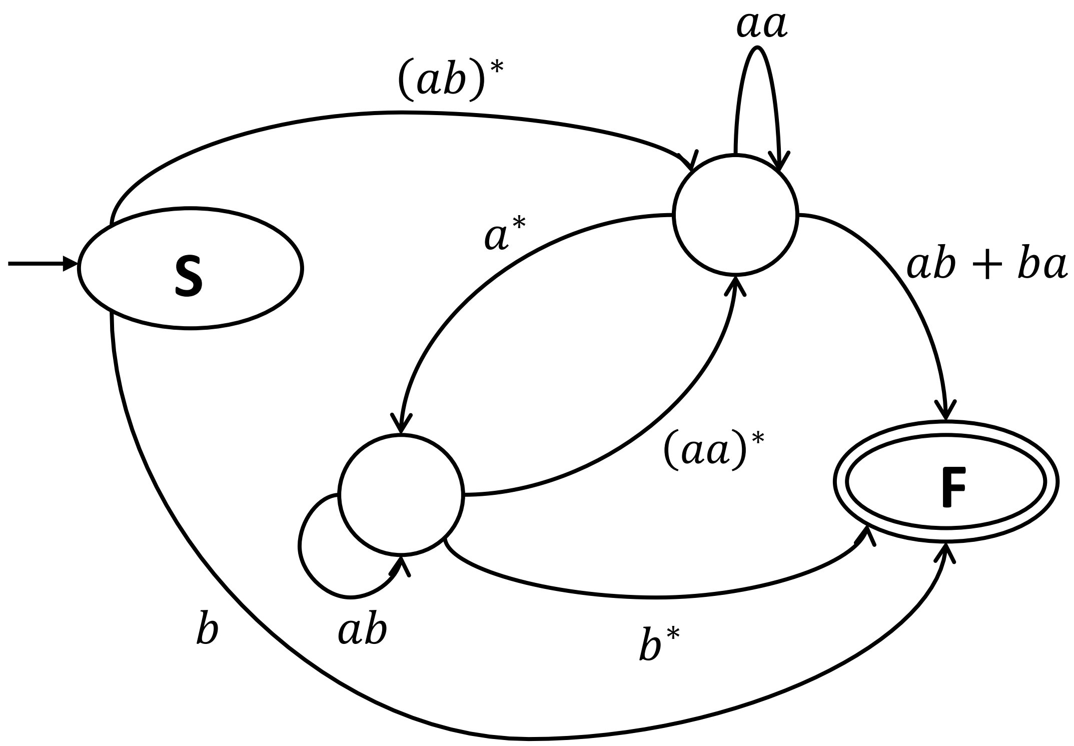
- A start state and a final state
- A single arrow goes from the start state to the final state
- The label of this arrow is the regular expression corresponding to the language accepted by the DFA M.



DFA to Regular Expressions: GNFA

What are GNFAs? They are simply NFAs such that

- The transitions may have regular expressions
- A unique start state that has arrows going to other states, but has no incoming arrows
- A unique final state that has arrows incoming from other states, but has no outgoing arrows
- For an input string, **runs** on a GNFA are similar to that of an NFA, except now a block of symbols are read corresponding to the Regular Expressions on the transitions.
- $b,\ abababab, abaaaba$ are some input strings that have accepting runs for the GNFA on the right



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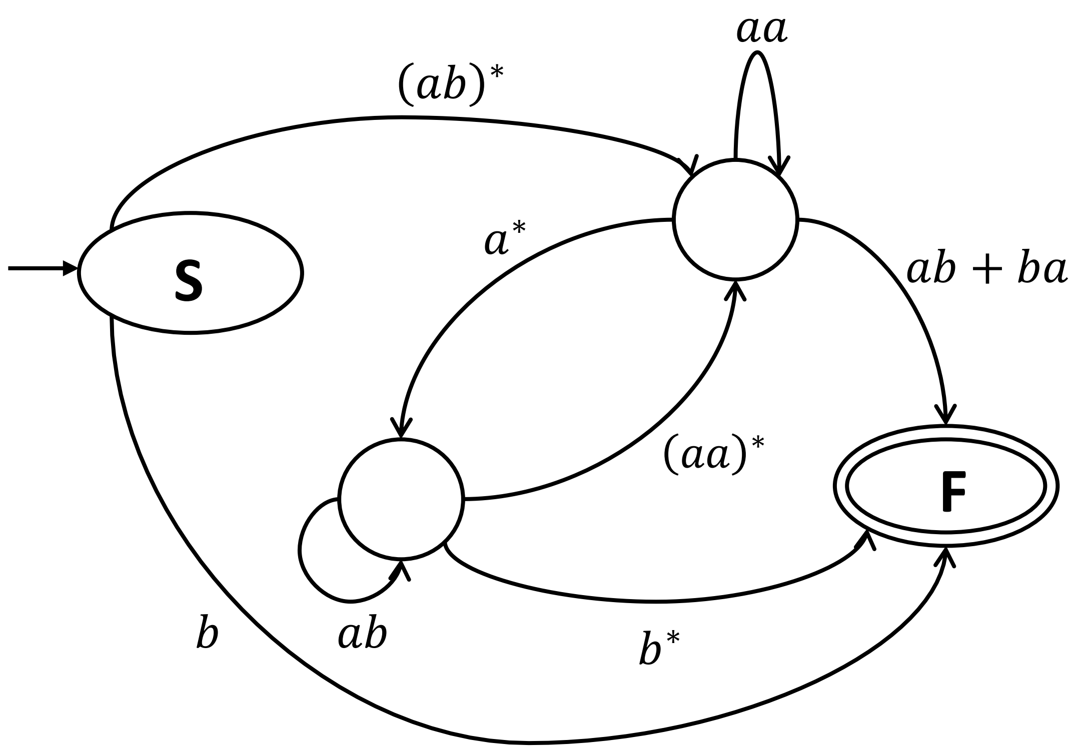
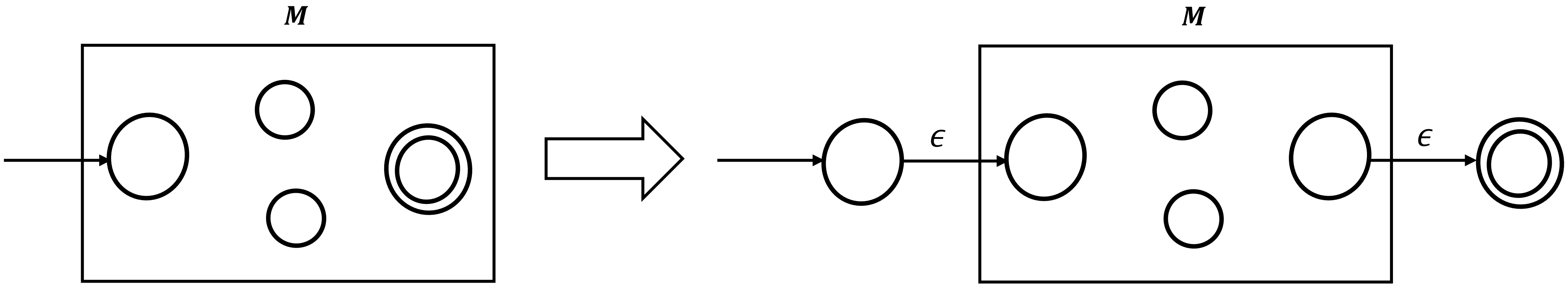


Starting from a DFA we will begin by constructing a GNFA with k states. We then outline a recursive procedure by which at each step, we will construct a GNFA with one less state. This step will be repeated until we obtain the **2-state GNFA**.

DFA to Regular Expressions: GNFA

Starting from the DFA M,

- Add a new start state with an ϵ arrow to the old start state.
- Add a new final state by with an ϵ arrow to the old final state.



DFA to Regular Expressions: GNFA

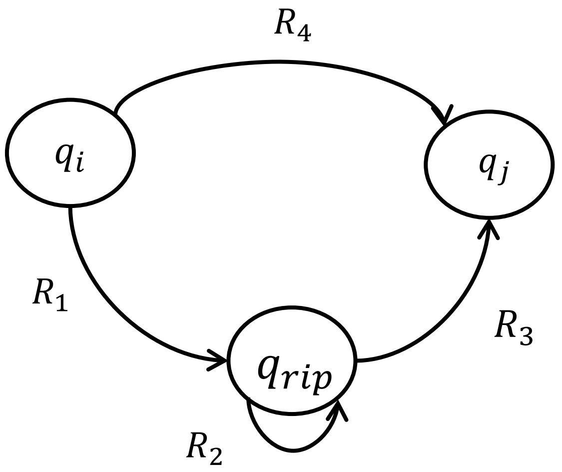
The crucial step is to convert a GNFA with k (>2) states to a GNFA with $k-1$ states. This is what we shall show next.

- Start by picking any state of the GNFA (except the new start and final states)

- Let us call this state q_{rip}. We "rip" q_{rip} out of the machine and create a GNFA with $k-1$ states.

- Of course, we need to "repair" the machine by altering the regular expressions that label each of the remaining arrows.

- The new labels compensate for the loss of q_{rip}.



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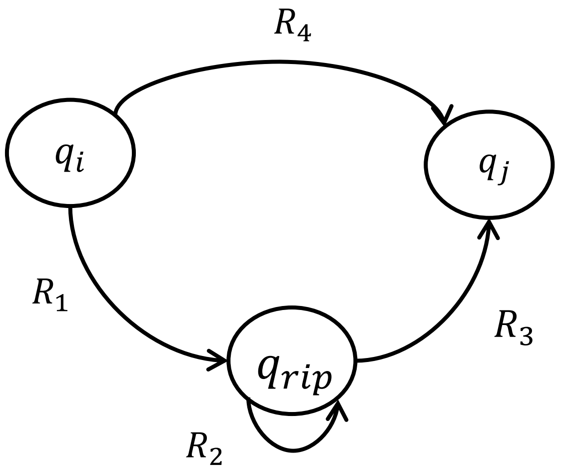
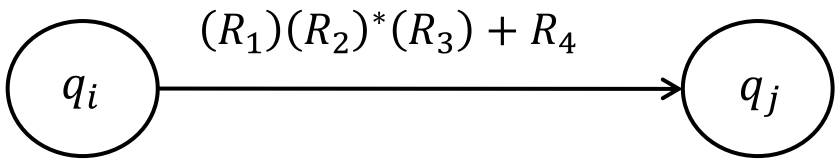




DFA to Regular Expressions: GNFA

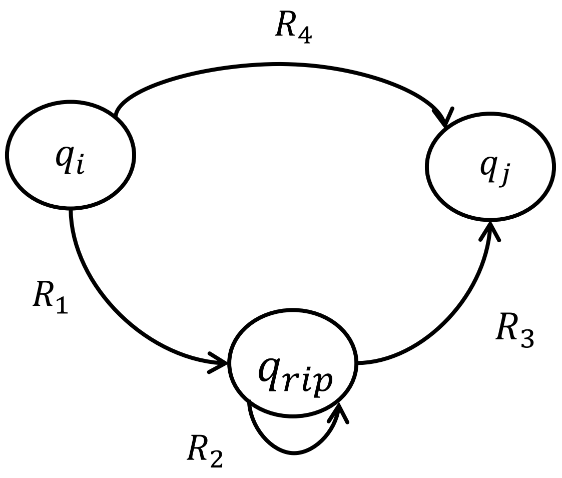
The crucial step is to convert a GNFA with k (>2) states to a GNFA with $k-1$ states.

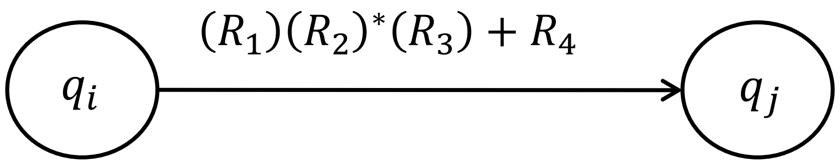
How do we remove q_{rip}? In the old machine if

- q_i goes to q_{rip} with an arrow labelled R_1
- q_{rip} goes to itself with an arrow labelled R_2
- q_{rip} goes to q_j with an arrow labelled R_3
- q_i goes to q_j with an arrow labelled R_4

Repeat this until $\boldsymbol{k = 2}$

then in the new machine, the arrow from q_i to q_j has the label $(R_1)(R_2)^*(R_3) + R_4$

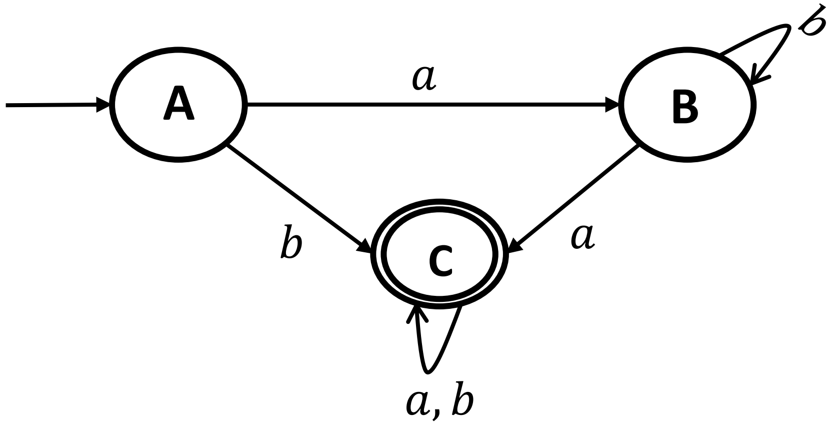




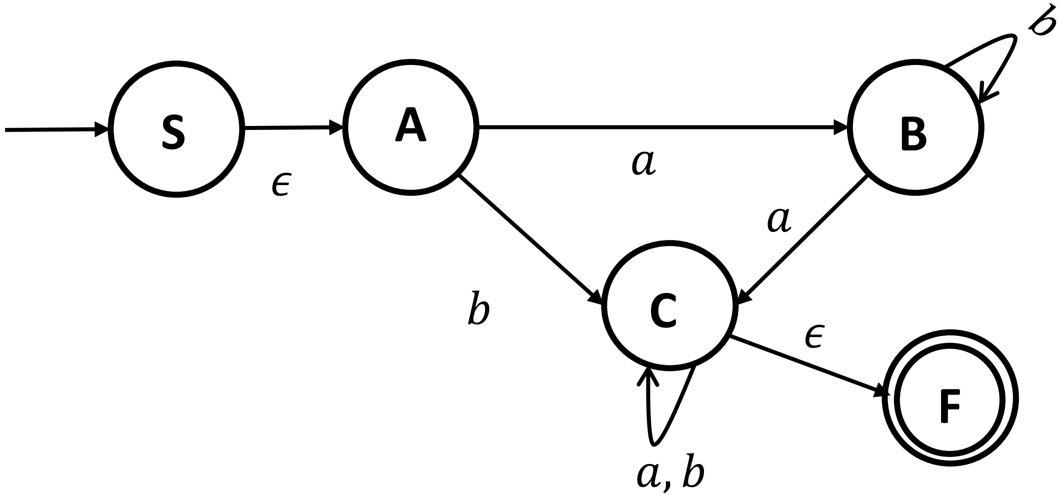
This should be done for **every pair** of arrows outgoing and incoming q_{rip}

DFA to Regular Expressions: GNFA

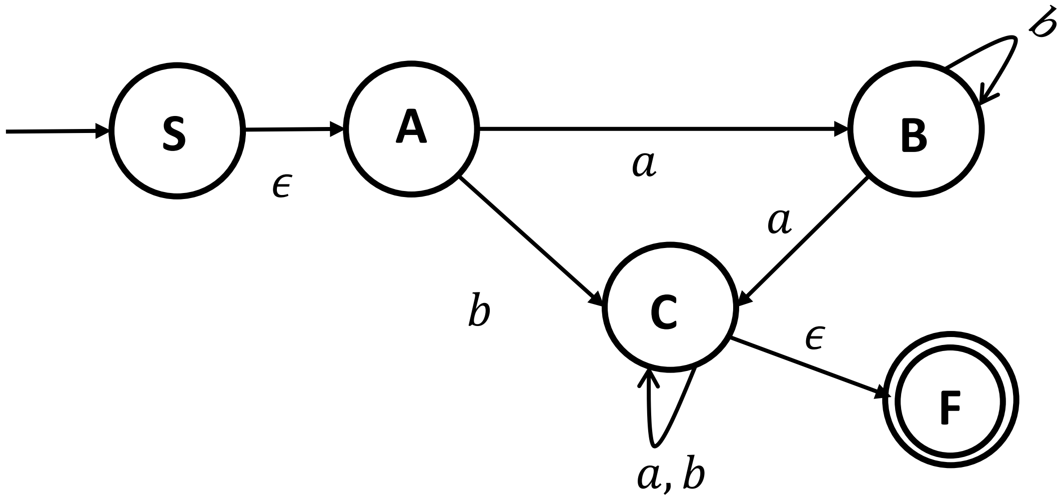
Let us look at an example. Consider the original DFA M below and find the regular expression corresponding to $L(M)$.



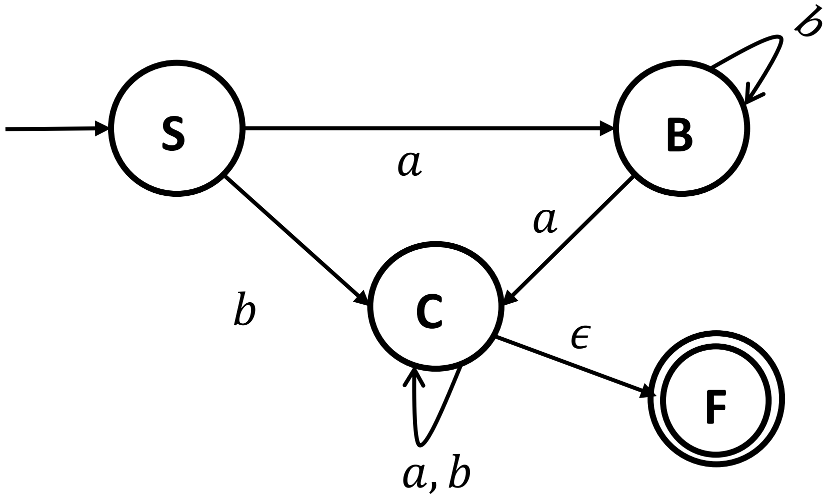
Step 1: Add new start and final states



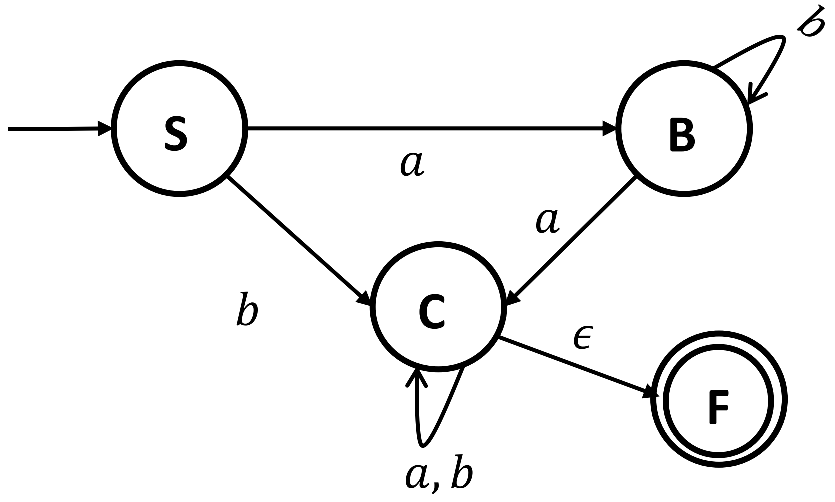
DFA to Regular Expressions: GNFA



Step 2: Eliminate A

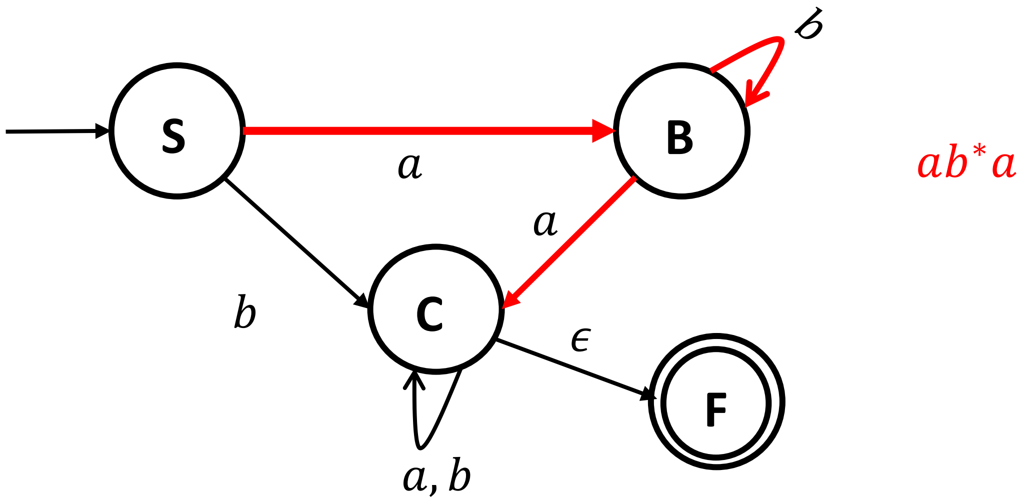


DFA to Regular Expressions: GNFA

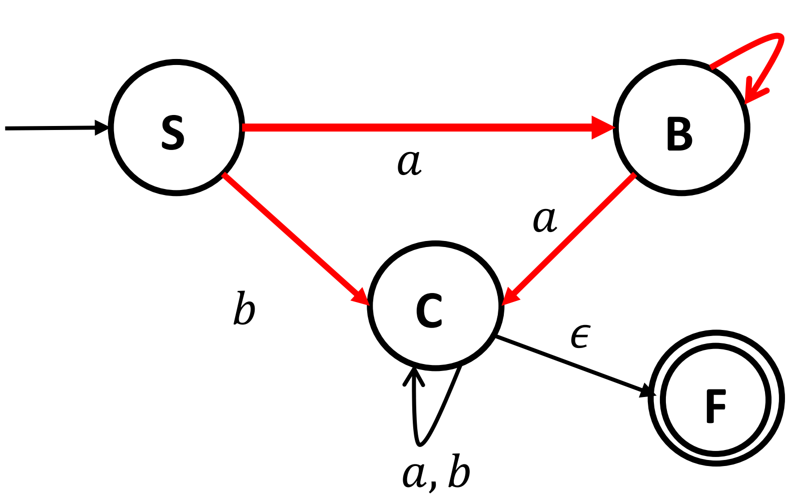


Step 2: Eliminate B

$S \to C$ via B, RE: ab^*a

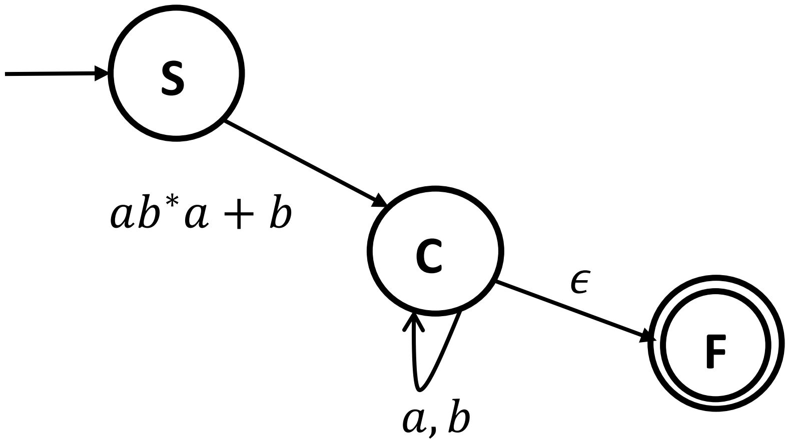


DFA to Regular Expressions: GNFA

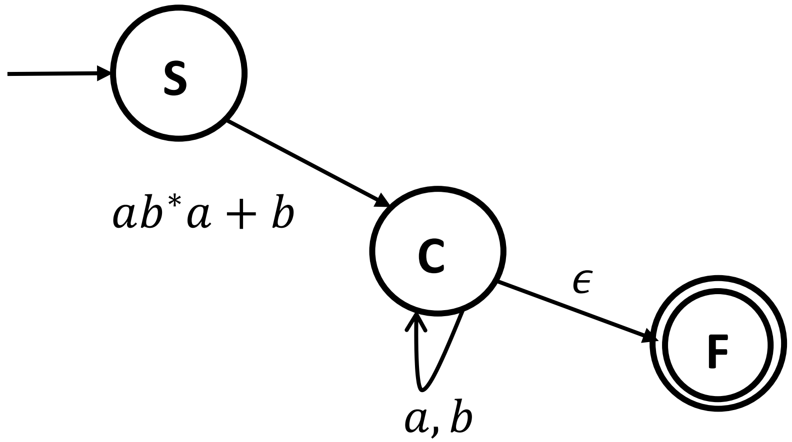


Step 2: Eliminate $\boldsymbol{B}$

$S \to C$ via B, RE: ab^*a
Overall RE for $S \to C$: $\boldsymbol{ab^*a + b}$

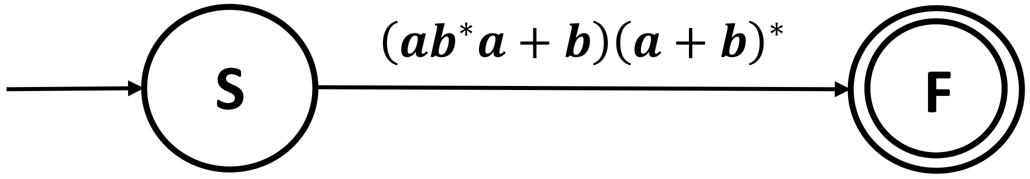


DFA to Regular Expressions: GNFA

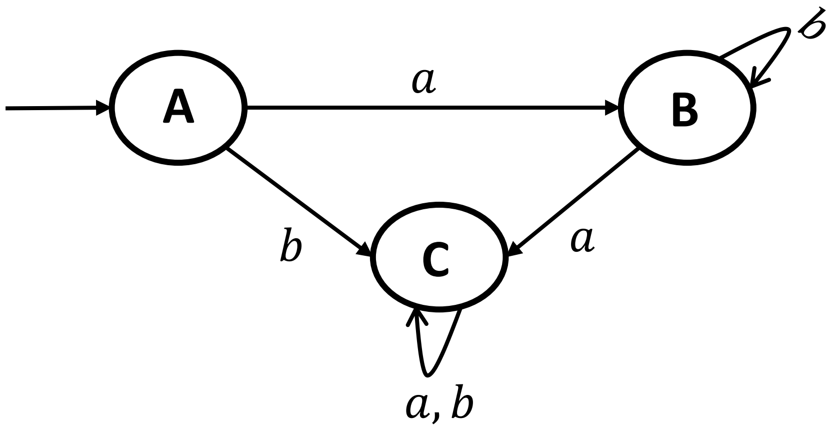


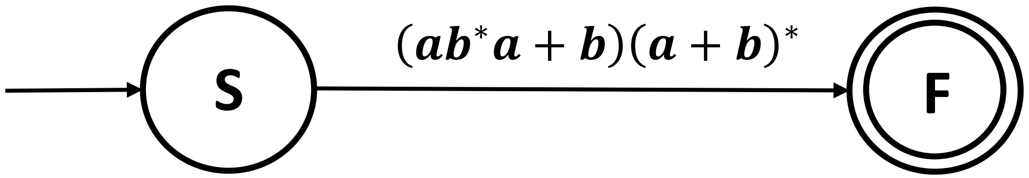
Step 2: Eliminate $\boldsymbol{C}$

$S \rightarrow F$ via C, RE: $(\boldsymbol{ab^*a + b})(\boldsymbol{a + b})^*$



DFA to Regular Expressions: GNFA





Recursively, we managed to convert the DFA M to a 2-state GNFA such that the label from of the arrow from the start state to the final state of the GNFA is the Regular Expression corresponding to $L(M)$.

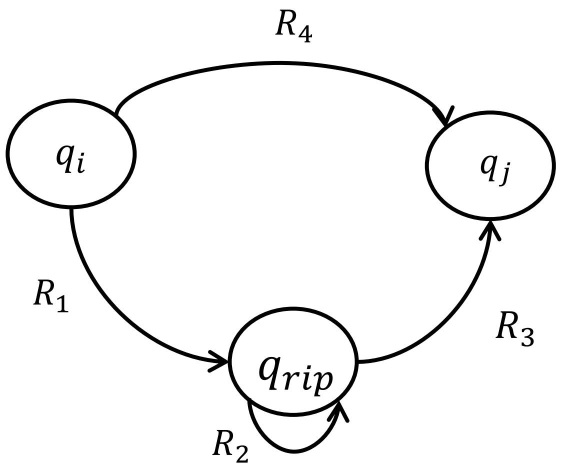
DFA to Regular Expressions: GNFA

Formally, a GNFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states.
- Σ is the input alphabet.
- $\delta: Q - \{q_0\} \times Q - \{F\} \mapsto \mathcal{R}$ is the transition function.
- q_0 is the start state.
- F is the final state.

Convert k-state GNFA to a 2-state GNFA:

We provide a recursive algorithm CONVERT(G) for this.



CONVERT(G):

1. Let k be the number of states of G.
2. If $k = 2$, then return the label R of the arrow between the start and the final state.
3. If $k > 2$, select any state $q_{\text{rip}} \in Q$ different from q_0 and F and let G' be the GNFA$(Q', \Sigma, \delta', q_0, F)$, where

$$Q' = Q - \{q_{rip}\},$$

and for any $q_i \in Q' - \{q_0\}$ and any $q_j \in Q' - \{q_0\}$, let

$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) + R_4,$$

for $R_1 = \delta(q_i, q_{rip})$, $R_2 = \delta(q_{rip}, q_{rip})$, $R_3 = \delta(q_{rip}, q_j)$ and $R_4 = \delta(q_i, q_j)$

4. Compute CONVERT(G') and return its value.

DFA to Regular Expressions: GNFA

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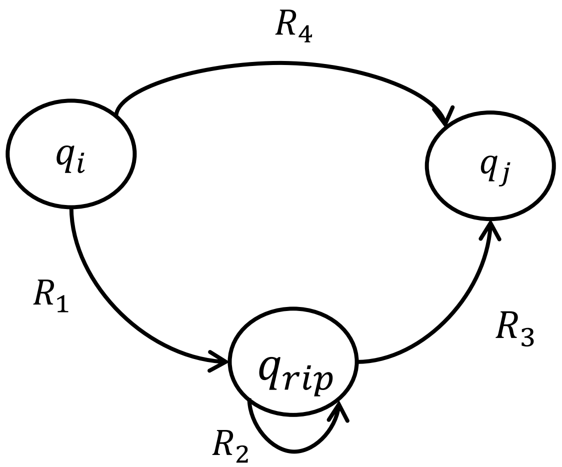


DFA, NFA, Regular Expressions have equal power and all of them correspond to Regular Languages

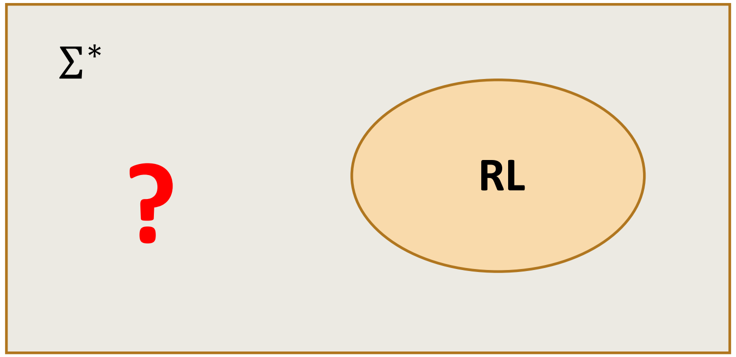
How do Non-regular languages look like?
How can we prove that certain languages are not regular?

Pumping Lemma

Recall that so far, we have proven that the following statements are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathcal{L}(D) = L$.
- There is an NFA N such that $\mathcal{L}(N) = L$.
- There is a regular expression R such that $\mathcal{L}(R) = L$.

- Not all languages are regular.



Thank You!