Lecture 5

(16 August 2024)

Recap.

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \quad P(B) > 0$$

Bayes' Theorem:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)},$$

A_i's form a partition of Ω.

Multiplication Rule

$$P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap A_2)P(A_3|A_1 \cap A_2)$$

$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$$

By induction we have

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P\left(A_i \,\Big|\, \bigcap_{j=1}^{i-1} A_j\right),$$

or

$$P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

$$= P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2) \cdots P(A_n | A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

Conditional Independence

The events A and B are conditionally independent given C with $P(C)>0$ if

$$P(A \cap B | C) = P(A|C)\,P(B|C).$$

$$P(A\cap B\mid C) = \frac{P(A\cap B\cap C)}{P(C)}$$

$$= \frac{\cancel{P(C)}\, P(B\mid C)\, P(A\mid B\cap C)}{\cancel{P(C)}}$$

If $P(B\mid C) > 0$ this implies that the conditional independence is equivalent to

$$P(A\mid B\cap C) = P(A\mid C).$$

Exercise. Show that the conditional independence of A and B given C neither implies nor is implied by the independence of A and B. Also for which events C is it the case that

$$P(A\cap B\mid C) = P(A\mid C)\,P(B\mid C) \iff P(A\cap B) = P(A)\,P(B)$$
$$\forall A, B \;?$$

Solution. Consider two independent fair coin tosses.

$A = \{1^{st}$ toss is a head$\}$

$B = \{2^{nd}$ toss is a head$\}$

$C = \{$the two tosses have different results$\}$

$P(A \cap B) = P(A)P(B)$

$P(A \cap B | C) = 0$, $P(A|C) = P(B|C) = 1/2$.

$\therefore P(A \cap B | C) \neq P(A|C)P(B|C)$.

Consider two coins, a blue and a red one. We choose one of the two at random, each being chosen with probability 1/2, and proceed with two independent tosses. The coins are biased: with the blue coin, the probability of heads in any given toss is 0.99, whereas for red coin it is 0.01.

Let B be the event that the blue coin was selected. Also let H_i be the event that the ith toss resulted in heads. Then

$$P(H_1 \cap H_2 \mid B) = P(H_1 \mid B)\, P(H_2 \mid B)$$
$$= 0.99 \times 0.99.$$

$$P(H_1 \cap H_2) \sim \tfrac{1}{2} \neq P(H_1)\, P(H_2) = \tfrac{1}{4}.$$

Conditional independence is equivalent to independence if $P(C) = 1$ (verify).

Review of counting

Permutations: Given n distinct objects and let $k \leq n$, we wish to count the number of different ways that we can pick k out of these n objects and arrange them in a sequence, i.e., the number of distinct k-object sequences

$$= n \cdot (n-1) \cdot \,--\, (n-k+1)$$

$$= {}^{n}P_{k} = \frac{n!}{(n-k)!}.$$

Combinations: Count the number of k-element subsets of a given n-element set. Notice that forming a combination is different than forming a permutation because in a combination there is no ordering of the selected elements.

$${}^{n}C_{k} = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}.$$

Partitions: Consider n and $n_1, \ldots, n_r$ s.t. $n = n_1 + n_2 + \cdots + n_r$.

No. of Partitions of n distinct elements into r disjoint subsets with the ith subset containing exactly n_i elements

$$= \frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1 \cdots n_{r-1}}{n_r}$$

$$= \frac{n!}{\cancel{(n-n_1)!}\, n_1!} \cdot \frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\, n_2!} \cdots \frac{(n-n_1 \cancel{-} \cdots - n_{r-1})!}{(n-n_1-n_2-\cdots-n_r)!\, n_r!}$$

$$= \frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

Module 2 (Discrete Random Variables)

- The Concept of a Random Variable
- Probability Distribution Function
- Types of RVs: Discrete & Continuous
- Expectation, Variance, Functions of RVs
- Multiple RVs, Conditioning, Independence

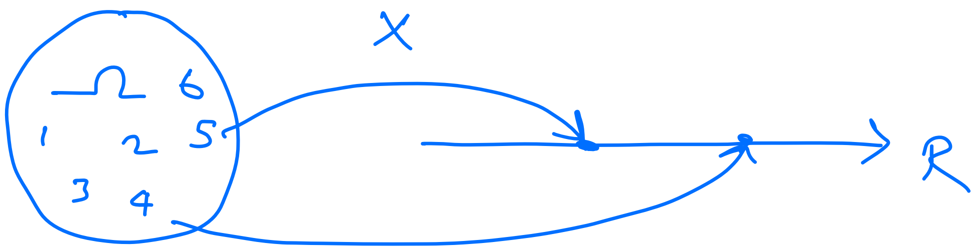
Random Variable

We may not be always interested in the actual outcome of a random experiment, but rather in some consequence of the random outcome.

A random variable is a function

from sample space to real numbers.

$$X : \Omega \longrightarrow R$$



$$X(1) = X(3) = X(5) = 1$$

$$X(2) = X(4) = X(6) = 0$$

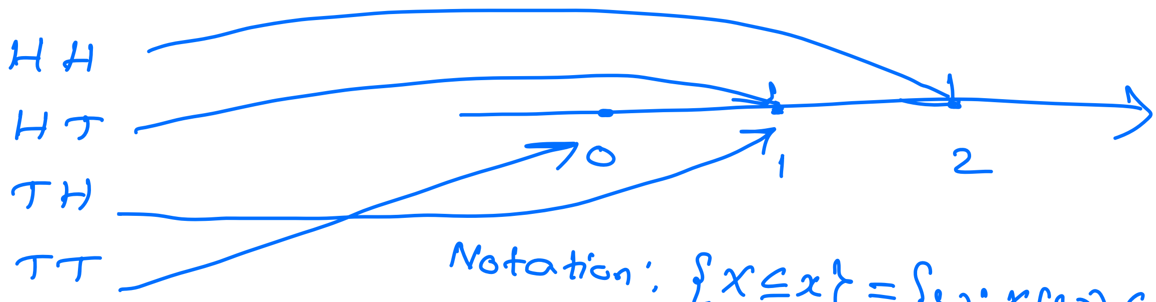
$$\Omega = \{ HH, TH, HT, TT \}$$

$X(\omega)$ = no. of heads in ω

$$X(HH) = 2 \quad X(TH) = 1 \quad X(HT) = 1$$

$$X(TT) = 0$$

we would like to speak about events of the form $X \leq x$, $x \in R$.



Notation: $\{X \le x\} = \{\omega : X(\omega) \le x\}$

$-\infty < c < 0$, $\{X \le c\} = \phi$

$0 \le c < 1$, $\{X \le c\} = \{TT\}$

$1 \le c < 2$, $\{X \le c\} = \{TT, TH, HT\}$

$c \ge 2$, $\{X \le c\} = \Omega$

<u>Definition</u>. A random variable is a function $X : \Omega \to R$ with the property that

$$\{\omega : X(\omega) \le x\} \in \mathcal{F} \text{ for each } x \in R,$$

for a given probability space $(\Omega, \mathcal{F}, P)$.

$$\{\omega : X(\omega) \le x\} = X^{-1}((-\infty, x]).$$

$\Omega = \{HH, TT, HT, TH\}$

$\mathcal{F} = \{\phi, \Omega, \{HT, TH\}, \{HH, TT\}\}$

The function $X : \Omega \to \mathbb{R}$ defined as $X(\omega) =$ no. of heads is not a random variable because

$$X^{-1}((-\infty, 1]) = \{TT, HT, TH\} \notin \mathcal{F}.$$

However it is a RV with respect to the power set event space.

<u>Theorem</u>: Given a probability space $(\Omega, \mathcal{F}, P)$ let $X : \Omega \to \mathbb{R}$ be a random variable. Then the following holds.

(i) $X^{-1}((-\infty, x)) \in \mathcal{F}$

(ii) $X^{-1}([x_1, x_2]) \in \mathcal{F}$

(iii) $X^{-1}(\{x\}) \in \mathcal{F}$

(iv) $X^{-1}((x_1, x_2)) \in \mathcal{F}$

<u>Proof</u>: Let $A_i = X^{-1}((-\infty, x_i])$

where $x_i = x - \frac{1}{i}$.

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} X^{-1}((-\infty, x_i])$$

$$= X^{-1}\left(\bigcup_{i=1}^{\infty} (-\infty, x_i]\right)$$

$$= X^{-1}\left(\bigcup_{i=1}^{\infty} \left(-\infty, x - \tfrac{1}{i}\right]\right)$$

$$= X^{-1}((-\infty, x)) \in \mathcal{F}$$

(ii) $X^{-1}([x_1,\infty)) \in \mathcal{F}$

$X^{-1}((-\infty, x_2]) \in \mathcal{F}$

$X^{-1}([x_1,\infty) \cap (-\infty, x_2]) = X^{-1}([x_1, x_2])$

$= X^{-1}([x_1,\infty)) \cap X^{-1}((-\infty, x_2])$

$\in \mathcal{F}$.

(iii) $X^{-1}(\{x\}) = X^{-1}((-\infty, x] \cap (-\infty, x))$

$= X^{-1}((-\infty, x]) \cap X^{-1}((-\infty, x))$

$\in \mathcal{F}$.

(iv) $X^{-1}((x_1, x_2)) =$

$X^{-1}\left(\bigcup_{i=1}^{\infty} \left[x_1 + \frac{1}{i}, x_2 - \frac{1}{i}\right]\right)$

$= \bigcup_{i=1}^{\infty} X^{-1}\left(\left[x_1 + \frac{1}{i}, x_2 - \frac{1}{i}\right]\right) \in \mathcal{F}$.