End-Sem solutions

1. sol:- (a) A and $B \cup C$ are not independent.

Consider the following counterexample.

$\Omega = \{HH, HT, TH, TT\}$, $P(\omega) = 1/4$, $\omega \in \Omega$.

$A = \{HH, HT\}$, $B = \{HH, TH\}$, $C = \{TH, HT\}$.

The events A and B are independent and the events A and C are independent.

$$P(A) = 1/2, \quad P(B \cup C) = 3/4, \quad P(A \cap (B \cup C)) = 1/2$$

$$P(A \cap (B \cup C)) \neq P(A)\, P(B \cup C)$$

$$1/2 \neq \frac{1}{2} \times 3/4$$

(b) $P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$

$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$

$= P(A)\left[P(B) + P(C) - P(B)P(C)\right]$

$= P(A)\left[P(B) + P(C) - P(B \cap C)\right]$

$= P(A) \cdot P(B \cup C)$. $\Rightarrow$ A and $B \cup C$ are independent.

2. sol:- $P_X(K) = (1-p)^{K-1} p \qquad K = 1, 2, \ldots$

$$E[X] = \sum_{K=1}^{\infty} K \, P_X(K)$$

$$= \sum_{K=1}^{\infty} K (1-p)^{K-1} p$$

$$= p \cdot \frac{1}{(1-1+p)^2} \qquad \left(\because \frac{d}{d\gamma}[\gamma + \gamma^2 + \ldots] = \frac{d}{d\gamma}\left(\gamma/1-\gamma\right) = \frac{1}{(1-\gamma)^2} \right)$$

$$= 1/p.$$

$$E[X] = 1/p.$$

$$E[X^2] = \sum_{K=1}^{\infty} K^2 (1-p)^{K-1} p$$

$$= p \, \frac{2-p}{p^3}$$

$$\left[\text{because } \sum_{K=1}^{\infty} K^2 \gamma^{K-1} = \frac{d}{d\gamma}\left(\gamma \sum_{K=1}^{\infty} K \gamma^{K-1}\right) \right.$$

$$\left. = \frac{d}{d\gamma}\left(\frac{\gamma}{(1-\gamma)^2}\right) = \frac{1+\gamma}{(1-\gamma)^3} \right]$$

$$= \frac{2-p}{p^2}$$

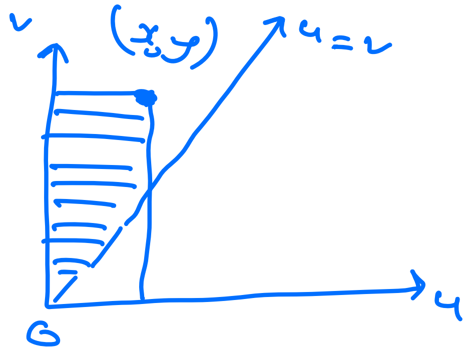
$$\operatorname{var}(x) = E[X^2] - E[X]^2 = \frac{1-p}{p^2}.$$

3. sol:-

Let $x \le y$.

$$F_{X,Y|X<Y} = \frac{P(X \le x, Y \le y, X<Y)}{P(X<Y)}$$

$P(X<Y) = \frac{1}{2}$ as X and Y are independent and identically distributed.



$P(X \le x, Y \le y, X<Y)$

$$= \int_{u=0}^{x} \int_{v=u}^{y} e^{-u} e^{-v} \, dv \, du$$

$$= \int_{u=0}^{x} e^{-u} \left[-e^{-v}\right]_{u}^{y} du$$

$$= \int_{u=0}^{x} e^{-u} \left[e^{-u} - e^{-y}\right] du$$

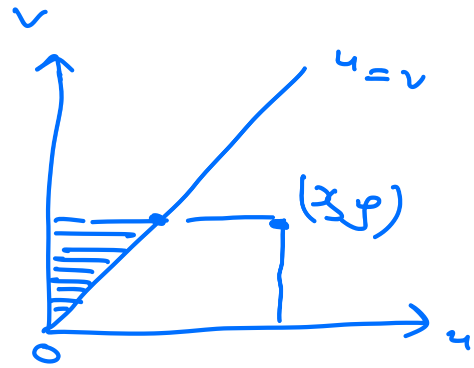
$$= \int_{u=0}^{x} \left(e^{-2u} - e^{-u-y}\right) du$$

$$= \left[\frac{-e^{-2u}}{2}\right]_{0}^{x} + e^{-y}\left[e^{-u}\right]_{0}^{x}$$

$$= \frac{1}{2}\left(1-e^{-2x}\right) + e^{-y}\left(e^{-x}-1\right).$$

$$\therefore\; F_{X,Y|X<Y}(x,y) = 1-e^{-2x}+2e^{-y}\left(e^{-x}-1\right)$$

for $x \le y$.



Let $x > y$.

$P(X \le x, Y \le y, X < Y)$

$$= \int_{u=0}^{y} \int_{v=u}^{y} e^{-u} e^{-v}\, dv\, du$$

$$= \int_{u=0}^{y} e^{-u} \cdot \left[e^{-u} - e^{-y}\right] du$$

$$= \frac{1}{2}\left(1-e^{-2y}\right) + e^{-y}\left(e^{-y}-1\right).$$

$$\therefore\; F_{X,Y|X<Y}(x,y) = 1-e^{-2y}+2e^{-y}\left(e^{-y}-1\right)$$

for $x > y$.

4. sol'n:- (a) Let $\mu = E[X]$ and $\sigma^2 = var(X)$.

For $t, s > 0$,

$$P(X-\mu \geq t) = P(X-\mu+s \geq t+s)$$

$$= P\left((X-\mu+s)^2 \geq (t+s)^2\right)$$

$$\leq \frac{E\left[(X-\mu+s)^2\right]}{(t+s)^2}$$

(By Markov's inequality)

$$= \frac{E\left[(X-\mu)^2\right] + s^2 + 2s\,E[X-\mu]}{(t+s)^2}$$

$$= \frac{\sigma^2 + s^2}{(t+s)^2}.$$

so $P(X-\mu \geq t) \leq \inf_{s>0} f(s)$,

where $f(s) = \frac{\sigma^2+s^2}{(s+t)^2}$.

$$f'(s) = (s+t)^2(2s) - (\sigma^2+s^2)\,2(s+t) = 0$$

$$\Rightarrow \cancel{s^2} + st = \sigma^2 + \cancel{s^2} \Rightarrow s = \frac{\sigma^2}{t} \text{ is optimal.}$$

check $f''(s) > 0$.

$\therefore P(X-\mu \geq t) \leq \dfrac{\sigma^2}{\sigma^2+t^2}$.

(b) $X = \begin{cases} a & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$

$\Longrightarrow P(X \geq a) = p = \dfrac{a \cdot p}{a} = \dfrac{E[X]}{a}$.

(c) $E[Y_n] = (E[X_1])^n = 0,$

$$\mathrm{Var}(Y_n) = E[Y_n^2] - E[Y_n]^2$$

$$= E\left[\prod_{i=1}^{n} X_n^2\right]$$

$$= \prod_{i=1}^{n} \mathrm{Var}(X_n)$$

$$= \left(\frac{(2)^2}{12}\right)^n = \left(\frac{1}{3}\right)^n.$$

$$P(|Y_n - 0| > \varepsilon \leq \frac{\cdot}{\varepsilon^2}$$

$$= \left(\frac{1}{3}\right)^n \cdot \frac{1}{\varepsilon^2}$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

$\therefore$ Y_n converges to 0 in probability.

5.sol:- $\mu_{X_t} = E[X_t]$

$$= E[A\cos(\omega_c t + \Theta)]$$

$$= E[A]\; E[\cos(\omega_c t + \Theta)]$$

$$E[\cos(\omega_c t + \Theta)] = \int_0^{2\pi} \cos(\omega_c t + \theta) \cdot \frac{1}{2\pi}\, d\theta$$

$$= 0.$$

$$R_X(t_1 t_2) = E[X_{t_1} X_{t_2}]$$

$$= E[A\cos(\omega_c t_1 + \Theta)\, A\cos(\omega_c t_2 + \Theta)]$$

$$= E[A^2]\; E[\cos(\omega_c(t_1 - t_2)) + \cos(\omega_c(t_1 + t_2 + 2\Theta)]$$

$$= E[A^2]\; E[\cos(\omega_c(t_1 - t_2))]$$

$R_X(t_1 t_2)$ is only a function of $t_1 - t_2$.

so X_t is wss.

$$X_t = A\cos(\omega_c t + \Theta)$$

$$X_{t+T} = A\cos(\omega_c t + \tilde{\Theta}),$$

where $\tilde{\Theta} = (\Theta + \omega_c t) \bmod 2\pi$.

$\tilde{\Theta}$ has the same uniform distribution as Θ, i.e., uniform over $[0, 2\pi]$.

Now since A & Θ are independent, the above observation implies that (A, Θ) and $(A, \tilde{\Theta})$ have the same joint distribution.

$\Rightarrow$ F_{X_t}, $F_{X_{t+\tau}}$ have same distribution.

$F_{X_{t_1}, X_{t_2}}(x_1, x_2) = g(f_{A,\Theta})$, for some function g.

$F_{X_{t_1+\tau}, X_{t_2+\tau}}(x_1, x_2) = g(f_{A, \tilde{\Theta}})$.

Now since $f_{A\Theta} = f_{A, \tilde{\Theta}}$, we have $F_{X_{t_1}, X_{t_2}}$ and $F_{X_{t_1+\tau}, X_{t_2+\tau}}$ have the same distribution.

Similarly, any nth order distribution is same.

$\therefore X_t$ is SSS.

6. sol:-

(a) True

(b) False

(c) False

(d) False

(e) False

(f) False

(g) True

(h) True

(i) False

(j) False