All Pairs Shortest Paths.

Assumptions: No negative cycle.

$\underline{(i,j)} \rightarrow$ For each vertex apply Djikstra's

$O(n)$

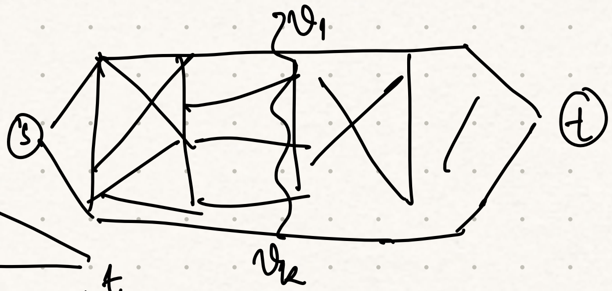
$O(|V| \cdot |E| \log |V|)$.

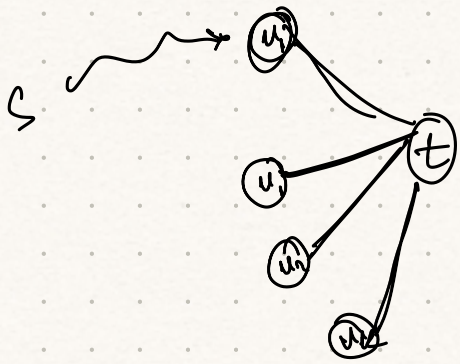
$\hookrightarrow O(n^3 \log n)$ in the worst case.

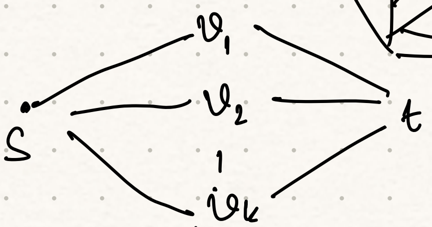
$$|E| = \frac{n^2}{1000}$$

<u>Qn</u>: Can we do better?

For the sake of simplicity, let us consider DAGs.







$$\text{Shortest } s\text{-}t \text{ path} = \min_{j \in [1,k]} \{ \text{shortest } (s-u_j) \text{ path} + w_{u_j - t} \}$$

$$= \min_{j \in [1,k]} \{ \text{Shortest } (s-v_j) \text{ path} + \text{shortest } (v_j - t) \text{ path} \}$$

<u>Claim</u>: A shortest path between any pair of vertices has length at most $|V|$.

$$d(i,j) = \min_{\underline{u \in V}} \{ d(i,u) + d(u,j) \}$$

Attempt 1.

Fine

$\hat{d}(i,j,V)$: Shortest distance between i & j using all V vertices.

$V = \{1, \dots, |V|\}$.

if G is DAG
but if G is not a DAG.
this could lead to cyclic dependence.

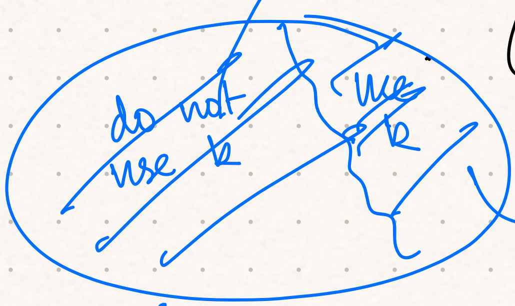
$\hat{d}(i,j,k)$: shortest distance between i & j using vertices $\{1, \dots, k\}$.

Shortest paths from i & j using $\{1, \dots, k-1\}$.

Shortest path from k to j using $\{1, k-1\}$.

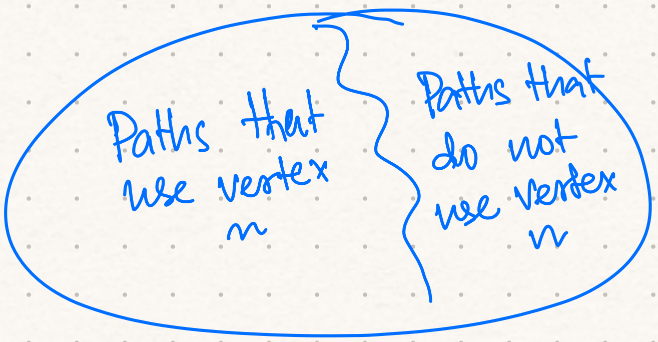
$$\hat{d}(i,j,k) = \min \begin{cases} \hat{d}(i,j,k-1) \\ \hat{d}(i,k,k-1) + \hat{d}(k,j,k-1) \end{cases}$$

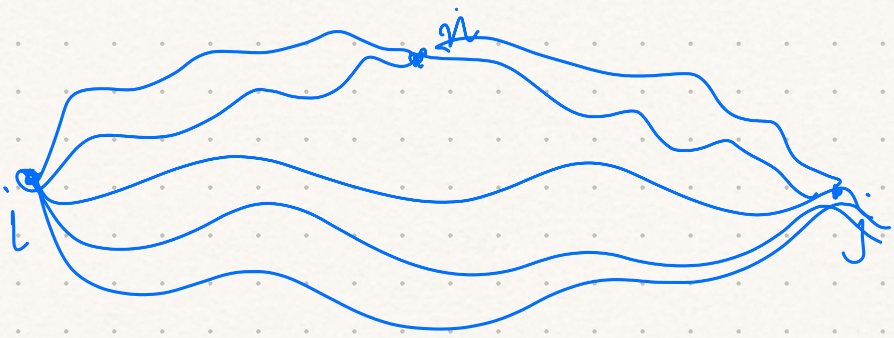
Shortest path from i to k using $\{1, \dots, k-1\}$.



Paths from i to j over $\{1, \dots, k\}$.

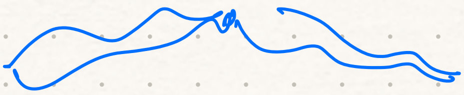
$i \longrightarrow j$





$\{1, \dots, n-1\}$

$$\hat{d}(i,j,k) = \min \begin{cases} \hat{d}(i,j,k-1) \\ \text{Shortest distance among all paths that go through } k \text{ and use } \{1,\dots,k\}. \end{cases}$$



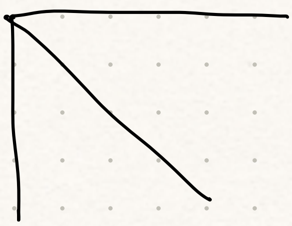
$\hat{d}(i,k,k-1) + \hat{d}(k,j,k-1)$ = Shortest distance among all paths from i to k over $\{1,\dots,k-1\}$. + Shortest distance among all paths from k to j over $\{1,\dots,k-1\}$.

# of all possible sub problems is $O(|V|^3)$.

$\hat{d}(i,j,k)$

also init $d(i,j,k) = w_{ij}$ if $(i,j) \in E$

$k \leftarrow$



Init: $\hat{d}(i,j,k) = \infty \ \forall i,j,k$.

For k in $[1,|V|]$:

  for i in V:

    for $j \neq i$ in V:

      $\tilde{d}(i,j,k) = \min\{\hat{d}(i,j,k-1), \hat{d}(i,k,k-1) + \hat{d}(k,j,k-1)\}$.

      Update $\hat{d}(i,j,k)$ if $\tilde{d}(i,j,k)$ is smaller.

When can $\hat{d}(i,j,1)$ make sense?

- If 1 is neighbour of both i & j.
  or i or $j = 1$ and 1 is a neighbour of the other.

$$\hat{d}(i,j,2) = \min \begin{cases} \overset{w_{i,j}}{\hat{d}(i,j,1)} \\ \hat{d}(i,2,1) + \hat{d}(2,j,1) \end{cases} \quad \forall i,j$$

$\hat{d}(i,j,3)$

Each entry of this 3 dimensional array can be filled with $O(1)$ many lookups of already filled values.

Revisiting Attempt 1

$d(i,j,\ell) :=$ Shortest path between i and j with at most ℓ edges.

$$d(i,j,\ell) = \min \begin{cases} \min\limits_{u:(u,j)\in E} \{ d(i,u,\ell-1) + w_{u,j} \} \\ d(i,j,\ell-1) \end{cases}$$

$|V| = n$

3-dim array, we would have $O(|V|^3)$ entries
n^3

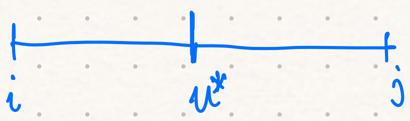
For each entry, (i,j,ℓ), we do $(d_j + 1)$ lookups.
$\leq n$

Worst case: $O(|V|^4)$

shortest amongst all $i \to u$ paths w at most $\frac{\ell}{2}$ edges.

$$d(i,j,\ell) = \min_{u\in V} \left\{ d(i,u,\tfrac{\ell}{2}) + d(u,j,\tfrac{\ell}{2}) \right\}$$

shortest amongst all $u \to j$ paths w/ at most $\frac{\ell}{2}$ edges.



Obs: If π is a shortest path from i to j and

u is on it then $\pi|_{i \to u}$ is also the shortest path from i to u.

Suppose not. $\exists$ path σ from i to u s.t $d(\sigma_{i \to u}) < d(\pi|_{i \to u})$

$\Rightarrow$ This contradicts that π is a shortest path from i to j as we can construct a path $\sigma_{i \to u} \circ \pi|_{u \to j}$ which has shorter dist.

$$d(\underbrace{i, j}_{n \text{ options each}}, \overbrace{\tfrac{n}{2^k}}^{\log n})$$

3-dim array has $n^2 \log n$ entries.

And for each entry we are looking at $2n$ entries.

$\hookrightarrow O(n^3 \log n)$

INIT $\leftarrow$ <FILL IT>

for k in $[1, \log n]$:
    for i in V:
        for j in V:
            Min value $= \infty$
            For u in V:
                val $= d(i, u, 2^{k-1}) + d(u, j, 2^{k-1})$
                If val < Min value:
                    Min value $\leftarrow$ val.
            $d(i, j, 2^k) \leftarrow$ Min value

n

Shortest (i, j, ℓ):
    For all u
        Shortest $(i, u, \frac{\ell}{2})$
        Shortest $(u, j, \frac{\ell}{2})$.