Lecture 23

(7 November 2024)

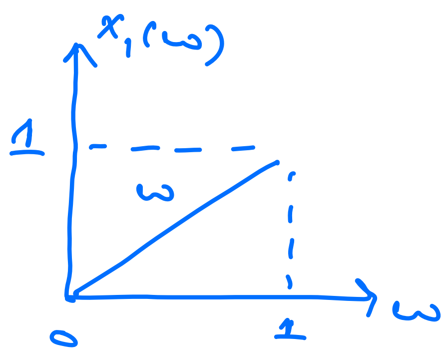
Almost sure Convergence

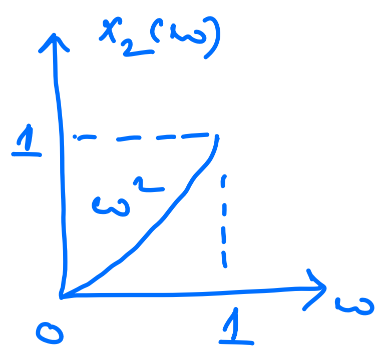
A sequence of random variables $X_1, X_2, \ldots$ is said to converge almost surely if

$$P\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Example. Let $\Omega = [0,1]$. Consider a probability law defined by $P([a,b]) = b-a$, for all $0 \le a \le b < 1$.

Define $X_n(\omega) = \omega^n$, for $n \in \mathbb{N}$.





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Note that

$$\lim_{n\to\infty} X_n(\omega) = \begin{cases} 0, & \text{if } 0 \le \omega < 1 \\ 1, & \text{if } \omega = 1. \end{cases}$$

so, $\{\omega : \lim_{n\to\infty} X_n(\omega) = 0\} = [0, 1)$.

$$\Longrightarrow P\left(\{\omega : \lim_{n\to\infty} X_n(\omega) = 0\}\right) = P([0,1)) = 1$$

since the singleton set $\{1\}$ has zero probability.

$\therefore$ X_n converges to 0 almost surely.

Strong Law of Large Numbers (SLLN)

Similar to WLLN, SLLN also deals with convergence of the sample mean to the true mean. However, SLLN ensures convergence in almost sure sense, which is stronger than convergence in probability.

SLLN. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. RVs with mean μ. Then

$\dfrac{\sum_{i=1}^{n} X_i}{n}$ converges almost surely to μ, i.e.,

$$P\left(\left\{\omega : \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i(\omega)}{n} = \mu\right\}\right) = 1.$$

Proof. Assume that $E[X_i^4] = k < \infty$.

$S_n = \sum_{i=1}^{n} X_i$. $E[S_n^4] = E\left[\left(\sum_{i=1}^{n} X_i\right)^4\right]$.

This will have terms of the form

$X_i^4,\ X_i^3 X_j,\ X_i^2 X_j^2,\ X_i^2 X_j X_k,\ X_i X_j X_k X_l$

where i, j, k, l are different.

Assume $\mu = 0$. Then because of independence it follows that

$$E[X_i^3 X_j] = E[X_i^3]E[X_j] = 0$$

$$E[X_i^2 X_j X_k] = E[X_i^2]E[X_j]E[X_k] = 0$$

$$E[X_i X_j X_k X_l] = E[X_i]E[X_j]E[X_k]E[X_l] = 0.$$

so $E[S_n^4] = nE[X_i^4] + 6\binom{n}{2}E[X_i^2 X_j^2]$

$$= nE[X_i^4] + 3n(n-1)\underbrace{E[X_i^2]E[X_j^2]}_{=(E[X_i^2])^2 \le E[X_i^4]}$$

as $\mathrm{Var}(X_i^2) \ge 0$

$$\le nk + 3n(n-1)k \le 3n^2 k$$

$$\Rightarrow E[S_n^4/n^4] \le 3k/n^2$$

$$\Rightarrow E\left[\sum_{n=1}^{\infty} S_n^4/n^4\right] = \sum_{n=1}^{\infty} E\left[S_n^4/n^4\right]$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot (3K) < \infty.$$

This implies that
$$P\left(\sum_{n=1}^{\infty} S_n^4/n^4 < \infty\right) = 1$$

$$\sum_{n=1}^{\infty} \frac{S_n^4}{n^4} < \infty \Rightarrow \lim_{n\to\infty} \frac{S_n^4}{n^4} = 0$$

$$\Rightarrow 1 = P\left(\sum_{n=1}^{\infty} \frac{S_n^4}{n^4} < \infty\right) \le P\left(\lim_{n\to\infty} \frac{S_n^4}{n^4} = 0\right)$$

$$\Rightarrow P\left(\lim_{n\to\infty} \frac{S_n}{n} = 0\right) = 1.$$

In the above proof we have used $E\left[\sum_{i=1}^{\infty} Z_i\right] = \sum_{i=1}^{\infty} E[Z_i]$. This is not necessarily true always for any Z_i's.

However this holds true when all Z_i are non-negative random variables. This is because of monotone convergence theorem (not covered in this course).

Hirearchy of Convergence Notions

Recall the four notions of convergence of RVs we have seen,

(1) Almost sure convergence

$$X_n \xrightarrow{a.s.} X \quad \text{if} \quad P\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

(2) Convergence in mean-square sense

$$X_n \xrightarrow{m.s.} X \quad \text{if} \quad \lim_{n\to\infty} E\left[(X_n - X)^2\right] = 0.$$

(3) Convergence in Probability

$$X_n \xrightarrow{P} X \quad \text{if} \quad \lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0 \quad \forall \varepsilon > 0.$$

(4) Convergence in distribution

$$X_n \xrightarrow{D} X \quad \text{if} \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \text{for all points } x$$

at which $F_X(x)$ is continuous.

Theorem. The following implications hold.

$$\left(X_n \xrightarrow{a.s.} X\right) \Rightarrow \left(X_n \xrightarrow{P} X\right) \Rightarrow \left(X_n \xrightarrow{D} X\right).$$

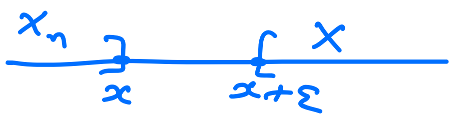
$$\left(X_n \xrightarrow{m.s.} X\right) \Rightarrow \left(X_n \xrightarrow{P} X\right)$$

No other implications hold in general.

Proof. $X_n \xrightarrow{m.s.} X \Rightarrow X_n \xrightarrow{P} X$ is proved in the last class.

We prove that $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$.

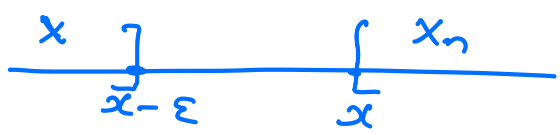
Suppose $X_n \xrightarrow{P} X$, i.e., $P(|X_n - X| > \varepsilon) \to 0$ as $n \to \infty$, for every $\varepsilon > 0$.



$$F_{X_n}(x) = P(X_n \le x) = P(X_n \le x,\ X \le x+\varepsilon) + P(X_n \le x,\ X > x+\varepsilon)$$

$$\le F_X(x+\varepsilon) + P(|X_n - X| > \varepsilon).$$

Similarly



$$F_X(x-\varepsilon) = P(X \le x-\varepsilon)$$

$$= P(X \le x-\varepsilon,\ X_n \le x) + P(X \le x-\varepsilon,\ X_n > x)$$

$$\le F_{X_n}(x) + P(|X_n - X| > \varepsilon).$$

Thus

$$F_X(x-\varepsilon) - P(|X_n - X| > \varepsilon) \leq F_{X_n}(x) \leq F_X(x+\varepsilon) + P(|X_n - X| > \varepsilon).$$

As $n \to \infty$ we get

$$F_X(x-\varepsilon) \leq \lim_{n\to\infty} F_{X_n}(x) \leq F_X(x+\varepsilon) \quad \text{for every } \varepsilon > 0$$

since $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$

As $\varepsilon \to 0$ $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ since for the points x at which $F_X(x)$ is continuous we have

$$\lim_{\varepsilon\to 0} F_X(x-\varepsilon) = \lim_{\varepsilon\to 0} F_X(x+\varepsilon) = F_X(x).$$

Now we prove that $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X.$

Suppose $X_n \xrightarrow{a.s.} X$ i.e.,

$$P\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Define a sequence of events A_n by

$$A_n = \{\omega : |X_n(\omega) - X(\omega)| < \varepsilon\}, \text{ for some } \varepsilon > 0$$

To prove that $X_n \xrightarrow{P} X$, it suffices to show that $\lim_{n\to\infty} P(A_n) = 1$.

We define B_n by

$$B_n = \{ \omega : |X_k(\omega) - X(\omega)| < \varepsilon \text{ for all } k \geq n \}.$$

Note that $B_n \subseteq A_n$ and $B_1 \subseteq B_2 \subseteq \cdots$.

So, $\lim_{n\to\infty} P(B_n) = P\left(\bigcup_{n=1}^{\infty} B_n\right)$.

Also, we have $\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} \subseteq \bigcup_{n=1}^{\infty} B_n$.

To see this consider ω s.t. $X_n(\omega) = X(\omega)$.

$\exists\, n_0$ s.t. $|X_n(\omega) - X(\omega)| < \varepsilon$, $\forall n \geq n_0$

$$\Rightarrow \omega \in B_{n_0}$$

$$\Rightarrow \omega \in \bigcup_{n=1}^{\infty} B_n.$$

Since $P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$, we have $P\left(\bigcup_{n=1}^{\infty} B_n\right) = 1$. Thus $\lim_{n\to\infty} P(B_n) = 1$.

$P(B_n) \leq P(A_n) \leq 1 \Rightarrow \lim_{n\to\infty} P(A_n) = 1$ by squeeze theorem.

Module 5 (Random Processes)

Random Process

A random process is a collection of random variables usually indexed by time.

Discrete-time random process:

$$(X_t ; t \in N)$$

Continuous-time random process:

$$(X_t : t \in R)$$

For each t, X_t is a random variable.

$$\omega \longmapsto X_1(\omega), X_2(\omega), ---$$

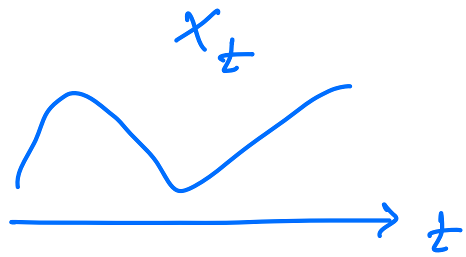
Discrete-time

$$\omega \longmapsto X_t(\omega), t \in R$$

Continuous-time

For a fixed $\omega \in \Omega$, $(X_t(\omega)\ t \in T)$ is called the sample path at ω.

E.g. Bernoulli process



E.g. Stock value

Mean Function of a Random Process

For a random process $(X_t, t \in T)$, the mean function is defined as

$$\mu_X(t) = E[X_t].$$

Example. $X_t = A + Bt$, $A, B \sim N(1, 1)$ and A, B are independent.

$$\mu_X(t) = E[X_t] = E[A] + tE[B] = t + 1,$$

$$t \in [0, \infty).$$

The mean function $\mu_X(t)$ gives us the expected value of X_t at time t but it does not give us any information about how X_{t_1} and X_{t_2} are related. To get some insight on the relation between X_{t_1} and X_{t_2}, we define correlation and covariance functions.

Correlation function

$$R_X(t_1, t_2) = E[X_{t_1} X_{t_2}]$$

Covariance function

$$C_X(t_1, t_2) = Cov(X_{t_1}, X_{t_2})$$

$$= E[X_{t_1} X_{t_2}] - E[X_{t_1}] E[X_{t_2}]$$

$$= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2).$$

<u>Exercise</u>. $X_t = A + Bt$, A, B are independent and $N(1, 1)$. Show that

(i) $R_X(t_1, t_2) = 2 + t_1 + t_2 + 2t_1 t_2$

(ii) $C_X(t_1, t_2) = 1 + t_1 t_2$.