CS 302.1 - Automata Theory

Lecture 13

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Quick Recap

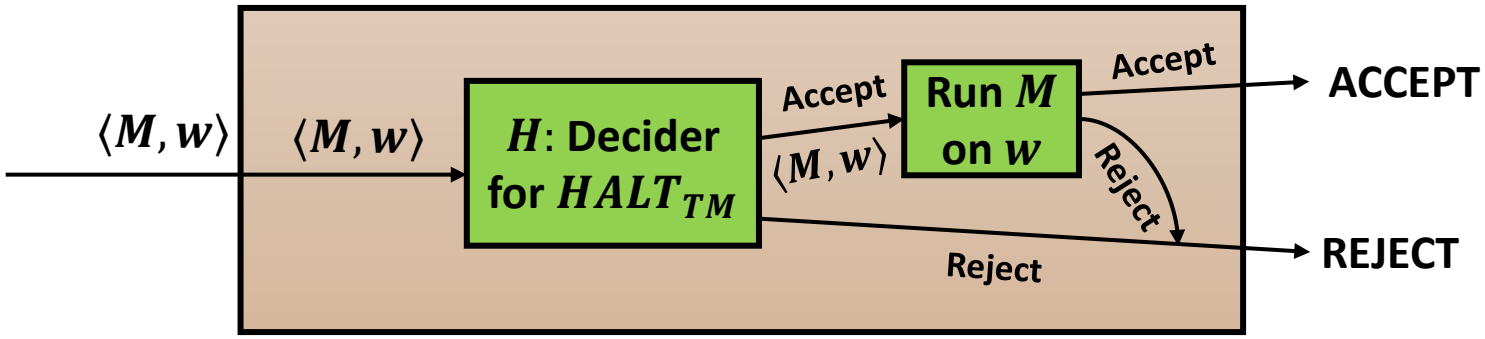
$HALT_{TM} = \{\langle M, w\rangle | M$ halts on input $w\}$. Is $HALT_{TM}$ decidable?

The Halting Problem: Does there exist a Total Turing Machine H that accepts as input a Turing Machine M and an input string w and outputs YES, if $M(w)$ halts (accepts or rejects) and NO, if $M(w)$ does not halt (loops forever), i.e.

$$H(\langle M, w\rangle) = \begin{cases} \textbf{ACCEPTS, if } M(w) \textbf{ HALTS, i.e. accepts or rejects} \\ \textbf{REJECTS, if } M(w) \textbf{ does not HALT, i.e. loops infinitely} \end{cases}$$

NO! $A_{TM} \preccurlyeq HALT_{TM}$

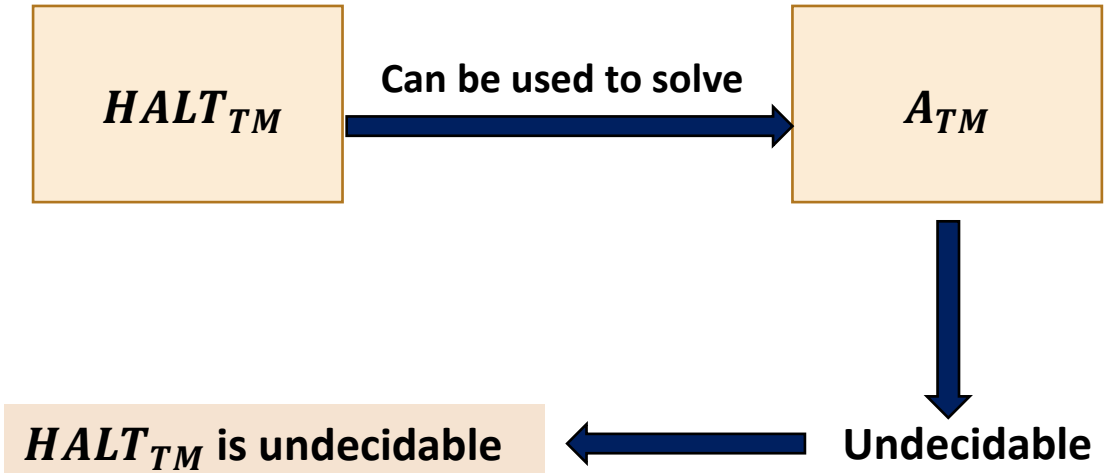
$A =$ On input $\langle M, w\rangle$
Run $H(\langle M, w\rangle)$
If H rejects, output $REJECT$
If H accepts,
Run $M(w)$
If $M(w)$ accepts, output $ACCEPT$
If $M(w)$ rejects, output $REJECT$



Quick Recap

Generally,

- A language $\boldsymbol{A}$ **reduces to** another language $\mathbf{B}$ ($\boldsymbol{A} \preccurlyeq \boldsymbol{B}$) iff we can build a **solver for** $\boldsymbol{A}$ using a **solver for** $\boldsymbol{B}$
- In terms of computability, suppose using B we can compute A. Then, if $\boldsymbol{A}$ **is undecidable then so is** $\boldsymbol{B}$.
- We showed: $\boldsymbol{A_{TM} \preccurlyeq HALT_{TM}}$ to prove that $\boldsymbol{HALT_{TM}}$ is undecidable.



Suppose, $\boldsymbol{A \preccurlyeq B}$

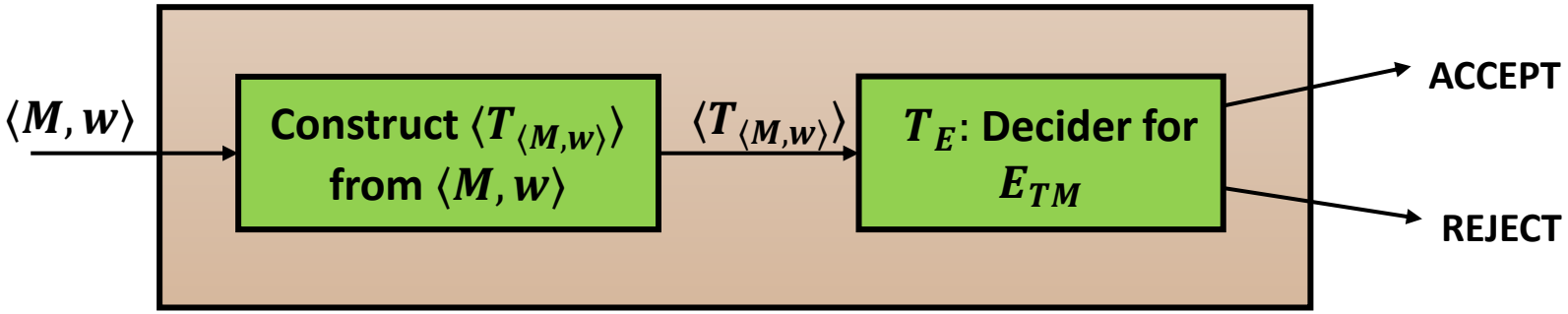
- If A is undecidable then B is undecidable.
- **If $\boldsymbol{B}$ is decidable then $\boldsymbol{A}$ is decidable.**

Quick Recap

$\boldsymbol{E_{TM}} = \{\langle M\rangle | M$ is a Turing Machine and $L(M) = \Phi\}$. **Is $\boldsymbol{E_{TM}}$ decidable?**

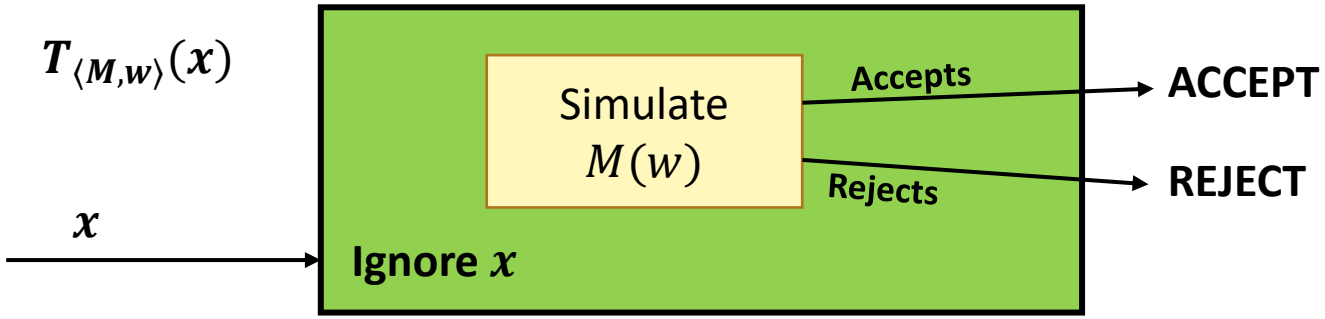
NO! $\boldsymbol{\overline{A}_{TM} \leqslant E_{TM}}$

Proof: Let T_E be the Turing Machine that decides E_{TM}. We shall prove that $\boldsymbol{\overline{A_{TM}} \leqslant E_{TM}}$ by constructing a Turing Machine N for $\boldsymbol{\overline{A_{TM}}}$ using $\boldsymbol{T_E}$.



$\boldsymbol{T_{\langle M,w\rangle}(x) =}$

- Ignore x.
- Run M on w
  - If M accepts w, accept x
  - If M rejects w, reject x



$\boldsymbol{L(T) =}$ **Φ, if $M(w)$ doesn't accept**; **Σ^*, if $M(w)$ accepts**

Quick Recap

$\boldsymbol{E_{TM}} = \{\langle M \rangle | M \text{ is a Turing Machine and } L(M) = \Phi\}$

Proof: Let T_E be the Turing Machine that decides E_{TM}. We shall prove that $\boldsymbol{\overline{A_{TM}} \leqslant E_{TM}}$ by constructing a Turing Machine N that decides $\boldsymbol{\overline{A_{TM}}}$ using $\boldsymbol{T_E}$.

$N(\langle M, w \rangle) =$

- Construct $\langle T_{\langle M,w \rangle} \rangle$, the encoding of $T_{\langle M,w \rangle}$ such that for any input x it works as follows:
  - Ignore x.
  - Run M on w
    - If M accepts w, accept x
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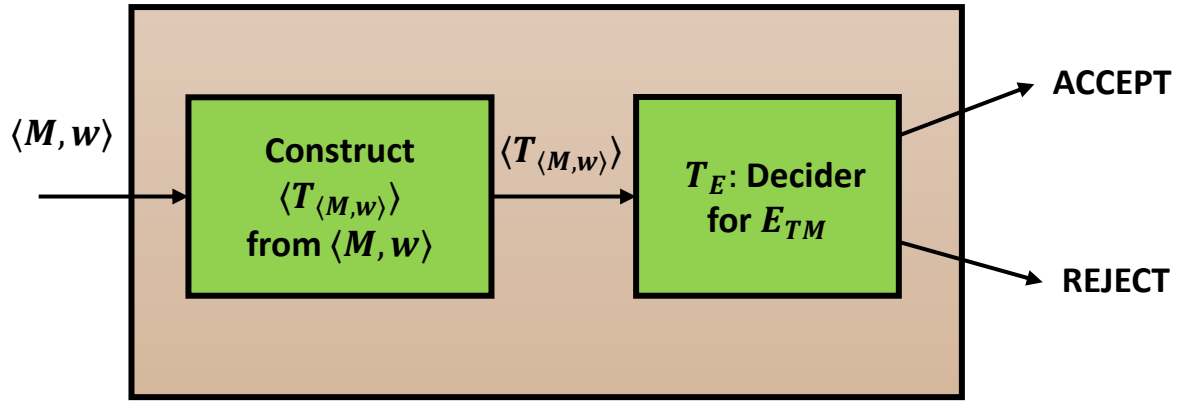
- Send $\langle T_{\langle M,w \rangle} \rangle$ to T_E and Output
  ACCEPT if T_E accepts
  REJECT if T_E rejects



- $\boldsymbol{\overline{A_{TM}} \leqslant E_{TM}}$
- $\boldsymbol{\overline{A_{TM}}}$ **is undecidable**
- $\boldsymbol{E_{TM}}$ **is undecidable!**

Quick Recap

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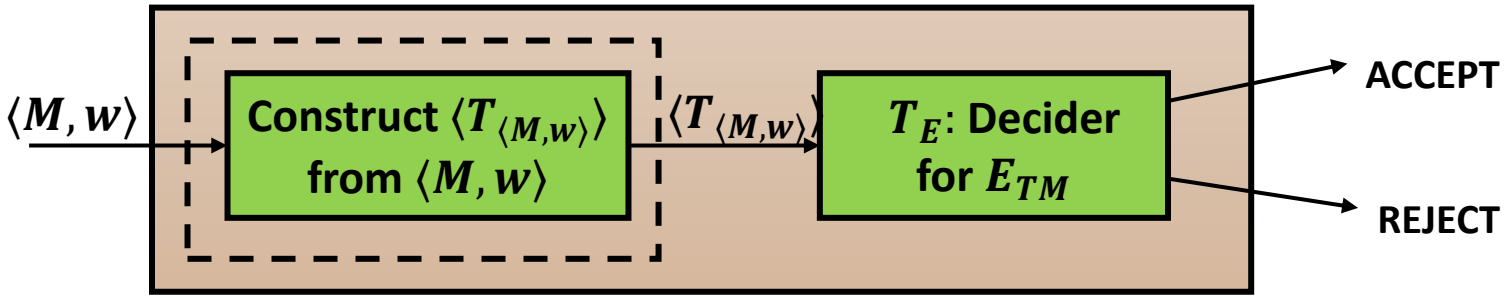

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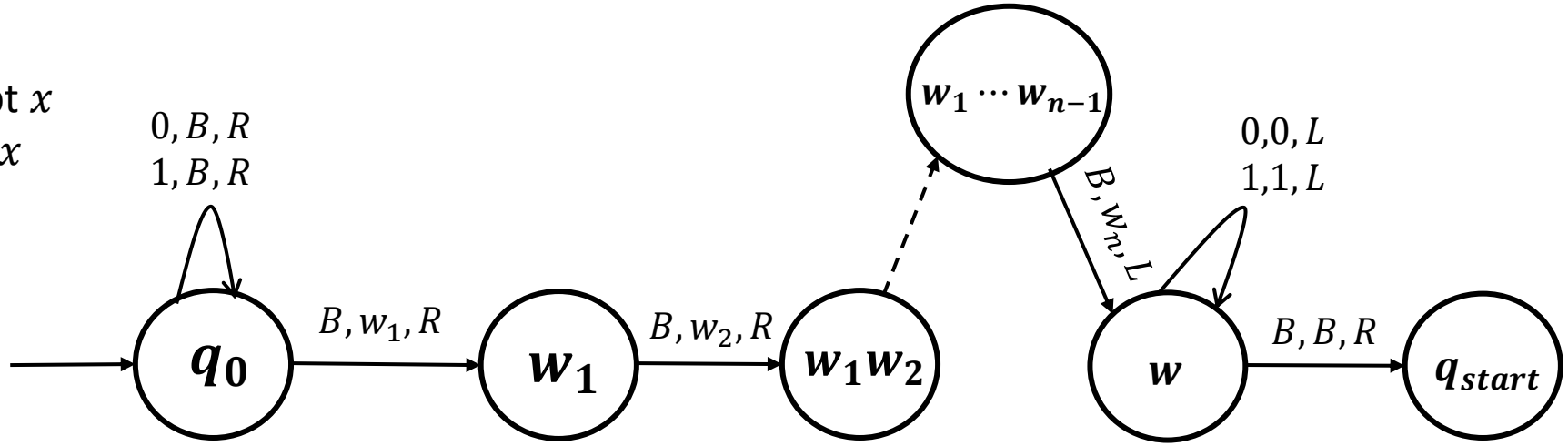
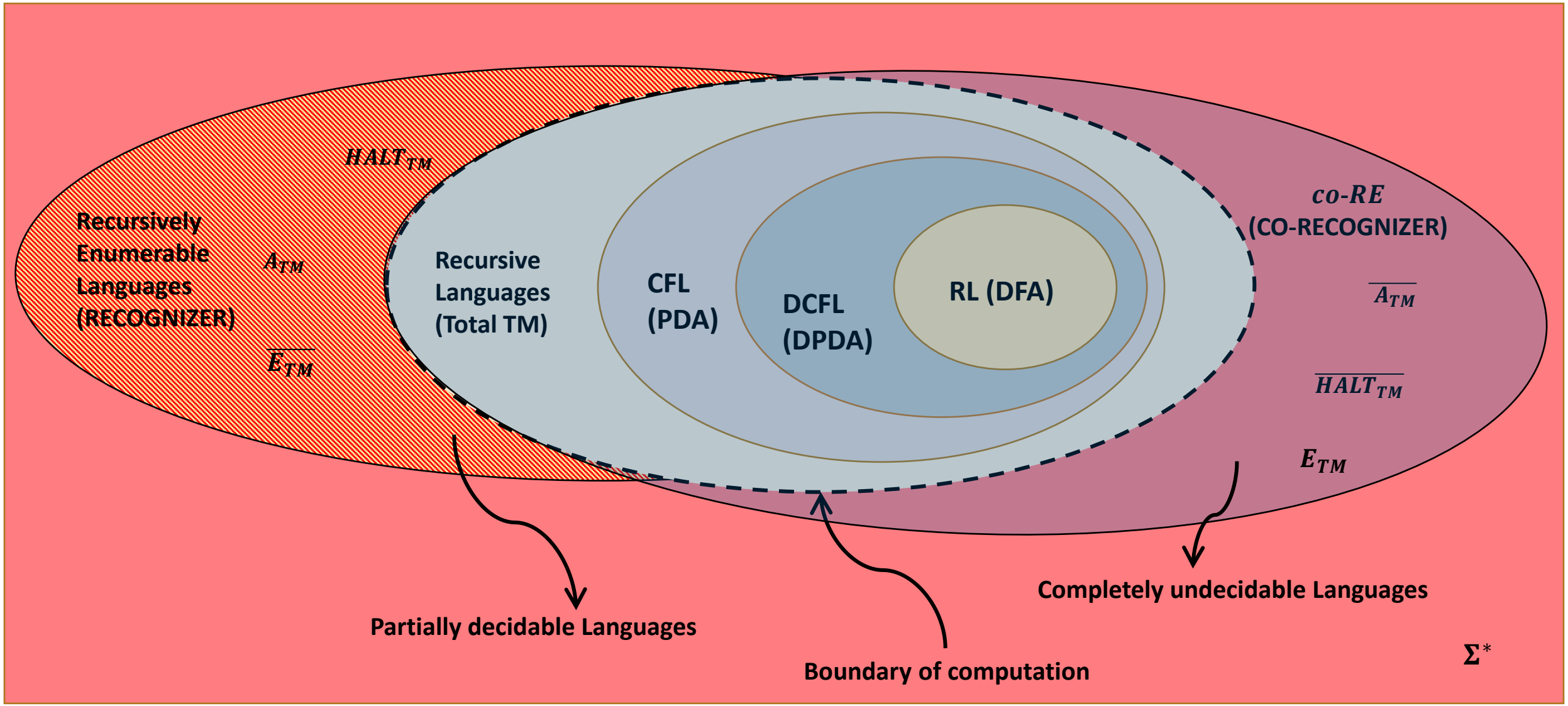


Everything in one slide



Closure properties

Are Recursive languages **closed under Union**? If R_1 and R_2 are recursive, is $R_1 \cup R_2$ recursive?

Proof:

- Let M_1 and M_2 be the Total Turing Machines corresponding to R_1 and R_2 respectively.
- Using M, we construct a Total Turing Machine M' such that M' decides $R_1 \cup R_2$.

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$M' =$ On input w
Run $M_1(w)$
Run $M_2(w)$
If either of them accept, $ACCEPT$
If both rejects, $REJECT$

Recursive languages are **CLOSED under Union.**

Closure properties

Are Recursive languages **closed under intersection**? If R_1 and R_2 are recursive, is $R_1 \cap R_2$ recursive?

Proof:

- Let M_1 and M_2 be the Total Turing Machines corresponding to R_1 and R_2 respectively.
- Using M, we construct a Total Turing Machine M' such that M' decides $R_1 \cap R_2$.

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Run $M_1(w)$
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If both of them accept, $ACCEPT$
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Recursive languages are **CLOSED under Intersection.**

Closure properties

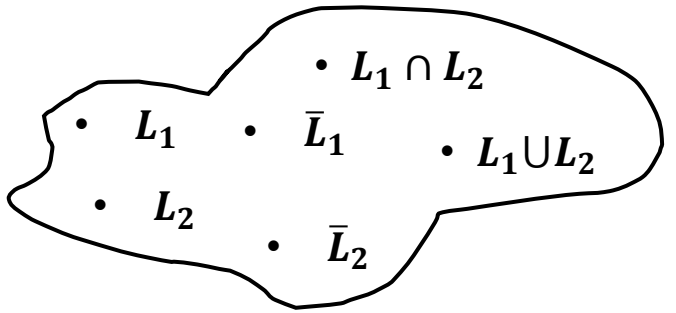
Are Recursive languages **closed under complementation**? If some language R is recursive, is $\bar{R}$ also recursive?

Proof:

- If R is recursive then there exists a total TM M for R.
- Using M, we construct a Total Turing Machine $\bar{M}$ such that $\bar{M}$ decides $\bar{R}$.

$\bar{M}$ = On input w
Run $M(w)$.
If M accepts, $REJECT$
If M rejects, $ACCEPT$

Recursive languages are **CLOSED under complementation**



Set of all recursive (decidable) languages

Closure properties

L and $\bar{L}$ are both Recursively Enumerable if and only if L is Recursive.

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For the other direction: Let M_1 be the TM for L and M_2 be the TM for $\bar{L}$. Then, if $w \in L$, $M_1(w)$ accepts and if $w \notin L$, $M_2(w)$ accepts.

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How do we build a Total Turing Machine for L? There is one problem: We can't run M_1 and M_2 one after the other as if some input M_1 gets stuck in an infinite loop and M_2 never gets control.

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$M' =$

For $i = 1, 2, \cdots$

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L is Recursive

Closure properties

Using Dovetailing it is easy to prove that:

- RE languages are closed under **union and intersection**
    - On input w, run $M_1(w)$ and $M_2(w)$ in parallel using dovetailing
    - **For union**: If either $M_1(w)$ or $M_2(w)$ accepts, $ACCEPT$
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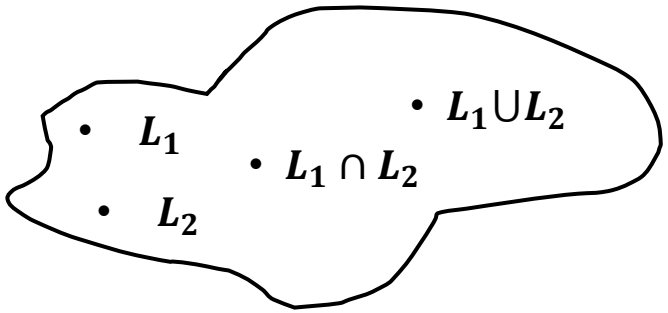

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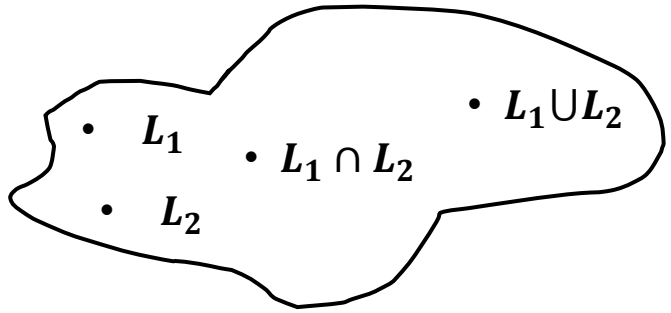

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- Well, we just proved that L and $\bar{L}$ are both Recursively Enumerable, **iff** L is Recursive.
- But we know that there exists problems that are in RE but are not Recursive (e.g: A_{TM}, $HALT_{TM}$,$\cdots$).
- So the complement of such problems are not in RE (e.g.: $\overline{A_{TM}}$, $\overline{HALT}_{TM}$, $\cdots$).

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So if,
w $\notin L$, $i.e., w \in \bar{L}$, $\bar{M}$ outputs $ACCEPT$ or loops forever
$w \in L, i.e., w \notin \bar{L}, \bar{M}$ outputs $REJECT$

Closure properties

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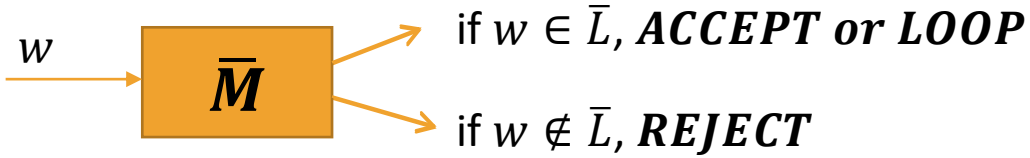

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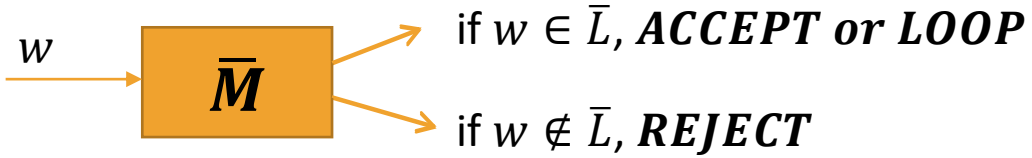

Co-Recursively Enumerable Language/co-Turing Recognizable (Co-RE/$\overline{RE}$/nRE): A language C is **Co-Recursively Enumerable (co-RE/$\overline{RE}$/nRE) or Co-Turing Recognizable** if

$\forall \omega \in C, M(\omega)$ **doesn't reject** (accepts or loops forever)
$\forall \omega \notin C, M(\omega)$ **rejects**

If $L \in RE$, $\bar{L} \in co\text{-}RE$ and vice versa

$co\text{-}RE$

Co-Recursively Enumerable Language (Co-RE/$\overline{RE}$/nRE): $C \in$ **Co-RE** if

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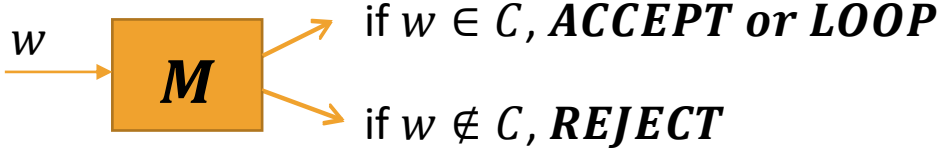

- $\boldsymbol{RE}$: Halts and accepts if $w \in L$. May loop when $w \notin L$.
- $\boldsymbol{co\text{-}RE}$: May loop when $w \in L$. Halts and rejects if $w \notin L$.

$co\text{-}RE$

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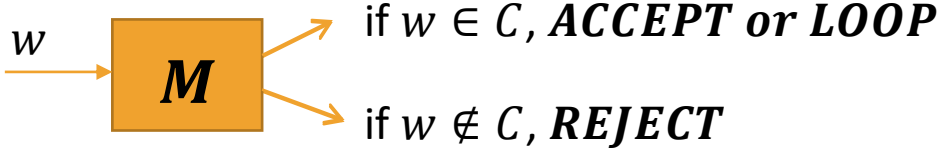

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Note: Every Recursive Language $\boldsymbol{R}$, is both in $\boldsymbol{RE}$ and $\boldsymbol{co\text{-}RE}$, i.e. $\boldsymbol{R \subseteq RE \cap co\text{-}RE}$

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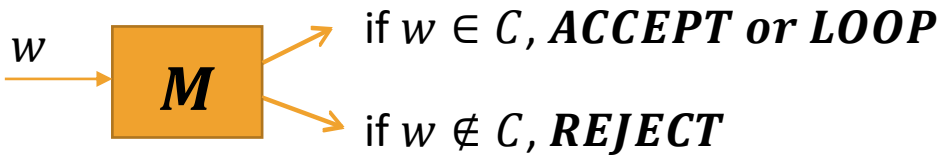

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Is $\boldsymbol{R = RE \cap co\text{-}RE}$?

$co\text{-}RE$

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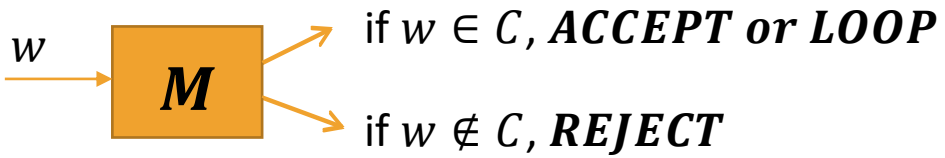

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Is $\boldsymbol{R = RE \cap co\text{-}RE}$?

We have to prove the following: **If $\boldsymbol{L \in RE}$ and $\boldsymbol{L \in co\text{-}RE}$, then $\boldsymbol{L}$ is Recursive**

$co\text{-}RE$

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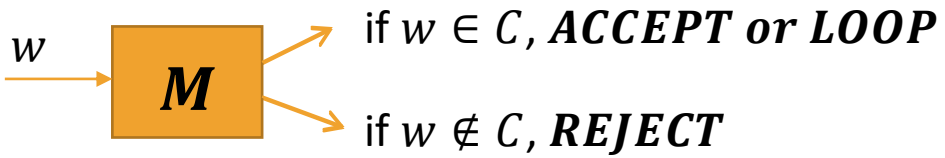

- $\boldsymbol{RE}$: Halts and accepts if $w \in L$. May loop when $w \notin L$.
- $\boldsymbol{co\text{-}RE}$: May loop when $w \in L$. Halts and rejects if $w \notin L$.

Note: Every Recursive Language R, is both in RE and $co\text{-}RE$, i.e. $R \subseteq RE \cap co\text{-}RE$

Is $R = RE \cap co\text{-}RE$?

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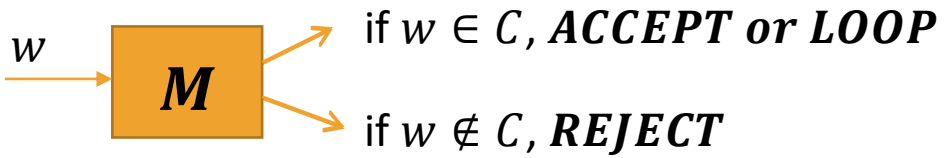
$co\text{-}RE$

Co-Recursively Enumerable Language (Co-RE/$\overline{RE}$/nRE): $C \in$ **Co-RE** if

$\forall \omega \in C, M(\omega)$ **doesn't reject**
$\forall \omega \notin C, M(\omega)$ **rejects**



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We need to run M and $\overline{M}$ in parallel using dovetailing.

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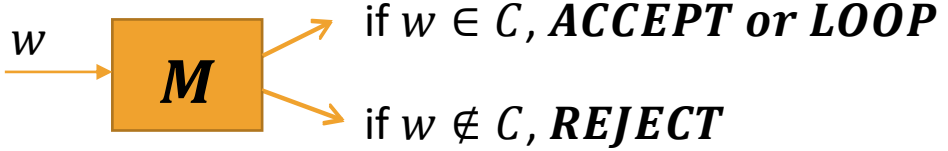

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$D =$ On input w
Run $M(w)$ and $\overline{M}(w)$ in parallel
If $M(w)$ accepts, output $ACCEPT$
If $\overline{M}(w)$ rejects, output $REJECT$

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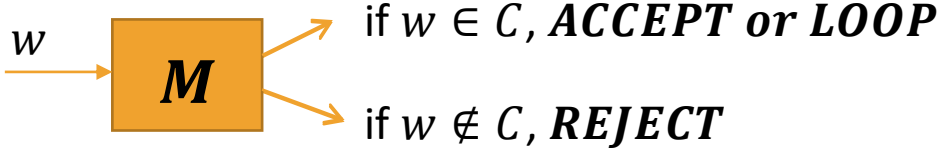

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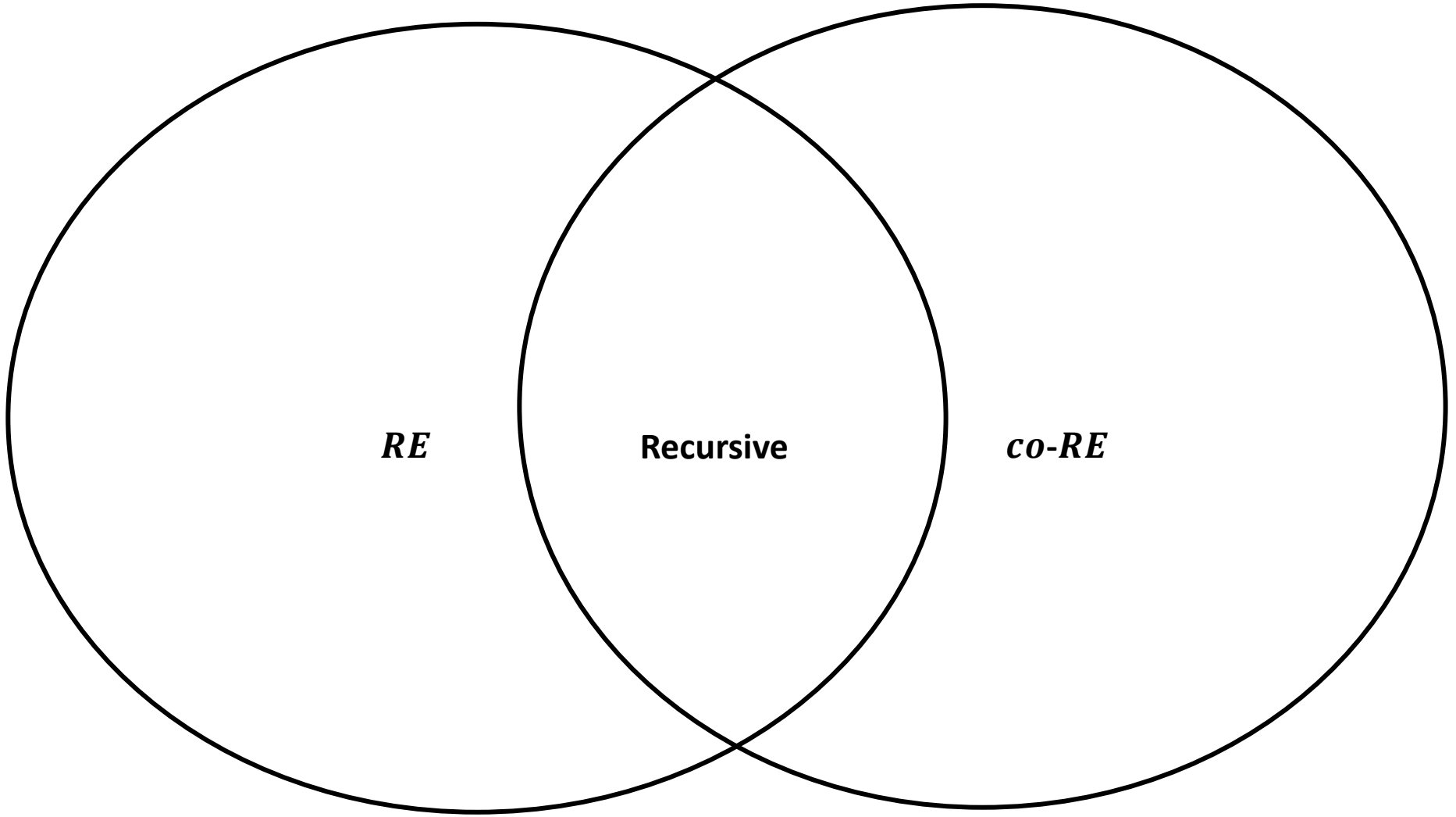
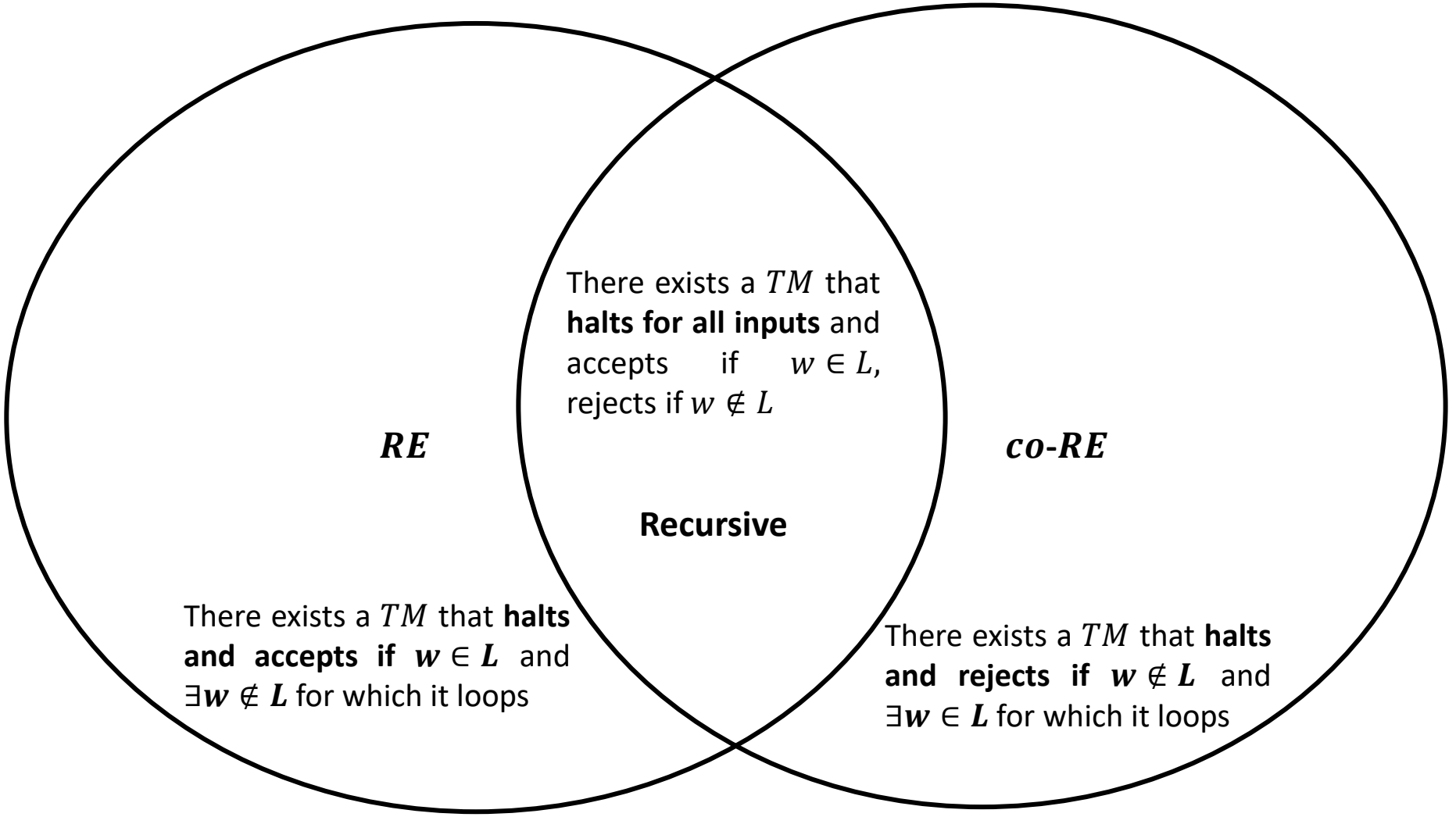
So $R = RE \cap co\text{-}RE$


RE
Recursive
$co\text{-}RE$


There exists a TM that **halts for all inputs** and accepts if $w \in L$, rejects if $w \notin L$
$\boldsymbol{RE}$
$\boldsymbol{co\text{-}RE}$
Recursive
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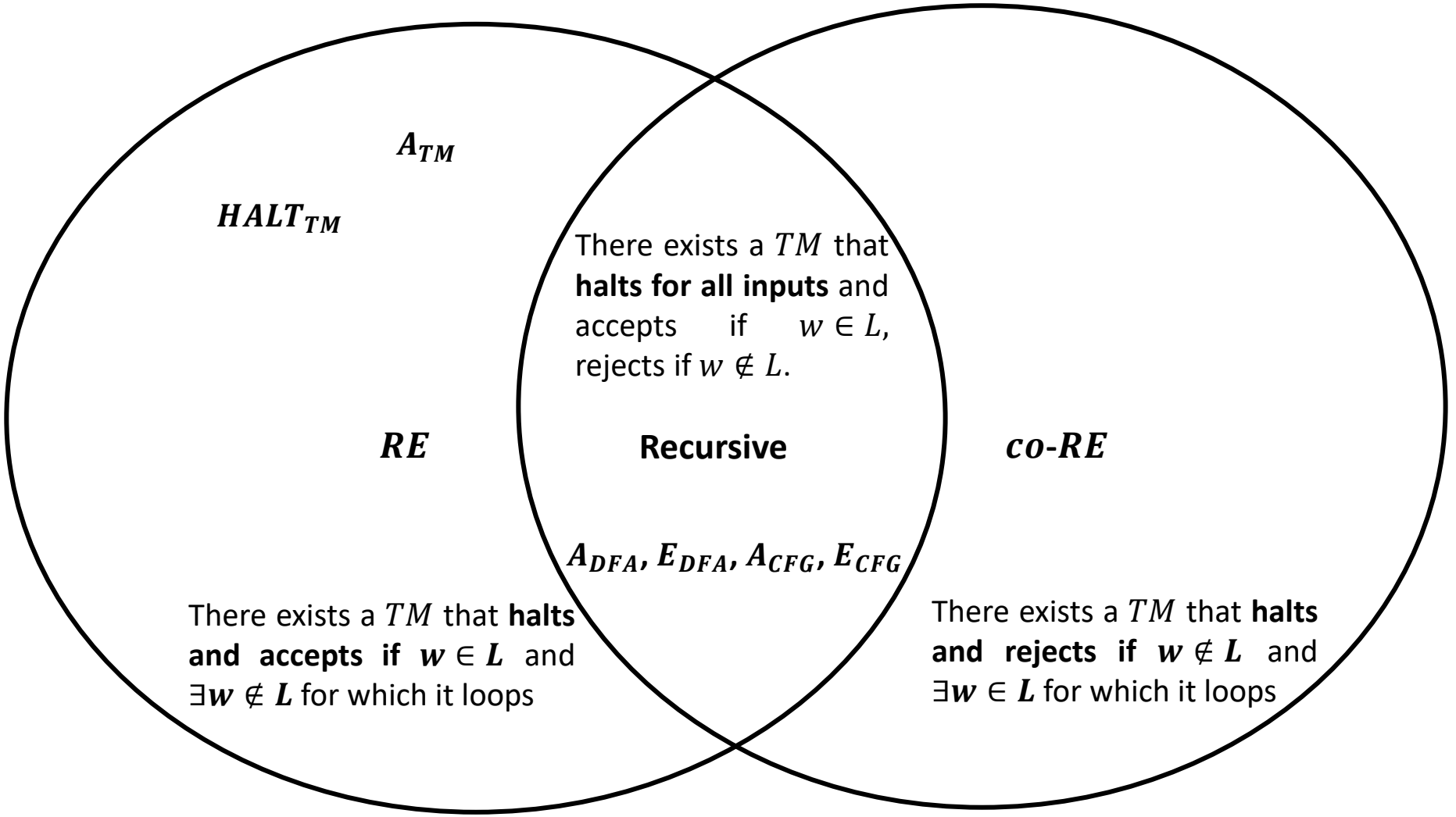
A_{TM}
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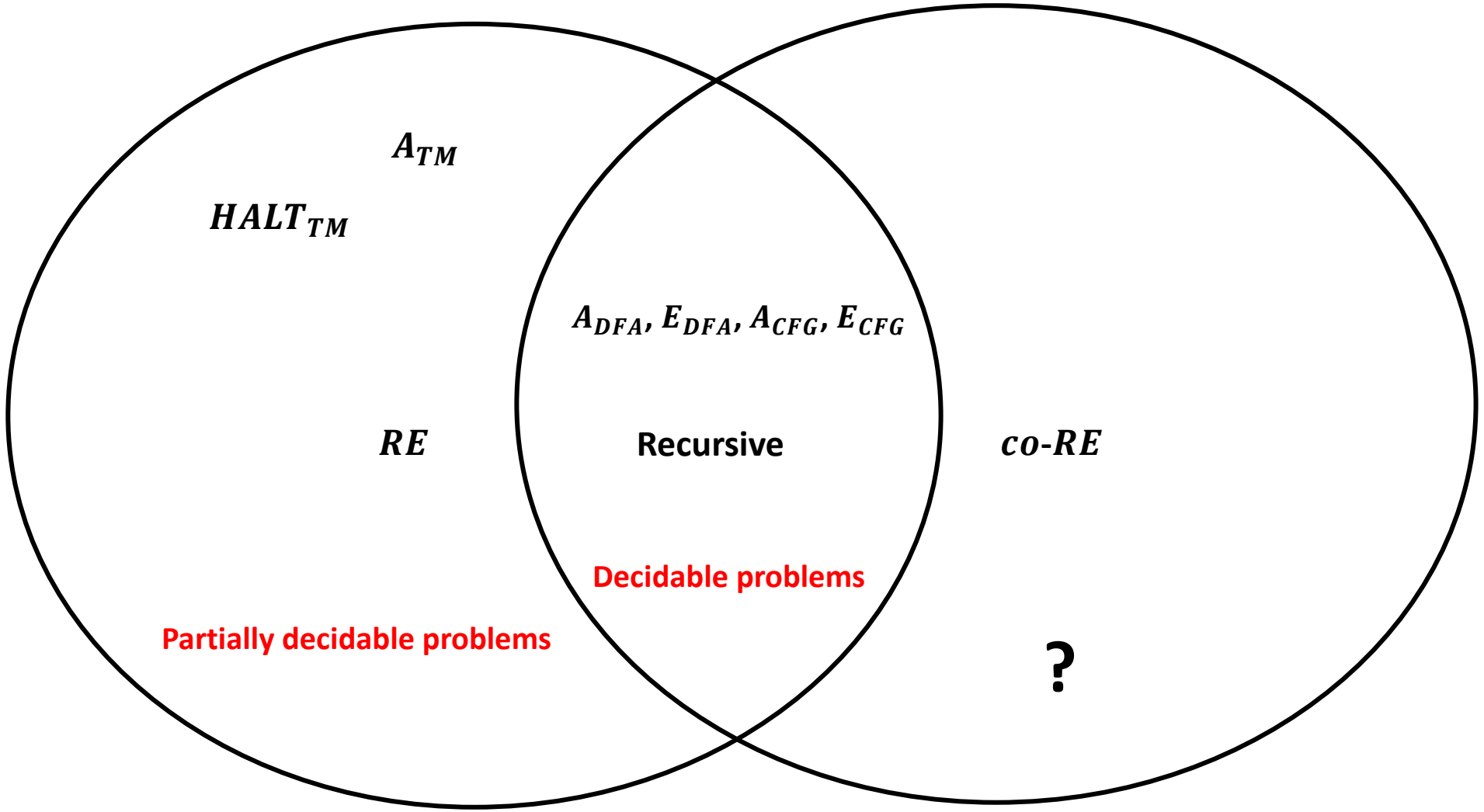
A_{TM}
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RE
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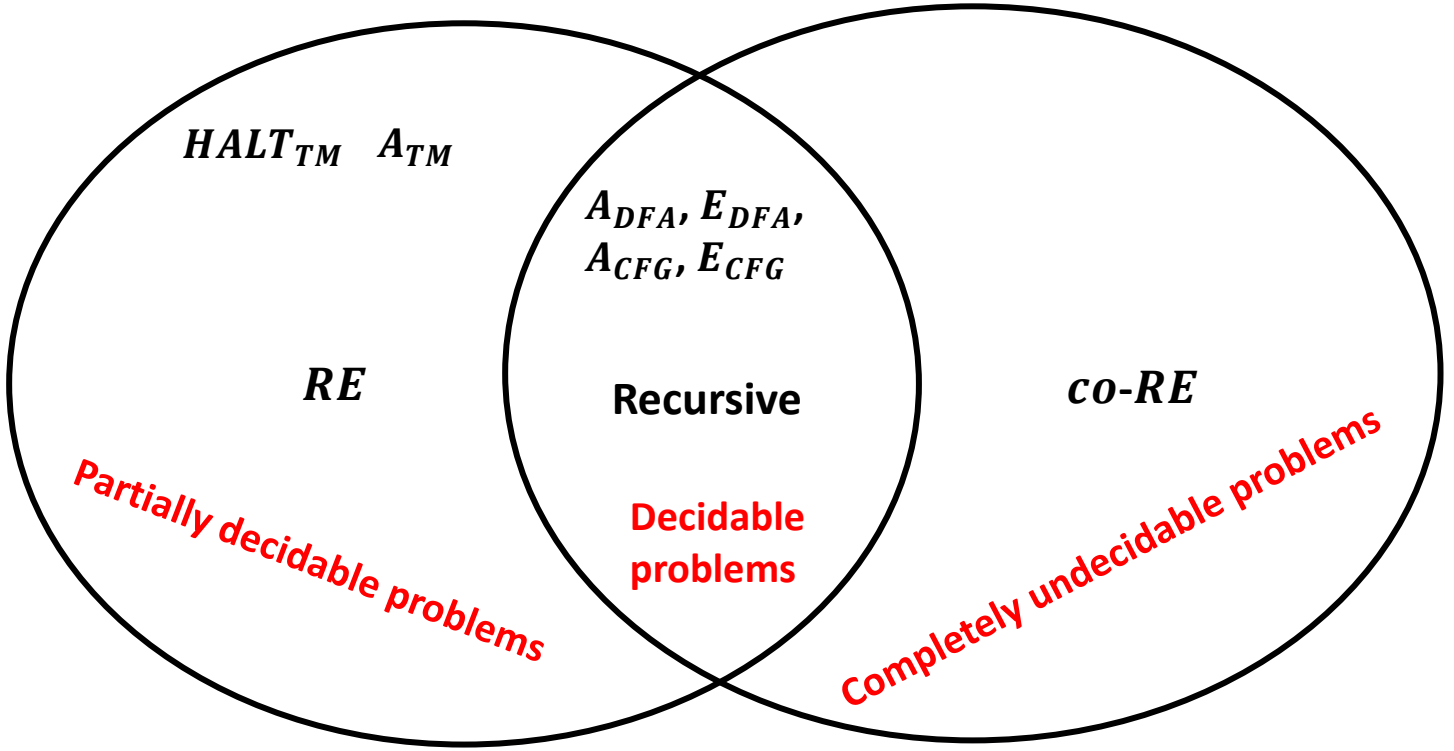
Decidable problems
Partially decidable problems
?

Closure properties

Completely undecidable languages: Languages L for which there exists at least one instance $w \in L$, for which the TM enters into an infinite loop.

So, languages **that are in $co\text{-}RE$ but are not recursive** are **completely undecidable.**



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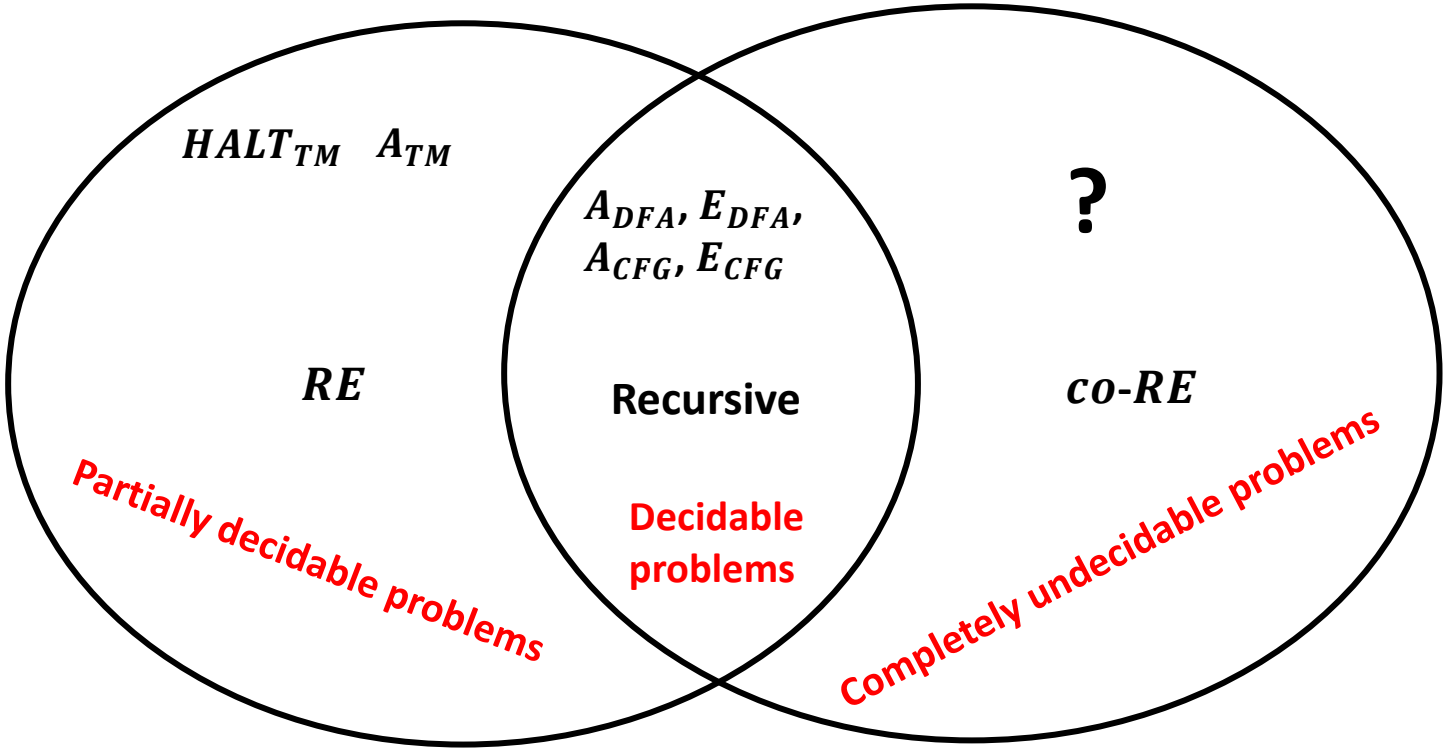

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If $\boldsymbol{L \in RE}$ but is not Recursive (partially decidable), then $\boldsymbol{\bar{L} \in co\text{-}RE}$ but is not recursive. So Complement of any partially decidable language is completely undecidable

- E.g.: $A_{TM} \in RE$ and so $\overline{A_{TM}} \in co\text{-}RE$ and is **completely undecidable**

$\boldsymbol{A_{TM} = \{\langle M, w\rangle | M}$ **accepts input** $\boldsymbol{w\}}$

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- Similarly, **$\overline{HALT_{TM}}$ is also completely undecidable**

$\overline{HALT_{TM}} = \{\langle M, w\rangle | M$ doesn't halt on input $w\}$

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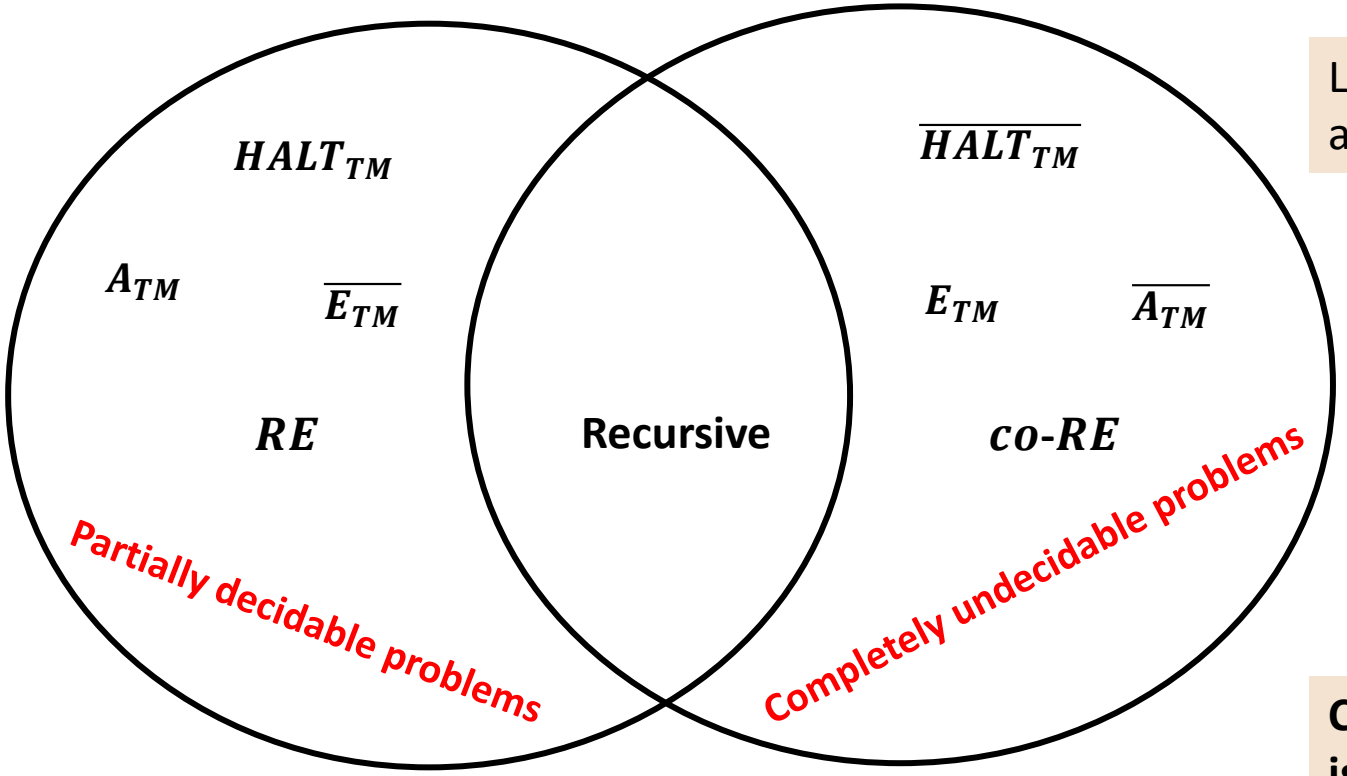

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Summing up

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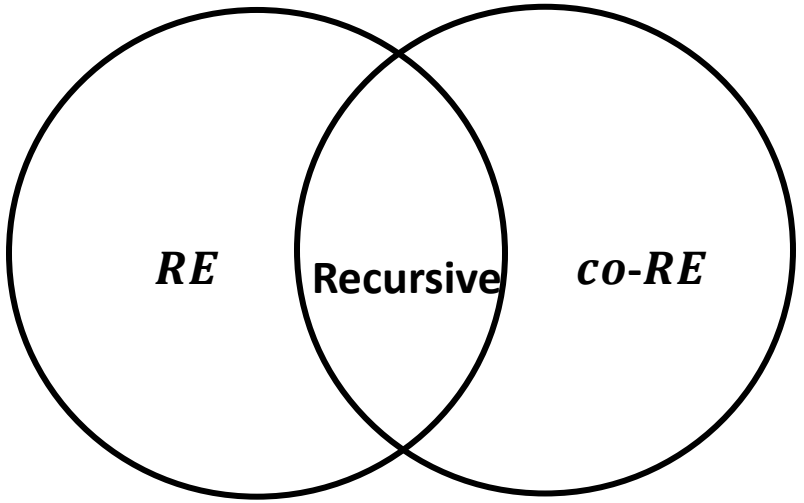
- **RE is closed under union & intersection but not complement**



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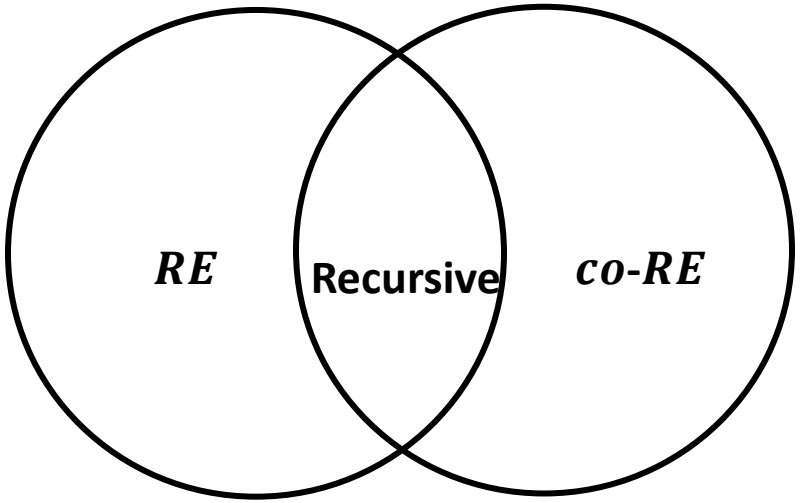

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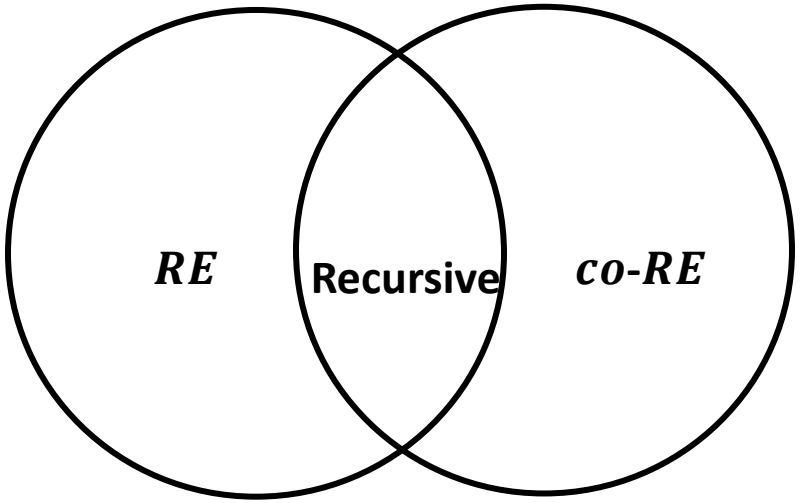

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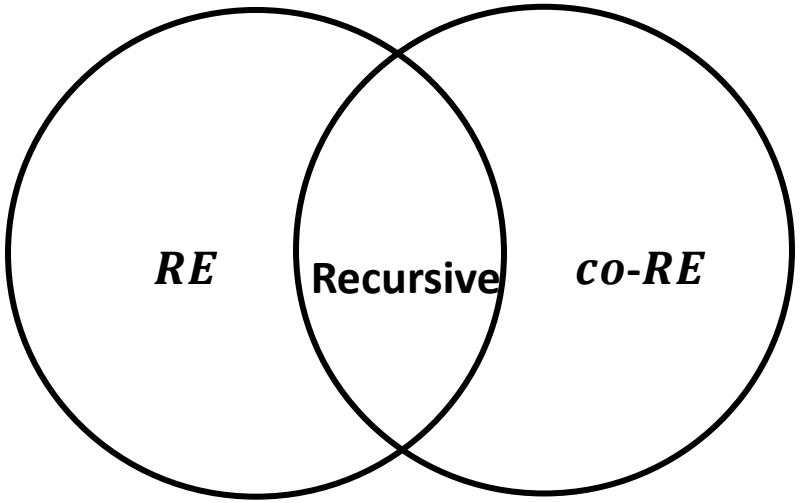

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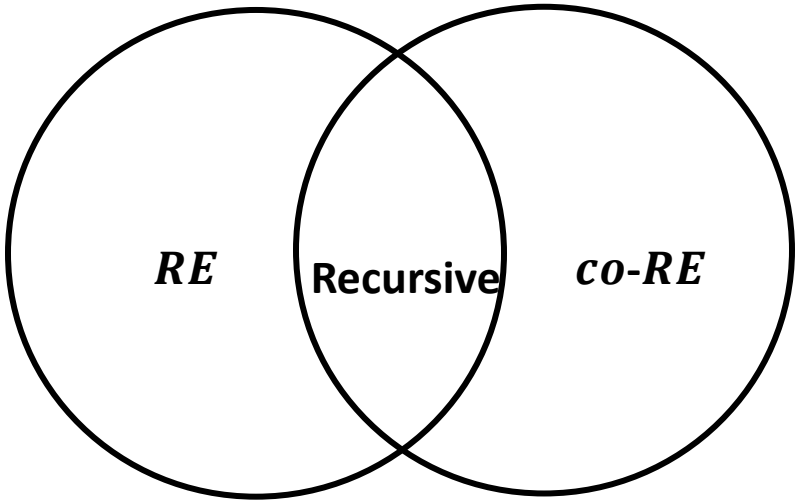

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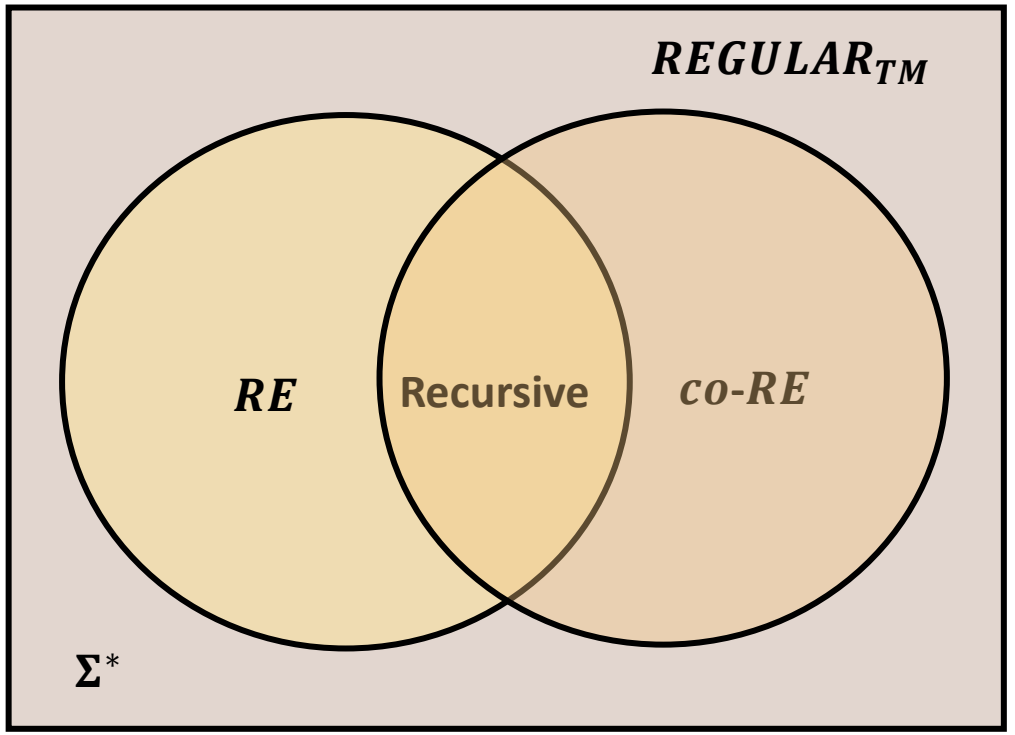
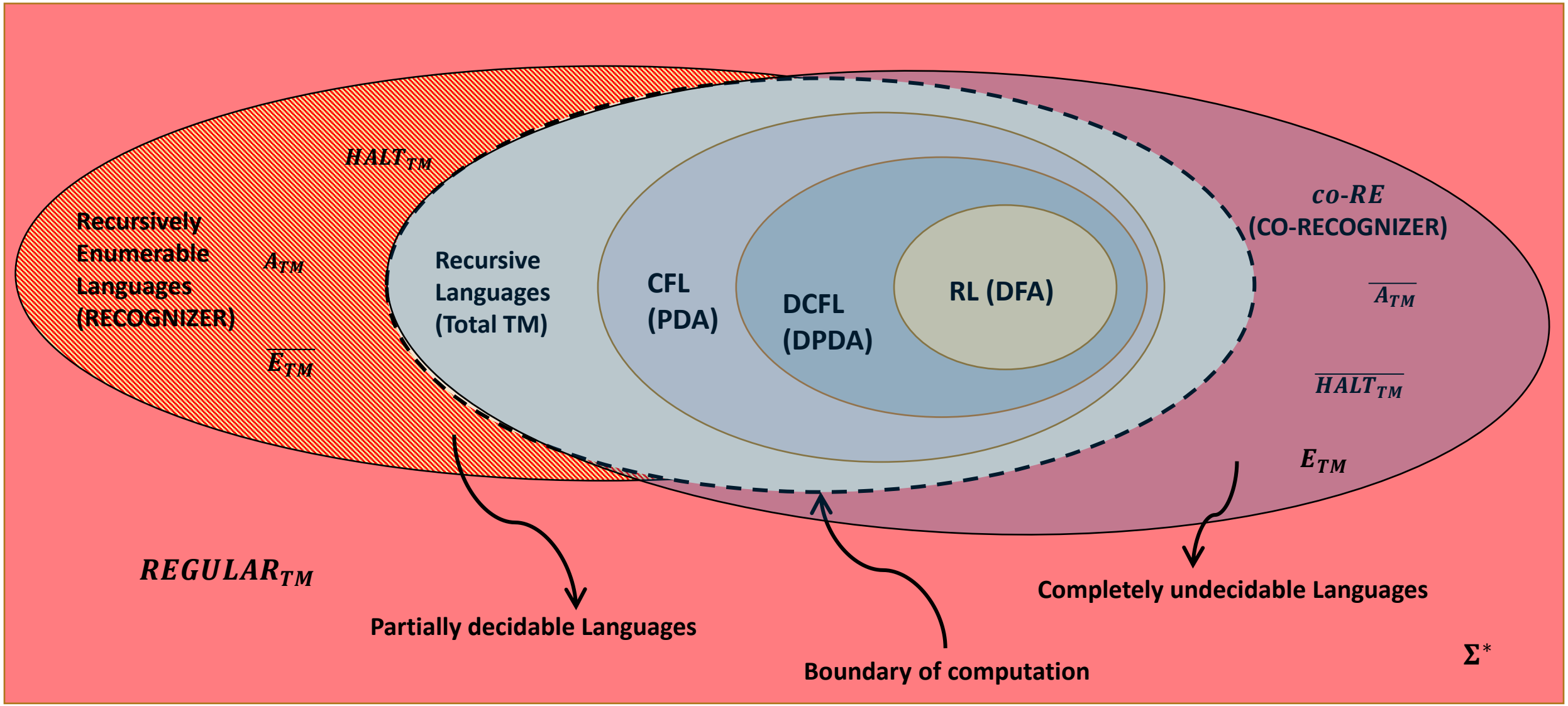
E.g.: $\boldsymbol{REGULAR_{TM}} = \{\langle \boldsymbol{M} \rangle | \boldsymbol{L(M)}$ **is regular**$\}$



Everything in one slide



The Road ahead to Complexity Theory…

- We finished up by looking at problems that are decidable/undecidable.
- There are many things that I couldn't cover:
  - Several cool problems that can be proven to be decidable/undecidable and classified to be in $R, RE, co\text{-}RE$ etc
  - Mapping reduction, Recursion Theorem, Rice's Theorem
- Problems that are not computable are highly likely to never be solved on feasible computational devices.
- In how much time/space can **computable problems** be solved in? Complexity Theory: classify problems according to their hardness.
- Million dollar problems waiting to be solved!
- E.g.: Quantum computers model how nature computes at the fundamental level: provably faster than classical machines on several problems and most likely violates the Extended Church Turing Thesis.

Thank You!