CS 302.1 - Automata Theory

Lecture 06

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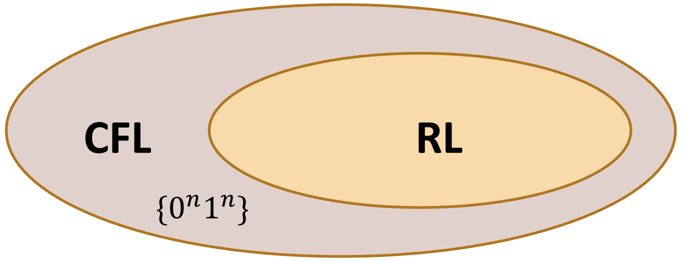
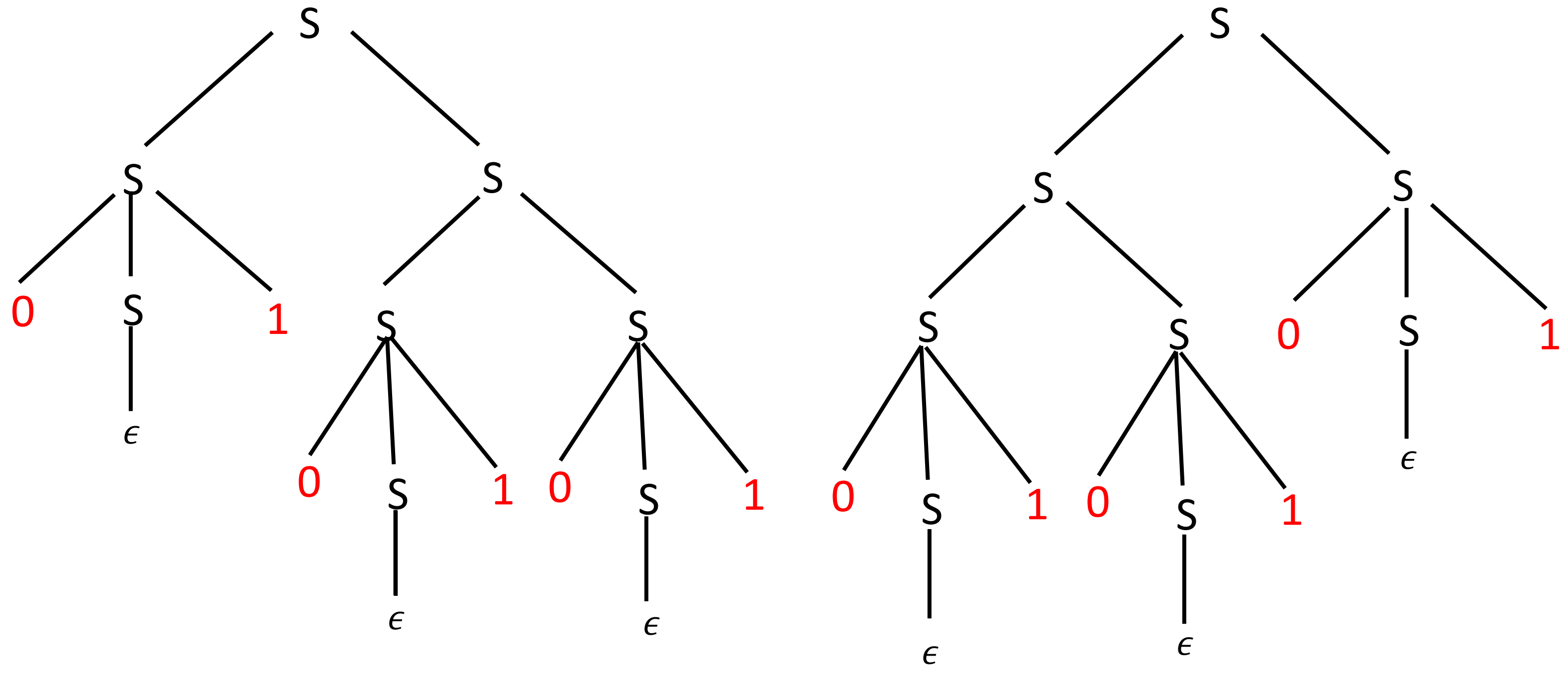
Quick Recap

Context-Free Grammars: If the *rules* of the underlying grammar G are of the form

$$\boldsymbol{V} \rightarrow (\boldsymbol{V} \cup \boldsymbol{T})^*$$

then such a grammar is called **Context-Free.**





Parse trees: These are ordered trees that provide alternative representations of the derivation of a grammar.

Ambiguous grammars: There exists $\boldsymbol{\omega} \in \boldsymbol{L}(\boldsymbol{G})$, such that there are **two or more leftmost derivations for $\boldsymbol{\omega}$** (or equivalently two or more rightmost derivations) or equivalently **two or more parse trees for $\boldsymbol{\omega}$. Ambiguity** may not be desirable

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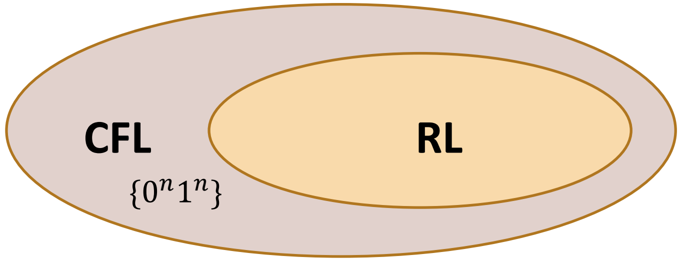

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Chomsky Normal Form: If every *rule* of the CFG is of the form

$\boldsymbol{A} \rightarrow \boldsymbol{BC}$ [B, C are not start variables]
$\boldsymbol{A} \rightarrow \boldsymbol{a}$ [a is a terminal]
$\boldsymbol{S} \rightarrow \boldsymbol{\epsilon}$ [S is the Start Variable]

- **Any CFG can be converted to a grammar in CNF that generates the same language.**
- The number of steps required to derive a string $w = 2|w| - 1$.
- Is crucial for **deciding** whether w is generated by a CFG G.

Chomsky Normal Form

Any CFL can be generated by a CFG written in Chomsky Normal Form.

Proof: The proof is constructive. Suppose we have a CFG G with a set of rules. To convert G into CNF, we do the following:

1. Add a new start variable $S' \to S$
2. Remove ϵ rules of the form $A \to \epsilon$
   - Remove nullable symbols/rules
3. Remove unit (short) rules of the form $A \to B$
   - Remove useless symbols/rules
4. Remove long rules of the form $A \to u_1 u_2 \cdots u_k$
   - Remove useless symbols/rules

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1. Add a new start variable $S' \rightarrow S$

2. **Remove ϵ rules of the form $A \rightarrow \epsilon$**

For each occurrence of A in the right side of the rule, we add a new rule with the occurrence of A deleted.

E.g.: Consider any rule $B \rightarrow uAvAw$ (u, v, w can be strings of variables and terminals)

Then new rules: $B \rightarrow uAvAw|uvAw|uAvw|uvw$

What if you had a rule such as $B \rightarrow A$? Then we would have needed to add a rule $B \rightarrow \epsilon$ (unless this rule has been already removed) as B is a **nullable variable.**

Repeat this procedure, until all ϵ-rules are removed.

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E.g.:

$$S \to 0|X0|ZYZ$$
$$X \to Y|\epsilon$$
$$Y \to 1|X$$

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E.g.: $S \to 0|X0|ZYZ$
$X \to Y|\epsilon$
$Y \to 1|X$

To remove $X \to \epsilon$, we add new rules: $S \to 0|X0|ZYZ$
$X \to Y$
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E.g.:

$$S \rightarrow 0|X0|ZYZ$$
$$X \rightarrow Y|\epsilon$$
$$Y \rightarrow 1|X$$

To remove $X \rightarrow \epsilon$, we add new rules:

$$S \rightarrow 0|X0|ZYZ$$
$$X \rightarrow Y$$
$$Y \rightarrow 1|X|\epsilon$$

To remove $Y \rightarrow \epsilon$, we add:

$$S \rightarrow 0|X0|ZYZ|ZZ$$
$$X \rightarrow Y$$
$$Y \rightarrow 1|X$$

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2. Remove ϵ rules of the form $A \rightarrow \epsilon$

3. **Remove unit rules of the form $\boldsymbol{A \rightarrow B}$**

We **remove the rule $\boldsymbol{A \rightarrow B}$** and **whenever a rule $\boldsymbol{B \rightarrow u}$ appears** (u is a string of terminals and variables), we **add a new rule $\boldsymbol{A \rightarrow u}$**, unless this rule was already removed.

Repeat these steps until all unit rules are removed.

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E.g.:

$$S \rightarrow A|11$$
$$A \rightarrow B|1$$
$$B \rightarrow S|0$$

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E.g.:

$$\begin{gathered} S \rightarrow A|11 \\ A \rightarrow B|1 \\ B \rightarrow S|0 \end{gathered}$$

Remove $\boldsymbol{S \rightarrow A}$	Remove $\boldsymbol{A \rightarrow B}$	Remove $\boldsymbol{B \rightarrow S}$	Remove $\boldsymbol{B \rightarrow B}$	Remove $\boldsymbol{S \rightarrow B}$	Remove $\boldsymbol{A \rightarrow S}$
$S \rightarrow 11\|\boldsymbol{B}\|\mathbf{1}$	$S \rightarrow 11\|B\|1$	$S \rightarrow 11\|B\|1$	$S \rightarrow 11\|B\|1$	$S \rightarrow 11\|\mathbf{0}\|1$	$S \rightarrow 11\|\mathbf{0}\|1$
$A \rightarrow B\|1$	$A \rightarrow 1\|\boldsymbol{S}\|\mathbf{0}$	$A \rightarrow 1\|S\|0$	$A \rightarrow 1\|S\|0$	$A \rightarrow 1\|S\|0$	$A \rightarrow 1\|\mathbf{11}\|0$
$B \rightarrow S\|0$	$B \rightarrow S\|0$	$B \rightarrow 0\|\mathbf{11}\|\mathbf{1}\|\boldsymbol{B}$	$B \rightarrow 0\|11\|1$	$B \rightarrow 0\|11\|1$	$B \rightarrow 0\|11\|1$

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$$\begin{aligned} S &\rightarrow A|11 \\ A &\rightarrow B|1 \\ B &\rightarrow S|0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \boldsymbol{S} &\rightarrow \mathbf{11|0|1} \\ \boldsymbol{A} &\rightarrow \mathbf{1|11|0} \\ \boldsymbol{B} &\rightarrow \mathbf{0|11|1} \end{aligned}$$

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2. Remove ϵ rules of the form $A \rightarrow \epsilon$
3. Remove unit rules of the form $A \rightarrow B$
4. **Remove long rules of the form $\boldsymbol{A \rightarrow u_1 u_2 \cdots u_k}$**

Note that each u_i could be a variable or a terminal. We do the following:

- Replace $A \rightarrow u_1 u_2 \cdots u_k$, $(k \geq 3)$ with the rules $A \rightarrow u_1 A_1,\ A_1 \rightarrow u_2 A_2, \cdots, A_{k-2} \rightarrow u_{k-1} u_k$
- We replace any terminal u_i in the preceding rules with the new variable U_i and add the rule $U_i \rightarrow u_i$

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Remove $\boldsymbol{\epsilon}$ rules of the form $\boldsymbol{A \to \epsilon}$ (For each occurrence of A in the right side of the rule, add a new rule with the occurrence of A deleted ; Remove nullable variables, Repeat the procedure until all ϵ rules are removed)**.**

Remove unit rules of the form $\boldsymbol{A \to B}$ (Whenever a rule $B \to u$ appears, we add a new rule $A \to u$, unless this rule was already removed. Repeat these steps until all unit rules are removed.)

Remove long rules of the form $\boldsymbol{A \to u_1 u_2 \cdots u_k}$ (Replace $A \to u_1 u_2 \cdots u_k$, $(k \geq 3)$ with the rules $A \to u_1 A_1$, $A_1 \to u_2 A_2, \cdots, A_{k-2} \to u_{k-1} u_k$; Replace any terminal u_i in the preceding rules with the new variable U_i and add the rule $U_i \to u_i$).

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CNF: $\boldsymbol{A \rightarrow BC}$ [B, C are not start variables]
$\boldsymbol{A \rightarrow a}$ [a is a terminal]
$\boldsymbol{S \rightarrow \epsilon}$ [S is the Start Variable]

Convert the CFG

$$\boldsymbol{S \rightarrow ASA|aB}$$
$$\boldsymbol{A \rightarrow B|S}$$
$$\boldsymbol{B \rightarrow b|\epsilon}$$

to CNF.

1. Add a new start variable

$$\boldsymbol{S' \rightarrow S}$$
$$S \rightarrow ASA|aB$$
$$A \rightarrow B|S$$
$$B \rightarrow b|\epsilon$$

2a. Remove ϵ rules ($B \rightarrow \epsilon$)

$$S' \rightarrow S$$
$$S \rightarrow ASA|aB|\boldsymbol{a}$$
$$A \rightarrow B|S|\boldsymbol{\epsilon}$$
$$B \rightarrow b$$

2b. Remove ϵ rules (A→ ϵ)

$$S' \rightarrow S$$
$$S \rightarrow ASA|aB|a|\boldsymbol{AS}|\boldsymbol{SA}|\boldsymbol{S}$$
$$A \rightarrow B|S$$
$$B \rightarrow b$$

Chomsky Normal Form

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Convert the CFG

$$\boldsymbol{S \rightarrow ASA|aB}$$
$$\boldsymbol{A \rightarrow B|S}$$
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to CNF.

3a. Remove $\boldsymbol{S \rightarrow S}$

$$S' \rightarrow S$$
$$S \rightarrow ASA|aB|a|AS|SA$$
$$A \rightarrow B|S$$
$$B \rightarrow b$$

3b. Remove $\boldsymbol{S' \rightarrow S}$

$$S' \rightarrow \boldsymbol{ASA|aB|a|AS|SA}$$
$$S \rightarrow ASA|aB|a|AS|SA$$
$$A \rightarrow B|S$$
$$B \rightarrow b$$

3c. Remove A$\rightarrow$ $\boldsymbol{B}$

$$S' \rightarrow ASA|aB|a|AS|SA$$
$$S \rightarrow ASA|aB|a|AS|SA$$
$$A \rightarrow S|\boldsymbol{b}$$
$$B \rightarrow b$$

3d. Remove A$\rightarrow$ $\boldsymbol{S}$

$$S' \rightarrow ASA|aB|a|AS|SA$$
$$S \rightarrow ASA|aB|a|AS|SA$$
$$A \rightarrow b|\boldsymbol{ASA|aB|a|AS|SA}$$
$$B \rightarrow b$$

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$$S' \to ASA|aB|a|AS|SA$$
$$S \to ASA|aB|a|AS|SA$$
$$A \to b|ASA|aB|a|AS|SA$$
$$B \to b$$

4a. Remove long rules

$$S' \to A\boldsymbol{SA}|aB|a|AS|SA$$
$$S \to ASA|aB|a|AS|SA$$
$$A \to b|ASA|aB|a|AS|SA$$
$$B \to b$$

There are other rules of the form: Var → ASA

4b. Remove long rules

$$S' \to A\boldsymbol{U}|aB|a|AS|SA$$
$$S \to A\boldsymbol{U}|aB|a|AS|SA$$
$$A \to b|A\boldsymbol{U}|aB|a|AS|SA$$
$$U \to SA$$
$$B \to b$$

4c. Remove long rules

$$S' \to AU|\boldsymbol{V}B|a|AS|SA$$
$$S \to AU|\boldsymbol{V}B|a|AS|SA$$
$$A \to b|AU|VB|a|AS|SA$$
$$U \to SA$$
$$\boldsymbol{V \to a}$$
$$B \to b$$

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$$\boldsymbol{S \to ASA|aB}$$
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to CNF.

$$\boldsymbol{S' \to AU|VB|a|AS|SA}$$
$$\boldsymbol{S \to AU|VB|a|AS|SA}$$
$$\boldsymbol{A \to b|AU|VB|a|AS|SA}$$
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- Any automata that recognizes **ALL** context free languages will need unbounded memory.

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- Any automata that recognizes **ALL** context free languages will need unbounded memory.

Intuition to build an Automata for CFL

- It should be some **Finite State Machine** that has access to a memory device with infinite memory, i.e.

Automata for CFL = FSM + Memory device

- **FSM may choose to ignore the memory device** completely in which case it behaves like a DFA/NFA.
- FSM makes use of the Memory device to recognize "non-Regular" CFLs.

E.g.: $\{0^n 1^n, n \in \mathbb{N}\}$

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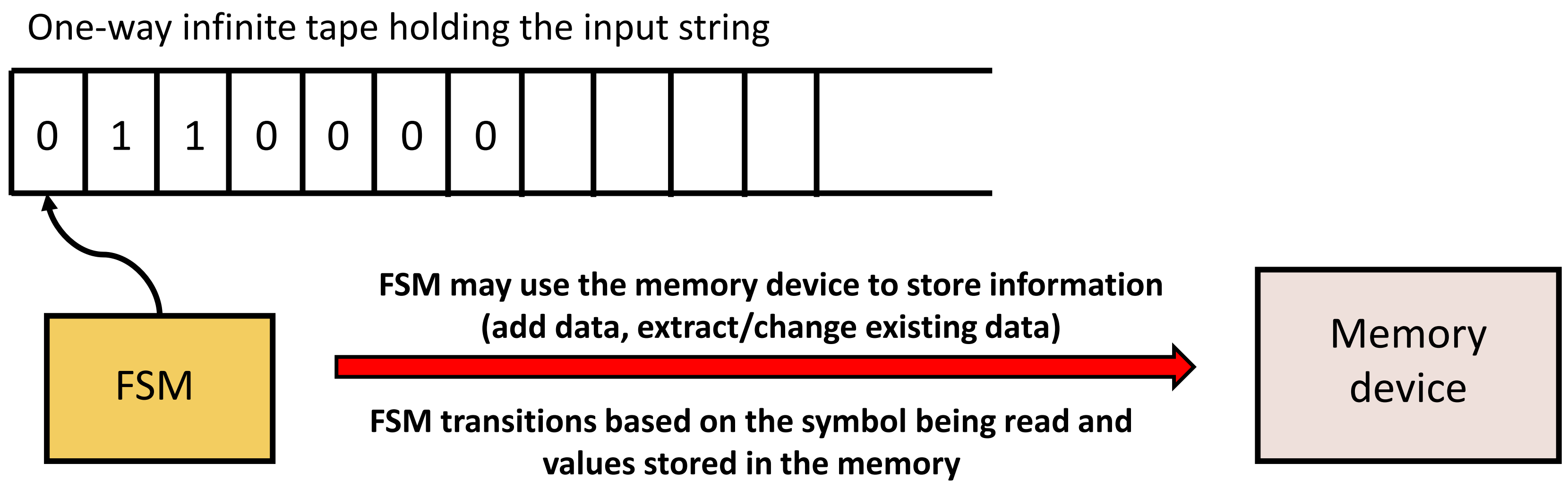

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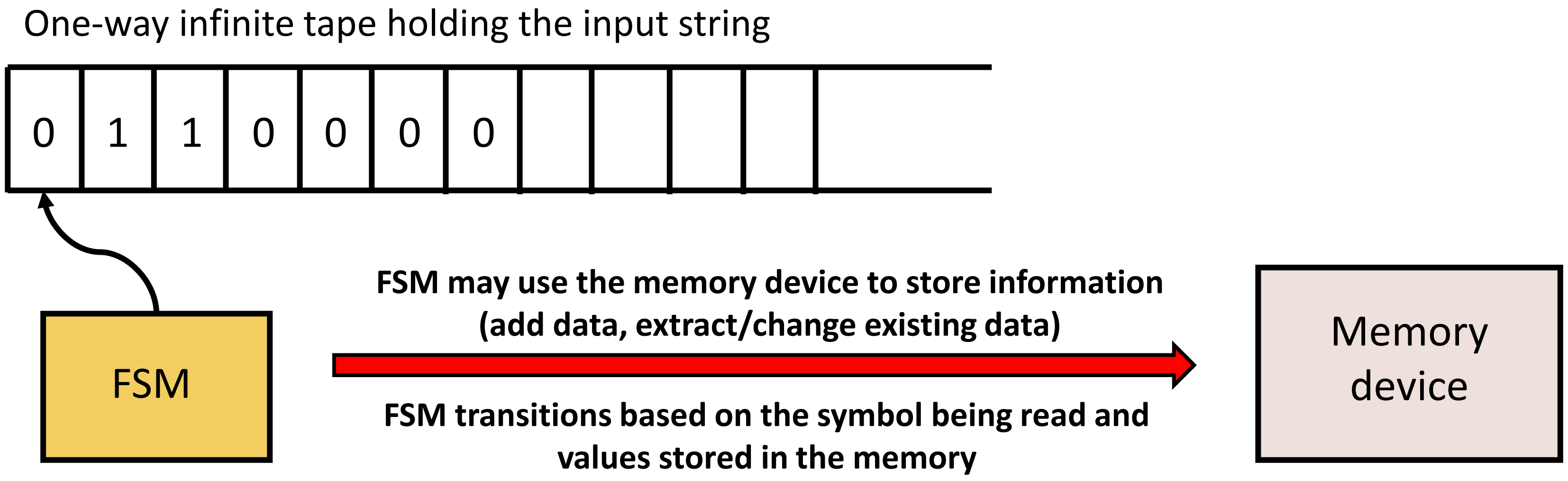


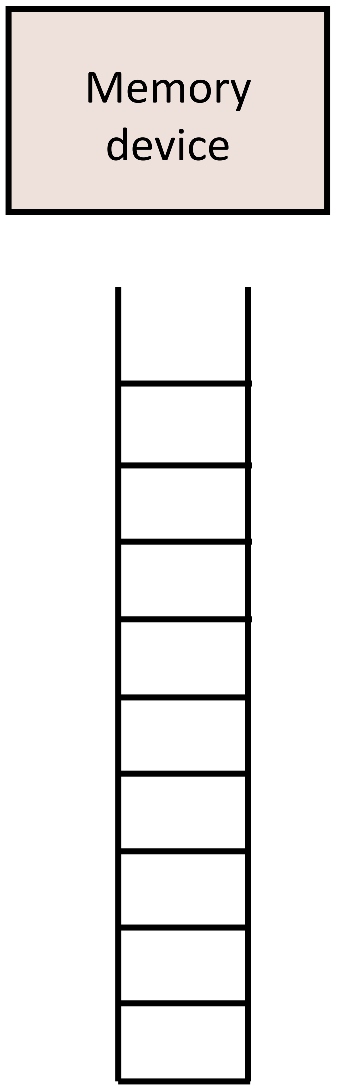
The memory device

- Simple memory device with unbounded memory.

Pushdown Automata

The memory device

- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, new elements can be added to the Stack (**PUSH**).
- At any stage, the element at the **top** of the STACK can be read by removing it from the stack (**POP**).



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- Simple memory device with unbounded memory.
- Consider a **STACK**
- At any stage, elements can be pushed or popped.

PUSH

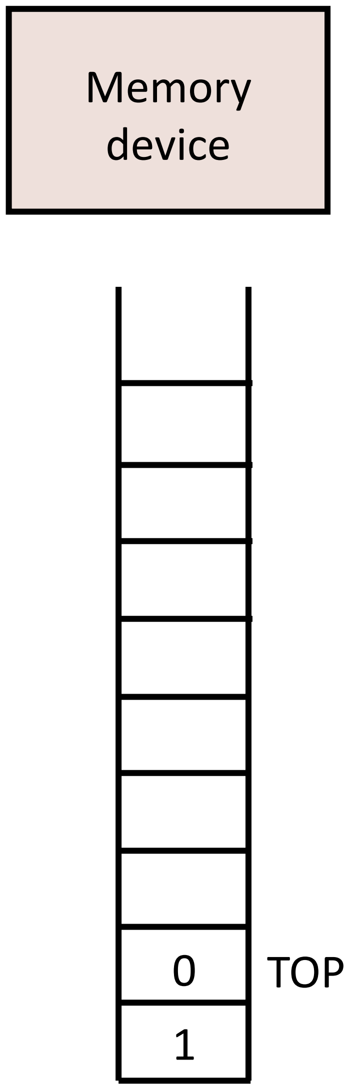
- New symbols can be **pushed** in to the STACK.

  E.g: PUSH 1

- The Top of the STACK now covers the old stack top, i.e.

  TOP = TOP + 1

- The size of the stack keeps growing.



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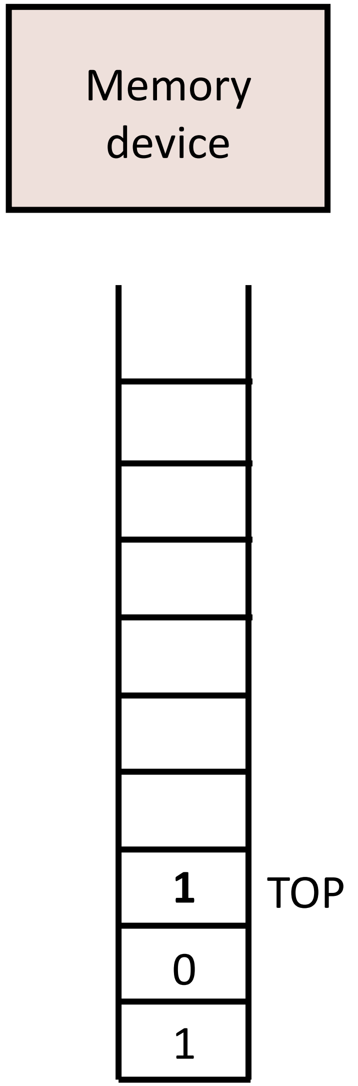

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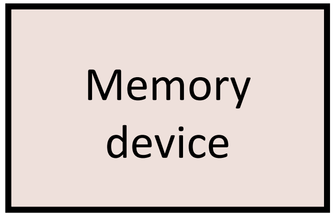

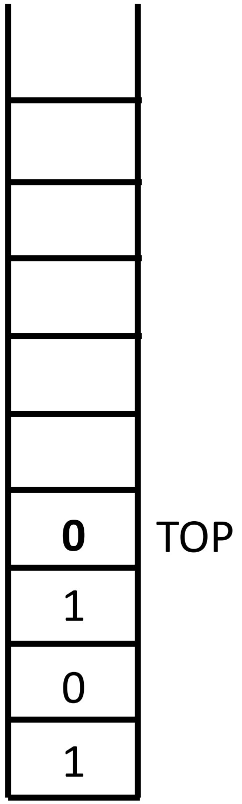

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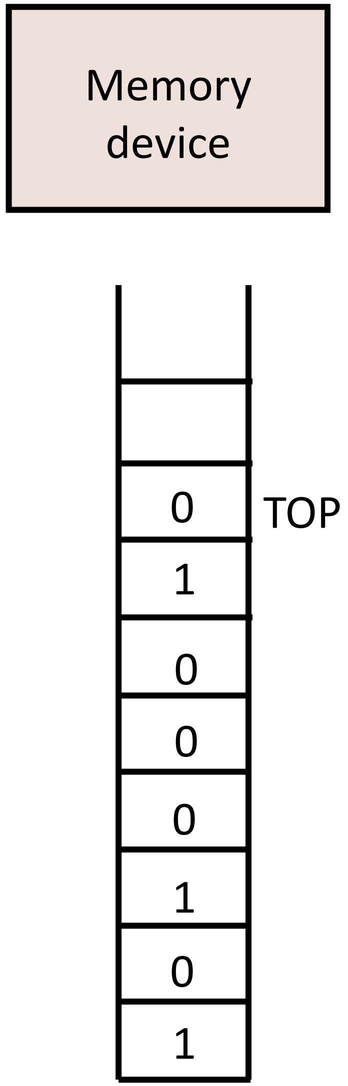

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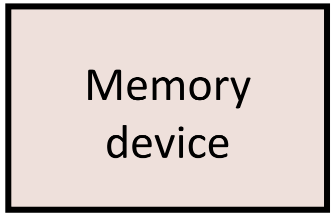


POP

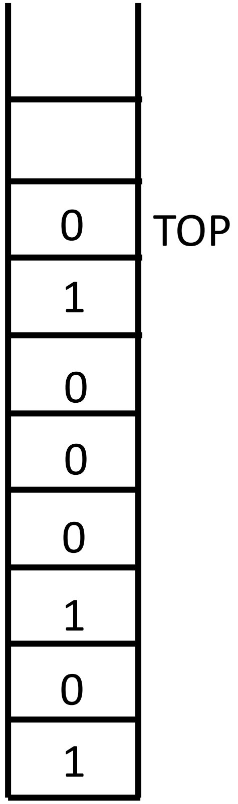
- The element from the TOP of the stack can be **popped** out

E.g.: **POP 0**

- The Top of the STACK moves to the element below.

TOP = TOP - 1

- Successive **POP** operations shrink the stack size. Elements can be popped until EMPTY.
- **Last In First Out (LIFO)**: The last element that was pushed is the first to be popped out



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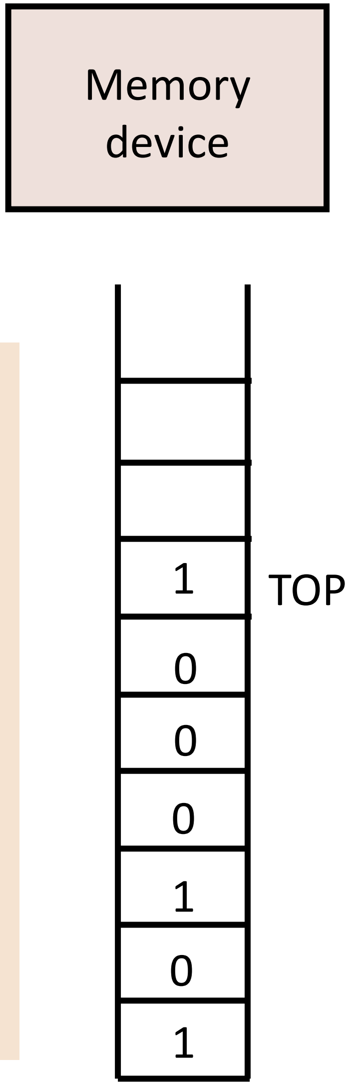

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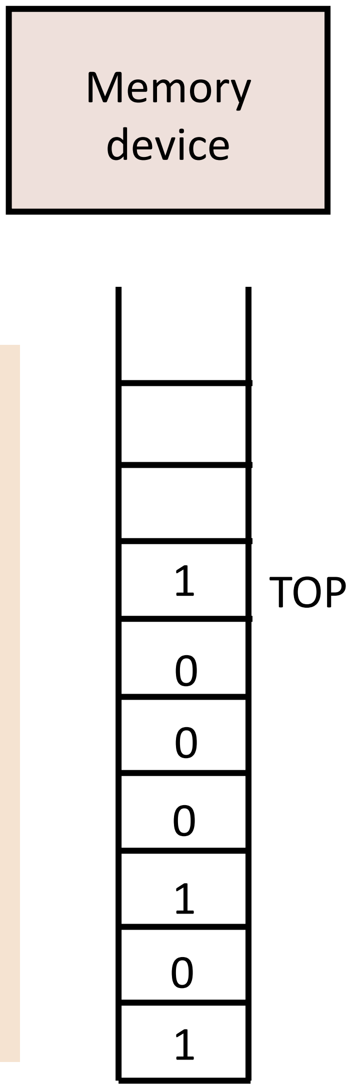

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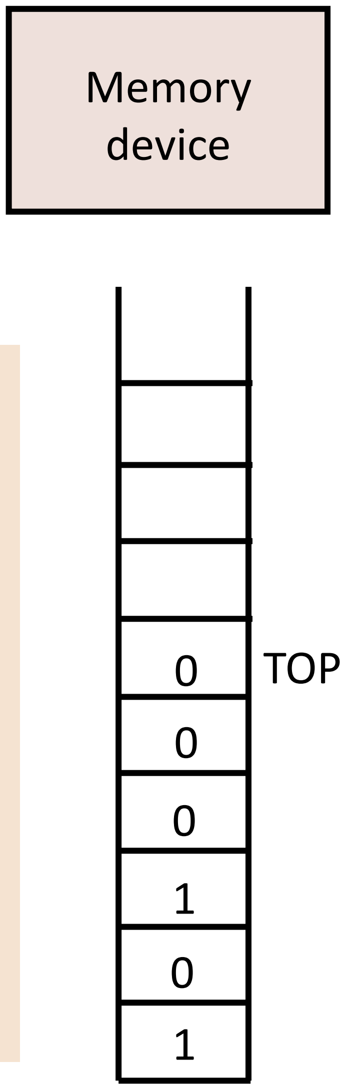

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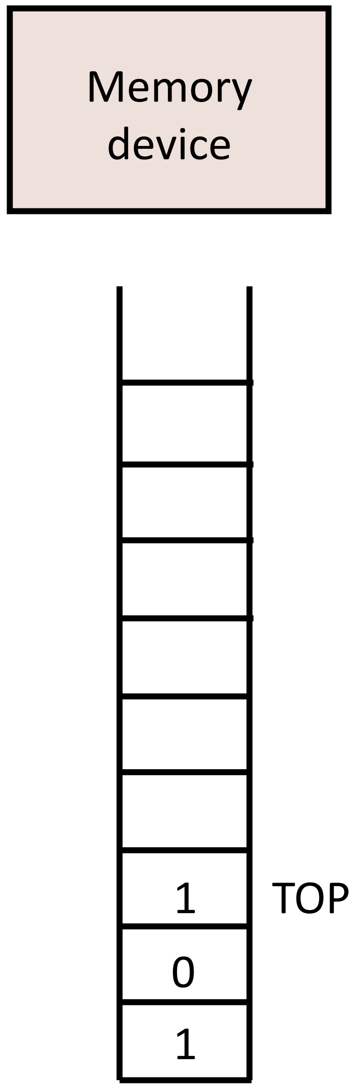
- Consider a **STACK**
- Last In First Out (**LIFO**)

POP

- The element from the TOP of the stack can be **popped** out.
- TOP = TOP - 1
- Elements can be popped until STACK is EMPTY.

- How would you know that the STACK is EMPTY?



Pushdown Automata

The memory device

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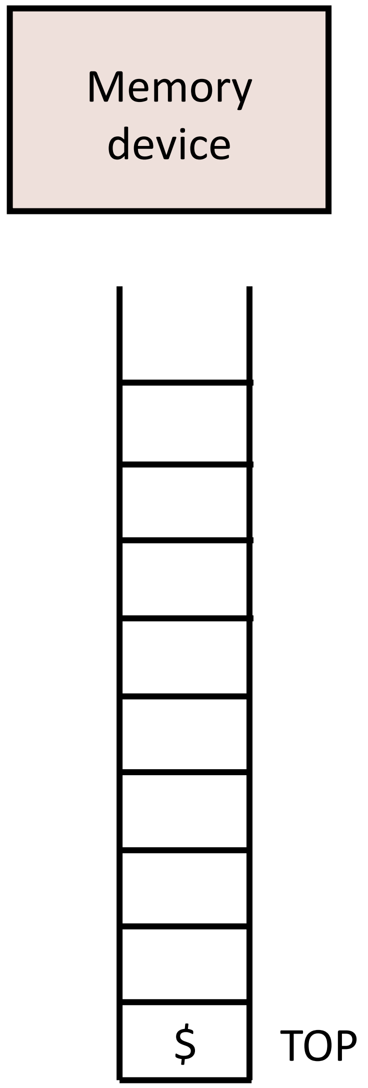
- How would you know that the STACK is EMPTY?

- There is generally some special symbol (say $) that demarcates the bottom of the STACK.

- This element is Pushed at the very beginning. Whenever the popped element = $, the STACK is EMPTY.



Pushdown Automata

Memory device of PDA: STACK

- STACK is a **LIFO** data structure of unbounded memory
- Only the TOP element can be read from the STACK.
- The bottom of the STACK contains a special symbol ($)
- Characterized by two operations:

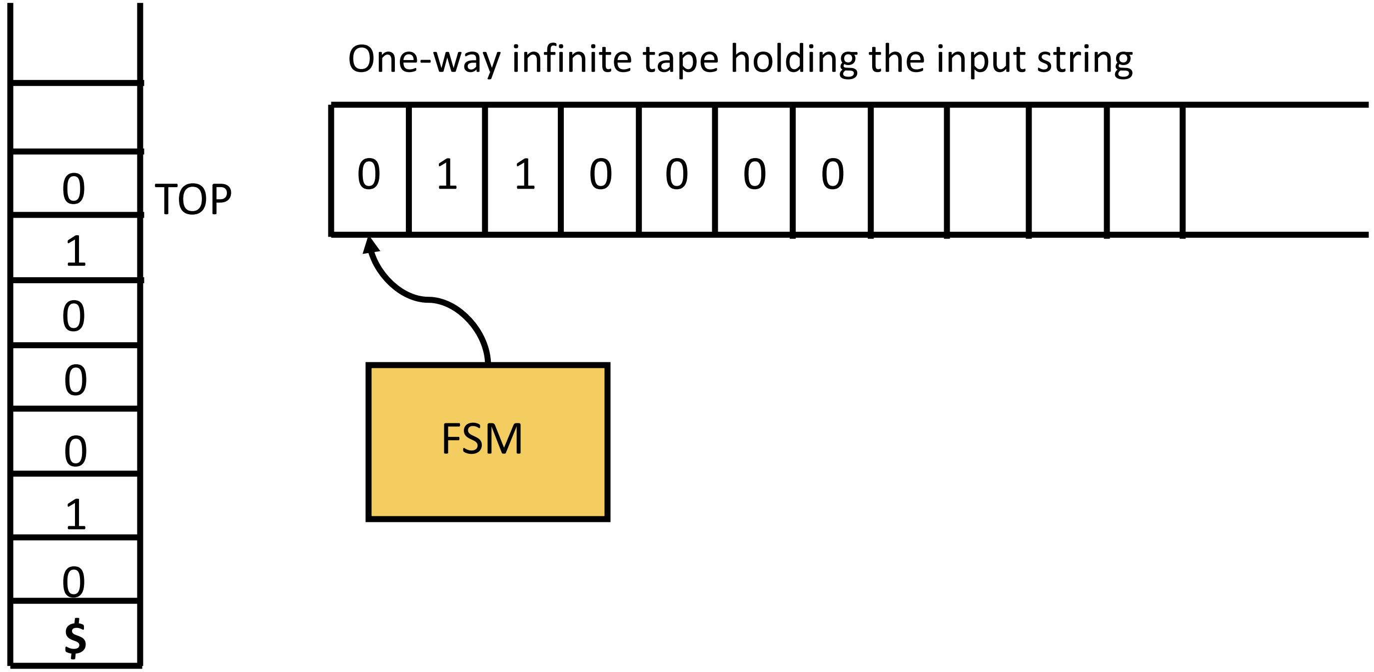
PUSH

- New symbols can be **pushed** in to the STACK.
- TOP = TOP + 1

POP

- The element from the TOP of the stack can be **popped** out.
- TOP = TOP - 1
- Elements can be popped until STACK is EMPTY.

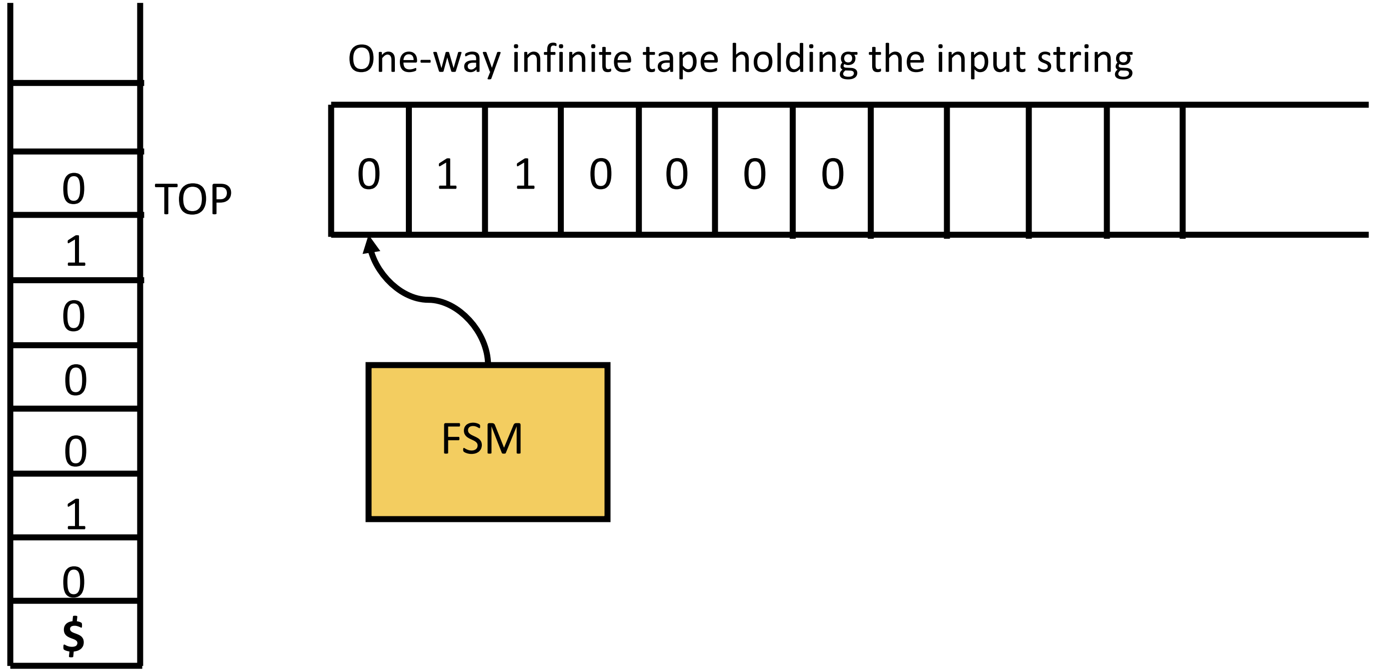
Pushdown Automata



- A Pushdown Automata (PDA) is a finite automaton that has access to a stack.

- The FSM:

Pushdown Automata



- A Pushdown Automata (PDA) is a finite automaton that has access to a stack.
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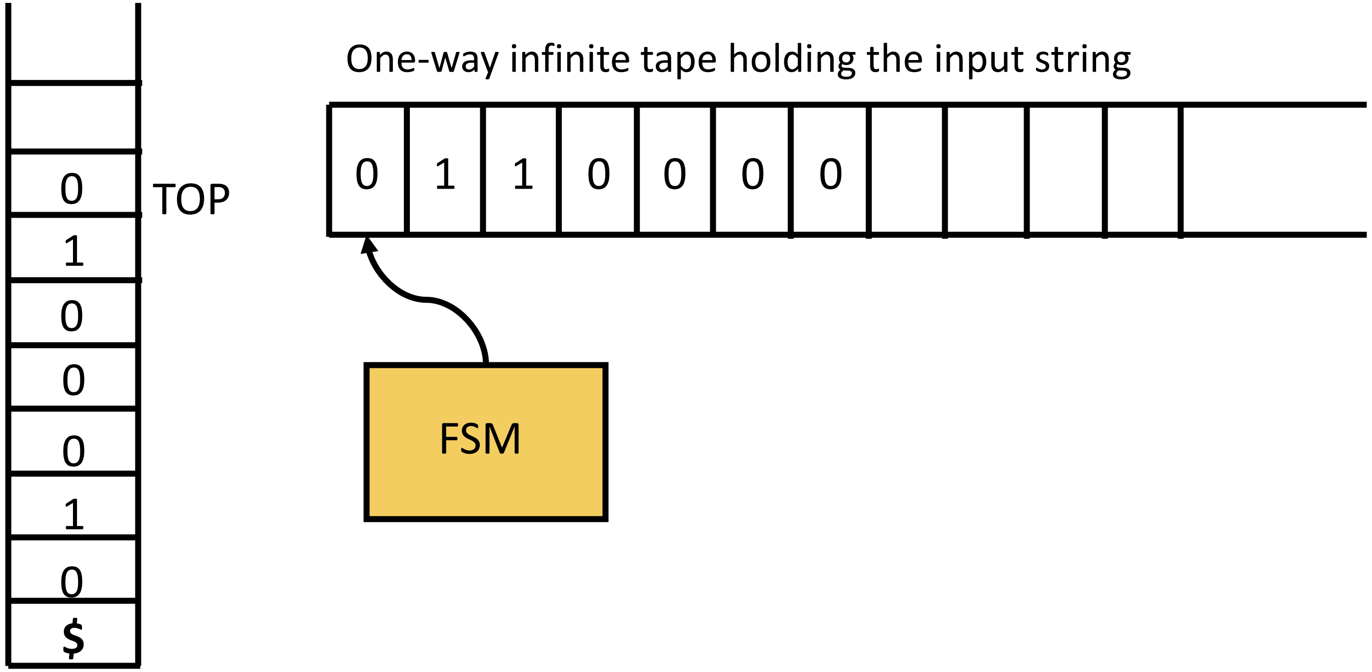
Pushdown Automata



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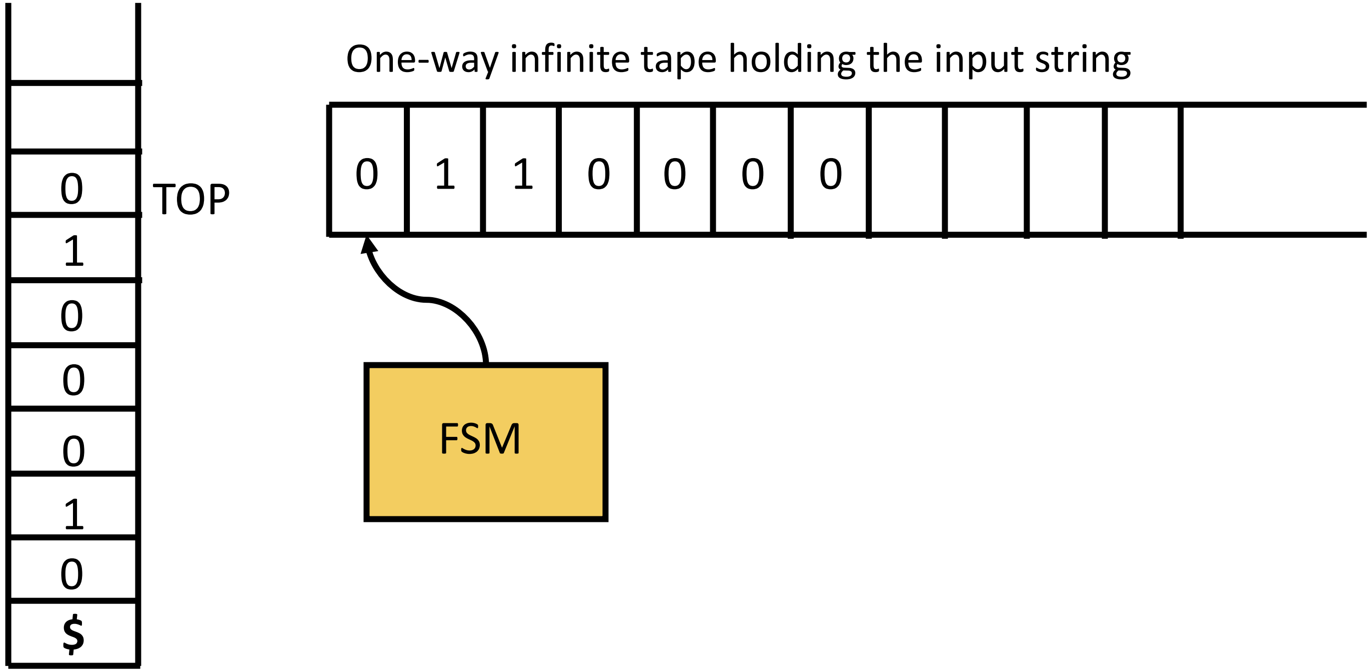
  - **How can we read the TOP? By popping**

Pushdown Automata



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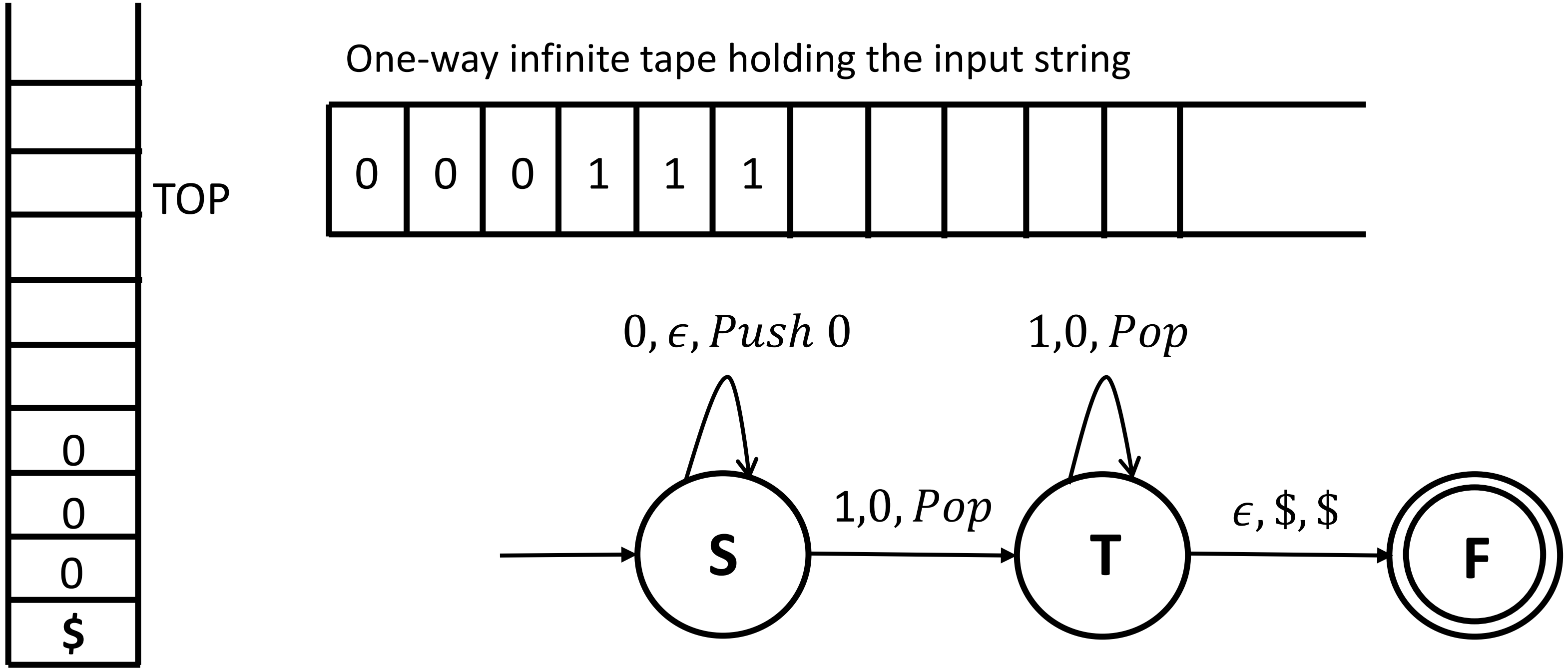
- The FSM:
  - Transitions based on the Input symbol and the element at the top of the stack (e.g.: If I/P symbol = 0 & TOP = 0, transition from i to j) – **Pop 0**

  - Pushes new elements into the Stack (e.g.: If I/P symbol = 0, PUSH 0, transition from i to j).

Pushdown Automata



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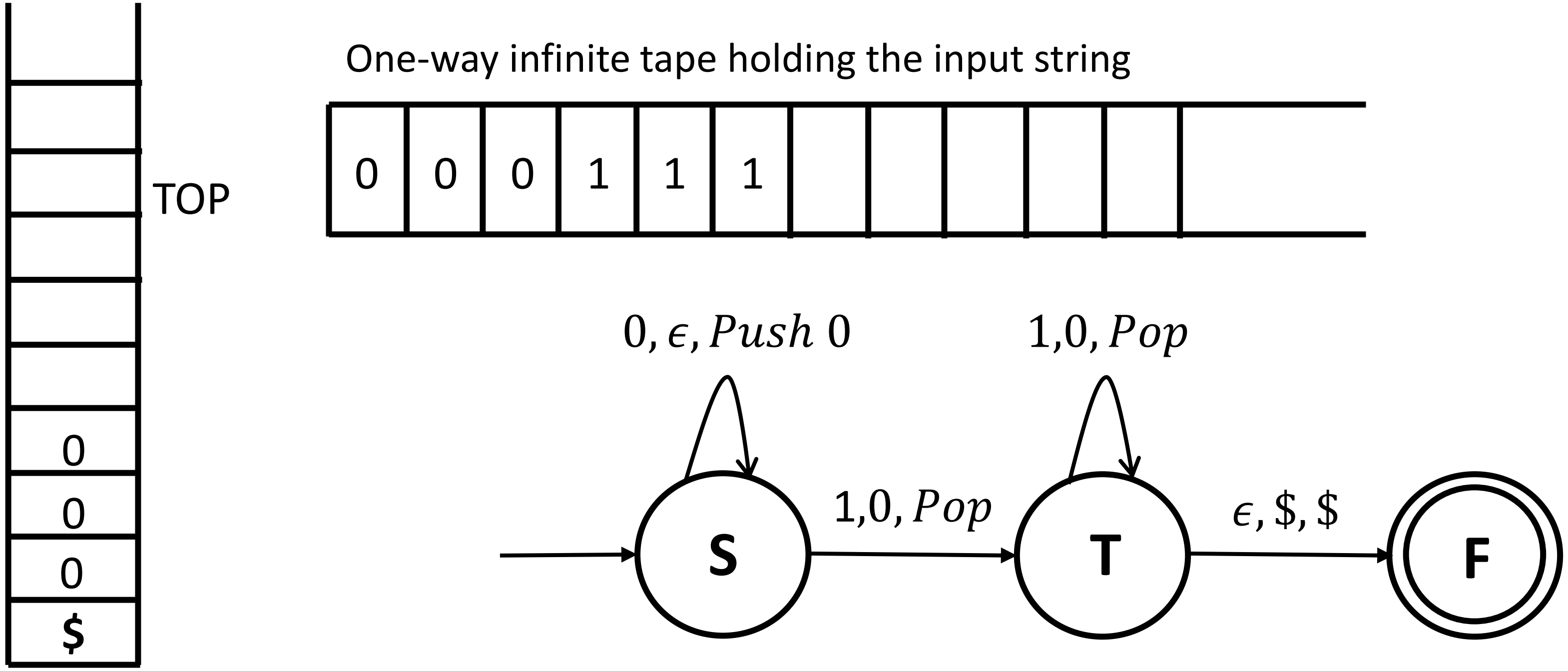
Pushdown Automata

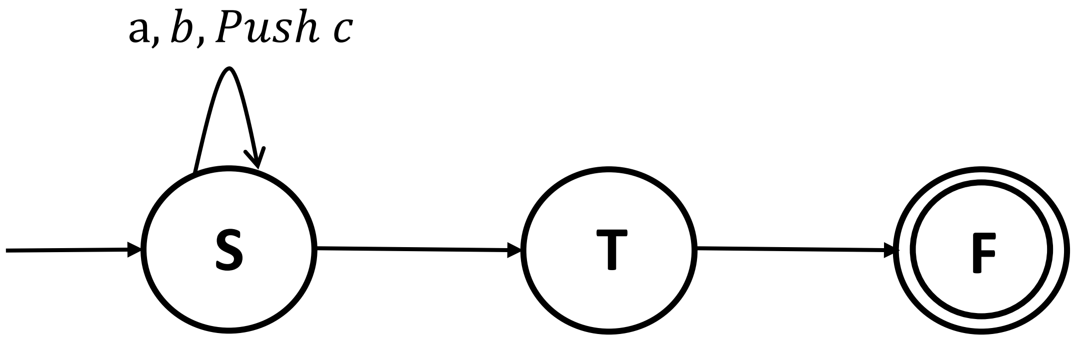


PDAs are **non-deterministic**.

- ϵ-transitions
- Multiple transitions/input symbol possible

Pushdown Automata

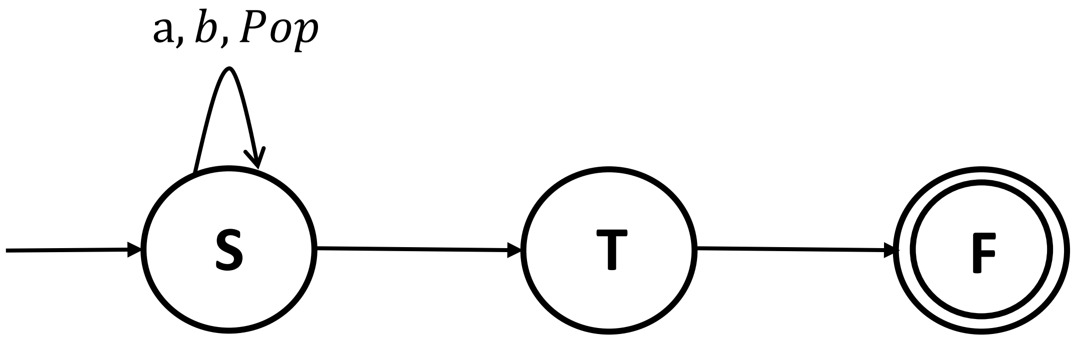
- How to represent a transition in a PDA?



If input symbol = a, Stack top = b (if b is popped) and Push c onto the Stack

Pushdown Automata

- How to represent a transition in a PDA?

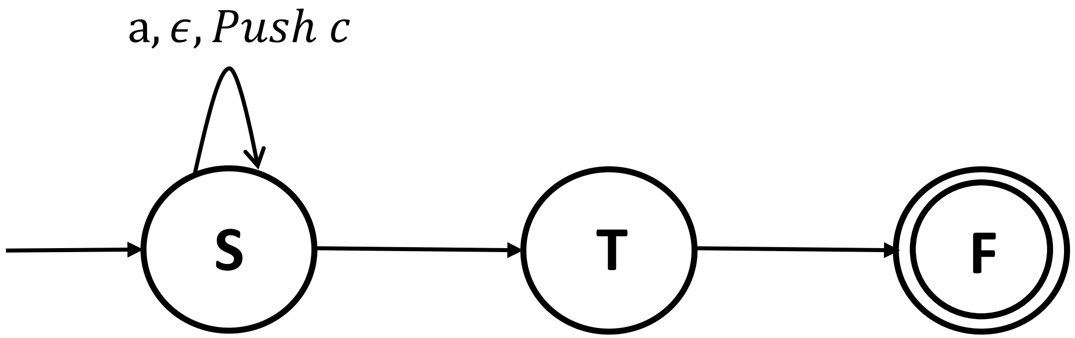


If input symbol = a, and b is popped, remain in S.

(If the symbol read is a and the stack TOP $= b$, then remain in S)

Pushdown Automata

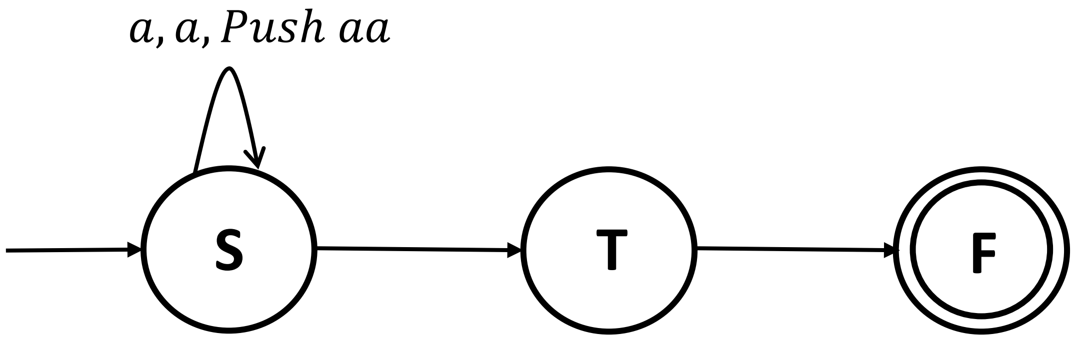
- How to represent a transition in a PDA?



If input symbol = a, then Push c

Pushdown Automata

- How to represent a transition in a PDA?



If input symbol = a, and a is popped, then Push aa and remain in S.

So effectively, the PDA pushes a onto the stack if it reads a on the input tape and the stack top = a.

This is a “shorthand” for describing two operations:

Pushdown Automata

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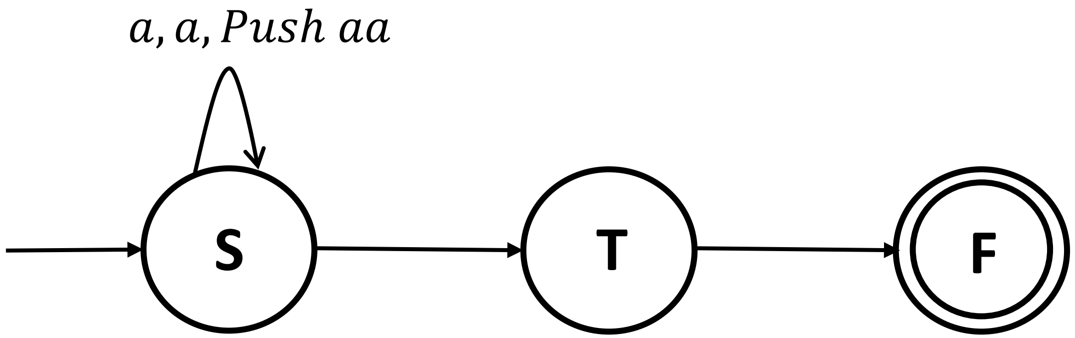

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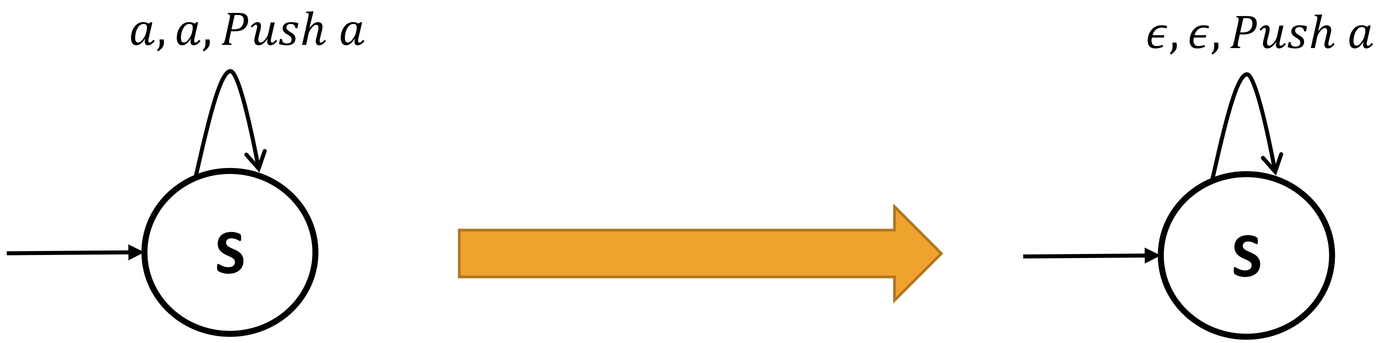
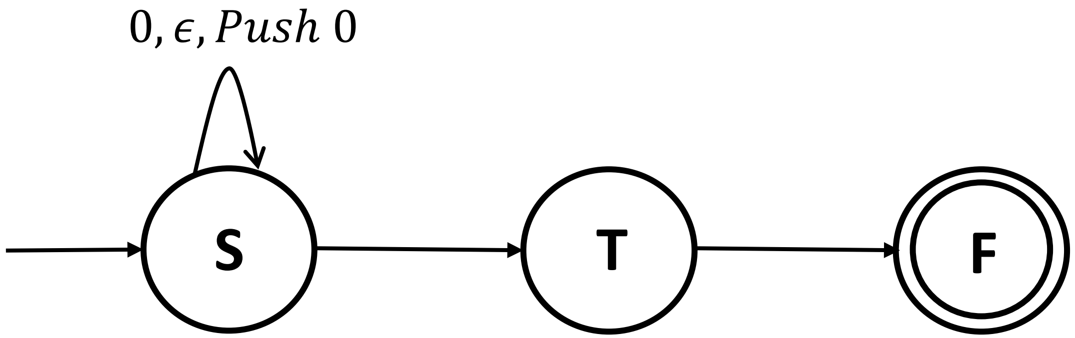


Pushdown Automata

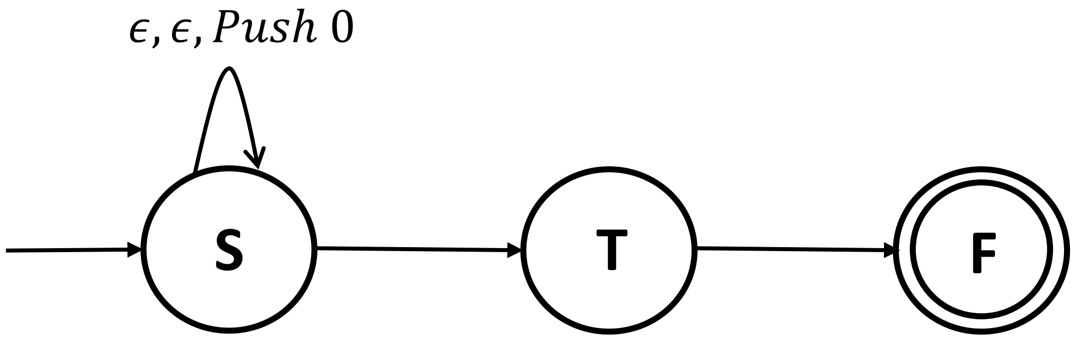
- How to represent a transition in a PDA?



If input symbol = 0, Push 0 onto the Stack irrespective of the element at the top of the stack

Pushdown Automata

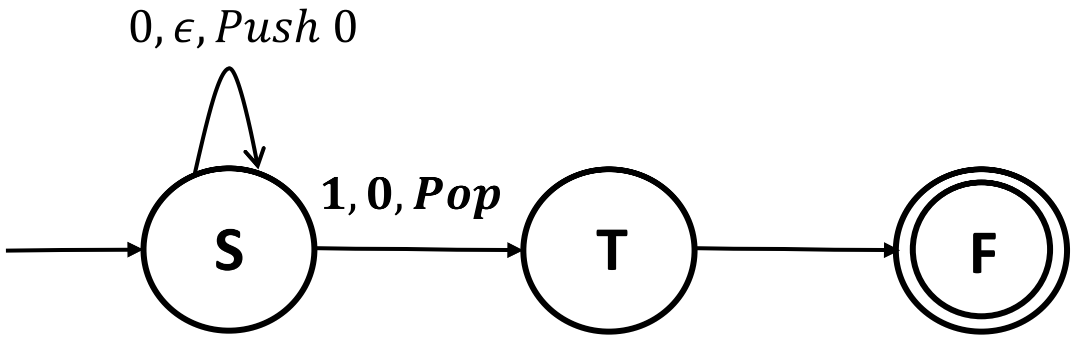
- How to represent a transition in a PDA?



Without reading the input symbol and the Stack top, Push 0 onto the Stack

Pushdown Automata

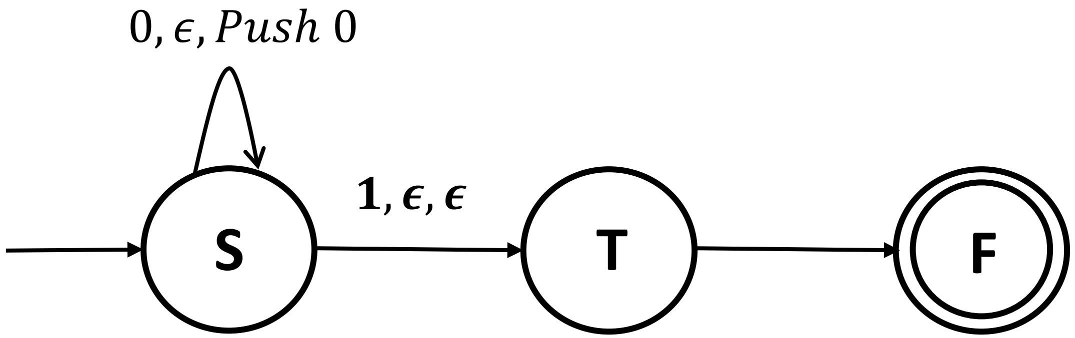
- How to represent a transition in a PDA?



If the input symbol is 1, and the element at the top of the stack is 0 **(Pop 0)**, then transition from S to T

Pushdown Automata

- How to represent a transition in a PDA?

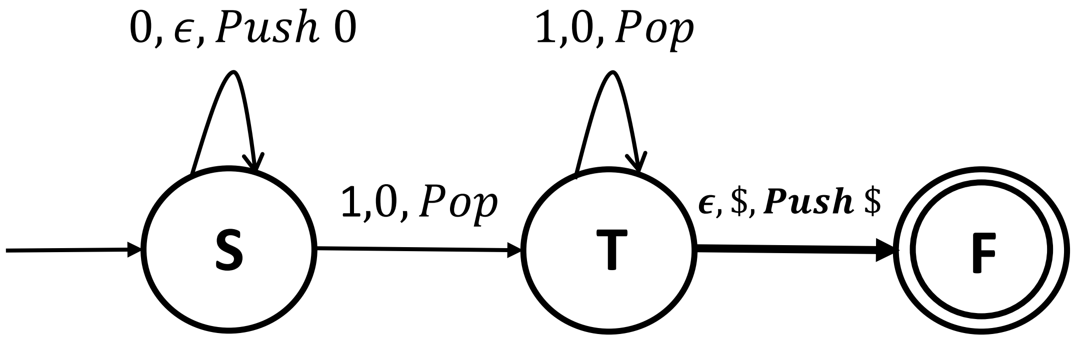


If the input symbol is 1, transition to T by ignoring the stack completely.

If this happens at every step of the execution of the PDA, then it is as powerful as an NFA.

Pushdown Automata

- How to represent a transition in a PDA?



If the Stack is empty, i.e. TOP = \$, transition to F from T, without reading the input

Pushdown Automata

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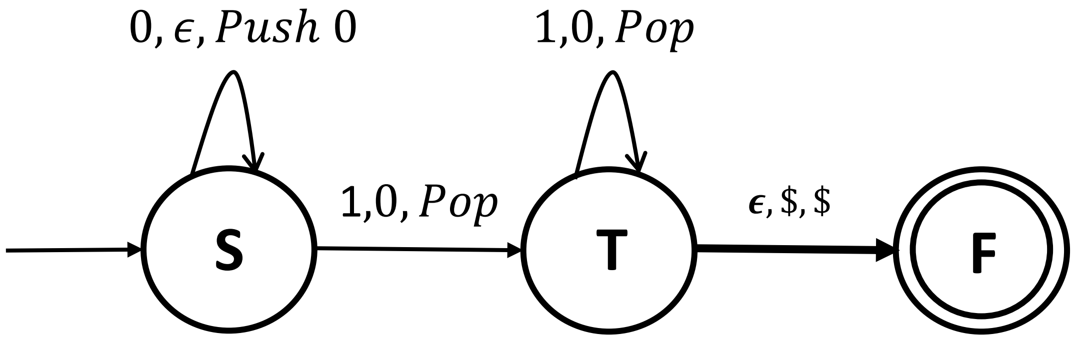

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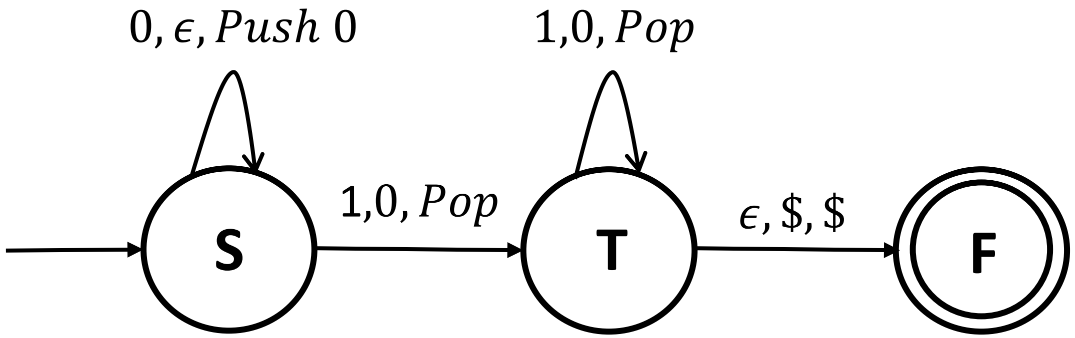
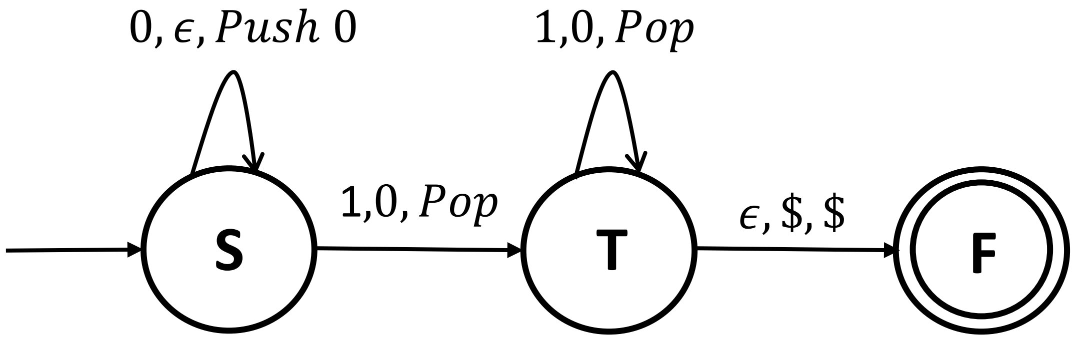


What is the language accepted by this PDA?

Pushdown Automata

- How to represent a transition in a PDA?

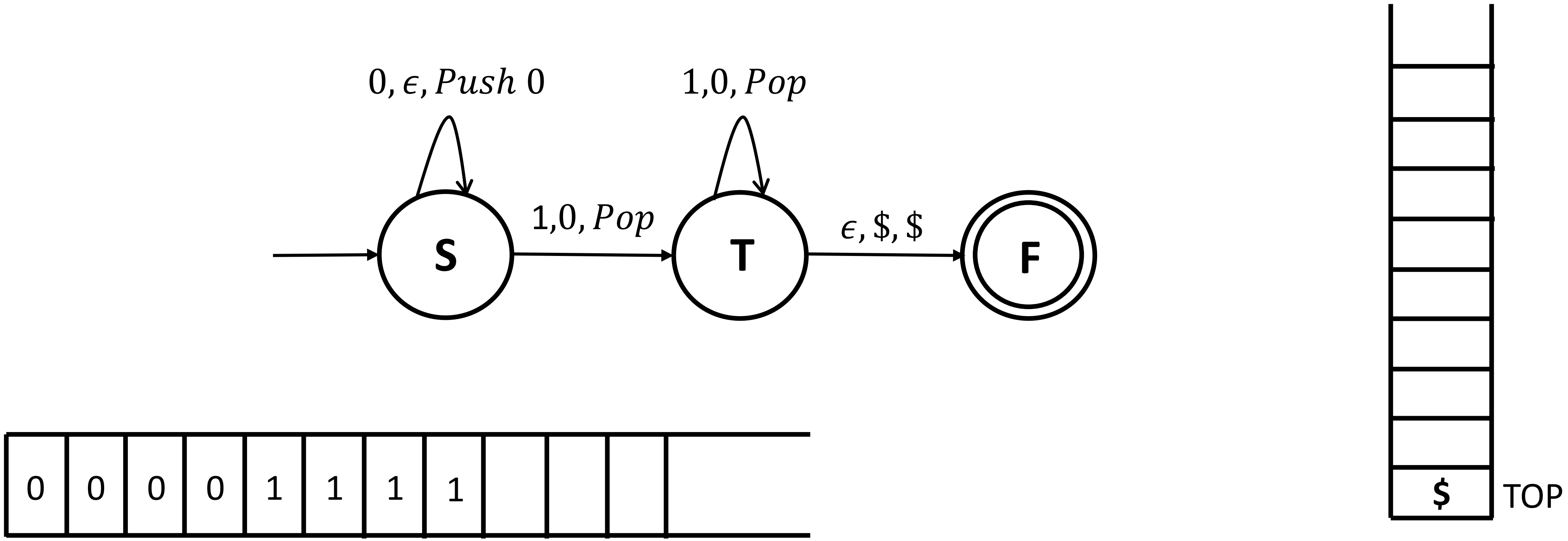


What is the language recognized by this PDA?

Verify that it is $\boldsymbol{L = \{0^n 1^n, n \geq 1\}}$

Pushdown Automata

What is the language recognized by this PDA?



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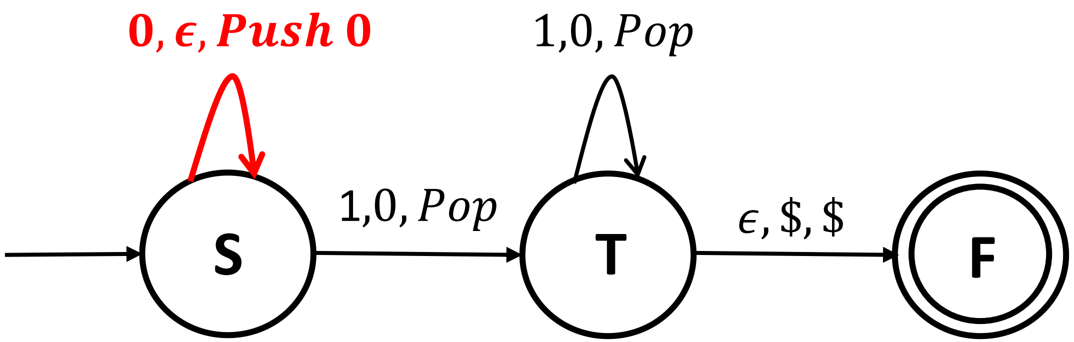

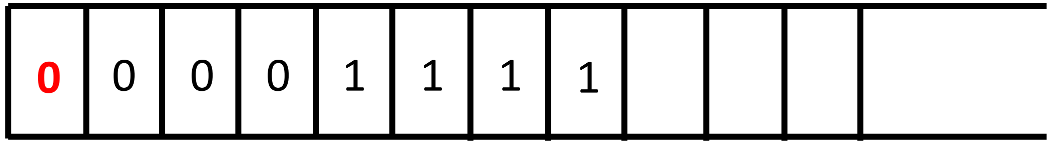

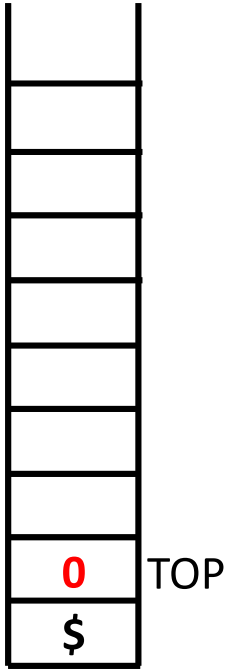

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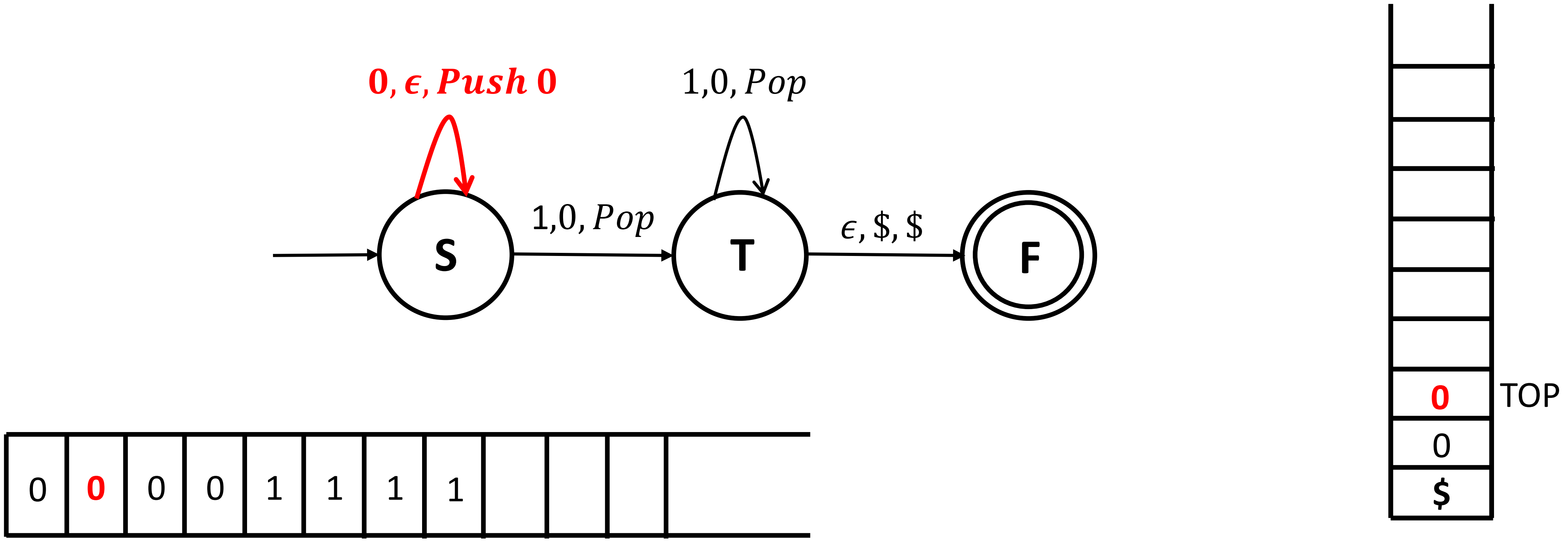

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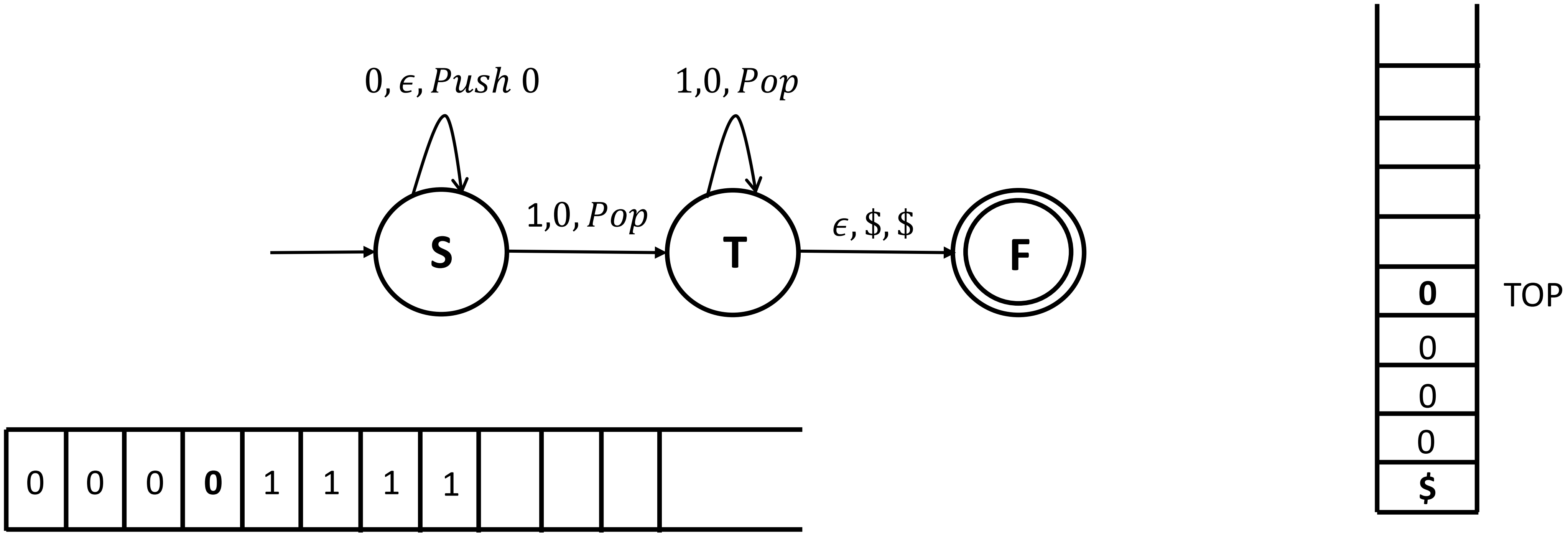

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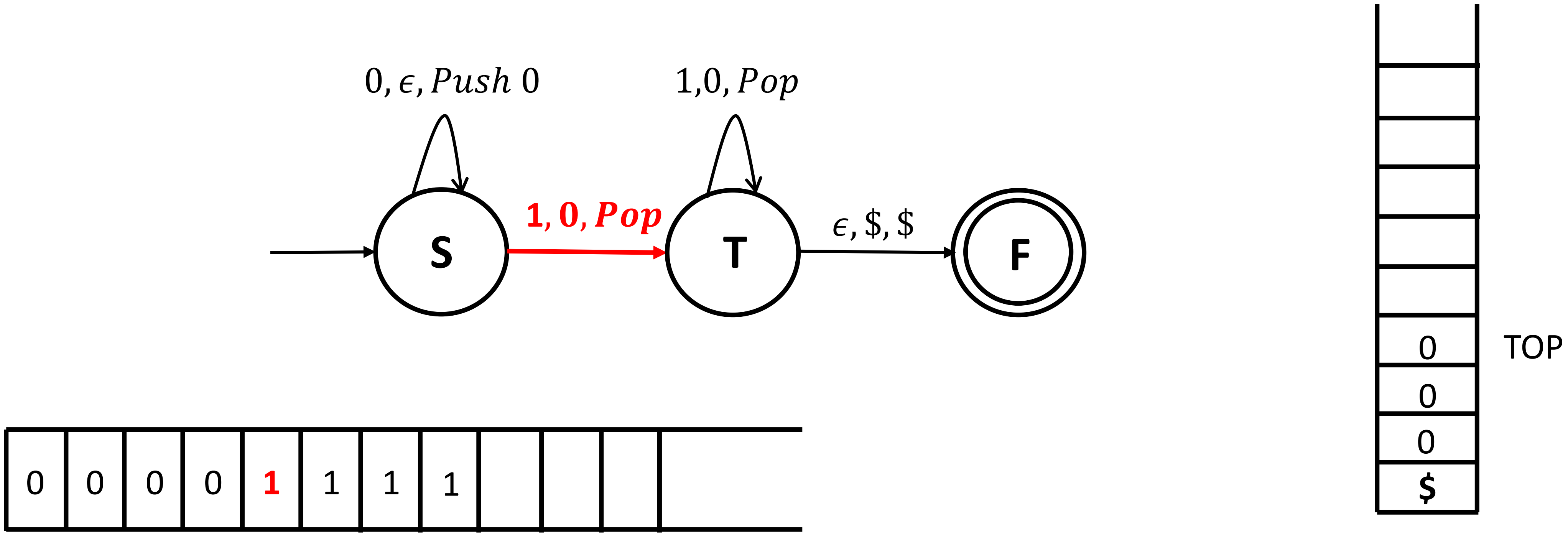

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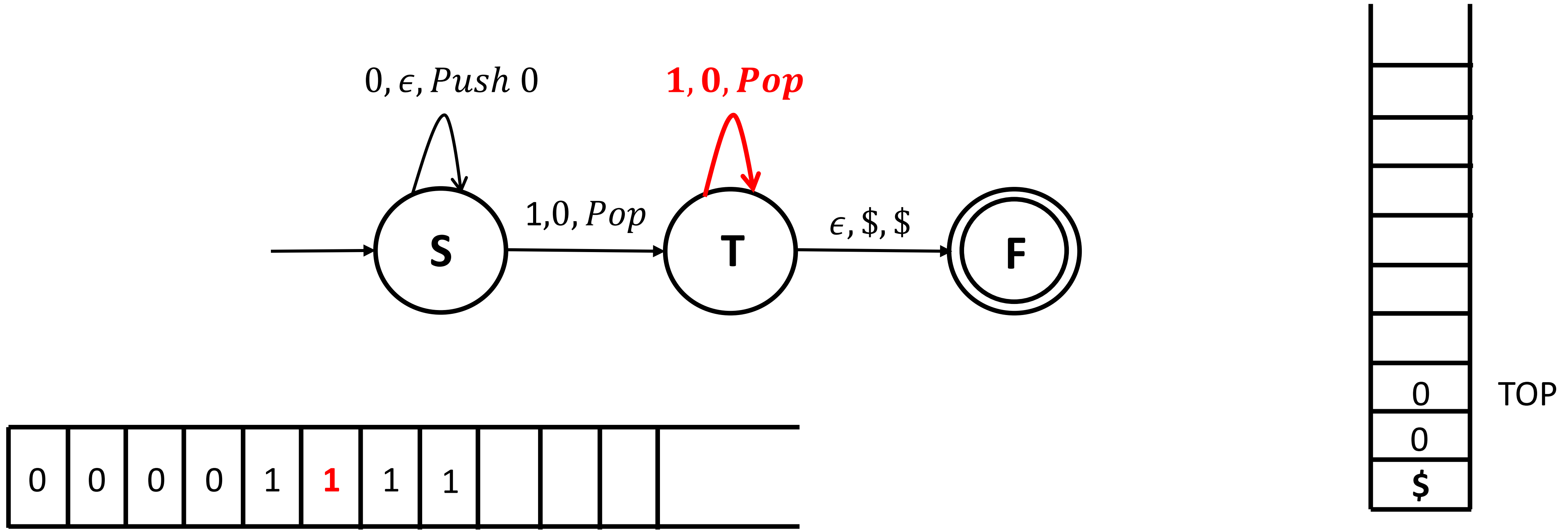

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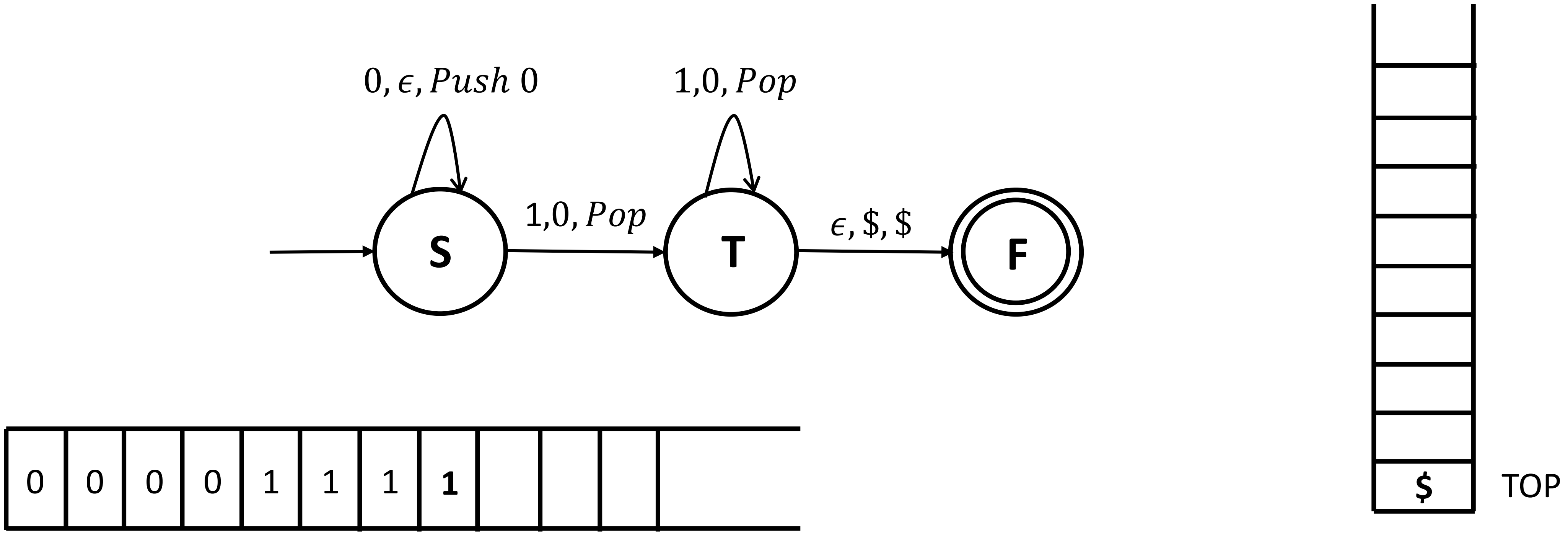

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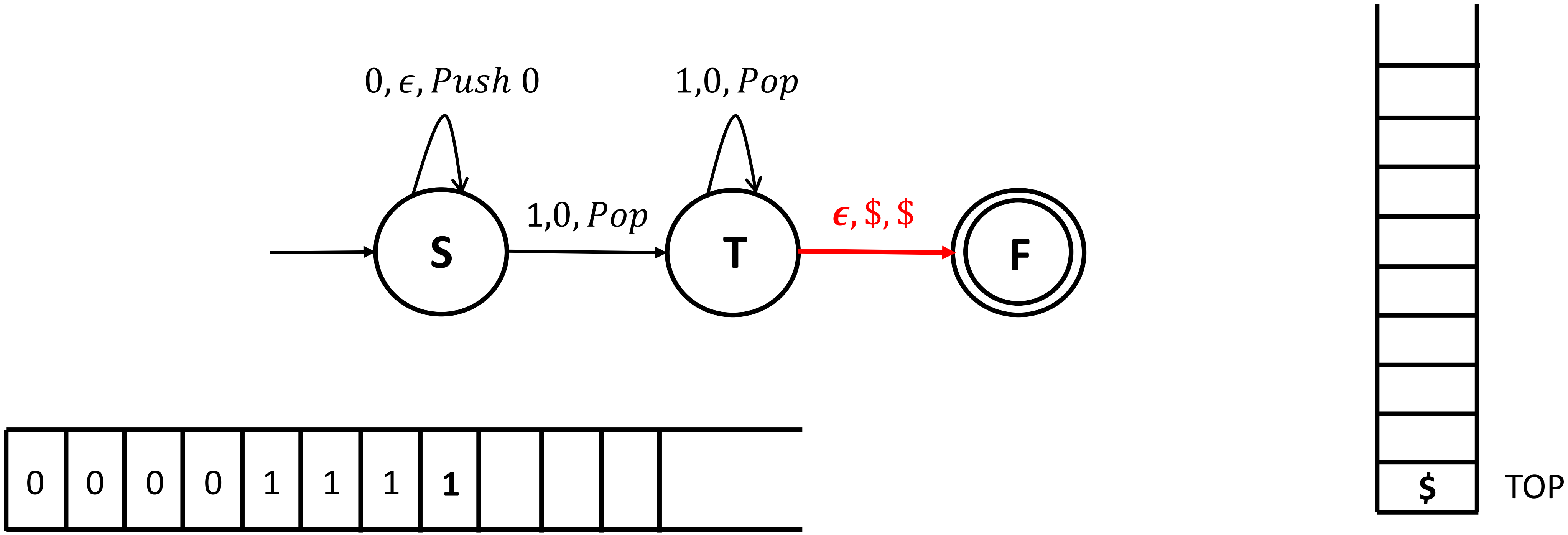

The language recognized by the PDA: $\boldsymbol{L = \{0^n1^n, n \geq 1\}}$

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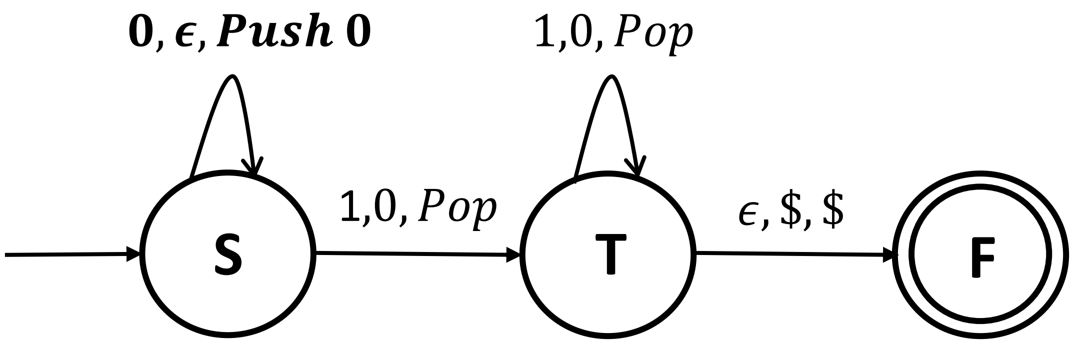

In some references (such as Sipser):

- The transitions of the PDA are labelled as "$\boldsymbol{a}, \boldsymbol{b} \rightarrow \boldsymbol{c}$", implying: If the input symbol read is $\boldsymbol{a}$, and the element at the top of the stack is $\boldsymbol{b}$ (pop $\boldsymbol{b}$), then push $\boldsymbol{c}$ on to the Stack.

Pushdown Automata

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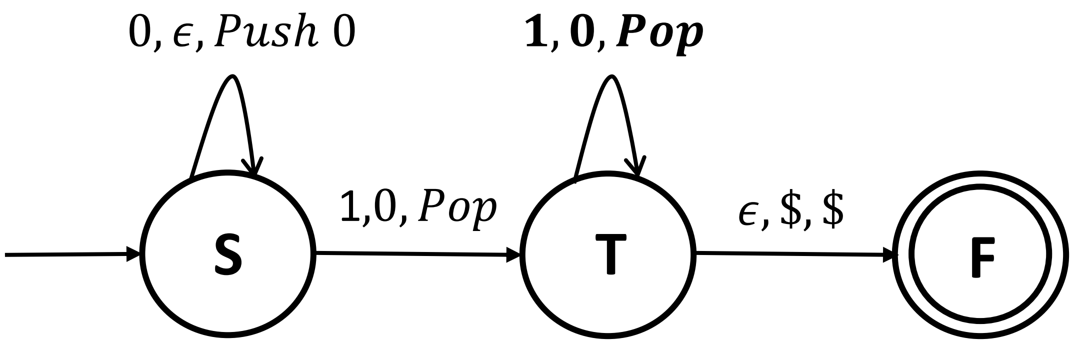

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- The label "$\boldsymbol{a, b \rightarrow \epsilon}$" implies that if the input symbol is $\boldsymbol{a}$ then **pop b**.

Pushdown Automata

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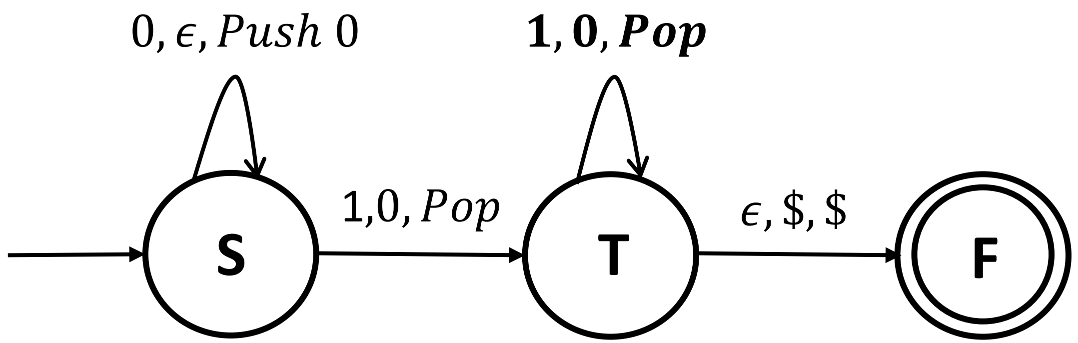

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- The label "$\boldsymbol{a, b \rightarrow \epsilon}$" implies that if the input symbol is $\boldsymbol{a}$ and $\boldsymbol{b}$ is **popped**.

- The symbol signifying the bottom of the Stack $ is pushed at the very beginning.

Pushdown Automata

Formally, a PDA M is a 6-tuple $(\mathrm{Q}, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q is a finite set called the ***states.***
- Σ is the set of input ***alphabets.***
- Γ is the set of ***Stack alphabets***
- $\delta: Q \times \Sigma_\epsilon \times \Gamma_\epsilon \mapsto \mathcal{P}(Q \times \Gamma_\epsilon)$ is the ***transition function*** [$\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$]
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A PDA accepts a string $w \in L$, if there exists a run such that

- It **reaches a final state** when the entire string is read.

OR

- The **stack is empty** when the entire string is read.

Pushdown Automata

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These two notions of acceptance are equivalent

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Transition function:

- $\boldsymbol{\delta(q_i, a, b) = (q_j, c)}$: If the input symbol read is a and b is popped, then push c onto the stack and transition from q_i to q_j

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- $\boldsymbol{\delta(q_i, a, b) = (q_j, \epsilon)}$: If the input symbol read is a, and the stack top = b (b is popped) then transition from q_i to q_j

- $\boldsymbol{\delta(q_i, \epsilon, \$) = (q_j, \$)}$:

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Transition function:

- $\boldsymbol{\delta(q_i, a, b) = (q_j, c)}$: If the input symbol read is a and b is popped, then push c onto the stack and transition from q_i to q_j

- $\boldsymbol{\delta(q_i, a, \epsilon) = (q_j, c)}$: If the input symbol read is a, then push c onto the stack and transition from q_i to q_j

- $\boldsymbol{\delta(q_i, a, b) = (q_j, \epsilon)}$: If the input symbol read is a, and the stack top = b, then pop b and transition from q_i to q_j

- $\boldsymbol{\delta(q_i, \epsilon, \$) = (q_j, \$)}$: Transition from q_i to q_j if the stack is empty.

Pushdown Automata

Formally, a PDA M is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q is a finite set called the ***states.***
- Σ is the set of input ***alphabets.***
- Γ is the set of ***Stack alphabets***
- $\delta: Q \times \Sigma_\epsilon \times \Gamma_\epsilon \mapsto \mathcal{P}(Q \times \Gamma_\epsilon)$ is the ***transition function*** [$\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$]
- $q_0 \in Q$ is the ***start state***.
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Transition function:

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- If the input symbol read is a and a is popped, then Push a and remain at q_i : ?

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Transition function:

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- If the input symbol read is a and a is popped, then Push a and remain at q_i : $\boldsymbol{\delta(q_i, a, a) = (q_i, a)}$

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[$\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$]

- The Language of the PDA P is the set of strings the PDA accepts, i.e.

$$L = \{w | P \text{ accepts } w\}$$

There exists an accepting run for $\boldsymbol{w}$ on $\boldsymbol{P}$

- If $\mathcal{L}(P) = L$, then the PDA P recognizes L

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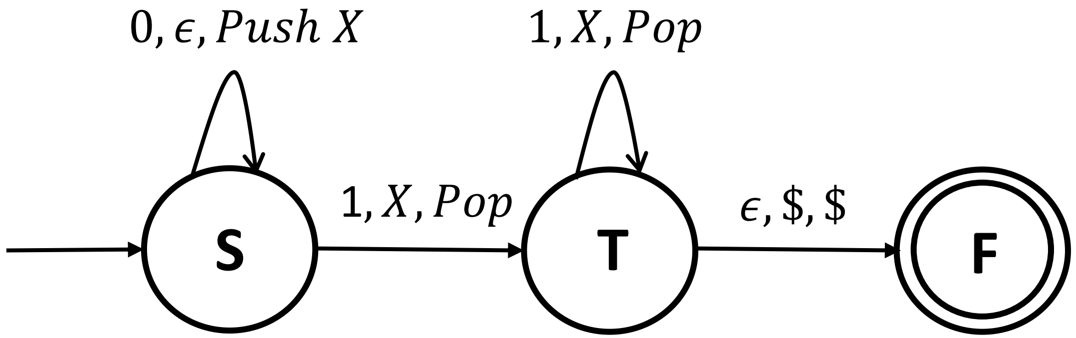
- If $\mathcal{L}(P) = L$, then the PDA P recognizes L
- Stack alphabet **can be different** from the input alphabet



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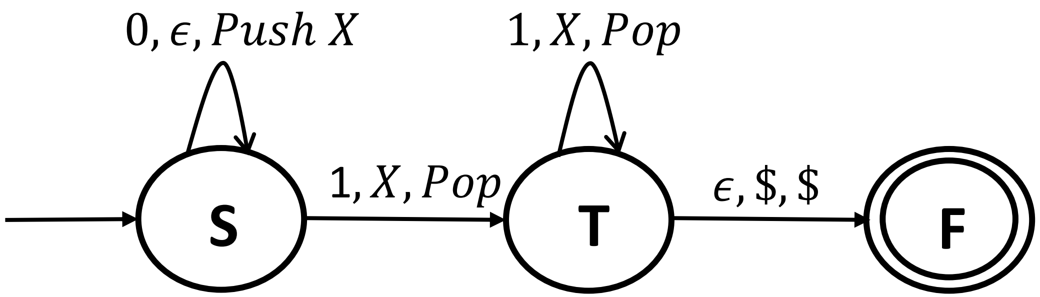

$$\delta(S, 0, \epsilon) = (S, X)$$

Pushdown Automata

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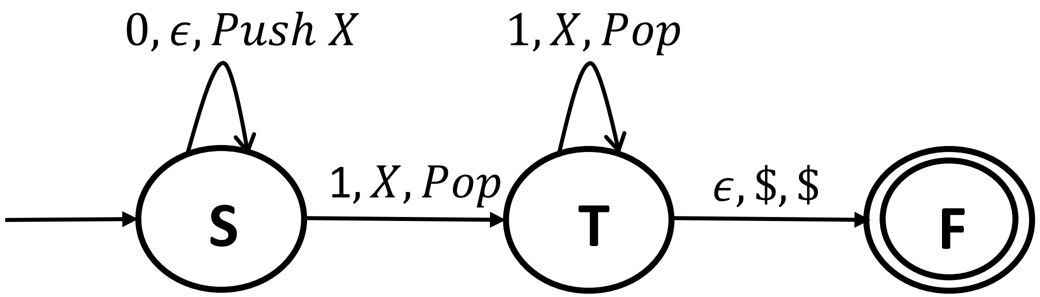
- Stack alphabet **can be different** from the input alphabet



$$\delta(S, 0, \epsilon) = (S, X)$$
$$\delta(S, 1, X) = (T, \epsilon)$$

Pushdown Automata

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- Σ is the set of input ***alphabets.***
- Γ is the set of ***Stack alphabets***
- $\delta: Q \times \Sigma_\epsilon \times \Gamma_\epsilon \mapsto \mathcal{P}(Q \times \Gamma_\epsilon)$ is the ***transition function*** $[\ \Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}\ ]$
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- The Language of the PDA P is the set of strings the PDA accepts, i.e.

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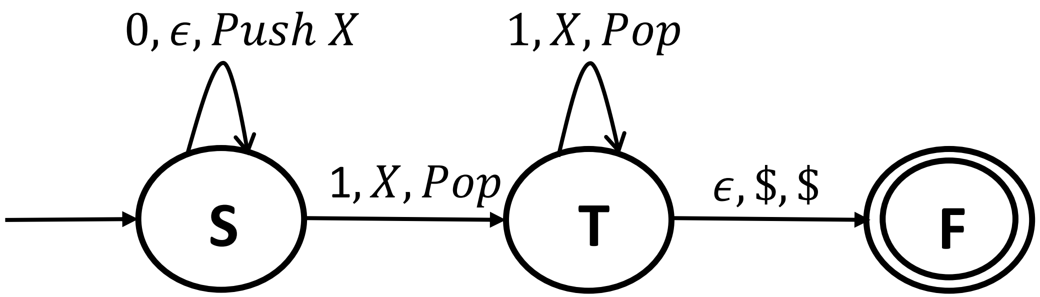
- If $\mathcal{L}(P) = L$, then the PDA P recognizes L

- Stack alphabet **can be different** from the input alphabet



$$\delta(S, 0, \epsilon) = (S, X)$$
$$\delta(S, 1, X) = (T, \epsilon)$$
$$\delta(T, 1, X) = (T, \epsilon)$$
$$\delta(T, \epsilon, \$) = (F, \$)$$

Thank You!