CS 302.1 - Automata Theory

Lecture 12

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Quick Recap

Recursive Language/Turing Decidable/Decidable: A language L is called Recursive or Turing decidable or Decidable if there exists a Total Turing Machine M and

$$\left.\begin{array}{l}\forall \omega \in L, M(\omega) \text{ accepts} \\ \forall \omega \notin L, M(\omega) \text{ rejects}\end{array}\right\} \text{Halts on all inputs}$$

The Church Turing thesis: An algorithm can be written for a problem if and only if it is decidable, i.e. there exists a Total Turing machine that solves the problem. **Total TM** $\Leftrightarrow$ **Algorithms!**

Recursively Enumerable Language/Turing Recognizable (RE): A language L is called Recursively Enumerable (RE) or Turing Recognizable if

$\forall \omega \in L, M(\omega)$ **accepts**

$\forall \omega \notin L, M(\omega)$ **doesn't accept** (rejects or runs infinitely)

Co-Recursively Enumerable Language/co-Turing Recognizable (Co-RE/$\overline{RE}$/nRE): A language L is **Co-Recursively Enumerable (co-RE/$\overline{RE}$) or Co-Turing Recognizable** if

$\forall \omega \in L, M(\omega)$ **doesn't reject** (accepts or loops)

$\forall \omega \notin L, M(\omega)$ **rejects**

Quick Recap

There exists a one-one mapping (bijective relationship) between the set of finite length binary strings and TMs.

Universal Turing Machine: A Universal Turing Machine, denoted as $\boldsymbol{U_{TM}}$ accepts as input (i) the encoding of a Turing Machine M and (ii) an input string w and **simulates $\boldsymbol{M}$ running on $\boldsymbol{w}$**, i.e.

$$\boldsymbol{U_{TM}(\langle M, w\rangle)} = \begin{cases} \textbf{ACCEPTS, if } \boldsymbol{M(w)} \textbf{ accepts} \\ \textbf{REJECTS, if } \boldsymbol{M(w)} \textbf{ rejects} \\ \textbf{LOOPS INFINITELY, if } \boldsymbol{M(w)} \textbf{ loops infinitely} \end{cases}$$

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$$\boldsymbol{U_{TM}}(\langle \boldsymbol{M}, \boldsymbol{w} \rangle) = \begin{cases} \textbf{ACCEPTS, if } \boldsymbol{M}(\boldsymbol{w}) \textbf{ accepts} \\ \textbf{REJECTS, if } \boldsymbol{M}(\boldsymbol{w}) \textbf{ rejects} \\ \textbf{LOOPS INFINITELY, if } \boldsymbol{M}(\boldsymbol{w}) \textbf{ loops infinitely} \end{cases}$$

Some examples of languages that are recursive/decidable:

- $\boldsymbol{A_{DFA}} = \{\langle \boldsymbol{DFA} \rangle, \boldsymbol{w} | \boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{DFA})\}$
- $\boldsymbol{E_{DFA}} = \{\langle \boldsymbol{DFA} \rangle | \boldsymbol{L}(\boldsymbol{DFA}) = \boldsymbol{\Phi}\}$
- $\boldsymbol{A_{CFG}} = \{\langle \boldsymbol{CFG}, \boldsymbol{w} \rangle | \boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{CFG})\}$
- $\boldsymbol{E_{CFG}} = \{\langle \boldsymbol{CFG} \rangle | \boldsymbol{L}(\boldsymbol{CFG}) = \boldsymbol{\Phi}\}$

An undecidable language:

- $\boldsymbol{A_{TM}} = \{\langle \boldsymbol{M}, \boldsymbol{w} \rangle | \boldsymbol{M} \textbf{ accepts input } \boldsymbol{w}\}$

Quick Recap

There exists a one-one mapping (bijective relationship) between the set of finite length binary strings and TMs.

An undecidable language:

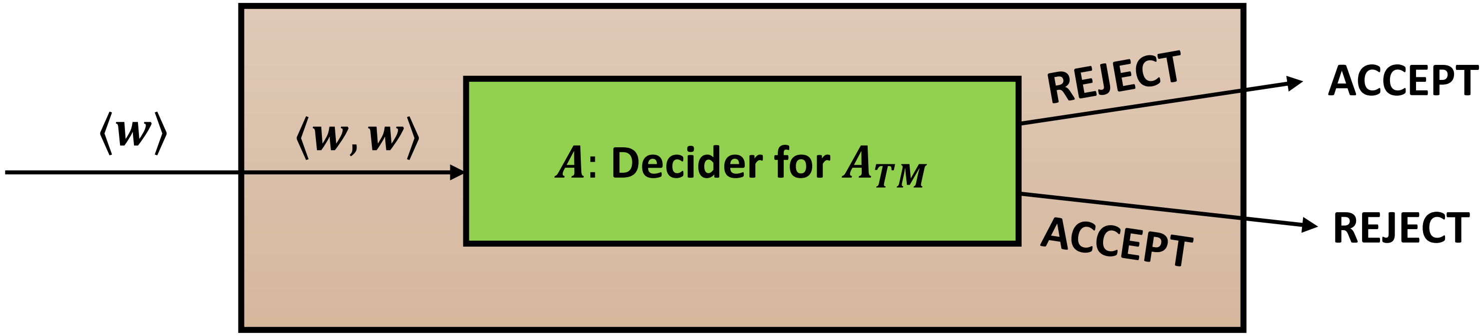
- $A_{TM} = \{\langle M, w\rangle | M$ **accepts input** $w\}$

- A_{TM} **is undecidable**
- $A_{TM} \in RE$ **but not recursive**
- A_{TM} **is partially decidable**

Proof strategy: By contradiction. We assume that a Total TM A exists that decides A_{TM}.

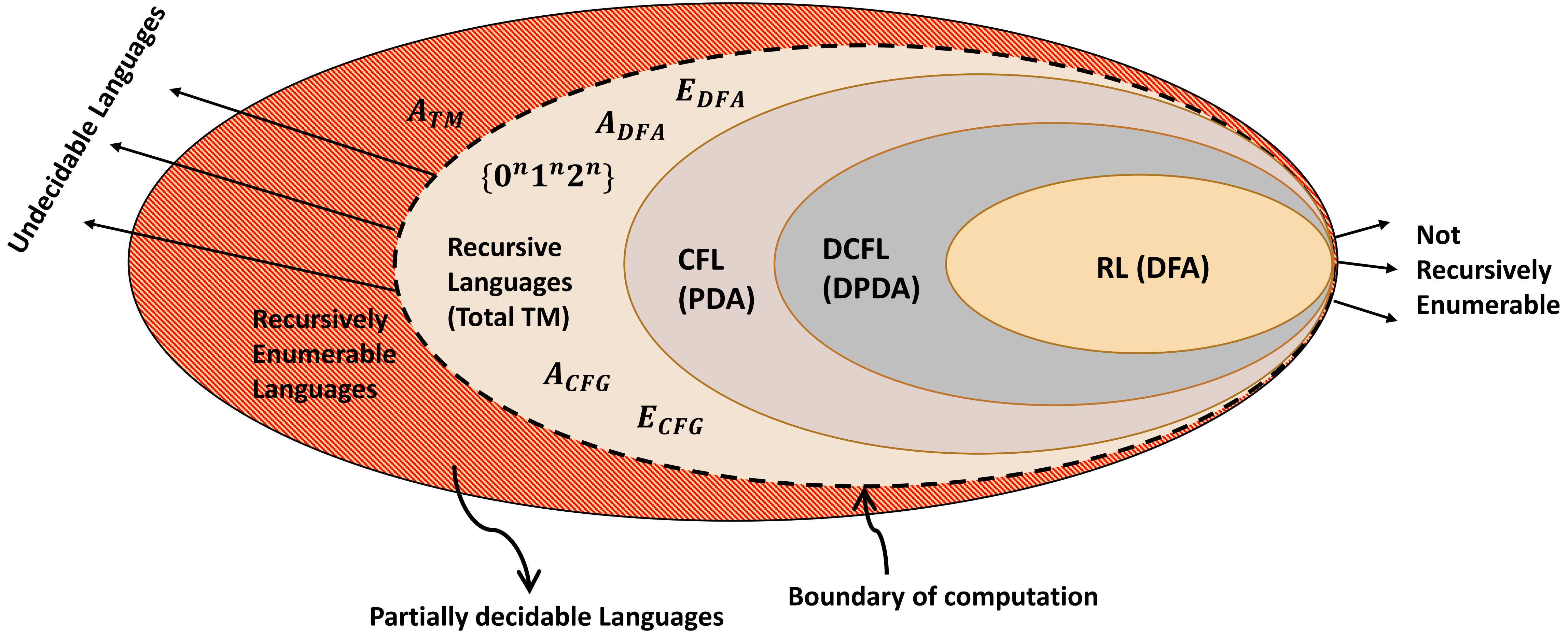
We build a Total TM D that accepts w as input and calls $A(\langle w, w\rangle)$ as a subroutine and outputs the opposite of A.

The TM D cannot decide the input $w = \langle D\rangle$. So A is undecidable.



For $w = \langle D\rangle$, there is a contradiction!

Quick Recap



Undecidable Languages
A_{TM}
E_{DFA}
A_{DFA}
$\{0^n1^n2^n\}$
Recursive Languages (Total TM)
CFL (PDA)
DCFL (DPDA)
RL (DFA)
Not Recursively Enumerable
Recursively Enumerable Languages
A_{CFG}
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Partially decidable Languages
Boundary of computation

An undecidable problem

$A_{TM} = \{\langle M, w\rangle | M$ **accepts input** $w\}$**.** A_{TM} **is undecidable**

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$$A(\langle M, w\rangle) = \begin{cases} \textbf{ACCEPTS, if } M(w) \textbf{ accepts} \\ \textbf{REJECTS, if } M(w) \textbf{ rejects or loops infinitely} \end{cases}$$

The proof uses a technique called **Diagonalization**.

First, recall that there exists a bijective map (one-one correspondence) between the set of all finite-length binary strings and Turing Machines.

We can list all the Turing Machines and write down the result of running any M_i on input $\langle M_j\rangle$.

An undecidable problem

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	$\langle M_0\rangle$	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$
M_0	Accept	Accept	Loops	Reject	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_2	Reject	Loops	Accept	Loops	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

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M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_2	Reject	Loops	Accept	Loops	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

How would this Table look for A?

An undecidable problem

$\boldsymbol{A_{TM} = \{\langle M, w\rangle | M}$ **accepts input** $\boldsymbol{w\}}$**.** $\boldsymbol{A_{TM}}$ **is undecidable**

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M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_2	Reject	Loops	Accept	Loops	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

$\boldsymbol{A}$	$\langle M_0\rangle$	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$
M_0	Accept	Accept	**Reject**	Reject	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_2	Reject	**Reject**	Accept	**Reject**	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

An undecidable problem

$A_{TM} = \{\langle M, w\rangle | M$ accepts input $w\}$. A_{TM} is undecidable

- **A_{TM} is undecidable**
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A	$\langle M_0\rangle$	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
M_0	Accept	Accept	**Reject**	Reject	Accept	$\cdots$	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$	Accept	$\cdots$
M_2	Reject	**Reject**	Accept	**Reject**	Accept	$\cdots$	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
D						$\cdots$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$

$D(w) = \{$ **Run A($\langle w, w\rangle$)**

If $A(\langle w, w\rangle)$ accepts, then $REJECT$
If $A(\langle w, w\rangle)$ rejects, then $ACCEPT$

$\}$

$$D(\langle M_w\rangle) = \begin{cases} \textbf{ACCEPTS, if } M_w(\langle M_w\rangle) \textbf{ doesn't accept} \\ \textbf{REJECTS, if } M_w(\langle M_w\rangle) \textbf{ accepts} \end{cases}$$

- Somewhere we will also have the TM D.
- Note that D by definition **computes the opposite of the diagonal entries** of the table.

An undecidable problem

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M_0	**Accept**	Accept	Reject	Reject	Accept	$\cdots$	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$	Accept	$\cdots$
M_2	Reject	Reject	Accept	Reject	Accept	$\cdots$	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
D	**Reject**					$\cdots$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$

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M_0	Accept	Accept	Reject	Reject	Accept	$\cdots$	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$	Accept	$\cdots$
M_2	Reject	Reject	Accept	Reject	Accept	$\cdots$	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
D	Reject	Accept				$\cdots$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$

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M_3	Accept	Reject	Reject	**Accept**	Reject	$\cdots$	Reject	$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
D	**Reject**	**Accept**	**Reject**	**Reject**	**Accept**	$\cdots$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$

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Note that D by definition **computes the opposite of the diagonal entries** of the table.

What will be the D^{th} diagonal entry??

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A	$\langle M_0\rangle$	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
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M_3	Accept	Reject	Reject	**Accept**	Reject	$\cdots$	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	**Reject**	$\cdots$	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
D	**Reject**	**Accept**	**Reject**	**Reject**	**Accept**	$\cdots$	**??**	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$

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Note that D by definition **computes the opposite of the diagonal entries** of the table.

What will be the D^{th} diagonal entry??

Contradiction!

The Halting problem

$\boldsymbol{HALT_{TM}} = \{\langle \boldsymbol{M}, \boldsymbol{w}\rangle | \boldsymbol{M}$ **halts on input** $\boldsymbol{w}\}$**. Is** $\boldsymbol{HALT_{TM}}$ **decidable?**

The Halting Problem: Does there exist a Total Turing Machine H that accepts as input a Turing Machine M and an input string w and outputs YES, if $M(w)$ halts (accepts or rejects) and NO, if $M(w)$ does not halt (loops forever), i.e.

$$\boldsymbol{H(\langle M, w\rangle)} = \begin{cases} \textbf{ACCEPTS, if } \boldsymbol{M(w)} \textbf{ HALTS, i.e. accepts or rejects} \\ \textbf{REJECTS, if } \boldsymbol{M(w)} \textbf{ does not HALT, i.e. loops infinitely} \end{cases}$$

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- Turing stated the Halting problem and demonstrated its undecidability in his famous 1936 paper.
- This provided a negative answer to Hilbert's Eintscheidungsproblem.
- **Proof strategy:** We will try to show that if we had such a Total TM H, we would be able to build a Total TM for A_{TM}
- But since we proved that A_{TM} is undecidable, this would mean that $\boldsymbol{H}$ **is undecidable**.

The Halting problem

$HALT_{TM} = \{\langle M, w\rangle | M$ halts on input $w\}$. Is $HALT_{TM}$ decidable?

Proof idea: We first assume that there exists such a Total Turing Machine H. Then, we use H as a subroutine to construct a Total Turing Machine A for A_{TM}

But A cannot be Total and so a total TM that decides H cannot exist

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Proof idea: We first assume that there exists such a Total Turing Machine H. Then, we use H as a subroutine to construct a Total Turing Machine A for A_{TM}

Outlining the steps for building A using H:

- A calls $H(\langle M, w\rangle)$
- If H rejects, then we know that $M(w)$ loops forever and so A would output $REJECT$.

The Halting problem

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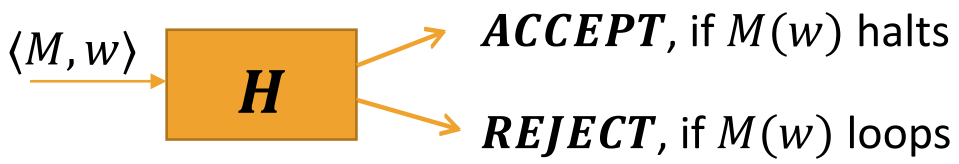

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- If H accepts,
    - $M(w)$ surely halts (either accepts or rejects).

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Outlining the steps for building A using H:

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- If H rejects, then we know that $M(w)$ loops forever and so A would output $REJECT$.
- If H accepts,
  - $M(w)$ surely halts (either accepts or rejects).
  - Simply run $M(w)$ and
    - $ACCEPT$ if $M(w)$ accepts
    - $REJECT$ if $M(w)$ rejects

The Halting problem

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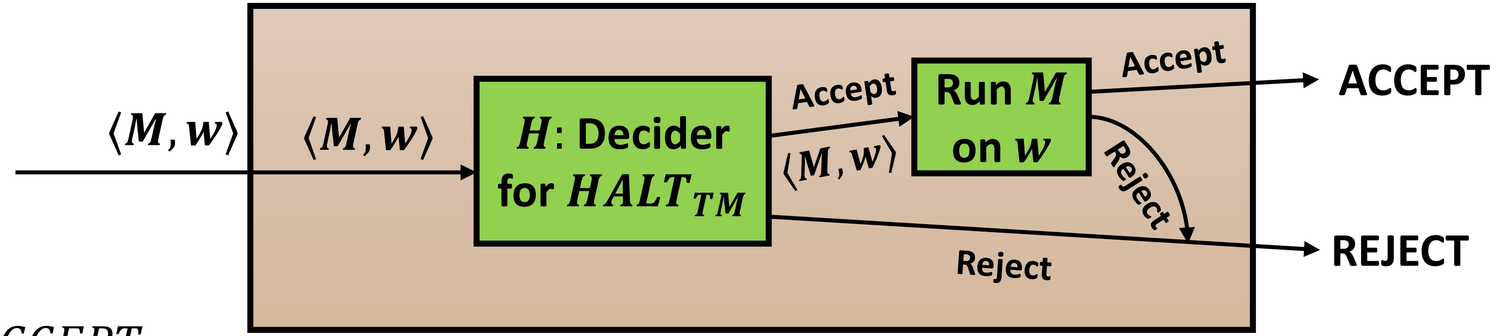
Proof: Assume that there exists such a Total Turing Machine H. Then, we use H as a subroutine to construct a Total Turing Machine A for A_{TM} as follows:

$A =$ On input $\langle M, w\rangle$
- Run $H(\langle M, w\rangle)$
- If H rejects, output $REJECT$
- If H accepts,
  - Run $M(w)$
  - If $M(w)$ accepts, output $ACCEPT$
  - If $M(w)$ rejects, output $REJECT$



https://www.udiprod.com/halting-problem/

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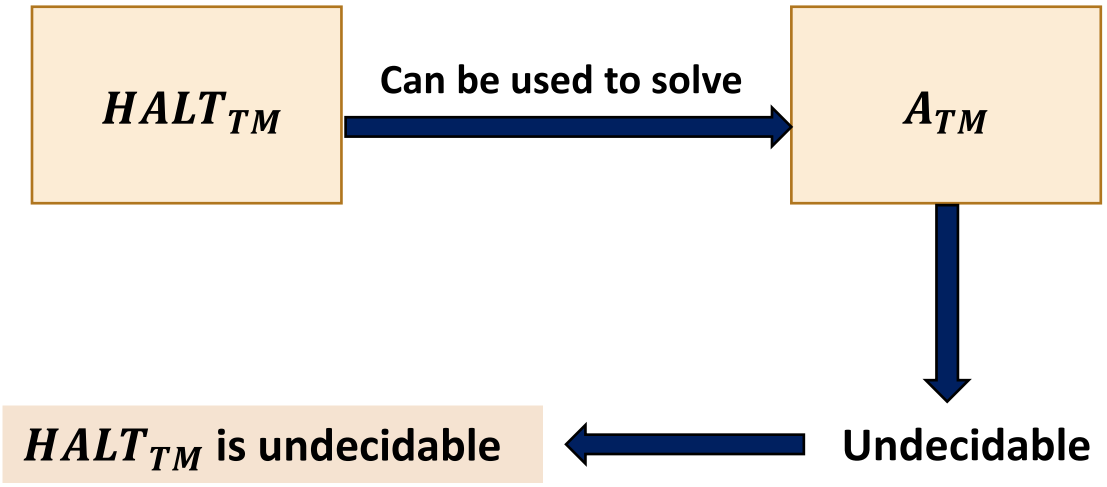

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$HALT_{TM} \in RE$ as H halts whenever M accepts or rejects w and so

$Q =$ On input $\langle M, w\rangle$:

- Simulate M on input w
- If M accepts w, $ACCEPT$; if M rejects w, $ACCEPT$

Q recognizes $HALT_{TM}$

The Halting problem

$HALT_{TM} = \{\langle M, w\rangle | M$ **halts on input** $w\}$. **Is** $HALT_{TM}$ **decidable?**

$$H(\langle M, w\rangle) = \begin{cases} \text{ACCEPTS, if } M(w) \text{ HALTS, i.e. accepts or rejects} \\ \text{REJECTS, if } M(w) \text{ does not HALT, i.e. loops infinitely} \end{cases}$$

$HALT_{TM} \in RE$ **as** H **halts whenever** M **accepts or rejects** w **and so**

$Q =$ **On input** $\langle M, w\rangle$**:**

- **Simulate** M **on input** w
- **If** M **accepts** w, $ACCEPT$; **if** M **rejects** w, $ACCEPT$

Q ***recognizes*** $HALT_{TM}$

- $HALT_{TM}$ **is undecidable**
- $HALT_{TM} \in RE$ **but not recursive**
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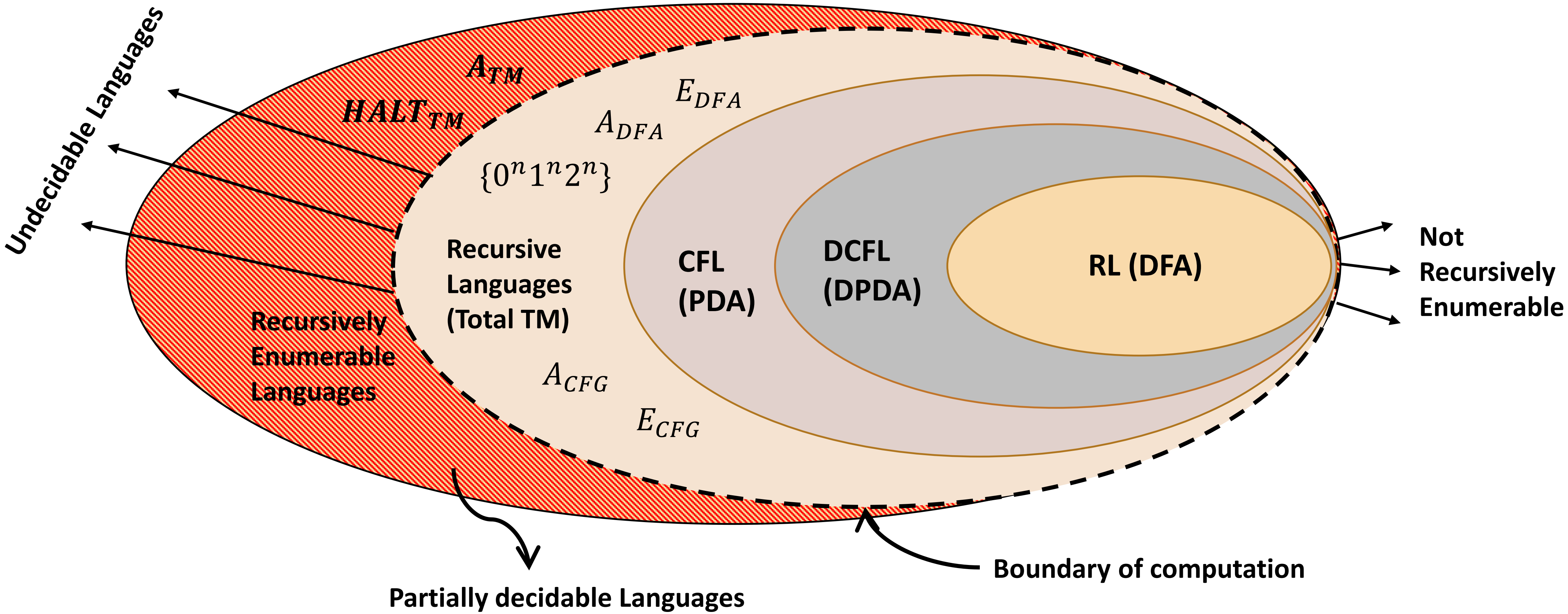
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Recursive Languages (Total TM)
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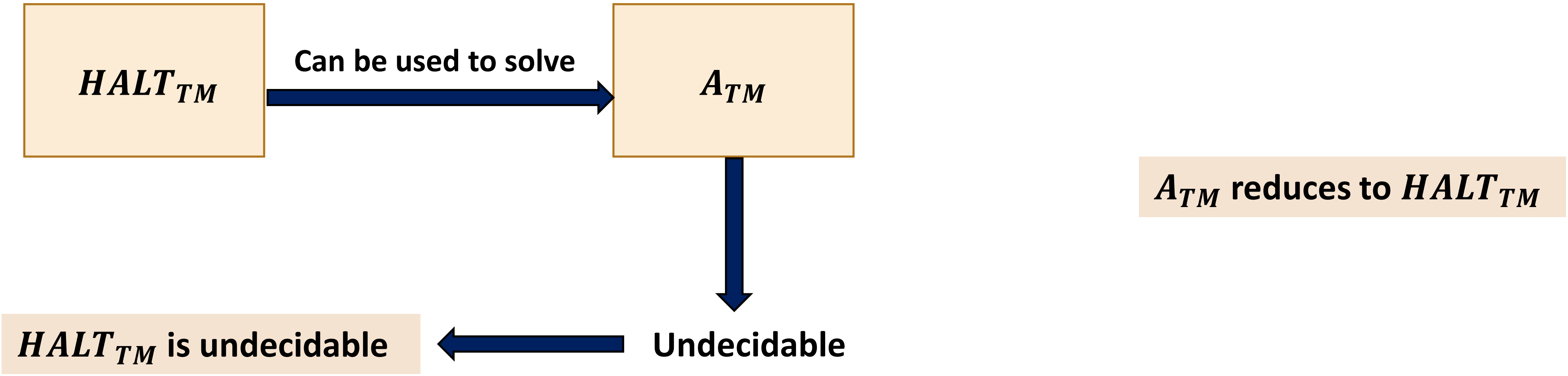
Partially decidable Languages
Boundary of computation

Reduction

Recall the proof of the undecidability of the Halting Problem

What did we do there?

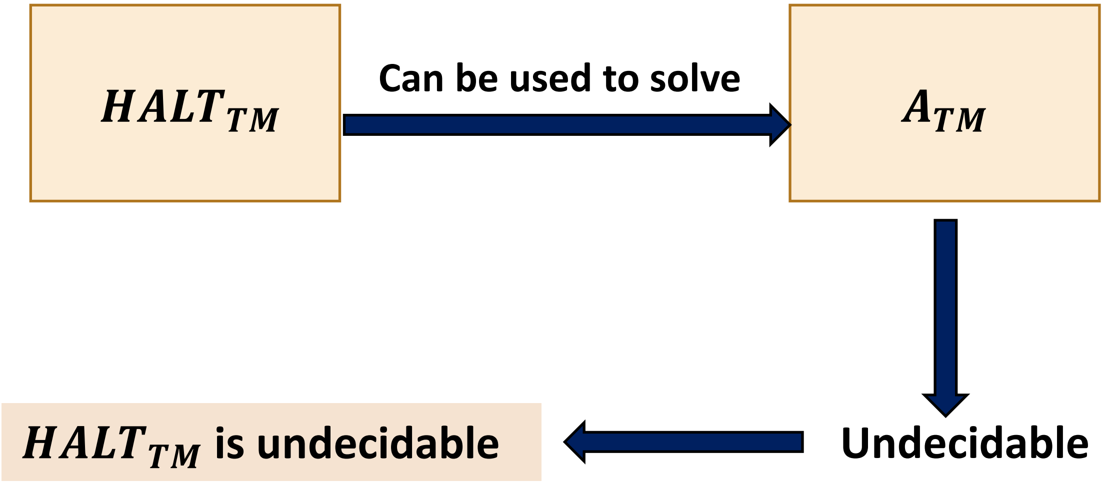
- We used a (supposed) decider for the Halting Problem to build a decider for A_{TM}.
- This established that $HALT_{TM}$ can be used to solve A_{TM}.
- As A_{TM} is undecidable, this established that $HALT_{TM}$ is undecidable too.
- The key underlying concept is an idea called **Reduction.**



Reduction

Generally,

- A language $\boldsymbol{A}$ **reduces to** another language $\mathbf{B}$ ($\boldsymbol{A} \preccurlyeq \boldsymbol{B}$) iff we can **build a solver for $\boldsymbol{A}$ using a solver for $\boldsymbol{B}$**
- In terms of computability, suppose using B we can compute A. Then, if **$\boldsymbol{A}$ is undecidable then so is $\boldsymbol{B}$**.
- So, in the last proof we showed: $\boldsymbol{A_{TM}} \preccurlyeq \boldsymbol{HALT_{TM}}$ to prove that $\boldsymbol{HALT_{TM}}$ is undecidable.



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- This is a common technique to show that certain problems are decidable/undecidable.

Suppose, $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ and

- **$\boldsymbol{A}$ is undecidable. Then $\boldsymbol{B}$ is undecidable.**

For example, we can prove that a problem P is undecidable by reducing the Halting problem to P.

$$\boldsymbol{HALT_{TM}} \preccurlyeq \boldsymbol{P} \Rightarrow \boldsymbol{P} \text{ is undecidable}$$

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For example, we can prove that a problem P is undecidable by reducing the Halting problem to P.

- Intuitively, this means **$\boldsymbol{B}$ is at least as hard as A**.
- So, if $\mathbf{A} \notin \boldsymbol{R} \Rightarrow \boldsymbol{B} \notin \boldsymbol{R}$
- $\boldsymbol{A} \notin \boldsymbol{RE} \Rightarrow \boldsymbol{B} \notin \boldsymbol{RE}$
- $\boldsymbol{A} \notin \boldsymbol{co\ RE} \Rightarrow \boldsymbol{B} \notin \boldsymbol{co\ RE}$

$\boldsymbol{HALT_{TM}} \preccurlyeq \boldsymbol{P} \Rightarrow \boldsymbol{P}$ is undecidable. Also, $\boldsymbol{P} \notin \boldsymbol{R}$

Reduction

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- A language $\boldsymbol{A}$ **reduces to** another language $\mathbf{B}$ ($\boldsymbol{A} \preccurlyeq \boldsymbol{B}$) iff we can **build a solver for $\boldsymbol{A}$ using a solver for $\boldsymbol{B}$**..
- In terms of computability, suppose using B we can compute A. Then, if **$\boldsymbol{A}$ is undecidable then so is $\boldsymbol{B}$**.
- So, in the last proof we showed: $\boldsymbol{A_{TM}} \preccurlyeq \boldsymbol{HALT_{TM}}$ to prove that $\boldsymbol{HALT_{TM}}$ is **undecidable**.
- This is a common technique to show that certain problems are decidable/undecidable.

Suppose, $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ and

- A is undecidable. Then B is undecidable.
- **$\boldsymbol{B}$ is decidable. Then $\boldsymbol{A}$ is decidable.**

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- So, if $\mathbf{B} \in \boldsymbol{R} \Rightarrow \boldsymbol{A} \in \boldsymbol{R}$
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Reduction

- A language $\boldsymbol{A}$ **reduces to** another language $\mathbf{B}$ ($\boldsymbol{A} \preccurlyeq \boldsymbol{B}$) iff we can **build a solver for A using a solver for B**.
- If $\boldsymbol{A}$ **is undecidable then so is** $\boldsymbol{B}$, i.e. ***B*** **is at least as hard as *A*.**
- If **B is decidable, then so is A.** Intuitively, this means $\boldsymbol{A}$ **is not harder than B**.
- This is a common technique to show that certain problems are decidable/undecidable.

This is how we can prove several problems are undecidable: by **reducing some known undecidable problem to these problems.**

Examples:

- $E_{TM} = \{\langle M\rangle | M$ is a Turing Machine and $L(M) = \Phi\}$
- $EQ_{TM} = \{\langle M_1, M_2\rangle | M_1$ and M_2 are Turing Machines having $L(M_1) = L(M_2)\}$
- $ALL_{TM} = \{\langle M\rangle | M$ halts on all inputs$\}$

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Generic strategy for proof:

- Consider that a Total TM for the given problem exists (say T_M).
- Build a total TM that can decide some undecidable problem (e.g. $HALT_{TM}$, A_{TM}) using T_M as a subroutine**.**

This reduction would prove that $\boldsymbol{E_{TM}, EQ_{TM}, ALL_{TM}}$ **are undecidable**.

Undecidability

$\boldsymbol{E_{TM}} = \{\langle M \rangle | M \text{ is a Turing Machine and } L(M) = \Phi\}$

Proof: Let T_E be the Turing Machine that decides E_{TM}. We shall prove that $\overline{\boldsymbol{A_{TM}}} \preccurlyeq \boldsymbol{E_{TM}}$ by constructing a Turing Machine N that decides $\overline{\boldsymbol{A_{TM}}}$ using $\boldsymbol{T_E}$.

What is $\overline{\boldsymbol{A_{TM}}}$?

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What is $\overline{\boldsymbol{A_{TM}}}$?

$\boldsymbol{A_{TM}} = \{\langle \boldsymbol{M}, \boldsymbol{w} \rangle | \boldsymbol{M}$ **accepts input** $\boldsymbol{w}\}$

$\overline{\boldsymbol{A_{TM}}} = \{\langle \boldsymbol{M}, \boldsymbol{w} \rangle | \boldsymbol{M}$ **DOES NOT accept input** $\boldsymbol{w}\}$

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$$\boldsymbol{A}(\langle \boldsymbol{M}, \boldsymbol{w} \rangle) = \begin{cases} \textbf{ACCEPTS, if } \boldsymbol{M}(\boldsymbol{w}) \textbf{ accepts} \\ \textbf{REJECTS, if } \boldsymbol{M}(\boldsymbol{w}) \textbf{ rejects/loops} \end{cases}$$

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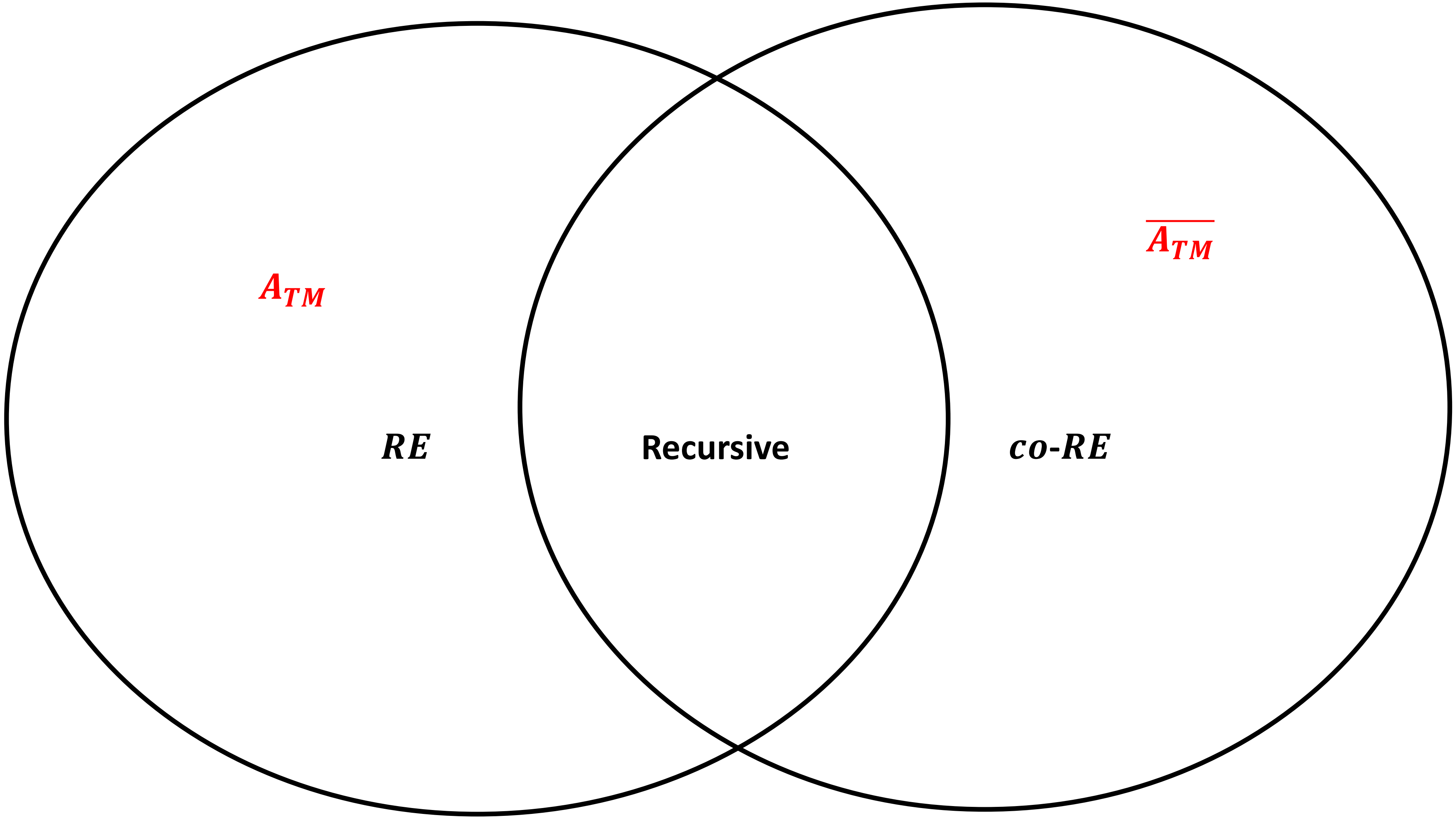

A_{TM}
$\overline{A_{TM}}$
RE
Recursive
$co\text{-}RE$

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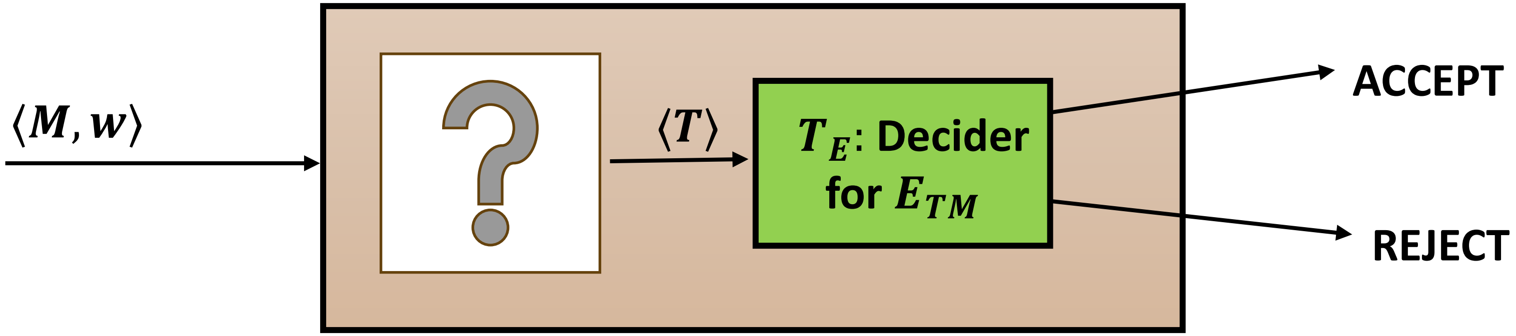

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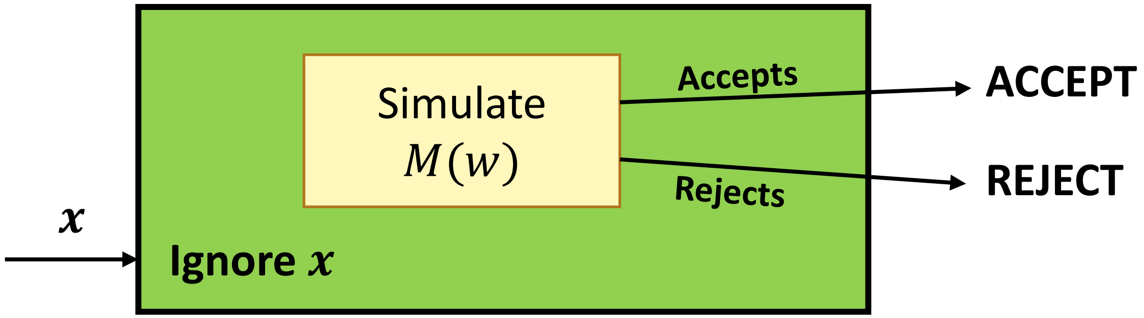
Key idea:

- $N(\langle M, w\rangle)$ first builds the encoding of a TM $T_{\langle M,w\rangle}$.
- $T_{\langle M,w\rangle}$ does not accept any string if and only if $M(w)$ does not accept.
- That is, running $T_{\langle M,w\rangle}(x)$ on **ANY** input x, runs M on w.

$\boldsymbol{T_{\langle M,w\rangle}(x)}$



What does this achieve??

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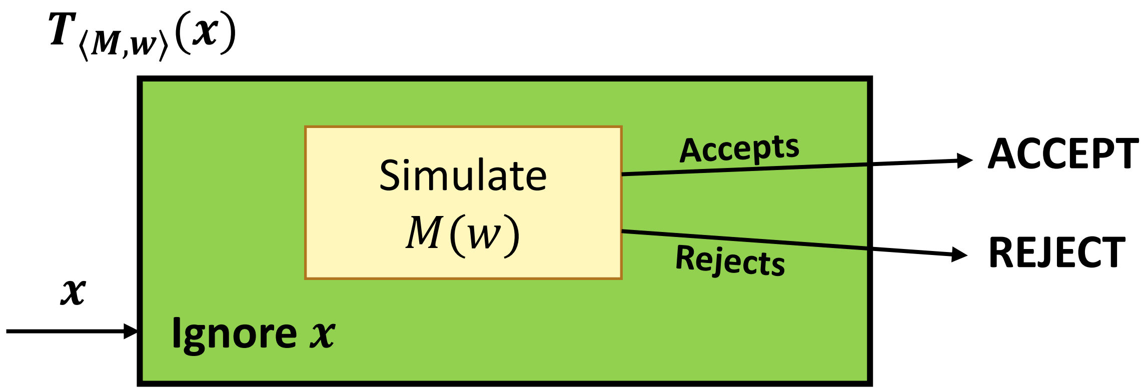

- This means, $L(T) = \Phi$ if M does not accept w and $L(T) \neq \Phi$ if M accepts w!
- This allows N to call $T_E(\langle T \rangle)$

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- **Deciding $\overline{\boldsymbol{A_{TM}}}$ is tied to whether $\boldsymbol{L(T) = \Phi}$!**

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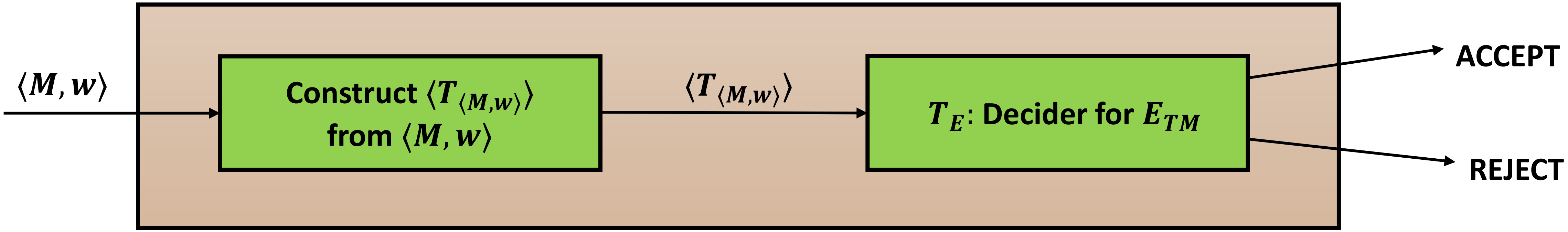

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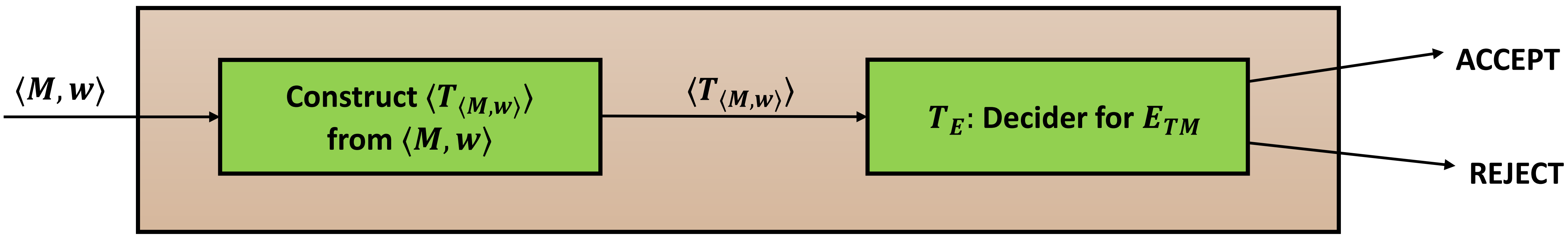

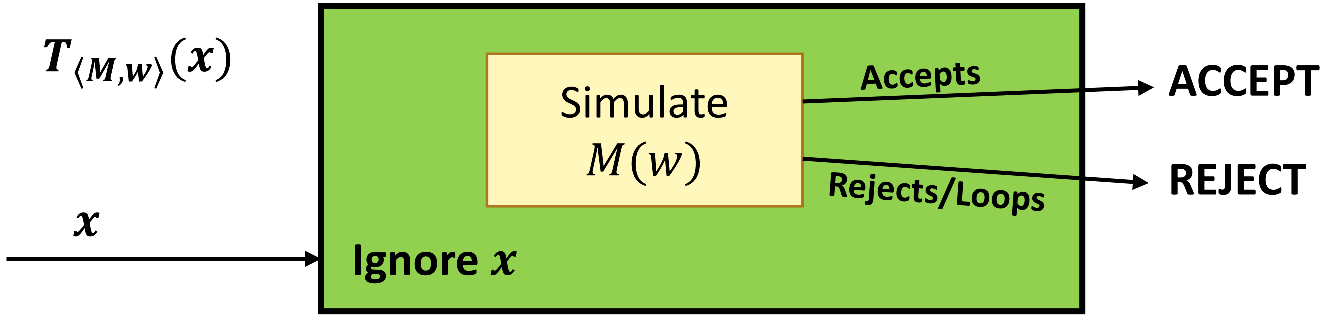

$T_{\langle M,w \rangle}(x) =$

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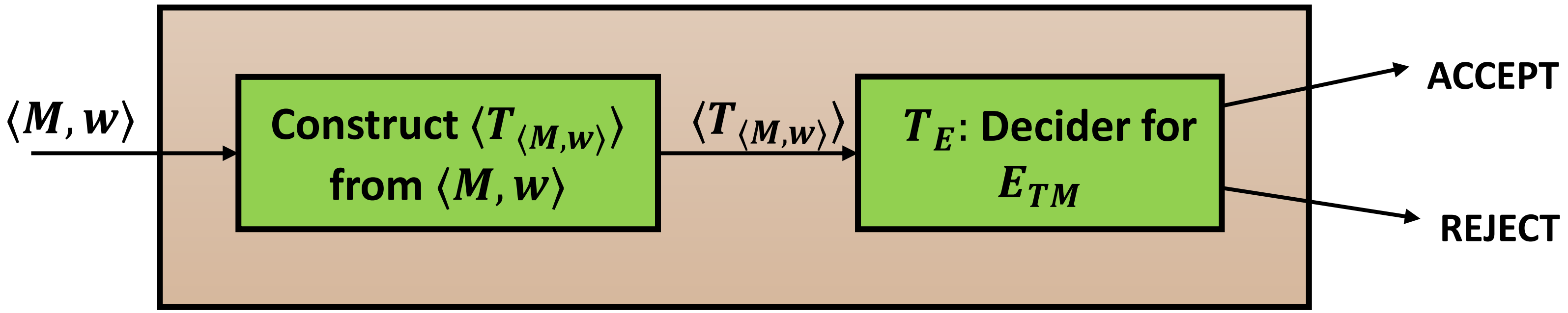

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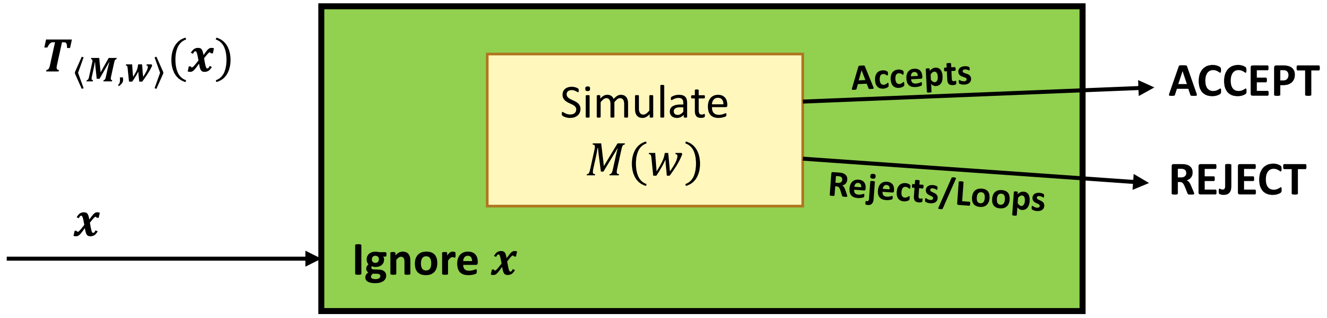

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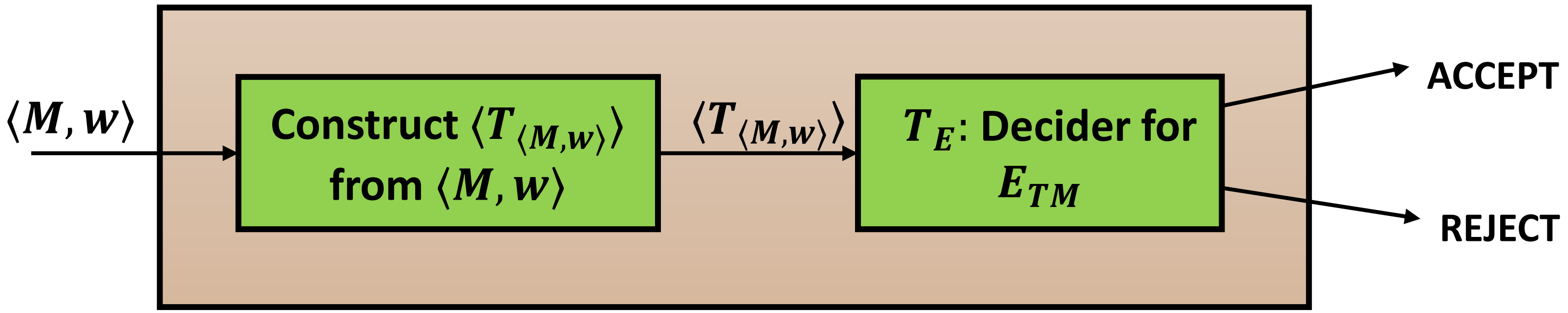

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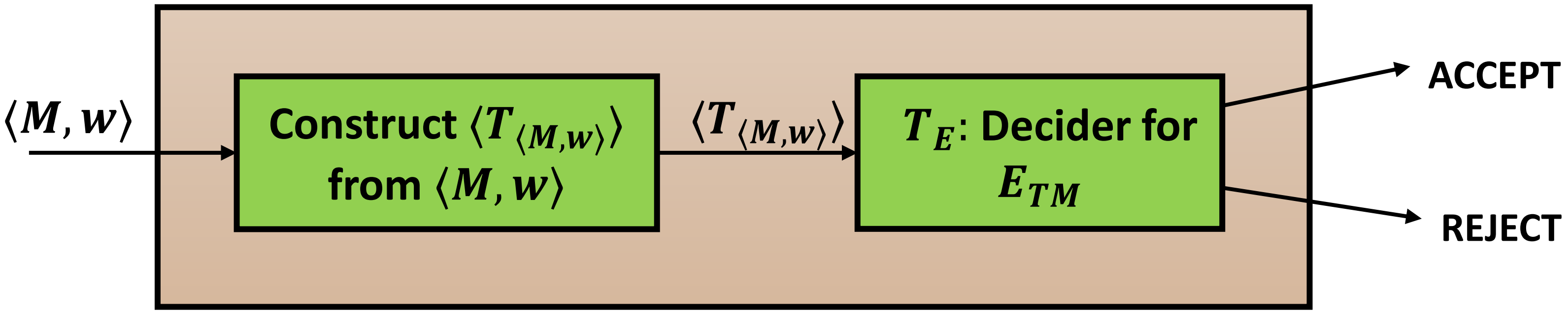

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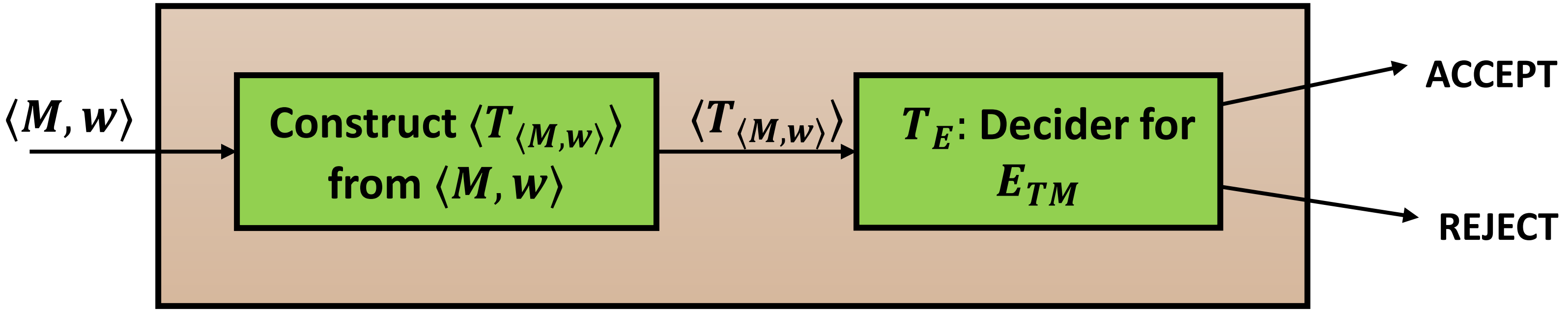

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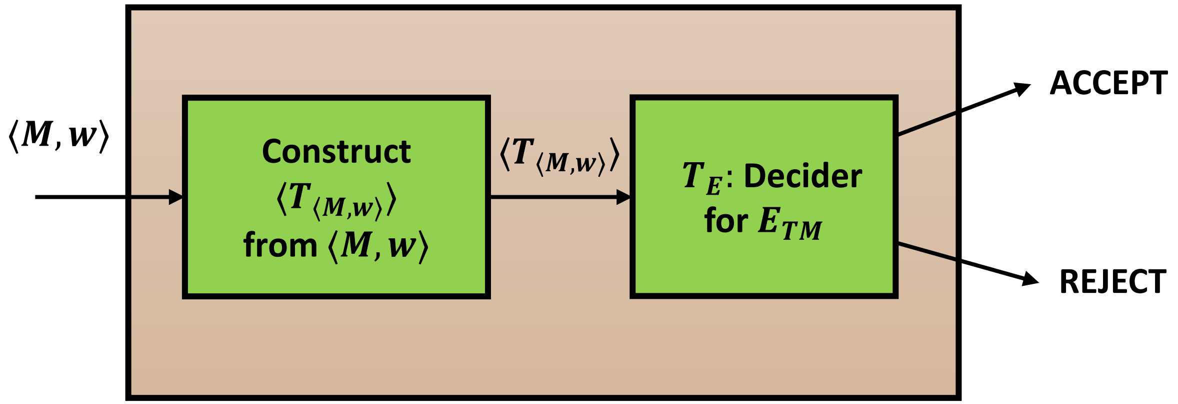

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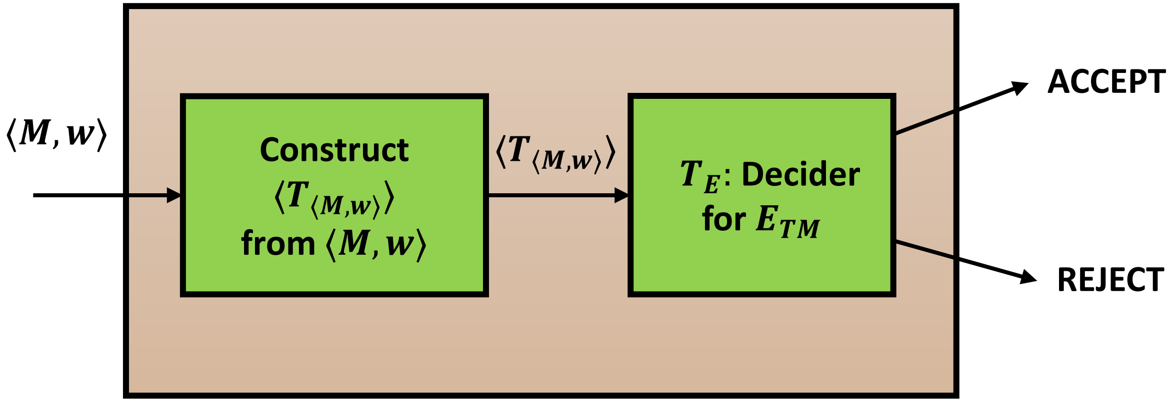
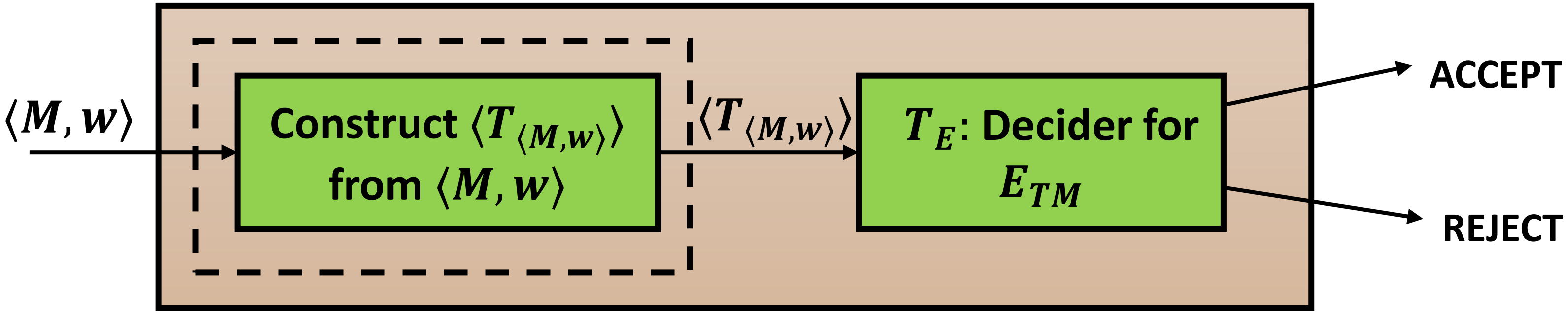


- $\overline{\boldsymbol{A_{TM}}} \preccurlyeq \boldsymbol{E_{TM}}$
- $\overline{\boldsymbol{A_{TM}}}$ **is undecidable**
- $\boldsymbol{E_{TM}}$ **is undecidable!**

Undecidability

$\boldsymbol{E_{TM}} = \{\langle M\rangle | M$ is a Turing Machine and $L(M) = \Phi\}$

Proof: Let T_E be the Turing Machine that decides E_{TM}. We shall prove that $\overline{\boldsymbol{A_{TM}}} \preccurlyeq \boldsymbol{E_{TM}}$ by constructing a Turing Machine N that decides $\overline{\boldsymbol{A_{TM}}}$ using $\boldsymbol{T_E}$.

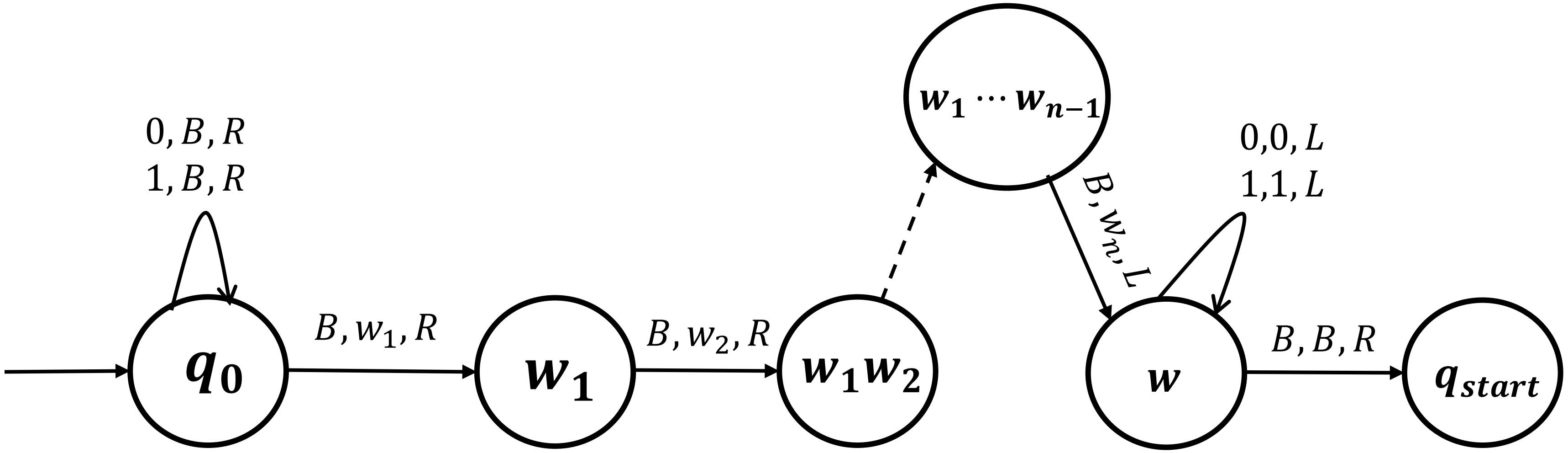


$\boldsymbol{T_{\langle M,w\rangle}}(x) =$

- Ignore x.
- Run M on w
  - If M accepts w, accept x
  - If M rejects w, reject x

$T_{\langle M,w\rangle}$ from $\langle M, w\rangle$:

- Remove i/p x from the tape of T
- `Embed' w into the TM of M



Undecidability

$E_{TM} = \{\langle M\rangle | M$ is a Turing Machine and $L(M) = \Phi\}$

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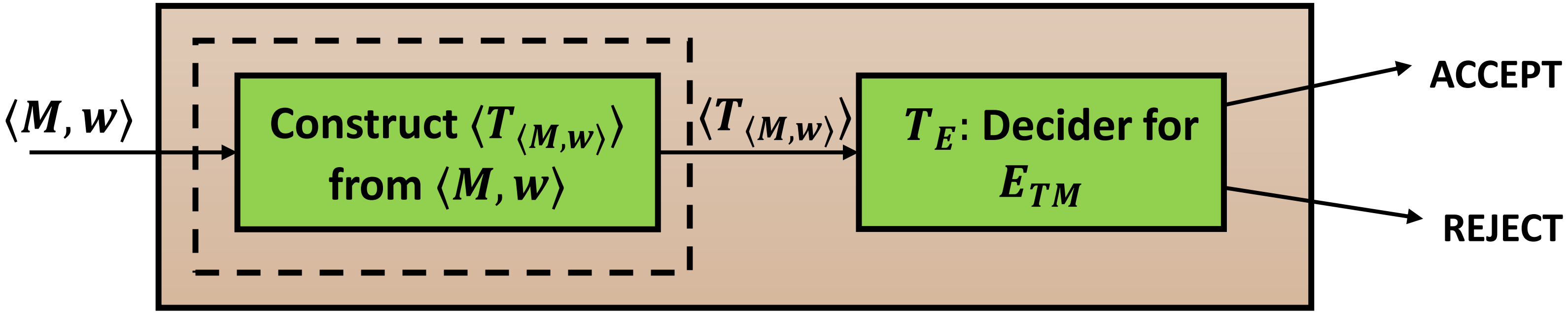

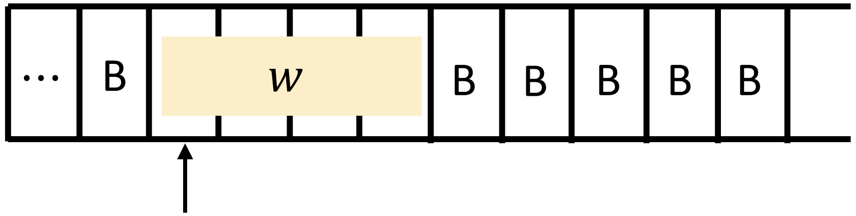

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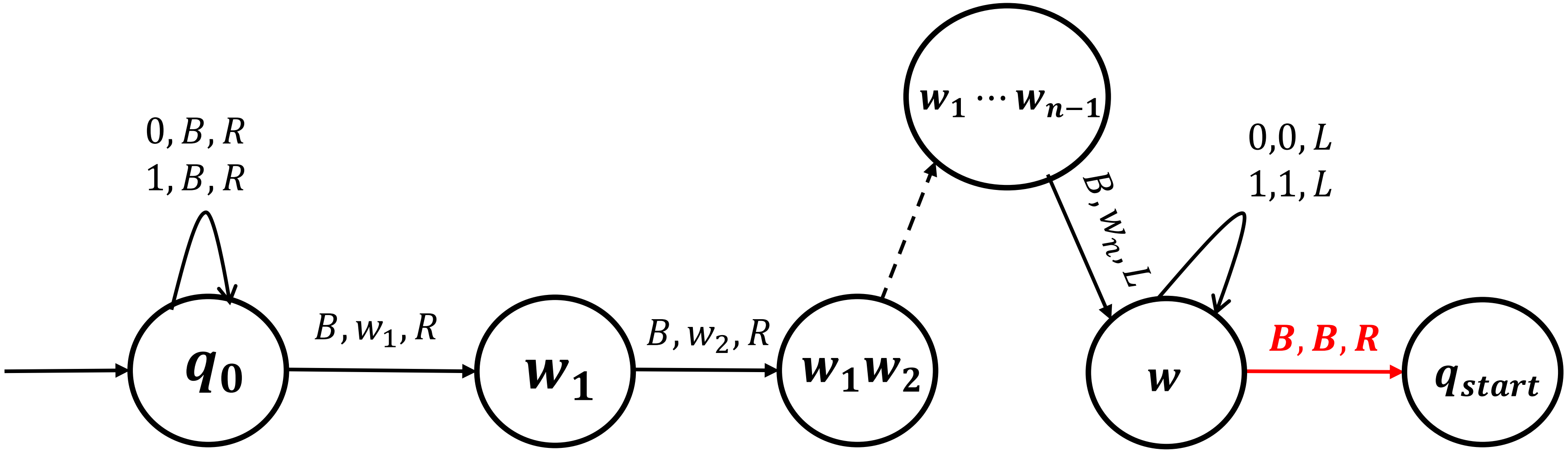

Undecidability

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Proof: Let T_E be the Turing Machine that decides E_{TM}. We shall prove that $\boldsymbol{\overline{A_{TM}}} \preccurlyeq \boldsymbol{E_{TM}}$ by constructing a Turing Machine N that decides $\boldsymbol{\overline{A_{TM}}}$ using $\boldsymbol{T_E}$.

$N(\langle M, w \rangle) =$

- Construct $\langle T_{\langle M,w \rangle} \rangle$, the encoding of $T_{\langle M,w \rangle}$ such that for any input x it works as follows:
  - Ignore x.
  - Run M on w
    - If M accepts w, accept x
    - If M rejects w, reject x

- Send $\langle T_{\langle M,w \rangle} \rangle$ to T_E and Output
  ACCEPT if T_E accepts
  REJECT if T_E rejects

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- $\boldsymbol{\overline{A_{TM}}}$ **is undecidable**
- $\boldsymbol{E_{TM}}$ **is undecidable!**

Claim: $\boldsymbol{E_{TM}} \in$ **co-RE**$-\boldsymbol{R}$

WHY?

Undecidability

$\boldsymbol{E_{TM}} = \{\langle M \rangle | M$ is a Turing Machine and $L(M) = \Phi\}$

Proof: Let T_E be the Turing Machine that decides E_{TM}. We shall prove that $\overline{\boldsymbol{A_{TM}}} \preccurlyeq \boldsymbol{E_{TM}}$ by constructing a Turing Machine N that decides $\overline{\boldsymbol{A_{TM}}}$ using $\boldsymbol{T_E}$.

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$\boldsymbol{E_{TM}} \in$ **co-RE**$-\boldsymbol{R}$

Proof idea: We can build a co-recognizer for E_{TM}.

$C =$ On input $\langle M \rangle$
- For $i = 1, 2, 3, \cdots$
  - For $j = 1, 2, 3, \cdots i$
    Run M on s_j for i steps.
    If M accepts s_j, REJECT.

Thank You!