Lecture 6

(19 August 2024)

Module 2 (Discrete Random Variables)

- The concept of a Random Variable
- Probability Distribution Function
- Discrete and Continuous RVs
- Expectation, Variance, Functions of RVs
- Multiple RVs, Conditioning, Independence

A random variable is a function $X : \Omega \to \mathbb{R}$ with the property that $\{\omega : X(\omega) \le x\} \in \mathcal{F}$ for each $x \in \mathbb{R}$.

<u>Notation</u>: $\{X \le x\} = \{\omega \in \Omega : X(\omega) \le x\}$

$$= X^{-1}((-\infty, x]).$$

$\Omega = \{HT, TH, HT, TT\}$

$\mathcal{F} = \{\phi, \Omega, \{HT, TH\}, \{TT, HH\}\}$

Consider a function $X : \Omega \to \mathbb{R}$:

$$X(\omega) = \text{no. of heads in } \omega$$

$$\{X \leq 1\} = \{\omega : X(\omega) \leq 1\}$$

$$= \{TH, HT, TT\} \notin \mathcal{F}$$

$\therefore X$ is not a random variable with respect to given Ω & $\mathcal{F}$.

<u>Theorem</u>. Given a sample space Ω an event space $\mathcal{F}$. Let $X : \Omega \to \mathbb{R}$ be a random variable. Then the following holds.

(i) $X^{-1}((-\infty, x)) \in \mathcal{F}$

or

$\{\omega \in \Omega : X(\omega) < x\} \in \mathcal{F}$

(ii) $X^{-1}([x_1, x_2]) \in \mathcal{F}$

or

$\{\omega \in \Omega : x_1 \le X(\omega) \le x_2\} \in \mathcal{F}$

(iii) $X^{-1}(\{x\}) \in \mathcal{F}$

or

$\{\omega \in \Omega : X(\omega) = x\}$

(iv) $X^{-1}((x_1, x_2)) \in \mathcal{F}$

or

$\{\omega \in \Omega : x_1 < X(\omega) < x_2\}$.

This also brings us to the consideration of Borel σ-field or Borel σ-algebra: the smallest σ-algebra on reals containing sets of form $(-\infty, x]$ $\forall x$,

$$\mathcal{B} = \{ (-\infty, x],\ (-\infty, x),\ (x, \infty),$$
$$[x_1, x_2],\ \{x\},\ (x_1, x_2), \dots \}.$$

The distribution function (or cumulative distribution function) of a random variable X is the function $F_X : \mathbb{R} \to [0, 1]$ given by

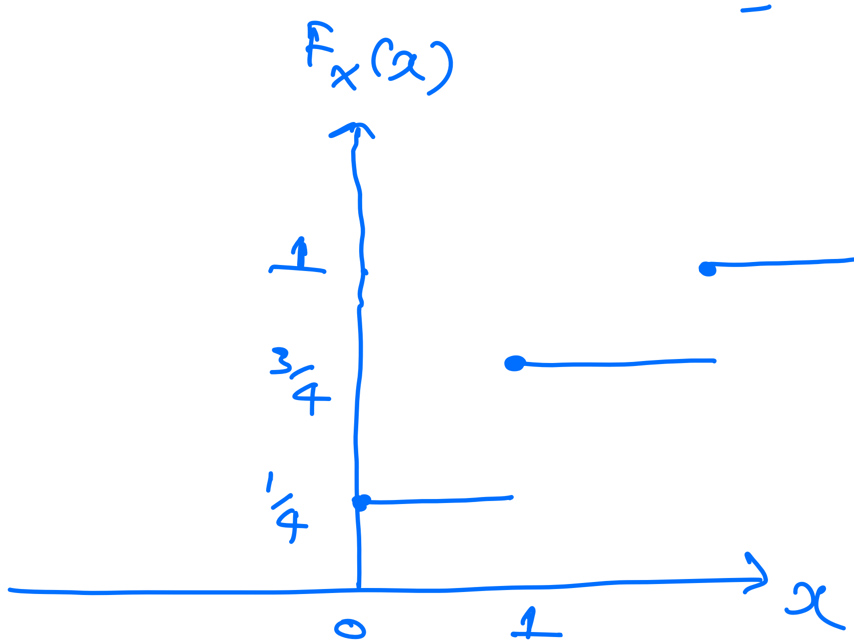
$$F_X(x) = P(\{\omega : X(\omega) \leq x\})$$
$$= P(X \leq x).$$

Examples.

1) $\Omega = \{HH, TT, HT, TH\}$, $\mathcal{F} = 2^{\Omega}$,
$P(\{\omega\}) = 1/4$.

$$\{X \le x\} = \begin{cases} \phi, & x < 0 \\ \{TT\}, & 0 \le x < 1 \\ \{TT, HT, TH\}, & 1 \le x < 2 \\ \Omega, & x \ge 2 \end{cases}$$

$$F_X(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \le x < 1 \\ 3/4, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

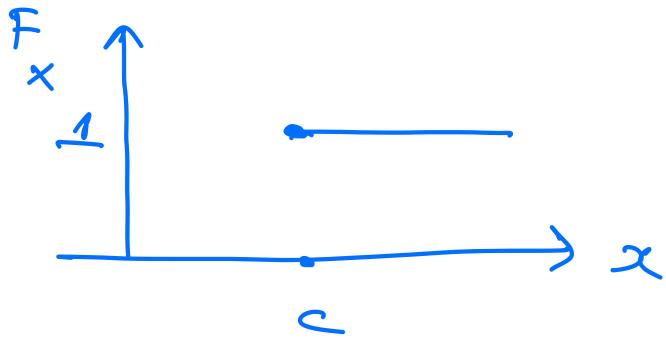


2) Constant RV

$X(\omega) = \underline{c}$ for all $\omega \in \Omega$

$$F_X(x) = \begin{cases} P(\phi), & x < c \\ P(\Omega), & x \geq c \end{cases}$$

$$= \begin{cases} 0, & x < c \\ 1, & x \geq c \end{cases}$$



3) Bernoulli random variable

$\Omega = \{H, T\}$, $\mathcal{F} = \{H, T, \phi, \Omega\}$,

$P(\{H\}) = p = 1 - P(\{T\})$.

$X(H) = 1$, $X(T) = 0$.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1-p, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

4) Indicator random variable

Given Ω, $\mathcal{F}$ and $A \in \mathcal{F}$.

Indicator random variable of an event A is defined by

$$I_A : \Omega \to \mathbb{R} :$$

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \in A^c . \end{cases}$$

Suppose $B_1, B_2, \dots, B_n$ forms a partition of Ω, we have

$$I_A = \sum_{i=1}^{n} I_{A \cap B_i} , \quad \text{i.e.,}$$

$$I_A(\omega) = \sum_{i=1}^{n} I_{A \cap B_i}(\omega) \quad \forall \omega \in \Omega .$$

The distribution function of X tells us about the values taken by X and the relative likelihoods, rather than about the sample space and the collection of events. For the time being we can forget all about probability spaces and concentrate on random variables and their distribution functions.

Theorem.

(a) If $x < y$ then $F_X(x) \le F_X(y)$.

(b) $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$.

(c) F_X is right continuous, that is

$$\lim_{\varepsilon \to 0^+} F_X(x+\varepsilon) = F_X(x) \quad \text{or}$$

$$F_X(x+) = F_X(x).$$

(d) $P(X > x) = 1 - F_X(x)$.

(e) $P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

(f) $P(X = x) = F_X(x) - \lim_{\varepsilon \to 0^+} F_X(x + \varepsilon)$

$$= F_X(x) - \lim_{y \to x^+} F_X(y)$$

$$= F_X(x) - F_X(x^-).$$

Proof. (a)

$$F_X(x) = P(X \le x)$$

$$\le P(X \le y)$$

(as $\{X \le x\} \subseteq \{X \le y\}$ for $x < y$)

$$= F_X(y).$$

(b) $\lim_{x \to -\infty} F_X(x) = 0$.

Let $A_n = \{\omega \in \Omega : X(\omega) \leq -n\}$.

$$\bigcap_{n=1}^{\infty} A_n = \phi$$

[Exactly along the same lines as a proof given in Lecture 1 in the discussion of mathematical induction]

$$A_1 \supseteq A_2 \supseteq \cdots$$

By the continuity of probability we have

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n)$$

$$\Rightarrow P(\phi) = \lim_{n \to \infty} P(X \leq -n)$$

$$0 = \lim_{x \to -\infty} F_X(x).$$

$$\lim_{x \to \infty} F_X(x) = 1.$$

$$B_n = \{\omega \in \Omega : X(\omega) \le n\}.$$

$$B_1 \subseteq B_2 \subseteq - -$$

$$\bigcup_{n=1}^{\infty} B_n = \Omega.$$

$$\lim_{x \to \infty} F_X(x) = \lim_{n \to \infty} P(X \le n)$$

$$= \lim_{n \to \infty} P(B_n)$$

$$= P\left(\bigcup_{n=1}^{\infty} B_n\right)$$

(by the continuity of probability)

$$= P(\Omega)$$

$$= 1.$$

(c) $A_n = \{\omega : X(\omega) \le x + 1/n\}$

$A_1 \supseteq A_2 \supseteq - - -$

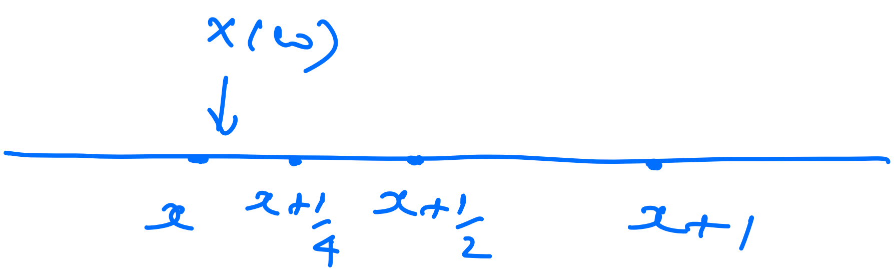
$$\bigcap_{n=1}^{\infty} A_n = \{\omega : X(\omega) \le x\}$$

Let $\omega \in \{\omega : X(\omega) \le x\}$

$\Rightarrow X(\omega) \le x + 1/n, \ \forall n \in \mathbb{N}$

$\Rightarrow \omega \in \bigcap_{n=1}^{\infty} A_n$.

Let $\omega \in \bigcap_{n=1}^{\infty} A_n \Rightarrow X(\omega) \le x + 1/n, \ \forall n \in \mathbb{N}$



$\Rightarrow X(\omega) \le x$

$$F_X(x^+) = \lim_{\varepsilon \to 0^+} P(X \le x + \varepsilon)$$

$$= \lim_{n\to\infty} P(X \le x + 1/n)$$

$$= \lim_{n\to\infty} P(A_n)$$

$$= P\left(\bigcap_{n=1}^{\infty} A_n\right) = P(X \le x).$$

(d) $P(X > x) = P(\{\omega : X(\omega) > x\})$

$$= 1 - P(\{\omega : X(\omega) \le x\})$$

$$= 1 - F_X(x).$$

(e) $P(x_1 < X \le x_2)$

$$= P(\{\omega : x_1 < X(\omega) \le x_2\})$$

$$= P(\{\omega : X(\omega) \le x_2\} \setminus \{\omega : X(\omega) \le x_1\})$$

$$= P(\{\omega : X(\omega) \le x_2\}) - P(\{\omega : X(\omega) \le x_1\})$$

$$= F_X(x_2) - F_X(x_1).$$

(f) $P(X=x) = F_X(x) - F_X(x^-)$

$$= F_X(x) - \lim_{\varepsilon \to 0^-} F_X(x+\varepsilon).$$

$$A_n = \{\omega : X(\omega) \leq x - 1/n\}$$

$$A_1 \subseteq A_2 \subseteq \cdots$$

$$\lim_{\varepsilon \to 0^-} F_X(x+\varepsilon) = \lim_{n \to \infty} P(X \leq x - 1/n)$$

$$= P\left(\bigcup_{n=1}^{\infty} \{X \leq x - 1/n\}\right)$$

$$= P(X < x).$$

$\therefore F_X(x) - P(X < x) = P(X = x).$