Assignment 3

(MA6.102) Probability and Random Processes, Monsoon 2024

Release date: 7 September 2024, Due date: 14 September 2024

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.
- Coding portion can be done in Python/Matlab. There will be a moss check, code copying will result in a straight 0.
- Submit a zipped folder (rollnumber.zip) containing your handwritten solutions (PDF), code and PDF of plots.

Problem 1. Let $X_1, X_2, \ldots, X_n$ be independent random variables and let $X = X_1 + X_2 + \cdots + X_n$. Suppose that each X_i is a Bernoulli random variable with parameter p_i, and that $p_1, p_2, \ldots, p_n$ are chosen so that the mean of X is a given $\mu > 0$. Show that the variance of X is maximized if the p_i values are chosen to be all equal to $\frac{\mu}{n}$.

Problem 2. Prove the following.
If X is a positive integer valued random variable satisfying

$$P(X > m + n | X > m) = P(X > n)$$

for any two positive integers m and n, then X is a geometric random variable.

Problem 3. (a) Give examples of two discrete random variables that are uncorrelated but not independent.
(b) Give examples of two discrete random variables where uncorrelatedness guarantees independence.

Problem 4. For any two random variables X and Y, Cauchy-Schwarz inequality states that

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

with equality if and only if $X = \alpha Y$, for some constant $\alpha \in \mathbb{R}$. Prove this, and use it to show that $|\rho(X, Y)| \leq 1$, where $\rho(X, Y)$ is the correlation coefficient of X and Y given by

$$\rho(X, Y) = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\text{var}(X)\text{var}(Y)}}.$$

[*Hint:* Observe that $\mathbb{E}[(X - \alpha Y)^2] \geq 0$, for all $\alpha \in \mathbb{R}$]

Problem 5. Let $\phi(Y) = \mathbb{E}[X|Y]$. For any function $g : \mathbb{R} \to \mathbb{R}$, show that

$$\mathbb{E}[\phi(Y)g(Y)] = \mathbb{E}[Xg(Y)].$$

Argue that the law of iterated expectations, ie., $\mathbb{E}\left[\mathbb{E}\left[X|Y\right]\right] = \mathbb{E}[X]$, is a special case of this.

Problem 6. (a) For any discrete random variable X and any event A such that $P(A) > 0$, show that

$$\mathbb{E}[X|A] = \frac{\mathbb{E}[\mathbb{1}_A X]}{P(A)},$$

where $\mathbb{1}_A$ is the indicator random variable of event A.

(b) X denotes the sum of outcomes obtained by rolling a die twice and A_i is the event that the first die shows i, for $i \in [1:6]$. Compute $\mathbb{E}[X|A_i]$, for $i \in [1:6]$.

Problem 7. You toss a fair coin 100 times. After each toss, either there have been more heads, more tails, or the same number of heads and tails. Let X be the number of times in the 100 tosses that there were more heads than tails. Estimate the PMF of X via simulation and plot it. Show that the most likely number of times you have more heads than tails when a coin is tossed 100 times is zero.

Now, once you have shown that the most likely number of times you have more heads than tails when a coin is tossed 100 times is zero, suppose you toss a coin 100 times.

(a) Let Y be the number of times in the 100 tosses that you have exactly the same number of heads as tails. Estimate the expected value of Y .

(b) Let Z be the number of tosses for which you have more heads than tails. Estimate the expected value of Z.