All Pairs Shortest Paths.

Assumptions: No negative cycle.

$\underline{(i,j)}$ → For each vertex apply Djikstra's
$O(n)$

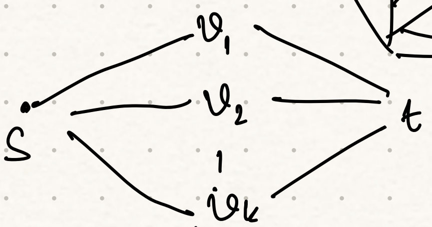
$O(|V| \cdot |E| \log |V|)$.

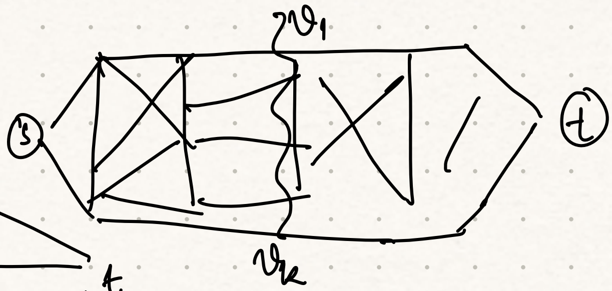
↳ $O(n^3 \log n)$ in the worst case.

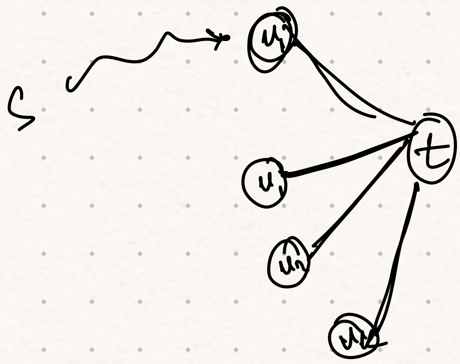
$$|E| = \frac{n^2}{1000}$$

<u>Qn:</u> Can we do better?

For the sake of simplicity, let us consider DAGs.







$$\text{Shortest } s\text{-}t \text{ path} = \min_{j \in [1,k]} \{ \text{shortest } (s\text{-}u_j) \text{ path} + w_{u_j\text{-}t} \}$$

$$= \min_{j \in [1,k]} \{ \text{Shortest } (s\text{-}v_j) \text{ path} + \text{Shortest } (v_j\text{-}t) \text{ path} \}$$

<u>Claim:</u> A shortest path between any pair of vertices has length at most $|V|$.

$$d(i,j) = \min_{\underline{u \in V}} \{ d(i,u) + d(u,j) \}$$

Attempt 1.

Fine

$\hat{d}(i,j,V)$: Shortest distance between i & j using all V vertices.

$V = \{1, \ldots, |V|\}$.

if G is DAG but if G is not a DAG. this could lead to cyclic dependence.

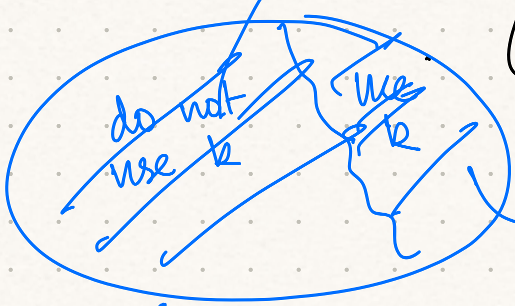
$\hat{d}(i,j,k)$: shortest distance between i&j using vertices $\{1, \ldots, k\}$.

$$\hat{d}(i,j,k) = \min \begin{cases} \hat{d}(i,j,k-1) \\ \hat{d}(i,k,k-1) + \hat{d}(k,j,k-1) \end{cases}$$

Shortest paths from i&j using $\{1, \ldots, k-1\}$.

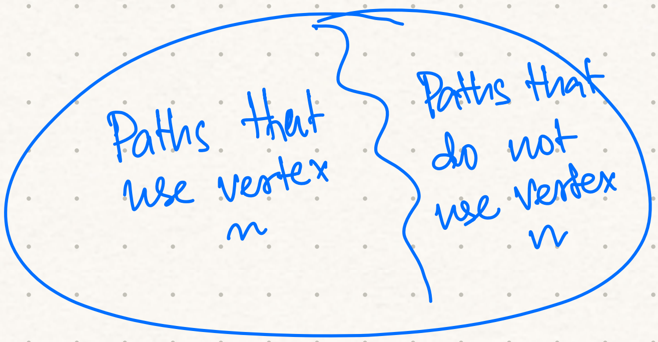
Shortest path from k to j using $\{1, .., k-1\}$.

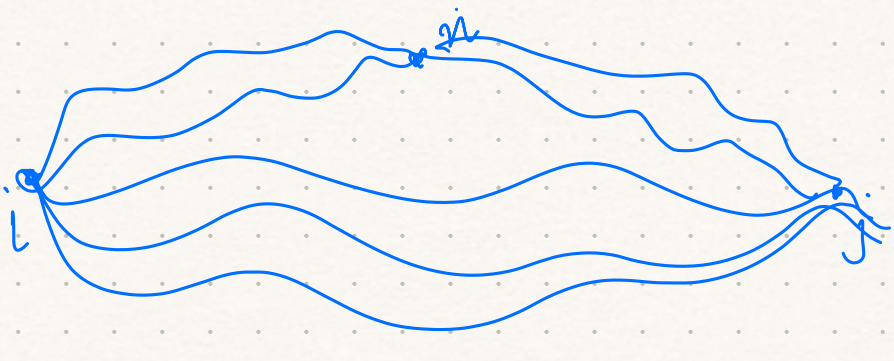
Shortest path from i to k using $\{1, \ldots, k-1\}$.



Paths from i to j over $\{1, \ldots, k\}$.

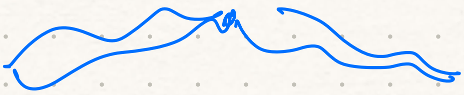
$i \to j$





$\{1, \ldots, n-1\}$

$$\hat{d}(i,j,k) = \min \begin{cases} \hat{d}(i,j,k-1) \\ \text{Shortest distance among all paths that go through } k \text{ and use } \{1,\cdots,k\}. \end{cases}$$



$$\left.\begin{array}{l} \hat{d}(i,k,k-1) \\ + \hat{d}(k,j,k-1) \end{array}\right\} = \begin{array}{l} \text{Shortest distance among all paths from } i \text{ to } k \text{ over } \{1,\cdots,k-1\}. \\ + \text{ Shortest distance among all paths from } k \text{ to } j \text{ over } \{1,\cdots,k-1\}. \end{array}$$

# of all possible sub problems is $O(|V|^3)$.

$\underline{\hat{d}(i,j,k)}$ also init $d(i,j,k) = w_{ij}$ if $(i,j) \in E$

Init: $\hat{d}(i,j,k) = \infty \ \forall\ i,j,k$.

For k in $[1,|V|]$:

  For i in V:

    For $j \neq i$ in V:

$$\tilde{d}(i,j,k) = \min \{ \hat{d}(i,j,k-1),\ \hat{d}(i,k,k-1) + \hat{d}(k,j,k-1) \}.$$

      Update $\hat{d}(i,j,k)$ if $\tilde{d}(i,j,k)$ is smaller.

When can $\hat{d}(i,j,1)$ make sense?

- If 1 is neighbour of both i & j.
  or i or $j = 1$ and 1 is a neighbour of the other.

$$\hat{d}(i,j,2) = \min \left\{ \begin{array}{l} \overset{w_{i,j}}{\hat{d}(i,j,1)} \\ \hat{d}(i,2,1) + \hat{d}(2,j,1) \end{array} \right\} \forall\ i,j$$

$\hat{d}(i,j,3)$

Each entry of this 3 dimensional array can be filled with $O(1)$ many lookups of already filled values.