Lecture 8

(29 August 2024)

Recap:

Random variable X

$X : \Omega \to R$ s.t. $\{X \le x\} \in \mathcal{F} \ \forall x \in R$

$$P_X(x) = P(X = x)$$

$$\sum_x P_X(x) = 1$$

$Y = g(X)$ is a random variable

$$P_Y(y) = \sum_{x : g(x) = y} P_X(x)$$

$$E[X] = \sum_x x P_X(x)$$

$$E[g(X)] = \sum_x g(x) P_X(x)$$

$$Var(X) = E[(X - EX)^2] = E[X^2] - (E[X])^2$$

Bernoulli, Binomial, Geometric RVs

Poisson Random Variable

A Poisson random variable takes values $0, 1, 2, \ldots$ with pmf

$$p_X(k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

for some $\lambda > 0$.

$$\sum_{k=0}^{\infty} p_X(k) = 1$$

In practice a Poisson random variable can be viewed as a limiting case of a binomial random variable.

Theorem: Consider a binomial RV Y with parameters n and p. As $n \to \infty$ while keeping $np = \lambda$ a constant, we have

$$\lim_{n \to \infty} p_Y(k) = e^{-\lambda} \cdot \lambda^k / k!\,.$$

Proof:

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n!}{(n-k)!\,k!} \left(\frac{\lambda}{n}\right)^k \left(1-\lambda/n\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \cdot \frac{n!}{(n-k)!} \cdot \frac{1}{n^k} \left(1-\lambda/n\right)^{n-k}$$

$$= \frac{n^k}{k!} \underbrace{\frac{(n-k+1)\cdot(n-k+2)\cdots n}{n^k}}_{\to 1} \underbrace{\left(1-\lambda/n\right)^{-k}}_{\to 1} \underbrace{\left(1-\lambda/n\right)^{n}}_{\to e^{-\lambda}}$$

$$\therefore \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{e^{-\lambda}\lambda^k}{k!}.$$

Mean of Poisson RV:

$$E[X] = \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{k \lambda^{k}}{k!}$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda .$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda} \cdot \lambda^{k}}{k!}$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{(k+1) \lambda^{k}}{k!}$$

$$= \lambda \sum_{k=0}^{\infty} \frac{k e^{-\lambda} \lambda^{k}}{k!} + \lambda = \lambda^2 + \lambda$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \lambda .$$

Two random variables X and Y on the probability space $(\Omega, \mathcal{F}, P)$ are called jointly discrete if (X, Y) takes values in some countable subset of R^2.

$$X : \Omega \longrightarrow R$$

$$Y : \Omega \longrightarrow R$$

Let $\text{Range}(X) = \mathcal{X}$, $\text{Range}(Y) = \mathcal{Y}$.

The associated joint pmf is given by

$$P_{XY}(x, y) = P(\{\omega : X(\omega) = x, Y(\omega) = y\})$$
$$= P(X = x, Y = y).$$

The pmfs of X and Y can be obtained from joint pmf using the formulas:

$$P_X(x) = \sum_{y \in \mathcal{Y}} P_{XY}(x, y)$$

$$P_Y(y) = \sum_{x \in \mathcal{X}} P_{XY}(x, y).$$

$$\begin{aligned}
P_X(x) &= P(X = x) \\
&= P(X = x \cap \Omega) \\
&= P\Big(X = x \cap \bigcup_{y \in \mathcal{Y}} (Y = y)\Big) \\
&= P\Big(\bigcup_{y \in \mathcal{Y}} \{X = x\} \cap \{Y = y\}\Big) \\
&= \sum_{y \in \mathcal{Y}} P(X = x, Y = y) \\
&\qquad\qquad \text{(by additivity)} \\
&= \sum_{y \in \mathcal{Y}} P_{XY}(x, y).
\end{aligned}$$

Similarly $P_Y(y) = \sum_{x \in \mathcal{X}} P_{XY}(x, y).$

Functions of Multiple Random Variables

Consider two jointly discrete random variables X and Y.

Let $Z = g(X, Y)$ i.e.,

$$Z(\omega) = g(X(\omega), Y(\omega)).$$

Analogous to the way we argued $g(X)$ is a random variable, $Z = g(X, Y)$ is also random variable.

$$P_Z(z) = \sum_{(x, y) : g(x, y) = z} P_{XY}(x, y).$$

Exercise

Prove that

$$E[g(X, Y)] = \sum_{x, y} g(x, y) P_{XY}(x, y).$$

Independence

Two discrete random variables X and Y are said to be independent if $p_{XY}(x,y) = p_X(x)\,p_Y(y)$ $\forall x, y$, i.e., the events $\{X=x\}$ and $\{Y=y\}$ are independent for all x, y.

Example. Two random variable X & Y take values in $\{0,1\}$ and p_{XY} is their joint PMF.

Suppose $p_{XY}(1,1) = p_X(1)\,p_Y(1)$.
Are X and Y independent?

$$p_{XY}(1,0) = p_X(1) - p_X(1,1)$$

$$= p_X(1) - p_X(1)\,p_Y(1)$$

$$= p_X(1)\,(1 - p_Y(1)) = p_X(1)\,p_Y(0).$$

Similarly $P_{XY}(x,y) = P_X(x)P_Y(y)$ $\forall x, y$.

Yes, X and Y are independent.

<u>Exercise</u>. Prove that the indicator random variables 1_A and 1_B are independent if and only if the events A and B are independent.