Lecture 13

(19 September 2024)

Some problems and In-class Queries

Q) Let F_1 and F_2 be two CDFs such that $F_1(x) < F_2(x)$ for all $x \in \mathbb{R}$. Assume that F_1 and F_2 are continuous and strictly increasing. Show that there exists RVs X_1 and X_2 with respective CDFs F_1 and F_2 defined on the same probability space such that $X_1 > X_2$.

Wrong Solution: Let $X_1 \ngtr X_2$

$$\Rightarrow X_1 \leq X_2$$

[This step is not correct]

$$F_{X_2}(x) = F_2(x)$$

$$= P(X_2 \le x)$$

$$\le P(X_1 \le x)$$

$$= F_{X_1}(x) = F_1(x)$$

$\therefore F_2(x) \ge F_1(x)$ — a contradiction

$\therefore F_1(x) < F_2(x) \;\forall x \Rightarrow X_1 > X_2$.

[of course this is wrong again]

The negation of $X_1 > X_2$ i.e.,

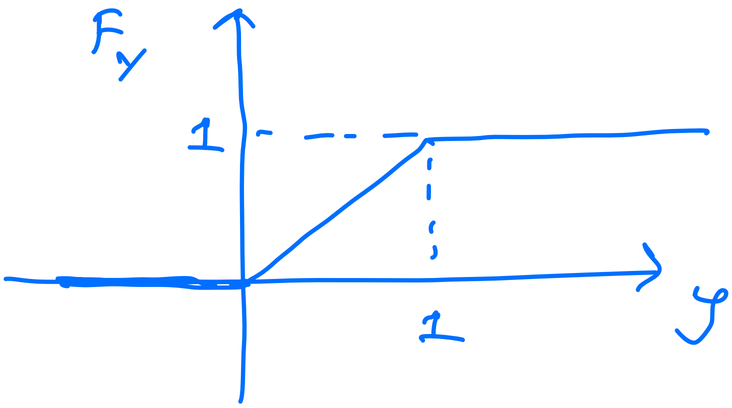
$X_1(\omega) > X_2(\omega) \;\forall \omega \in \Omega$ is

$$\exists \omega \text{ s.t. } X_1(\omega) \le X_2(\omega)$$

but not

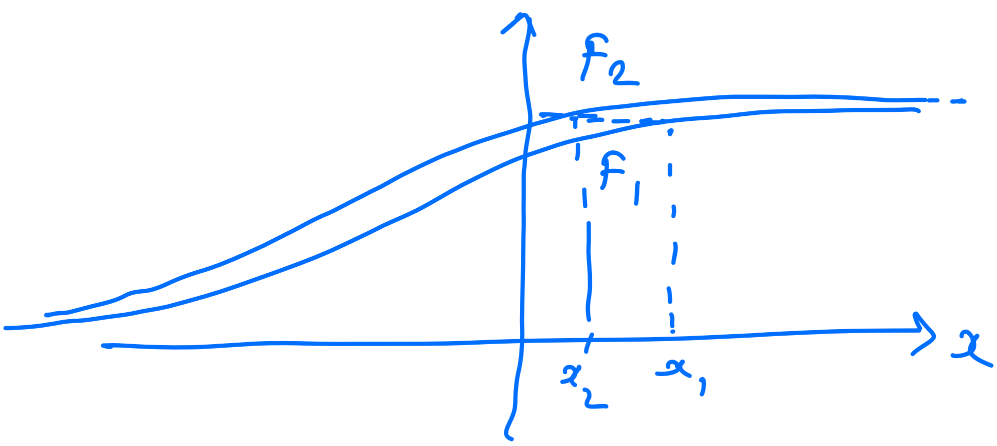
$$X_1(\omega) \le X_2(\omega) \;\forall \omega \in \Omega.$$

Correct Solution Consider a RV Y with the following CDF.



$$F_Y(y) = \begin{cases} 0, & y < 0 \\ y, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

Given two CDFs F_1 & F_2 which are continuous and strictly increasing.



$F_1(x_1) = F_2(x_2) = y$

$\Rightarrow x_1 > x_2$.

construct RV X_1 & X_2 as

$X_1 = F_1^{-1}(Y)$ $\quad X_2 = F_2^{-1}(Y)$.

we claim that CDF of $F_i^{-1}(Y)$ is F_i.

$$P(X_i \le x_i) = P(F_i^{-1}(Y) \le x_i)$$

$$= P(Y \le F_i(x_i))$$

$$= F_i(x_i), \quad i = 1, 2.$$

$F_1(x) < F_2(x) \ \forall x \implies F_1^{-1}(y) > F_2^{-1}(y) \ \forall y \in [0,1]$

Suppose $x_1 = F_1^{-1}(y) \le F_2^{-1}(y) = x_2$.

$$y = F_2(x_2) > F_1(x_2)$$

$$\ge F_1(x_1)$$

$$= y \longrightarrow \text{A contradiction.}$$

$$\therefore X_1 = F_1^{-1}(Y) > F_2^{-1}(Y) = X_2.$$

Q) In class we defined conditional variance as

$$var(X \mid Y=y) = E\left[(X - E[X \mid Y=y])^2 \mid Y=y \right].$$

The query is "why is the conditioning $Y=y$ required after the square"?

Not required i.e. the expression

$$E\left[(X - E[X \mid Y=y])^2 \right]$$

is a well-defined quantity and it is a real number.

However if we define variance in this way (without $Y=y$) then we will not have

$$Var(X \mid Y=y) = E[X^2 \mid Y=y] - E[X \mid Y=y]^2$$

analogous to $var(X) = E[X^2] - E[X]^2$.

$\therefore$ $Var(X \mid Y=y) = E\left[(X - E[X \mid Y=y])^2 \right]$ is apt.

Q) What is the intuition for the law of total variance?

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y]).$$

Before we look at the intuition let us first recall the definitions of $E[X|Y]$ and $\mathrm{Var}(X|Y)$.

$$E[X|Y=y] = \sum_x x\, p_{X|Y}(x|y)$$

Let $\phi(y) = E[X|Y=y]$.

$$E[X|Y] = \phi(Y) = \begin{cases} E[X|Y=y_1] & \text{w.p. } p_Y(y_1) \\ E[X|Y=y_2] & \text{w.p. } p_Y(y_2) \\ \vdots & \end{cases}$$

$$\mathrm{Var}(X|Y) = \psi(Y) = \begin{cases} \mathrm{Var}(X|Y=y_1) & \text{w.p. } p_Y(y_1) \\ \mathrm{Var}(X|Y=y_2) & \text{w.p. } p_Y(y_2) \\ \vdots & \end{cases}$$

We understand the intuition through an example. Consider students in three sections as below.

Section 1	Section 2	Section 3
St. 1	St. 1	St. 1
St. 2	St. 2	St. 2
⋮	⋮	⋮
St n_1	St n_2	St n_3

Let X = quiz score of a random student

Y = section of random student

$Y \in \{1, 2, 3\}$ $\quad X|Y=i$ takes n_i values.

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y]).$$

$\mathrm{Var}(X|Y=i)$ is the variance of quiz scores within section i.

St 1
St. 2
⋮
St n_1

St. 1
St. 2
⋮
St n_2

St. 1
St. 2
⋮
St n_3

$Var(X|Y=1)$ $\quad Var(X|Y=2)$ $\quad Var(X|Y=3)$

$E[Var(X|Y)]$ is the average of these variances

$$E[Var(X|Y)] = P_Y(1)\,Var(X|Y=1) + P_Y(2)\,Var(X|Y=2) + P_Y(3)\,Var(X|Y=3)$$

However this does not capture how the quiz scores vary across the sections. This is exactly captured by the second term in the law of total variance.

St. 1
St. 2
⋮
St. n_1

St. 1
St. 2
⋮
St. n_2

St. 1
St. 2
⋮
St. n_3

$E[X|Y=1]$ $\quad$ $E[X|Y=2]$ $\quad$ $E[X|Y=3]$

average quiz scores in each section

$Var(E[X|Y])$ is variance of these expected values.

$$Var(E[X|Y]) = \sum_y \left(E[X|Y=y] - E[X]\right)^2 p_Y(y)$$

Thus $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

= Avg. of variances within individual sections

+

variance of averages between sections.

Recap of Last Lecture:

A RV X is said to be continuous if its CDF can be expressed as

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du \quad x \in \mathbb{R}$$

for some integrable function $f: \mathbb{R} \to [0,\infty)$ called the probability density function (PDF).

$$f_X(x) = F_X'(x)$$

$$= \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}$$

$$P(X \in B) = \int_B f_X(x)\,dx$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

If $X \geq 0$ then

$$E[X] = \int_{0}^{\infty} P(X > x)\, dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Uniform RV:

$$f_X(x) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & \text{o.w.} \end{cases}$$

Exponential RV: $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

Exercise. Show that for an exponential RV

$$P(X > s+t \mid X > s) = P(X > t)$$

for $s, t \geq 0$.