Imp. formulae

→ RV X is continuous if $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

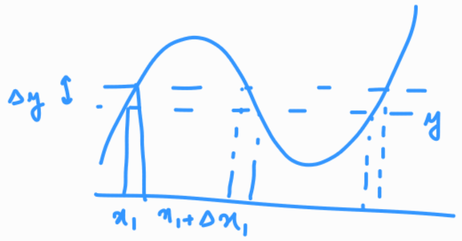
$= P(X \le x)$

→ $P(X \in B) = \int_B f_X(x)\,dx$

For infinitesimally small interval, $f_X(x)$ is assumed to be const.

$\therefore P(x \le X \le x + \Delta x) \cong f_X(x)\,\Delta x$

→ Jointly conti. when $F_{XY}(x,y) = \int_{v=-\infty}^{\infty} \int_{u=-\infty}^{\infty} f_{XY}(u,v)\,du\,dv$



$X \sim f_X$

f_Y ?

$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|}$, x_i are sol^n of $g(x) = y$.

$P(y < Y \le \Delta y) \cong f_X(x_1)|\Delta x_1| + f_X(x_2)|\Delta x_2| + f_X(x_3)|\Delta x_3|$

$= f_Y(y)\,\Delta y$

$\Rightarrow f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{\left|\frac{\Delta y}{\Delta x_i}\right|} \xrightarrow{\Delta y \to 0} \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|}$, $x_i = a_i(y)$ are roots of $g(x) = y$

→ If there are finite/countable points where $g(x)$ is not differentiable, still the above formula can be used.

⇒ Compute roots of $g(x) = y$ and then use the above formula.

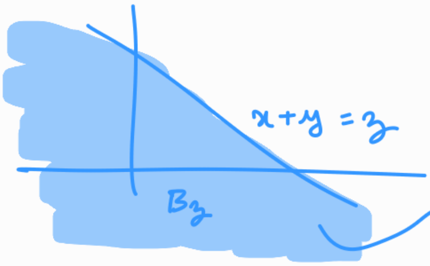
→ General procedure to get PDF, get CDF and differentiate.

Functions of two RV

X, Y are continuous ; independent.

$$P((X,Y) \in B) = \int_B f_{XY}(x,y)\,dx\,dy$$

$Z = X + Y$



$x \in (-\infty, \infty)$ and $y \in (-\infty, x)$

$$F_Z(z) = P(Z \le z)$$

$$= P(X + Y \le z)$$

$$= P((X,Y) \in B_z) \quad ; \quad B_z = \{(x,y) : x+y \le z\}.$$

$$= \iint_{(x,y) \in B_z} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_X(x) f_Y(y)\,dy\,dx \qquad (\because \text{Independent}).$$

– what is CONVOLUTION?

$$= \int_{x=-\infty}^{\infty} f_X(x) \left(\int_{y=-\infty}^{z-x} f_Y(y)\,dy \right) dx = \int_{x=-\infty}^{\infty} f_X(x) F_Y(z-x)\,dx$$

* Some similarity to convolution.

$$\Rightarrow f_Z(z) = \frac{d}{dz} \left(\int_{x=-\infty}^{\infty} f_X(x) F_Y(z-x)\,dx \right)$$

$$= \int_{x=-\infty}^{\infty} f_X(x) \frac{d}{dz}\left(F_Y(z-x)\right) dx = \int_{x=-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

Convolution of f_X and f_Y.

By symmetry,

$$\int_{x=-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx = \int_{y=-\infty}^{\infty} f_Y(y) f_X(z-y)\,dy$$

* For convolution, we need independence?

Exercise.

$X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$; X, Y are independent.

$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ using convolution.

• Y continuous, X discrete and X and Y are independent.

Claim: $Z = X + Y$ is conti.

$f_Z(z) = ?$

$F_Z(z) = P(X + Y \leq z)$ (∵ X is discrete Total prob. th^m)

$= \sum_x P(X+Y \leq z \mid X = x) P_X(x)$

$= \sum_x P(Y \leq z - x \mid X = x) P_X(x)$

$= \sum_x P(Y \leq z - x) P_X(x)$ (∵ X and Y are indep. $P(Y|X) = P(Y)$)

$= \sum_x F_Y(z-x) P_X(x)$

$\Rightarrow f_Z(z) = \sum_x f_Y(z-x) P_X(x)$

→ Sum of 2 discrete RV ✓
1 conti, 1 discrete RV ✓
2 conti RV ✓

• Functions of 2 RVs

$X, Y \sim$ Uniform $[0,1]$ and $X \perp\!\!\!\perp Y$ (Independent).

$Z = Y/X$.

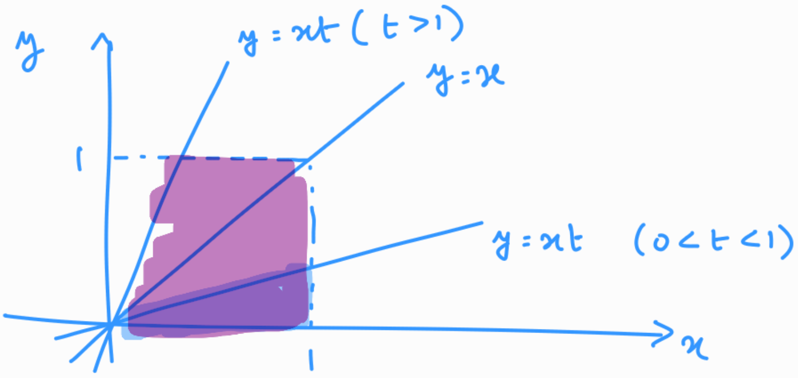
* If two RV are independent, we need not mention that it is jointly continuous. It is, by default, jointly conti.

$$f_{XY}(x,y) = \begin{cases} 1, & 0 \leq x, y \leq 1 \\ 0, & \text{o.w.} \end{cases}$$

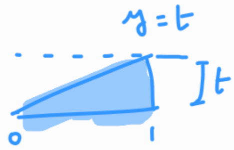
$F_Z(t) = P(Z \leq t)$

$= P(Y \leq Xt)$ (∵ X is +ve, inequality remains same).

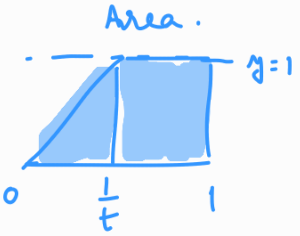
$= P(X, Y) \in B_t$ Here $B_t = \{(x,y) : y \leq xt\}$.



For $t<1$, $\quad F_Z(t) = \int_{B_t} 1 \, dx\, dy = \frac{1}{2}(t)(1) = \frac{t}{2}$



For $t>1$, $\quad F_Z(t) = \int_{B_t} 1 \, dx\, dy = (1)\left(1-\frac{1}{t}\right) + \frac{1}{2}\left(\frac{1}{t}\right)(1)$



$$= 1 - \frac{1}{t} + \frac{1}{2t} = 1 - \frac{1}{2t}$$

- <u>Two functions of two RVs</u>

$(X,Y) \sim f_{X,Y}$

$Z = g_1(X,Y)$, $\quad W = g_2(X,Y)$

$f_{ZW}(z,w) = ?$

General procedure:

1) Compute joint CDF.
2) Differentiate F to get f.

1) $F_{Z,W}(z,w) = P(Z \le z, W \le w)$

$= P(g_1(X,Y) \le z, \ g_2(X,Y) \le w)$

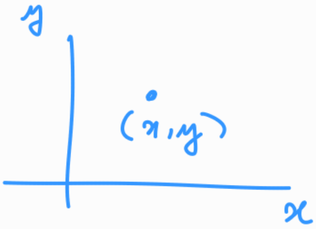
$= P((X,Y) \in B_{zw})$ $\quad$ ($\because$ We are writing in terms of (X,Y) because we only know f_{XY}).

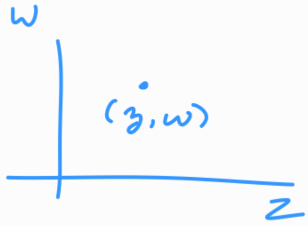
where $B_{zw} = \{(x,y) : g_1(x,y) \le z, \ g_2(x,y) \le w\}$.

$$F_{ZW}(z,w) = \int_{B_{ZW}} f_{XY}(x,y)\,dx\,dy \implies f_{ZW}(z,w) = \frac{\partial^2 F_{ZW}(z,w)}{\partial z\,\partial w}$$

Special cases.

$$g_1(X,Y) : aX + bY = z$$
$$g_2(X,Y) : cX + dY = w$$





$ad - bc \neq 0$.

So matrix is invertible.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} z \\ w \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^{-1}\begin{bmatrix} z \\ w \end{bmatrix}$$

Generalisation

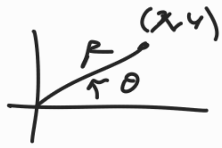
$g_1(x,y) = z$, $g_2(x,y) : w$

Continuous and differentiable

For such g_1, g_2 there is a formula

- The mapping $\underbrace{(g_1,g_2)}_{(g_1(x,y),\ g_2(x,y))} : (x,y) \longmapsto (z,w)$ is one-to-one.

Eg: $R = g_1(X,Y) = \sqrt{X^2+Y^2}$, $\theta = g_2(X,Y) = \tan^{-1}\left(\frac{Y}{X}\right)$



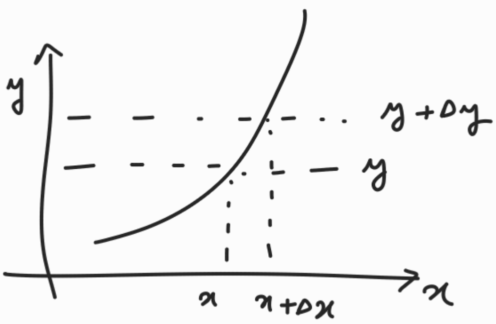
For one (x,y), there's only one (r,θ) & vice-versa.

Here θ is not differentiable only at finite no of points. So it's fine. still valid.

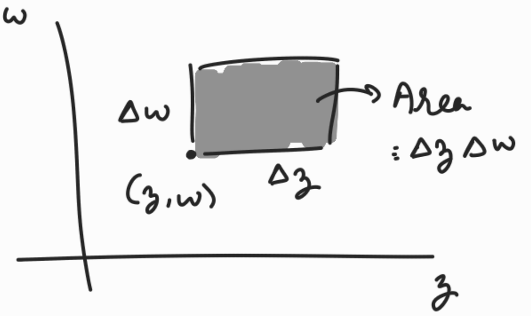
→ $z = g_1(x,y)$, $w = g_2(x,y)$

$\Rightarrow x = h_1(z,w)$, $y = h_2(z,w)$

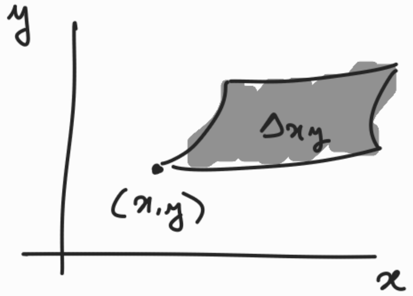
→



$$\frac{\Delta y}{\Delta x} \xrightarrow{\Delta y \to 0} \frac{dy}{dx}$$



$=$



$$\frac{\Delta z\, \Delta w}{\Delta_{xy}} \xrightarrow{\Delta z,\, \Delta w \to 0} \text{JACOBIAN.}$$

$\|^{lar}$ to derivative in the above case of $\frac{\Delta y}{\Delta x}$.

$$J(x,y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix}$$

Z, W.

$P(z < Z \le z + \Delta z,\ w < W \le w + \Delta w)$

$= f_{ZW}(z,w)\, \Delta z \Delta w.$ ($\because$ f is const. in that small area)

$= P((X,Y) \in \Delta_{xy})$ $=$ $\because$ there is one-one mapping of (z,w) to (x,y)

$= f_{XY}(x,y)\, \Delta_{xy}$ → Area of that region.

$$\Rightarrow f_{ZW}(z,w) = \frac{f_{XY}(x,y)}{\left(\frac{\Delta z\, \Delta w}{\Delta_{xy}}\right)} = \left.\frac{f_{XY}(x,y)}{|J(x,y)|}\right|_{\substack{x = h_1(z,w) \\ y = h_2(z,w)}}$$

Change of var. in Gaussian.

Change of var. thm (Calculus)

$$\int_A f_{XY}(x,y)\, dx\, dy = \int_B \frac{f_{XY}(z,w)}{|J(h_1(z,w), h_2(z,w))|}\, dz\, dw.$$

$r\, dr\, d\theta$.

$z = r\cos\theta, \quad w = r\sin\theta$

$\frac{1}{r}$ = Jacobian ↓ Exercise

$$\left(J(x,y)\right) J(z,w) = 1$$

Proof analogous to.

$$\vdots \quad \frac{dx}{dy} \cdot \frac{dy}{dx} = 1$$