Practice Problem Set 2

(MA6.102) Probability and Random Processes, Monsoon 2024

Problem 1. Consider a probabilistic model with sample space Ω and probability law P. Let $\{C_1, C_2, \ldots, C_n\}$ be a partition of Ω. For two events A and B, suppose we know that

- A and B are conditionally independent given C_i, i.e., $P(A \cap B | C_i) = P(A|C_i)P(B|C_i)$, for all $i \in \{1, 2, \ldots, n\}$;
- B is independent of C_i, i.e., $P(B \cap C_i) = P(B)P(C_i)$, for all $i \in \{1, 2, \ldots, n\}$.

Are A and B independent events?

Problem 2. Consider a positive integer-valued random variable Y whose CDF at integer vlaues is given by

$$F_Y(k) = 1 - \frac{2}{(k+1)(k+2)}, \text{ for integer values } k \geq 0.$$

(a) Compute $\mathbb{E}[Y]$ without finding the PMF P_Y.
(b) Let X be another integer-valued random variable with the conditional PDF given by

$$P_{X|Y}(x|y) = \frac{1}{y}, \text{ for } x \in \{1, 2, \ldots, y\}.$$

Find $\mathbb{E}[X]$.

Problem 3. Suppose that $M_X(s) < \infty$, for some $s > 0$. Show that $M_X(t) < \infty$, for all $t \in [0, s]$.

Problem 4. Let X be a random variable with mean μ and variance σ^2. Then prove that, for $c > 0$,

$$P(X - \mu \geq c) \leq \frac{\sigma^2}{\sigma^2 + c^2}.$$

Problem 5. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) random variables with $X_i \sim \text{Exponential}(\lambda)$, and let $N \sim \text{Geometric}(\beta)$ be an independent geometric random variable. Define $T = X_1 + X_2 + \cdots + X_N$, the sum of a random number N of i.i.d. exponential random variables. Show that $T \sim \text{Exponential}(\lambda\beta)$.

Hint: Use moment generating functions.

Problem 6. Let $X_1, Y_1, X_2, Y_2, \ldots$ be independent random variables, uniformly distributed in the unit interval $[0, 1]$, and let $W = \frac{\sum_{i=1}^{16} X_i - \sum_{i=1}^{16} Y_i}{16}$. Find a numerical approximation to the quantity $P(|W - \mathbb{E}[W]| < 0.001)$.

Problem 7. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables such that $X_n \sim \text{Geometric}(\frac{\lambda}{n})$, $n \in \mathbb{N}$, where $\lambda > 0$ is a constant. Define a new sequence Y_n as $Y_n = \frac{X_n}{n}$, $n \in \mathbb{N}$. Show that Y_n converges in distribution to $\text{Exponential}(\lambda)$.

Problem 8. Let $(N_t, t \in [0, \infty))$ be a Poisson process with rate λ. Find the probability that there are exactly two arrivals in $(0, 2]$ and exactly three arrivals in $(1, 4]$ (Note that the intervals $(0, 2]$ and $(1, 4]$ are not disjoint, so the number of arrivals in each interval are not independent).

Problem 9. Let $X_t = A \cos(\omega_c t + \Theta)$, where ω_c is a non-zero constant, A and Θ are independent random variables with $P(A > 0) = 1$ and $\mathbb{E}[A^2] < \infty$. If Θ is uniformly distributed over $[0, 2\pi]$, show that X_t is wide-sense stationary (WSS). Is X_t strict-sense stationary also?

Problem 10. Consider a WSS process X_t with autocorrelation $R_X(\tau) = \mathrm{e}^{-a|\tau|}$, where $a > 0$, for all $\tau \in \mathbb{R}$. Find the power spectral density of X_t.

All the best for end-semester examinations