Strong law of large nos. (SLLN)

$$\left.\begin{array}{l} a.s \\ \text{in } P \\ \text{in } D \\ M.S \end{array}\right\} \rightarrow \text{Hierarchy of conv.}$$

Convergence of functions : $f_n \longrightarrow f$.

$f, f_n : \mathbb{R} \to \mathbb{R}$.

$f_n(x) \to f(x) \ \forall \ (x)$. [Sure convergence].

• <u>Almost sure convergence</u> (a.s) [Strongest notion of convergence]

$X_1, X_2, \ldots$ converges to X almost surely if $(X_n : \Omega \to \mathbb{R})$.

$$P(\underbrace{\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}}) = 1$$

Need not be equal to Ω.

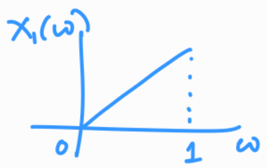
It's just a subset of ω with prob. 1.

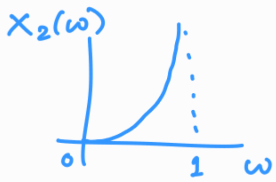
Ω, $A \subseteq \Omega$
$P(A) = 1$.

<u>Example</u> : $\Omega = [0,1]$.

$P([a,b]) = b-a$

$X_n(\omega) = \omega^n$, $\omega \in [0,1]$.





For a fixed ω

$$\lim_{n\to\infty} X_n(\omega) = \begin{cases} 0 & , \ \omega \in [0,1) \\ 1 & , \ \omega = 1 \end{cases}$$

Claim : $X_n \xrightarrow{a.s} 0$

Here $X(\omega) = 0$. ω is $[0,1)$ and $P([0,1)) = 1$.

($\because$ Continuity of prob and one point won't affect prob.).

WLLN : $X_1, X_2, \ldots$ be seq. of iid, then sample mean converges to true mean in prob.

i.e., $P\left(\left|\frac{\sum_{i=1}^{n} X_i}{n} - \mu\right| > \varepsilon\right) \to 0$ as $n \to \infty$ for every $\varepsilon > 0$.

All limit th^m, we have seq. to be iid.

• Strong law of large numbers :

$X_1, X_2, \ldots$ i.i.d RVs, mean μ, then

$$\underbrace{\frac{\sum_{i=1}^{n} X_i}{n}}_{= M_n} \xrightarrow{a.s} \mu \quad \text{i.e., } P\left(\left\{\omega : \lim_{n\to\infty} M_n(\omega) = \mu\right\}\right) = 1$$

In the proof of WLLN, we assumed $E[X^2] < \infty$.

|||^{ly} in the proof of SLLN, we need to assume $E[X^4] < \infty$. For ease of proof

(But SLLN also applies to cases where $E[X^4]$ is not less than ∞)

1) $X_n \xrightarrow{a.s} X$ if $P\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1$

Using this we stated SLLN.

2) Convergence in prob : $X_n \xrightarrow{P} X$ if $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$.

Using this, we stated WLLN.

3) Convergence in distribution : $X_n \xrightarrow{d} X$ if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Using this, we stated CLT.

4) Mean squared convergence / Convergence in mean square sense : $X_n \xrightarrow{m.s} X$ if $\lim_{n\to\infty} E[(X_n - X)^2] = 0$

• Hierarchy of convergence

$$\left(X_n \xrightarrow{a.s} X\right) \Rightarrow \left(X_n \xrightarrow{P} X\right) \Rightarrow \left(X_n \xrightarrow{D} X\right)$$

$$\left(X_n \xrightarrow{m.s} X\right) \Rightarrow \left(X_n \xrightarrow{P} X\right)$$

(Refer last class · last part).

In general, no other implication is true. But if $X = c$ then $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$ (Special case). So when X is const. conv. in $d \Leftrightarrow$ conv. in p

Proof of $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$.

Suppose $\lim\limits_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$.

To show $\lim\limits_{n\to\infty} F_{X_n}(x) = F_X(x)$

$F_{X_n}(x) = P(X_n \le x)$

$= P(X_n \le x, X > x+\varepsilon) + P(X_n \le x, X \le x+\varepsilon)$ $\quad (\because P(A) = P(A\cap B) + P(A \cap B^c).)$

This implies this. So a subset of this. So less prob. $\quad \because P(A\cap B) \le P(A)$ or $P(B)$.

$\le P(|X_n - X| > \varepsilon) + P(X \le x+\varepsilon)$.

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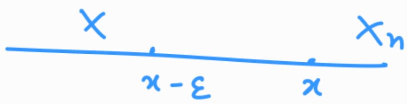
$F_X(x-\varepsilon) = P(X \le x - \varepsilon)$



$= P(X \le x-\varepsilon, X_n > x) + P(X \le x - \varepsilon, X_n \le x)$

$\le P(|X_n - X| > \varepsilon) + P(X_n \le x)$.

$\Rightarrow F_X(x-\varepsilon) - P(|X_n - X| > \varepsilon) \le P(X_n \le x) = F_{X_n}(x)$

$F_{X_n}(x) \le P(|X_n - X| > \varepsilon) + F_X(x+\varepsilon)$

$$\Rightarrow F_X(x-\varepsilon) - P(|X_n - X| > \varepsilon) \le F_{X_n}(x) \le F_X(x+\varepsilon) + P(|X_n - X| > \varepsilon)$$

Now as $n \to \infty$, given $X_n \xrightarrow{P} X$. So $\lim\limits_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$.

$$\Rightarrow \underbrace{F_X(x-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(x) \le F_X(x+\varepsilon)}_{\text{SANDWICH TH}^{M}.} \quad \forall\ \varepsilon>0.$$

Hence, proved $\left(X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X \right)$.

Proof of $X_n \xrightarrow{a.s} X \Rightarrow X_n \xrightarrow{P} X$

Suppose $P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$.

To show : $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$. i.e., $\lim_{n\to\infty} P(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) = 0$

Events.

Let $A_n = \{\omega : |X_n(\omega) - X(\omega)| < \varepsilon\}$.

$B_n = \{\omega : |X_k(\omega) - X(\omega)| < \varepsilon \ \forall\ k \ge n\}$.

$B_1 = A_1 \cap A_2 \cap A_3 \cap \ldots$, $B_2 = A_2 \cap A_3 \cap \ldots$

$\Rightarrow B_1 \subseteq B_2 \subseteq \ldots$

$\Rightarrow B_n = \bigcap_{k\ge n} A_k$

$\Downarrow$

$A_n \supseteq B_n$.

$\because \lim_{n\to\infty} P(B_n) = P\left(\bigcup_{n=1}^{\infty} B_n\right) \overset{?}{=} 1$

If we show this then $P(A_n) \to 1$ as $n \to \infty$.

(Because $A_n \supseteq B_n$ So $P(A_n) \ge P(B_n)$ But $P(A_n)$ can't exceed 1, so it's equal to 1).

$B = \bigcup_{n=1}^{\infty} B_n$, $B_{n_0} = \{\omega : |X_n(\omega) - X(\omega)| < \varepsilon, \ \forall\ n \ge n_0\}$

Claim: $\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} \subseteq B$.

$X_n(\omega) \xrightarrow{a.s} X(\omega)$

$\Rightarrow \exists\ n_0$ s.t $|X_n(\omega) - X(\omega)| < \varepsilon \ \forall\ n \ge n_0$

$\Rightarrow \omega \in B_{n_0}$

$\Rightarrow \omega \in \bigcup_{n=1}^{\infty} B_n$.

So if we show that $P(A_n) = 1$ then $P(A_n^c) = 0$

Here we are interchangeably using equivalence b/w def^n of conv. in P

$P(|X_n - X| > \varepsilon) \Leftrightarrow P(|X_n - X| \ge \varepsilon)$

$\Rightarrow P(B) = 1 \quad \Rightarrow \quad P(A) = 1.$

Hence proved.

These defn are equivalent in case of conv. in P.