CS 302.1 - Automata Theory

Lecture 05

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Quick Recap

Grammars: A Grammar is a 4-tuple (V, Σ, P, S), such that

- V is the set of **Variables**
- Σ is the set of **Terminals**
- P is the set of production **Rules** [$(\boldsymbol{V} \cup \boldsymbol{T})^{*}\boldsymbol{V}(\boldsymbol{V} \cup \boldsymbol{T})^{*} \rightarrow (\boldsymbol{V} \cup \boldsymbol{T})^{*}$]
- S is the **Start Variable** [The variable in the LHS of the first rule is generally the start variable]

- To show that a string $\boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{G})$, we show that there exists a **derivation ending up in $\boldsymbol{w}$** $(\boldsymbol{S} \overset{*}{\Rightarrow} \boldsymbol{w})$.
- The **language of the grammar, $\boldsymbol{L}(\boldsymbol{G})$** is $\{w \in \Sigma^{*} | S \overset{*}{\Rightarrow} w\}$

Right Linear grammar: If the *rules* of the underlying grammar G are of the form

$$Var \rightarrow Ter\ \boldsymbol{Var}$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then it is **Right-linear grammar.**

Left linear grammar: If the *rules* of the underlying grammar G are of the form

$$Var \rightarrow\ \boldsymbol{Var}\ Ter$$
$$Var \rightarrow Ter$$
$$Var \rightarrow \boldsymbol{\epsilon}$$

then such a grammar is called **Left-linear grammar.**

Left-linear grammar $\equiv$ Right-linear grammar $\equiv$ DFA $\equiv$ NFA $\equiv$ Regular Expressions

Quick Recap

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- P is the set of production **Rules** [$(\boldsymbol{V} \cup \boldsymbol{T})^* \boldsymbol{V} (\boldsymbol{V} \cup \boldsymbol{T})^* \rightarrow (\boldsymbol{V} \cup \boldsymbol{T})^*$]
- S is the **Start Variable** [The variable in the LHS of the first rule is generally the start variable]

- To show that a string $\boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{G})$, we show that there exists a **derivation ending up in $\boldsymbol{w}$** $(\boldsymbol{S} \overset{*}{\Rightarrow} \boldsymbol{w})$.
- The **language of the grammar, $\boldsymbol{L}(\boldsymbol{G})$** is $\{w \in \Sigma^* | S \overset{*}{\Rightarrow} w\}$

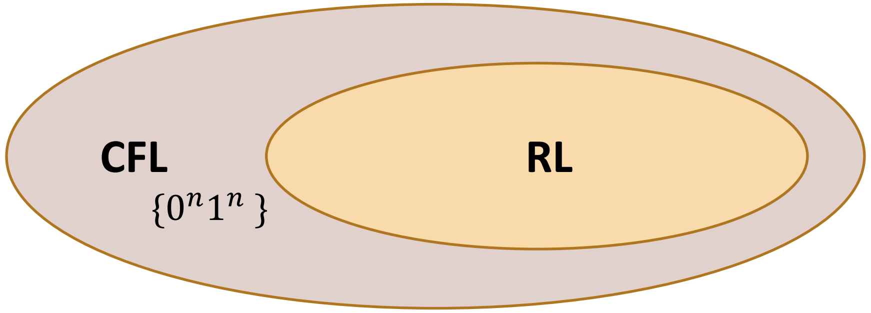
Context-Free Grammars: If the *rules* of the underlying grammar G are of the form

$$\boldsymbol{V} \rightarrow (\boldsymbol{V} \cup \boldsymbol{T})^*$$

then such a grammar is called **Context-Free.**

$$L(G) = \{\omega | \omega = 0^n 1^n, n \geq 0\}$$

So although $\boldsymbol{L}(\boldsymbol{G})$ is not regular, it is context-free.



Context free Grammars

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$$\boldsymbol{V} \to (\boldsymbol{V} \cup \boldsymbol{T})^*$$

then such a grammar is called **Context-Free.**

Regular languages $\subset$ **Context Free Languages**.

Consider the Grammar G with the following rules:

$$S \to 0S1|SS|\epsilon$$

Strings that can be derived by G:

$$S \to \epsilon$$

$$\{\epsilon\}$$

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$$S \rightarrow 0\boldsymbol{S}1 \rightarrow 0\boldsymbol{0S1}1 \ldots$$

$$\{\epsilon, 01, 0011, \ldots 0^n1^n\}$$

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$$S \rightarrow \boldsymbol{0S1} \rightarrow 0\boldsymbol{SS}1 \rightarrow 0\boldsymbol{0S1}S1 \rightarrow 001S1 \rightarrow 001\boldsymbol{0S1}1 \rightarrow 001011$$

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$$\{\epsilon, 01, 0011, \ldots 0^n 1^n, 001011, \ldots.\}$$

Show that the string $\mathbf{010101} \in \boldsymbol{L}(\boldsymbol{G})$.

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Strings that can be derived by G:

$$S \rightarrow \boldsymbol{SS} \rightarrow \boldsymbol{SS}S \rightarrow \mathbf{0}\boldsymbol{S}\mathbf{1}SS \rightarrow 0S1\mathbf{0}\boldsymbol{S}\mathbf{1}S \rightarrow 0S10S1\mathbf{0}\boldsymbol{S}\mathbf{1} \rightarrow 010101$$

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What is $\boldsymbol{L(G)}$?

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$$\{\epsilon, ()\}$$

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$$\{\epsilon, (\quad), ((\quad)), \dots ., \}$$

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$$\left\{\epsilon, (\ \ ), ((\ \ )), \ldots, \left(\left(\left((\ldots.)\right)\right)\right), ((\ \ )(\ \ )), ()()(), \ldots.\right\}$$

So, $L(G)$ is the language of all strings of properly nested parentheses.

$$L(G) = \{\omega | \omega \text{ is a correctly nested parenthesis}\}$$

Context free Grammars

Constructing CFG corresponding to a Language.

There is no fixed recipe for doing this. Requires some level of creativity.

Some tips might come in handy:

- Check if the CFL is a union of simpler languages. If $L(G) = L(G_1) \cup L(G_2)$ and G_1 and G_2 are known. If S_1 is the start variable for G_1 and S_2 is the start variable for G_2 then the the rules of G:

$$
\begin{aligned}
S &\rightarrow S_1 | S_2 \\
S_1 &\rightarrow \cdots\cdots \\
S_2 &\rightarrow \cdots\cdots
\end{aligned}
$$

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$$S \rightarrow S_1 | S_2$$
$$S_1 \rightarrow \cdots\cdots$$
$$S_2 \rightarrow \cdots\cdots$$

- Grammars with rules such as $S \rightarrow aSb$ help generate strings where the corresponding machine would need unbounded memory to *remember* the number of *a's* needed to verify that there are an equal number of *b's*. This was not possible with regular expressions/linear grammars.

Context free Grammars

Constructing CFG corresponding to a Language.

- Check if the CFL is a union of simpler languages.
- Grammars with rules such as $S \rightarrow aSb$ help generate where the portions of *a* and *b* are equal.

Example: Construct the grammar G such that $\boldsymbol{L(G) = \{\omega | \omega}$ **has equal number of** $\boldsymbol{0}$**'s and** $\boldsymbol{1}$**'s**$\}$

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- The first thing to notice is that $L_1 = \{0^n 1^n, n \geq 0\} \subset L(G)$. We know the grammar for this language.
- Any string $\omega \in L_1$ has a series of 0's followed by an equal number of 1's.
- Again, consider L_2 to comprise all strings that start with a series of 1's followed by an equal number of 0's, i.e.

$$L_2 = \{1^n 0^n, n \geq 0\}$$

- The grammar for L_2 is similar to that of L_1: replace the 0's with 1's and vice versa. Importantly, $L_2 = \{1^n 0^n, n \geq 0\} \subset L(G)$ also.

- Also, $L_1 \cup L_2 \subset L(G)$

Context free Grammars

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Example: Construct the grammar G such that $\boldsymbol{L(G)} = \{\boldsymbol{\omega}|\boldsymbol{\omega}$ **has equal number of 0's and 1's**$\}$

- So $\mathrm{L}'(\mathrm{G}') = \{0^n 1^n | n \geq 0\} \cup \{1^n 0^n | n \geq 0\} \subset L(G)$

- Grammar for L_1: $S \rightarrow 0S1|\epsilon$

- Grammar for L_2: $S \rightarrow 1S0|\epsilon$

- Grammar for $L_1 \cup L_2$:

$$S \rightarrow S_1|S_2$$
$$S_1 \rightarrow 0S_11|\epsilon$$
$$S_2 \rightarrow 1S_20|\epsilon$$

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- Grammar for $L_1 \cup L_2$:

$$S \rightarrow S_1 | S_2$$
$$S_1 \rightarrow 0S_1 1 | \epsilon$$
$$S_2 \rightarrow 1S_2 0 | \epsilon$$

- **Is that all? Is** $\boldsymbol{L_1 \cup L_2 = L(G)}$? $L_1 \cup L_2$ contains all strings that have equal number 0's followed by equal number of 1's or vice versa.

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- **Is that all? Is $\boldsymbol{L_1 \cup L_2 = L(G)}$?** $L_1 \cup L_2$ contains all strings that have equal number 0's followed by equal number of 1's or vice versa.
- What about strings such as $s_1 = 0101\cdots$ and $s_2 = 1010\cdots$? For this we need to be able to go from

$$0\boldsymbol{S_1}1 \rightarrow 0\boldsymbol{S_2}1 \rightarrow 0\boldsymbol{1S_20}1 \rightarrow \cdots$$

Context free Grammars

Constructing CFG corresponding to a Language.

Example: Construct the grammar G such that $\boldsymbol{L(G)} = \{\boldsymbol{\omega} | \boldsymbol{\omega}$ **has equal number of 0's and 1's**$\}$

- Grammar for $L_1 \cup L_2$:

$$S \to S_1 | S_2$$
$$S_1 \to 0S_11 | \epsilon$$
$$S_2 \to 1S_20 | \epsilon$$

- What about strings such as $s_1 = 0101 \cdots$ and $s_2 = 1010 \cdots$? Add transitions $S_1 \to S_2$ and $S_2 \to S_1$.

$$S \to S_1 | S_2$$
$$S_1 \to 0S_11 | \epsilon$$
$$S_2 \to 1S_20 | \epsilon$$
$$S_1 \to S_2$$
$$S_2 \to S_1.$$

Context free Grammars

Constructing CFG corresponding to a Language.

Example: Construct the grammar G such that $\boldsymbol{L(G)} = \{\boldsymbol{\omega}|\boldsymbol{\omega}$ **has equal number of $\mathbf{0}$'s and $\mathbf{1}$'s**$\}$

$$S \rightarrow S_1|S_2$$
$$S_1 \rightarrow 0S_11|\epsilon$$
$$S_2 \rightarrow 1S_20|\epsilon$$
$$S_1 \rightarrow S_2$$
$$S_2 \rightarrow S_1$$

- Can't we simplify this? We can replace S_1 and S_2 with a single Start variable as follows: $\boldsymbol{S \rightarrow 0S1|1S0|\epsilon}$
- What kind of strings does the grammar generate? Well if we use Rule $\boldsymbol{S \rightarrow 0S1}$, m times, we get to rules such as 0^mS1^m.
- Now applying the rule $S \rightarrow 1S0$, k times, we get $\boldsymbol{0^m1^kS0^k1^m}$.
- So the strings we obtain are of the form:

$$\{0^{m_1}1^{n_1}0^{m_2}1^{n_2}\cdots 0^{n_2}1^{m_2}0^{n_1}1^{m_1}\}\cup\{1^{m_1}0^{n_1}1^{m_2}0^{n_2}\cdots 1^{n_2}0^{m_2}1^{n_1}0^{m_1}\} \in L(G)$$

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- Simplified grammar:

$$S \rightarrow 0S1|1S0|\epsilon$$

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- Simplified grammar:

$$S \to 0S1 | 1S0 | \epsilon$$

- Is that all? What about strings such as $\{\mathbf{0110}, \mathbf{00111100}\}$?

- More generally, what about strings that are a concatenation of L_1 and L_2?

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- Simplified grammar:

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- Is that all? What about strings such as $\{\mathbf{0110}, \mathbf{00111100}\}$?
- More generally, what about strings that are a concatenation of L_1 and L_2?
- Adding transitions like $S \rightarrow S_1S_2$ incorporates this.

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Constructing CFG corresponding to a Language.

Example: Construct the grammar G such that $\boldsymbol{L(G)} = \{\boldsymbol{\omega} | \boldsymbol{\omega}$ **has equal number of 0's and 1's**$\}$

$$S \rightarrow S_1 | S_2 | S_1 S_2$$
$$S_1 \rightarrow 0 S_1 1 | \epsilon$$
$$S_2 \rightarrow 1 S_2 0 | \epsilon$$
$$S_1 \rightarrow S_2$$
$$S_2 \rightarrow S_1$$

- Simplify this further.

G: $\boldsymbol{S \rightarrow SS | 0S1 | 1S0 | \epsilon}$

Parse trees for CFG

Consider the Grammar G with the following rules:

$$S \to 0S1|SS|\epsilon$$

One derivation:

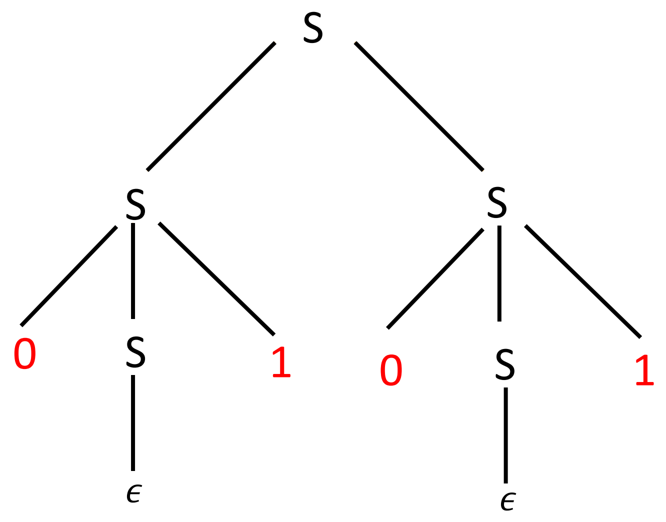
$$S \to \boldsymbol{SS} \to \mathbf{0}\boldsymbol{S}\mathbf{1}S \to 0S1\mathbf{0}\boldsymbol{S}\mathbf{1} \to \mathbf{0101}$$

Parse trees: These are ordered trees that provide alternative representations of the derivation of a grammar.

Parsing is a useful technique for compilers (Analysis of syntax eg: take sequence of tokens as input & output parse trees which provides structural representation of the input while checking for the correct syntax).

Features:

- The root node is the **Start variable**
- Branch out to nodes of the next level by following any of the rules of the grammar
- Stop when all the leaf nodes of the tree are terminals
- Read the terminals in the leaves from left to right.
- If w is the string obtained, then $\boldsymbol{S} \stackrel{*}{\Rightarrow} \boldsymbol{w}$ and $\boldsymbol{w} \in \boldsymbol{L}(\boldsymbol{G})$



Parse trees for CFG

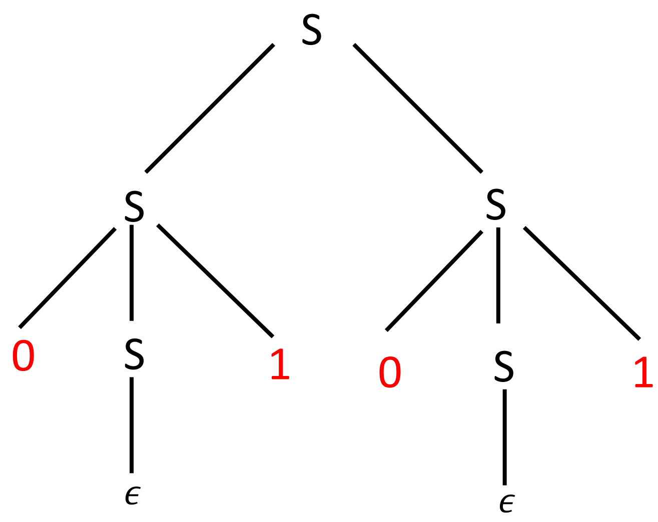
Consider the Grammar G with the following rules:

$$S \rightarrow 0S1|SS|\epsilon$$

Consider the following derivations for 0101:

1. $S \rightarrow \boldsymbol{SS} \rightarrow \mathbf{0}\boldsymbol{S}\mathbf{1}S \rightarrow 0S1\mathbf{0}\boldsymbol{S}\mathbf{1} \rightarrow \mathbf{0101}$
2. $S \rightarrow \boldsymbol{SS} \rightarrow \mathbf{0}\boldsymbol{S}\mathbf{1}S \rightarrow 01S \rightarrow 01\mathbf{0}\boldsymbol{S}\mathbf{1} \rightarrow \mathbf{0101}$
3. $S \rightarrow \boldsymbol{SS} \rightarrow S\mathbf{0}\boldsymbol{S}\mathbf{1} \rightarrow S01 \rightarrow \mathbf{0}\boldsymbol{S}101 \rightarrow \mathbf{0101}$

- The parse trees for all these derivations are the same.



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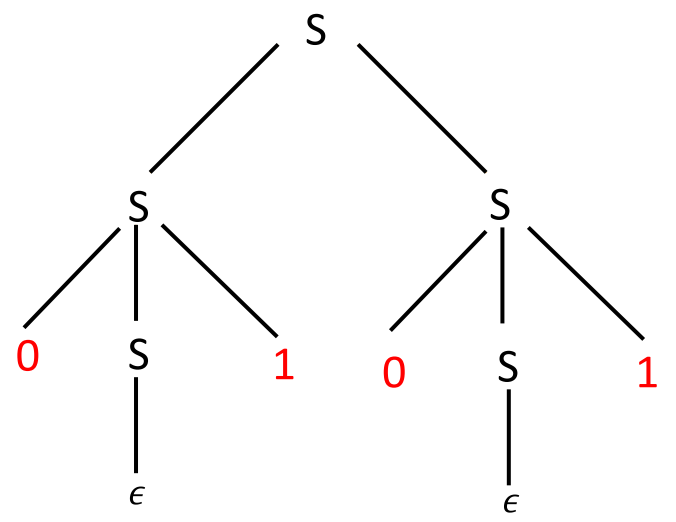
1. $S \to \boldsymbol{SS} \to \boldsymbol{0S1}S \to 0S1\boldsymbol{0S1} \to \mathbf{0101}$
2. $S \to \boldsymbol{SS} \to \boldsymbol{0S1}S \to 01S \to 01\boldsymbol{0S1} \to \mathbf{0101}$
3. $S \to \boldsymbol{SS} \to S\boldsymbol{0S1} \to S01 \to \boldsymbol{0S}101 \to \mathbf{0101}$

- The parse trees for all these derivations are the same.
- If a string is derived by replacing only the leftmost variable at every step, then the derivation is a **leftmost derivation.** (e.g. derivation 2.)
- ……………….rightmost variable = **rightmost derivation** (e.g. derivation 3.)
- Derivations may not always be **leftmost** or **rightmost** (e.g. derivation 1.)



Parse trees for CFG

Consider the Grammar G with the following rules:

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Consider the following derivations for 0101:

1. $S \to \boldsymbol{SS} \to \mathbf{0}\boldsymbol{S}\mathbf{1}S \to 0S1\mathbf{0}\boldsymbol{S}\mathbf{1} \to \mathbf{0101}$
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3. $S \to \boldsymbol{SS} \to S\mathbf{0}\boldsymbol{S}\mathbf{1} \to S01 \to \mathbf{0}\boldsymbol{S}101 \to \mathbf{0101}$

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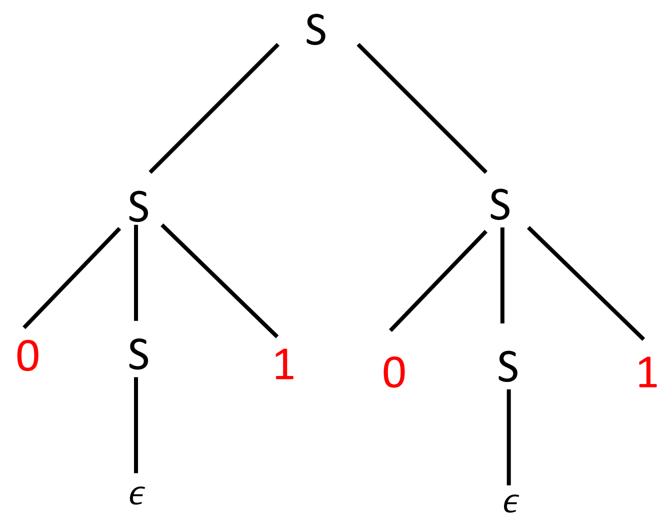

Ambiguous grammars: A CFG G is said to be **ambiguous** if there exists $\boldsymbol{\omega} \in \boldsymbol{L}(\boldsymbol{G})$, such that there are **two or more leftmost derivations for $\boldsymbol{\omega}$** (or equivalently two or more rightmost derivations) or equivalently **two or more parse trees for $\boldsymbol{\omega}$.**

Parse trees for CFG

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Consider the Grammar G with the following rules: $\boldsymbol{S \rightarrow 0S1|SS|\epsilon}$

Show that Grammar $\boldsymbol{G}$ is ambiguous, i.e. $\exists\, \omega \in L(G)$, such that there are two or more parse trees for ω.

Consider the string $\boldsymbol{\omega = 010101}$:

- Show that there exist two different parse trees for **010101**.
- Show that there exist two leftmost derivations for **010101.**

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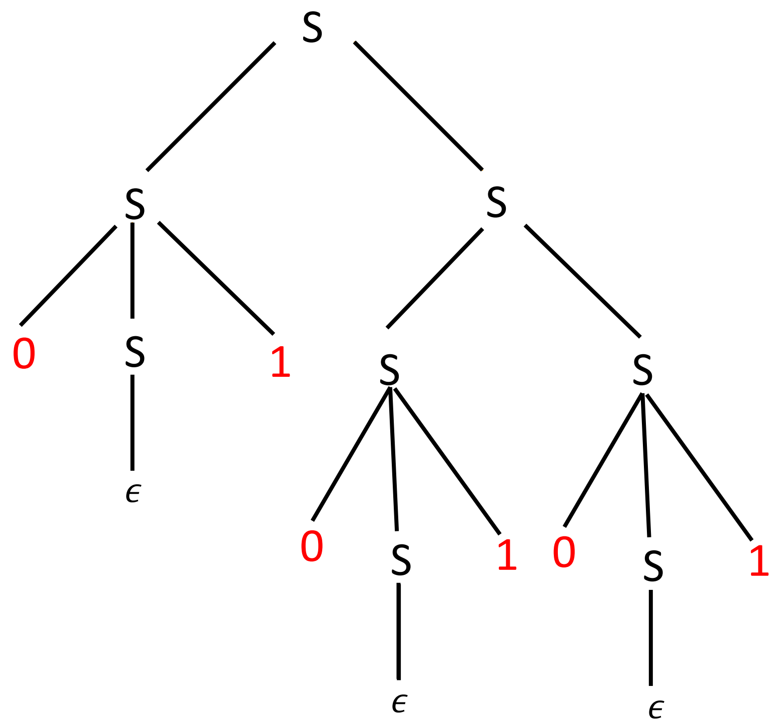

Leftmost Derivation: $S \rightarrow \boldsymbol{SS}$

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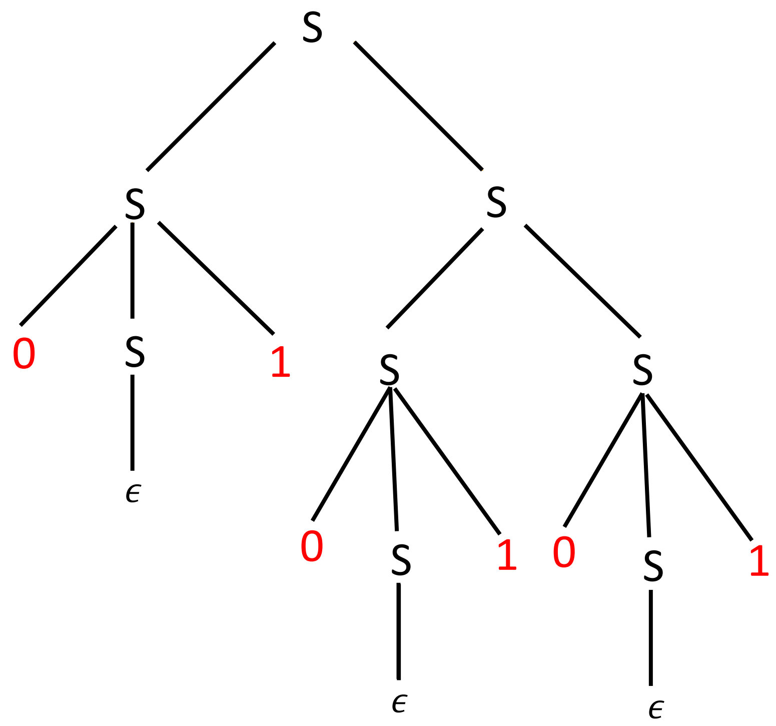

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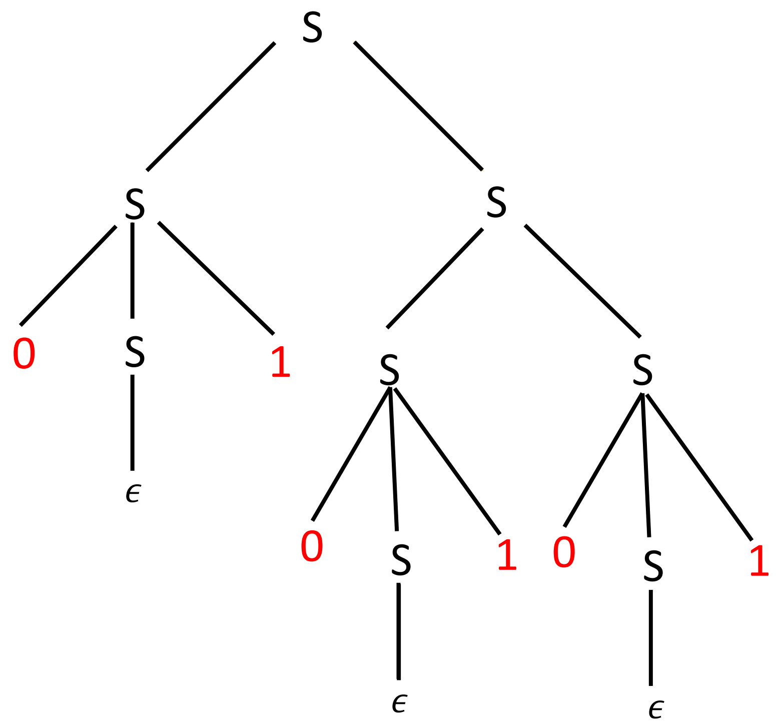

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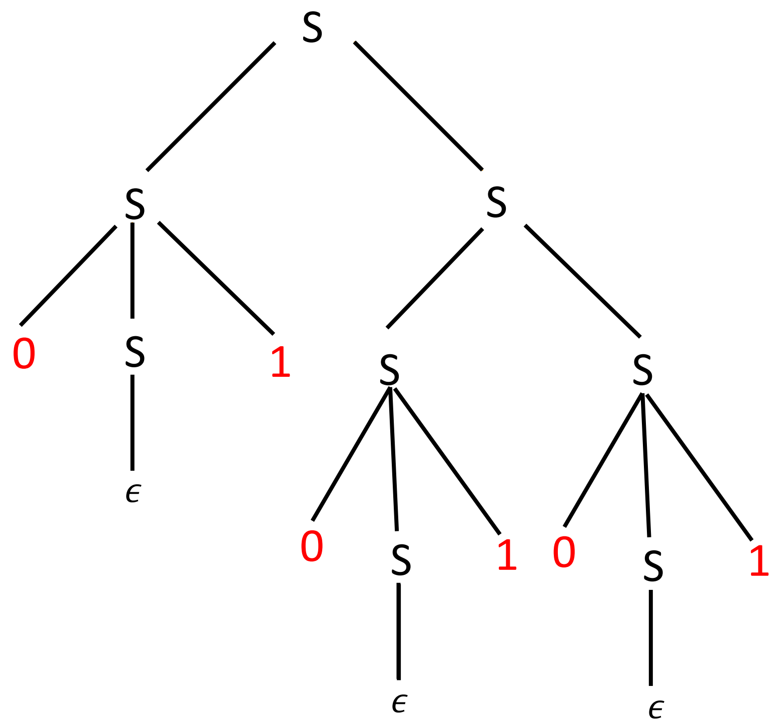

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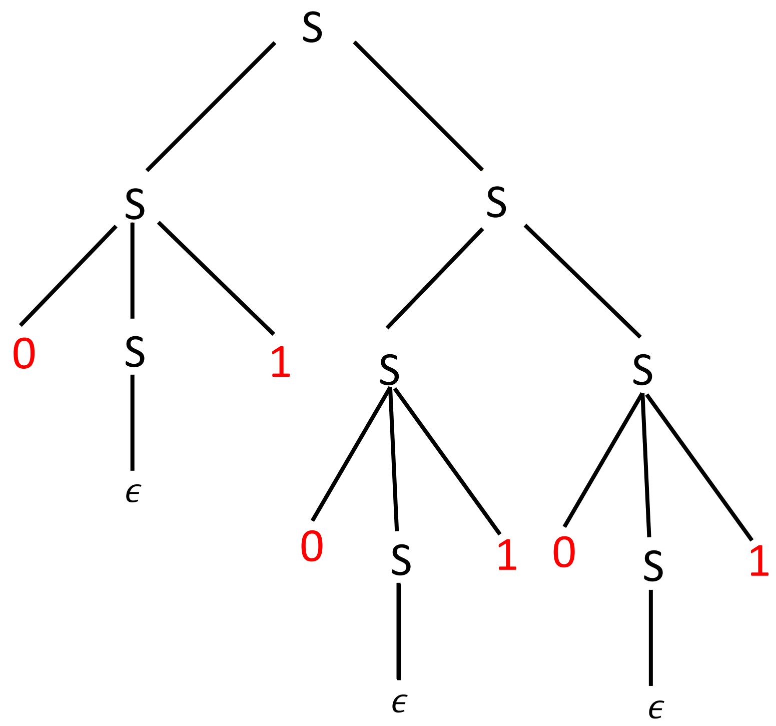

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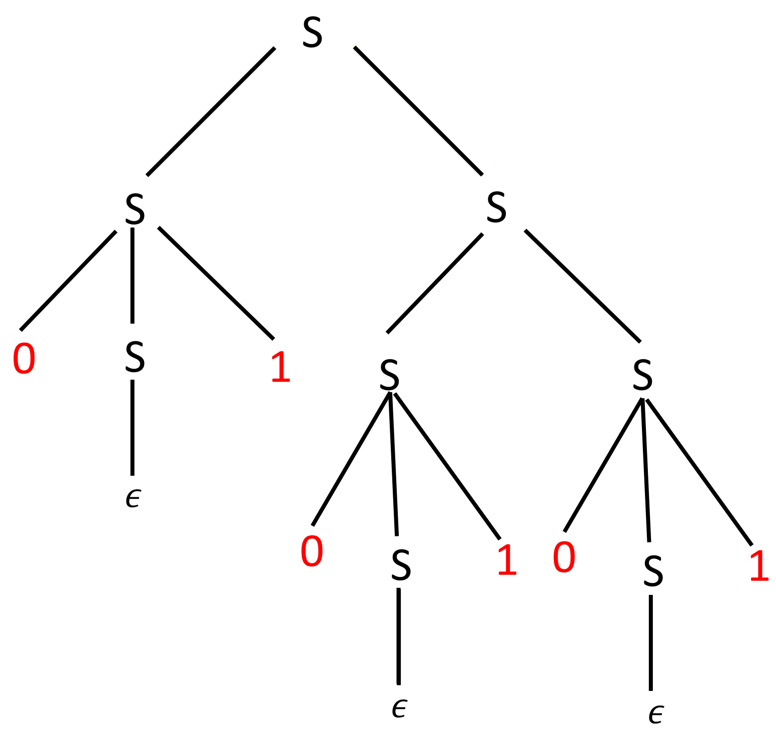

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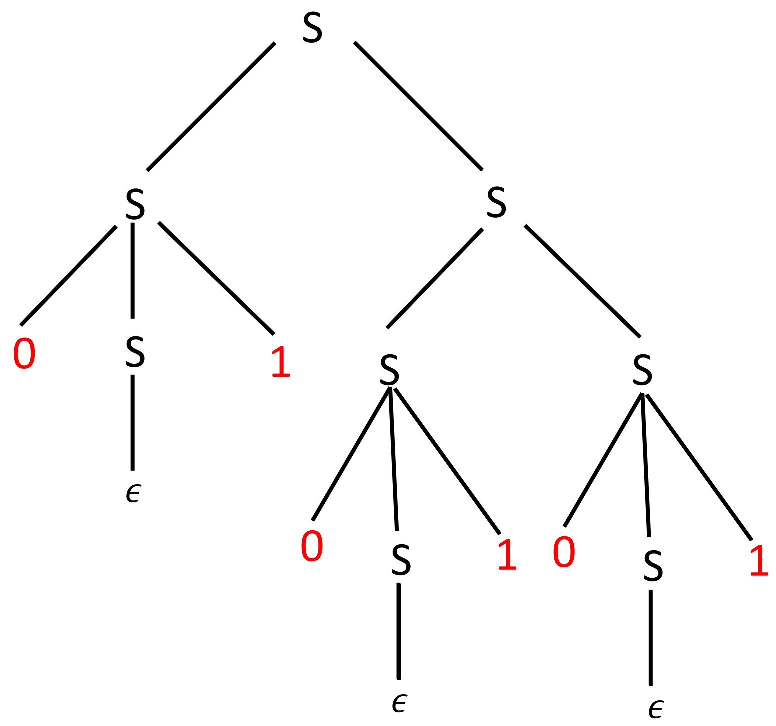

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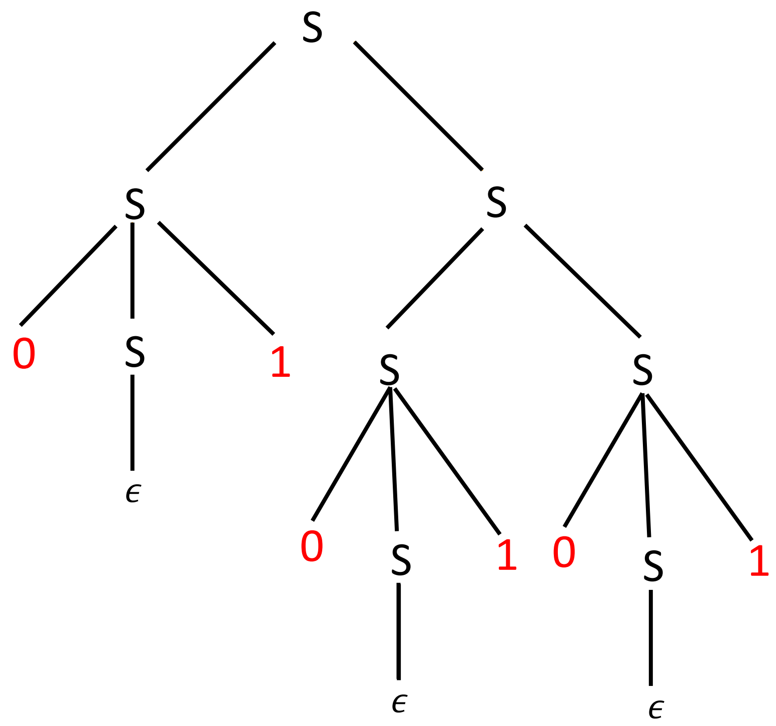

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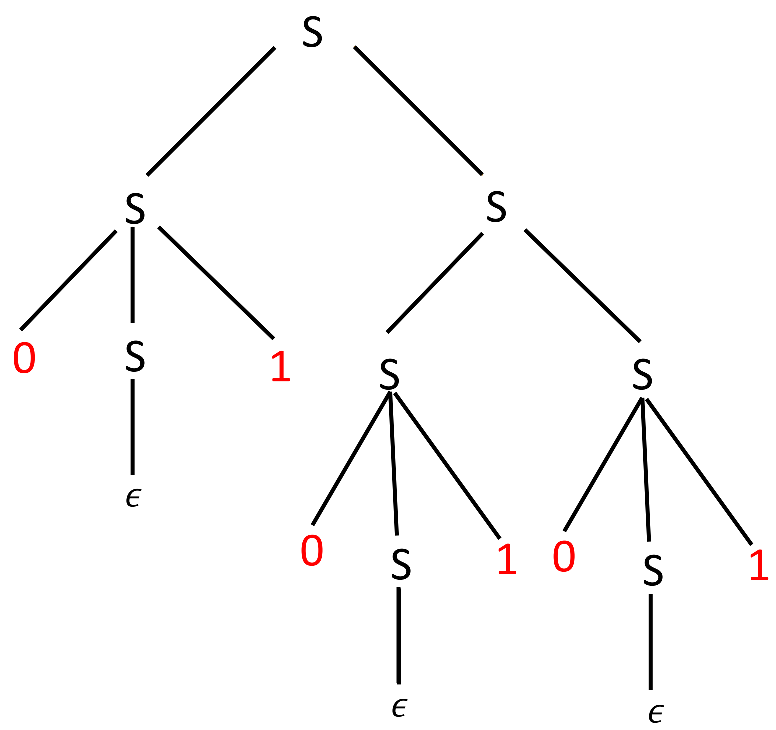

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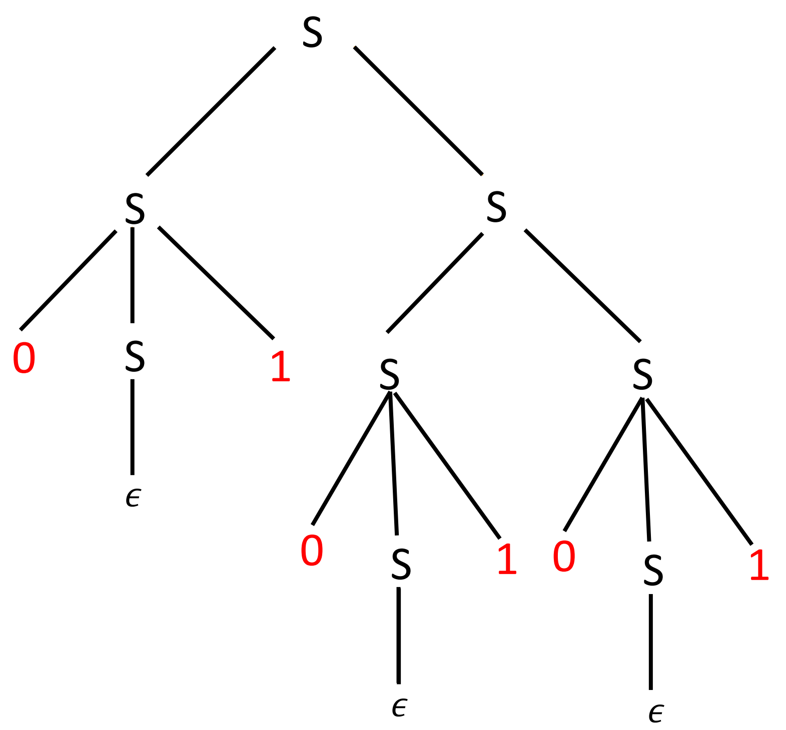

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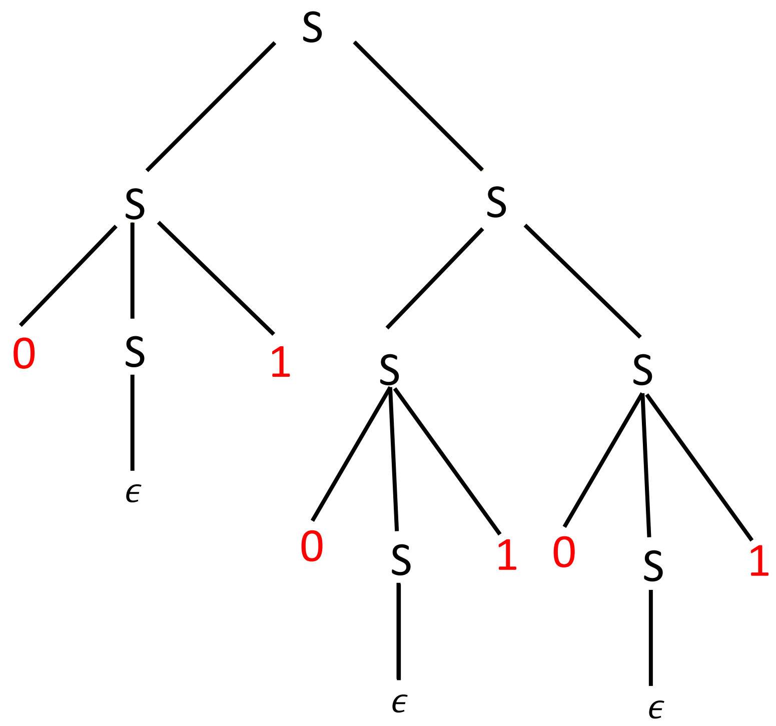

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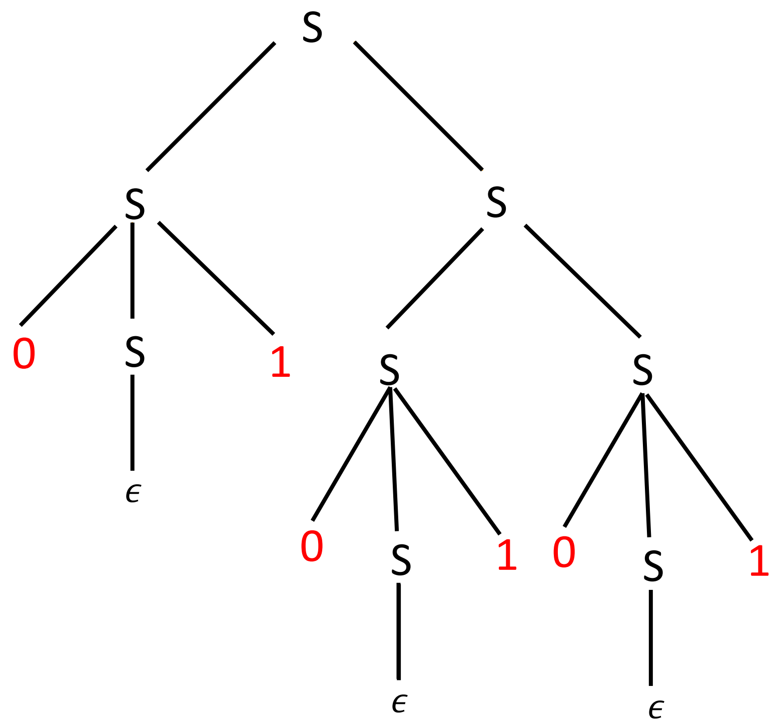

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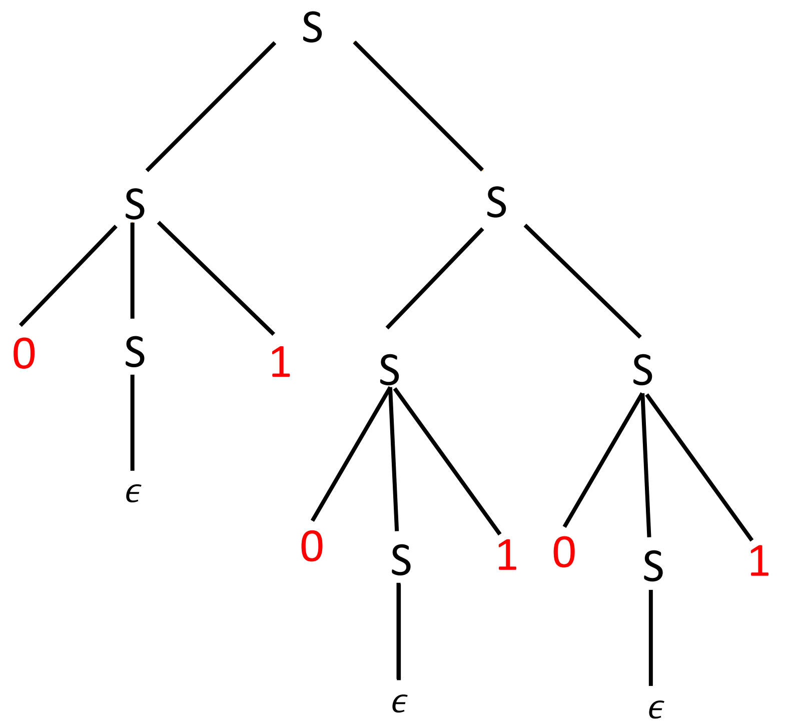

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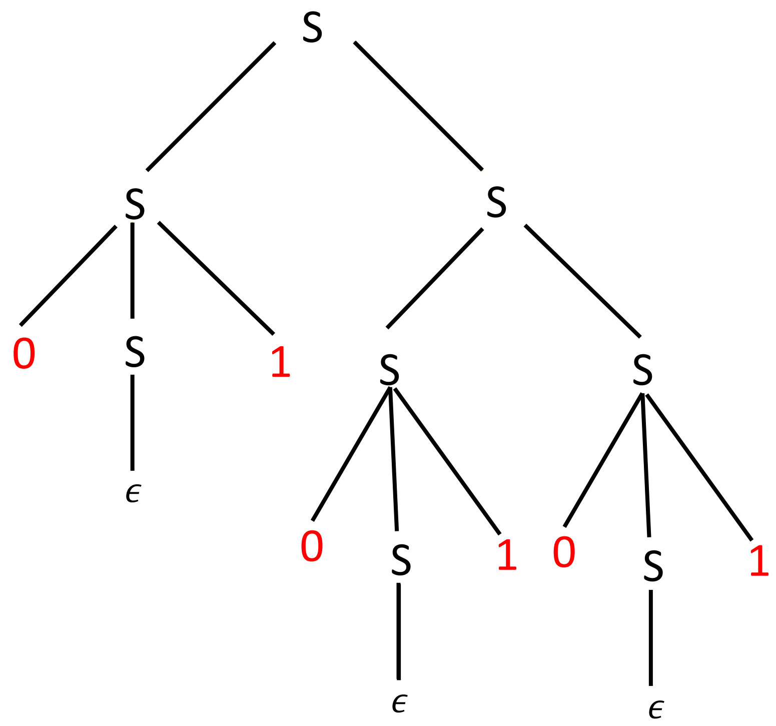

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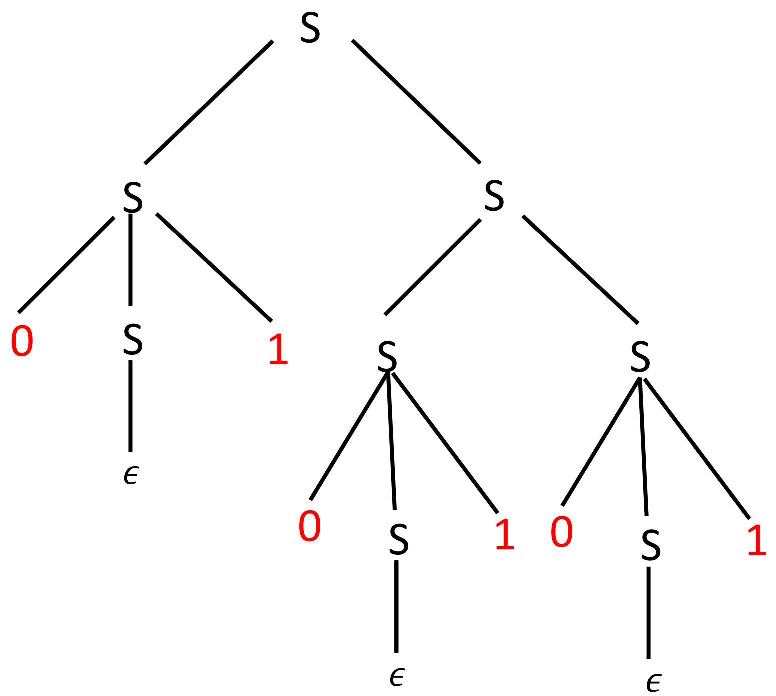
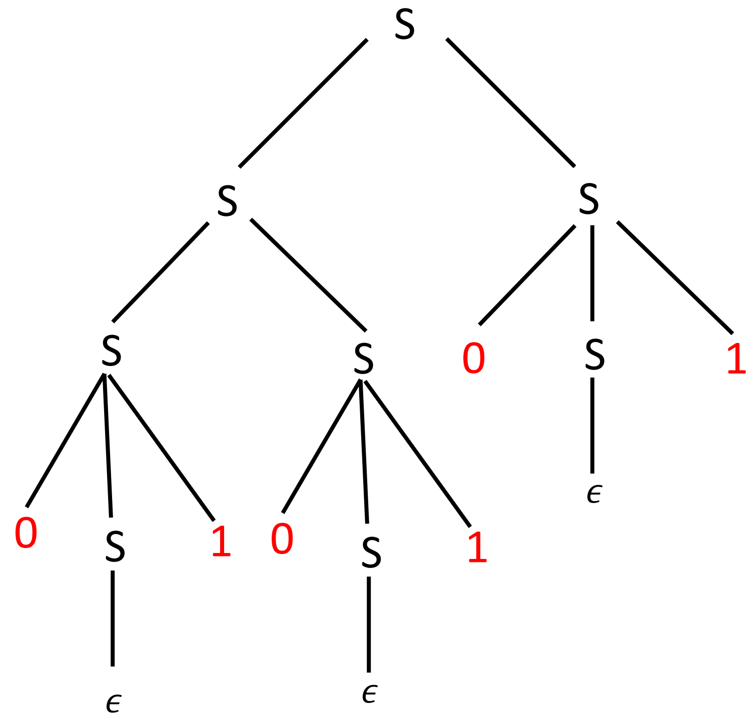


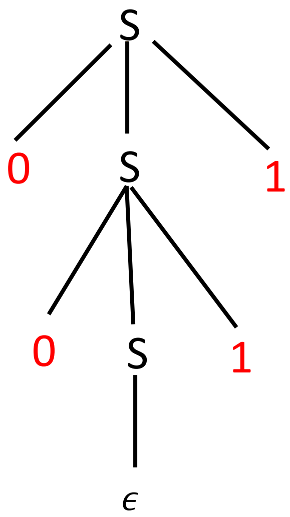


Leftmost Derivation: $S \to \boldsymbol{SS} \to \boldsymbol{SSS} \to \boldsymbol{0S1}SS \to 01SS \to 01\boldsymbol{0S1}S \to 0101\boldsymbol{S} \to 0101\boldsymbol{0S1} \to \mathbf{010101}$

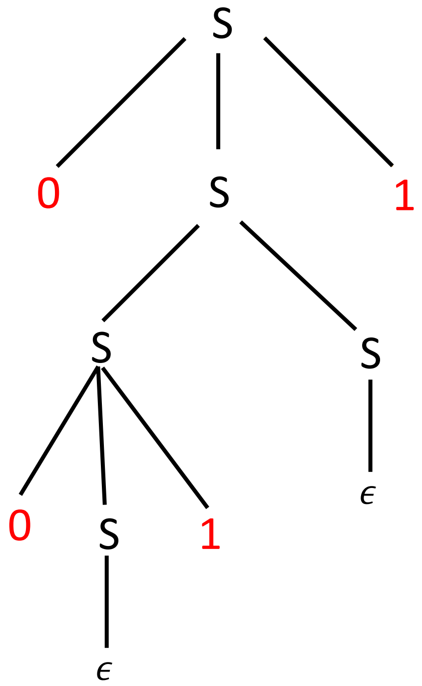
Parse trees for CFG

Show that the Grammar G with the following rules: $\boldsymbol{S \to 0S1 | SS | \epsilon}$ is ambiguous.

Consider string $\boldsymbol{\omega = 0011}$



LD: $S \to \mathbf{0}\boldsymbol{S}\mathbf{1} \to 0\mathbf{0}\boldsymbol{S}\mathbf{1}1 \to \mathbf{0011}$



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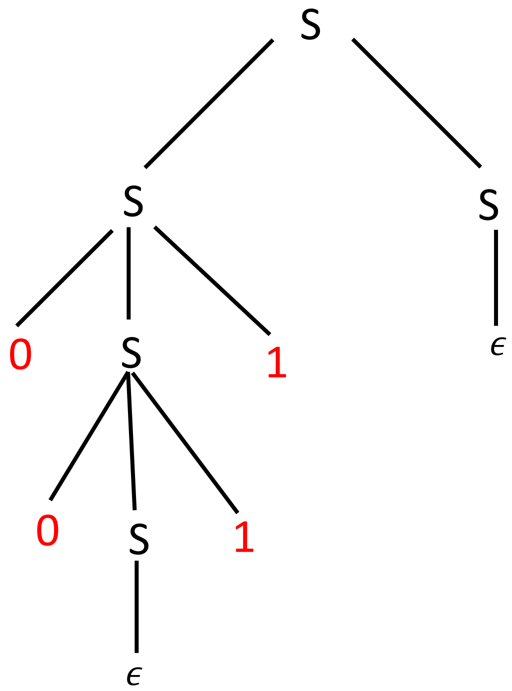

LD: $S \to \boldsymbol{SS} \to \mathbf{0}\boldsymbol{S}\mathbf{1}S \to 0\mathbf{0}\boldsymbol{S}\mathbf{1}1S \to 0011\boldsymbol{S} \to \mathbf{0011}$

Ambiguity

Unique structures are important. For example:

- The syntax of a programming language can be represented by a CFG.

- A compiler
  - translates the code written in the programming language into a form that is suitable for execution.
  - checks if the underlying programming language is syntactically correct.

- Parse trees are data structures that represent such structures.
- Parse tree for the code helps analyze the syntax. So ambiguity might lead to different interpretations and hence, different outcomes for the same code.

Ambiguity may not be desirable.

Ambiguity

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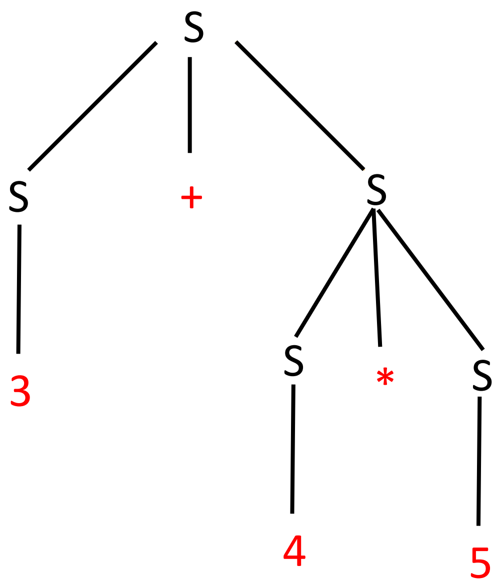
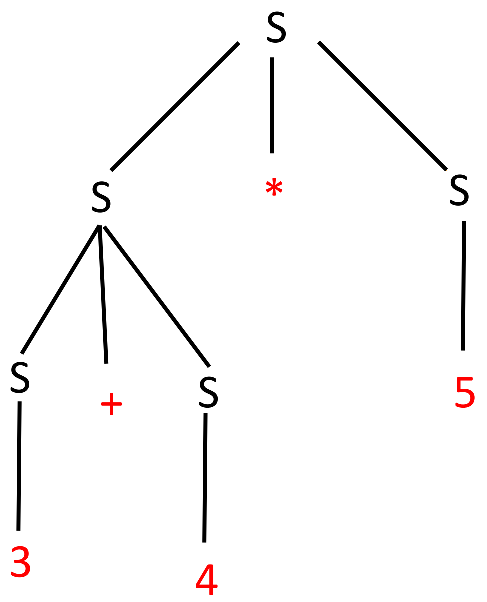
Consider the grammar: $\boldsymbol{S \to S + S \mid S * S \mid 0 \mid 1 \mid 2 \mid \cdots \mid 9}$

and the derivation of the string $\mathbf{3 + 4 * 5}$





- The grammar contains no information on the precedence relations of the various arithmetic operations.
- The grammar may group $+$ before $*$

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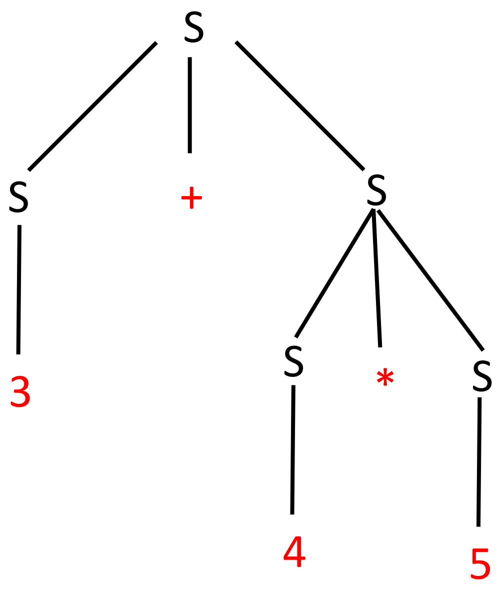

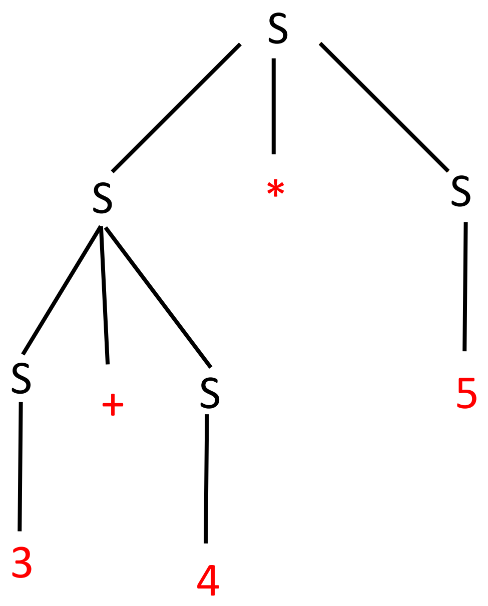

- What will be the result obtained from each of these ***parsings***?

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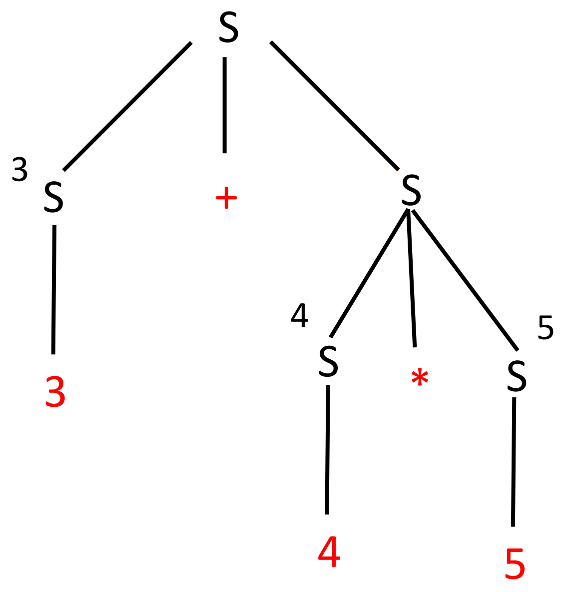

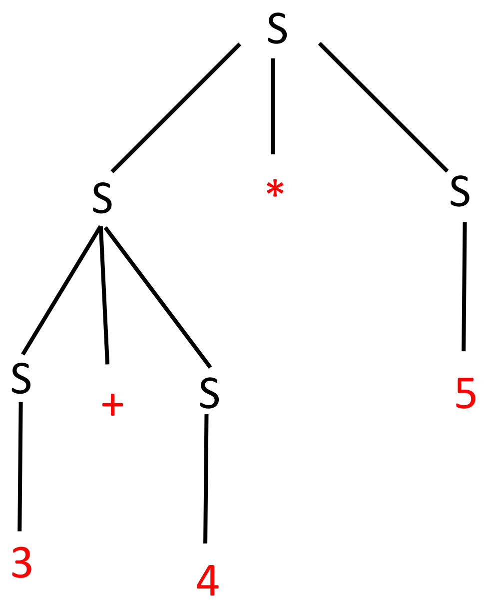

- If the compiler compiles the left parse tree

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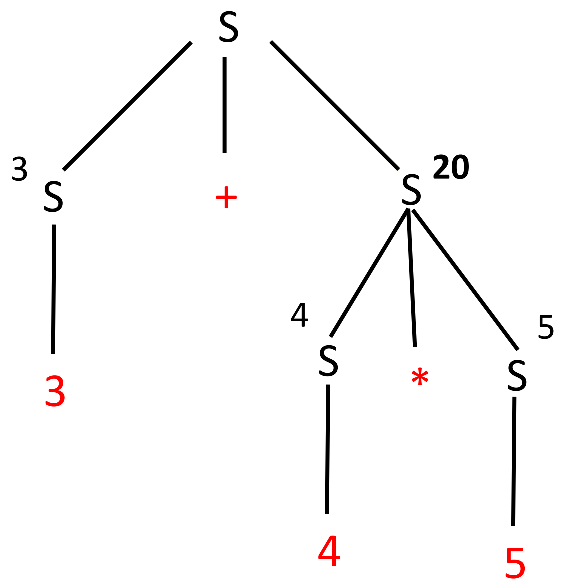

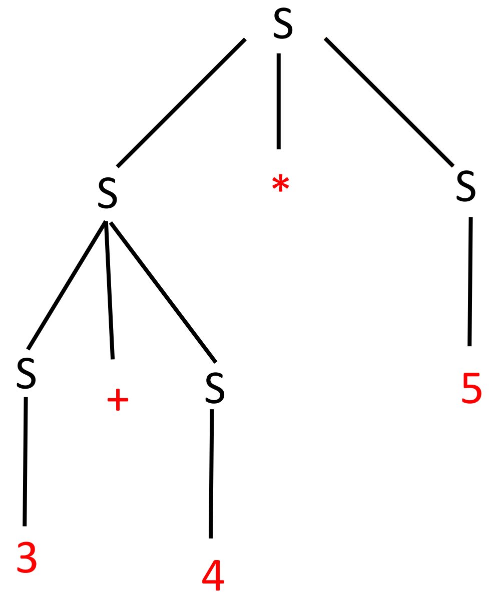

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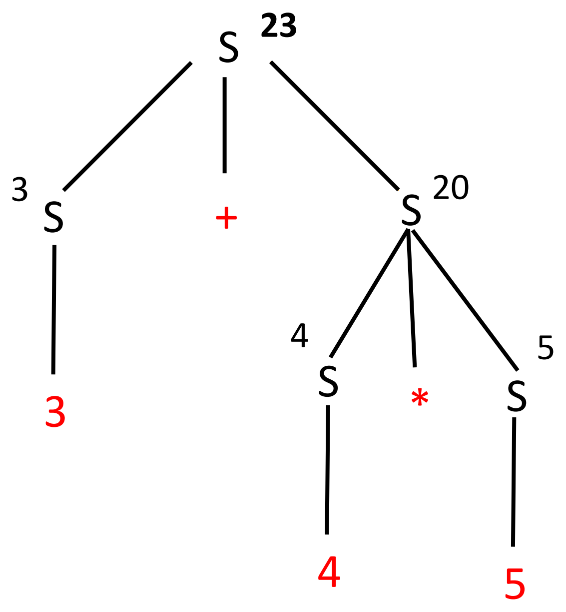

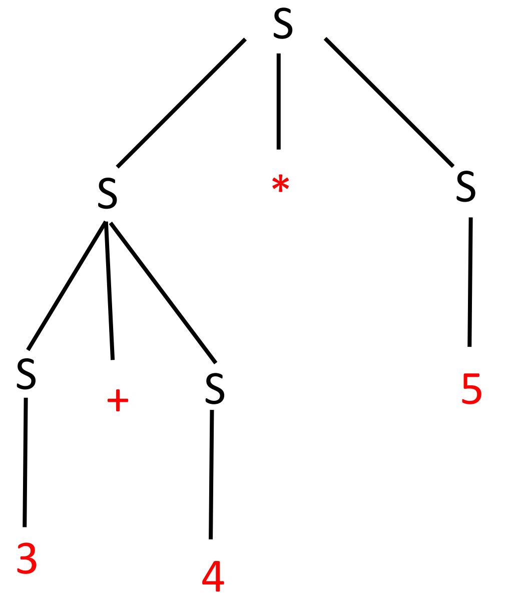

- If the compiler compiles the left parse tree. Outcome $= \mathbf{23}$

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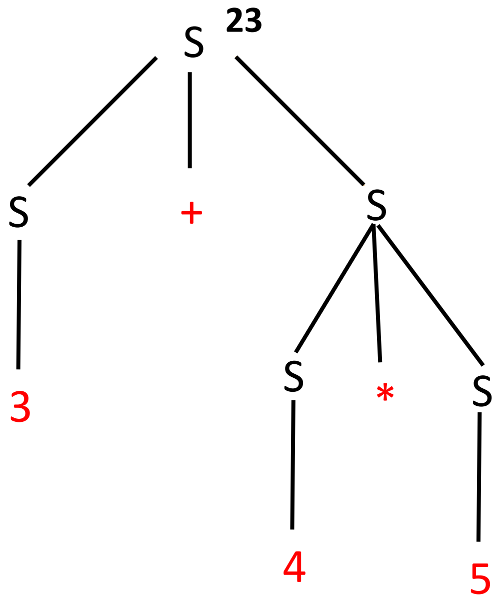
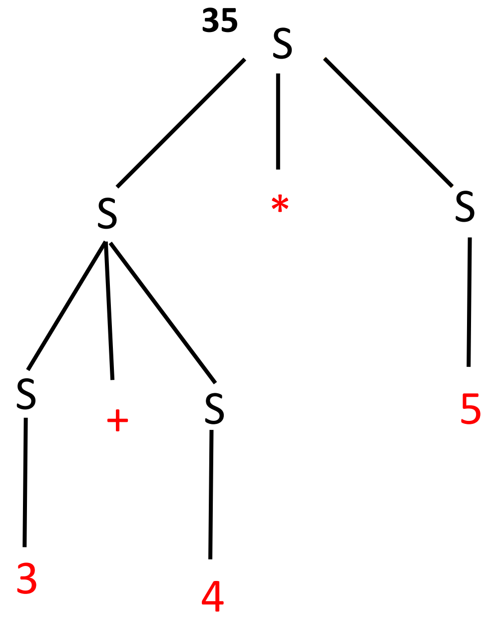
and the derivation of the string $\mathbf{3 + 4 * 5}$





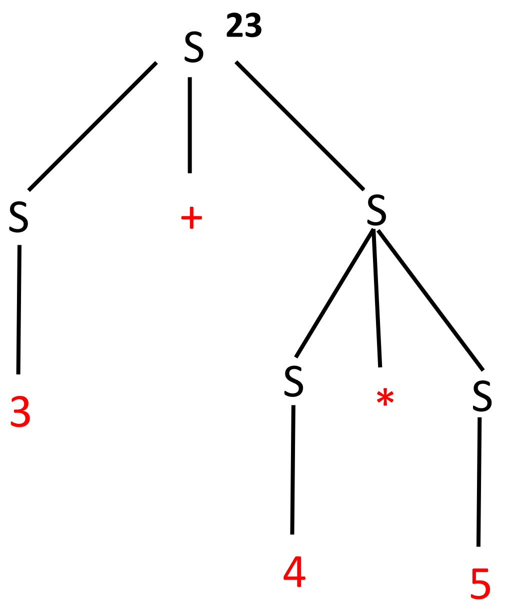
- If the compiler compiles the **right** parse tree. Outcome $= \mathbf{35}$

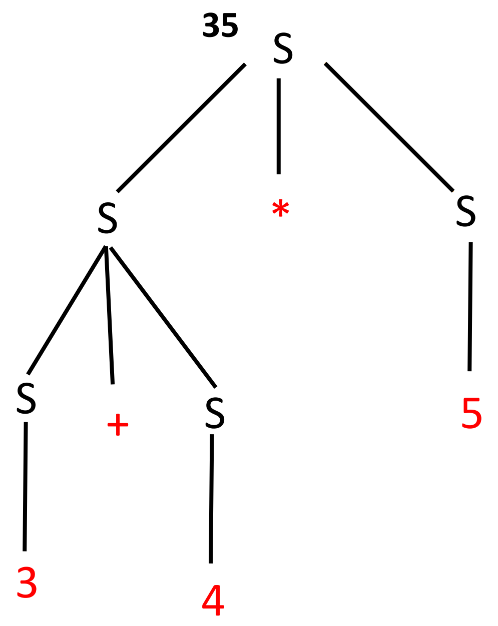
Ambiguity

Ambiguity may not be desirable.

Consider the grammar: $\boldsymbol{S} \rightarrow \boldsymbol{S} + \boldsymbol{S} \mid \boldsymbol{S} * \boldsymbol{S} \mid \mathbf{0} \mid \mathbf{1} \mid \mathbf{2} \mid \cdots \mid \mathbf{9}$

and the derivation of the string $\mathbf{3} + \mathbf{4} * \mathbf{5}$





- **How can we get rid of this ambiguity?**

Ambiguity

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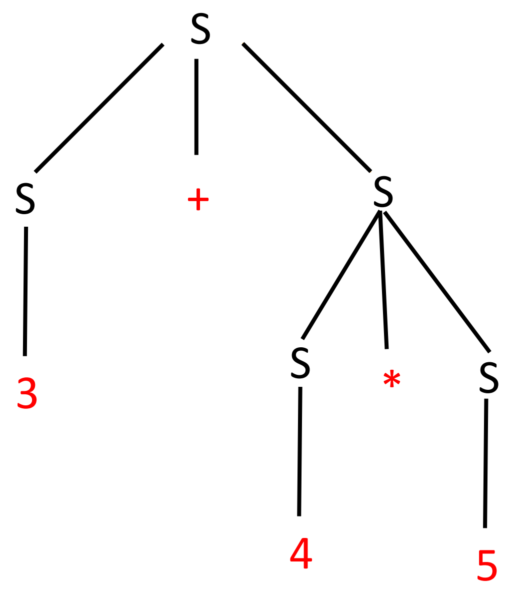
How can we get rid of this ambiguity? Change the production rules

1) Add parenthesis

New Grammar: $\boldsymbol{S} \rightarrow (\boldsymbol{S} + \boldsymbol{S}) \mid (\boldsymbol{S} * \boldsymbol{S}) \mid \mathbf{0} \mid \mathbf{1} \mid \mathbf{2} \mid \cdots \mid \mathbf{9}$



Old Parse tree (before adding parenthesis)

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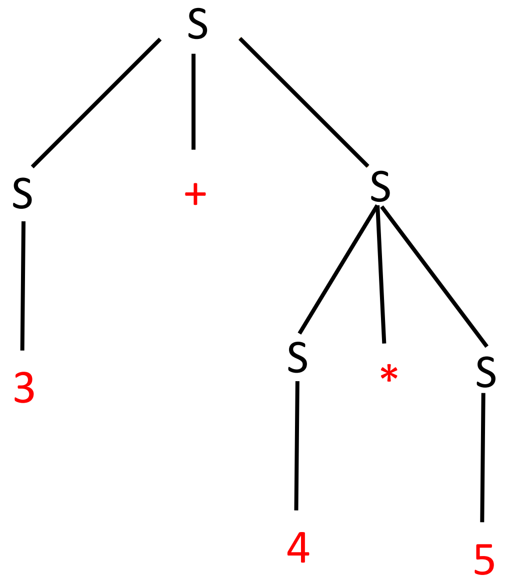

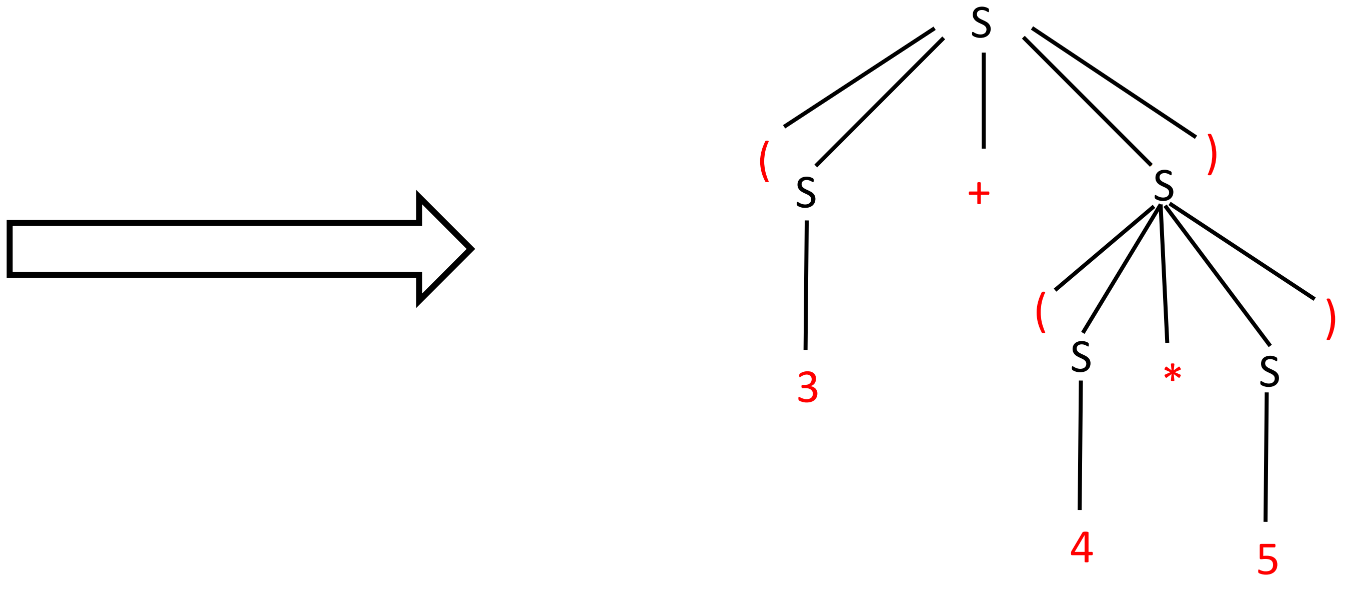

$\big(3 + (4 * 5)\big) = 23$

Ambiguity

Consider the grammar: $\boldsymbol{S \rightarrow S + S \mid S * S \mid 0 \mid 1 \mid 2 \mid \cdots \mid 9}$

How can we get rid of this ambiguity? Change the production rules

1) **Add parentheses**
2) **Add new variables**

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$$\boldsymbol{E \rightarrow E + T \mid T}$$
$$\boldsymbol{T \rightarrow T * F \mid F}$$
$$\boldsymbol{F \rightarrow 0 \mid 1 \mid 2 \mid \cdots \mid 9 \mid E}$$

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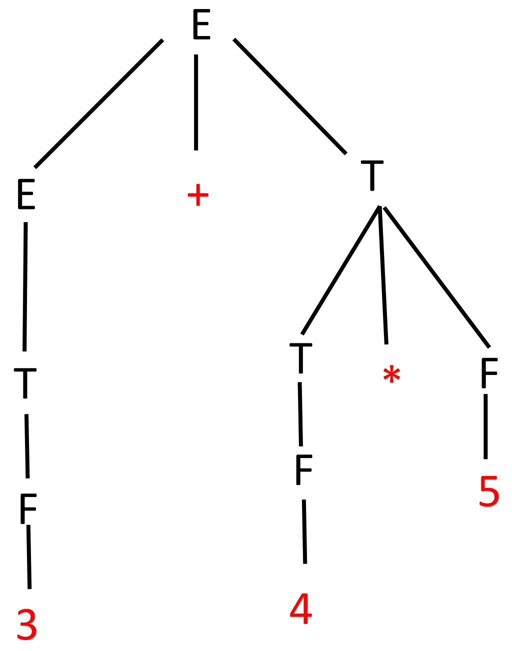
Parse tree to derive: $\mathbf{3 + (4 * 5)}$



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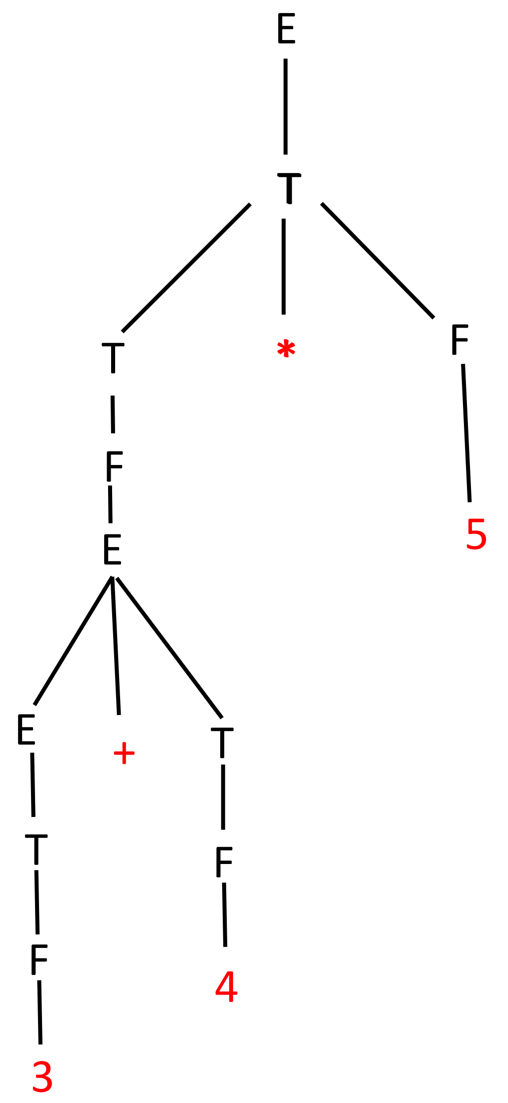
Parse tree to derive: $\boldsymbol{(3 + 4) * 5}$



Ambiguity

How can we get rid of this ambiguity? Change the production rules

1) **Add parentheses**
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- In general, it is not possible to write an algorithm that takes as input a grammar G and outputs, YES if G is ambiguous and NO, otherwise. **(Undecidable)**

- So removing ambiguity is impossible in general.

Chomsky Normal Form

Often it is easier to work with CFG in a simple standardized form - the Chomsky Normal Form (CNF) is one of them.

Chomsky Normal Form

A CFG G is in CNF if every rule of G is of the form

$$\boldsymbol{Var \rightarrow Var\,Var}$$
$$\boldsymbol{Var \rightarrow ter}$$
$$\boldsymbol{Start\,Var \rightarrow \epsilon}$$

where Var can be any variable, including the Start Variable, $Start\,Var$.

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Why are CNFs useful?

- Suppose you are given a CFG G and a string w as input and you have to write an algorithm that decides whether G generates w.
- Your algorithm outputs YES if G generates w and NO, otherwise.

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Why are CNFs useful?

- Suppose you are given a CFG G as and a string w as input and you have to **write an algorithm that decides whether $\boldsymbol{G}$ generates $\boldsymbol{w}$**.

- The algorithm outputs YES if G generates w and NO, otherwise.

❖ One idea is to go through ALL derivations one by one and output YES if any of them generates w.
❖ However, infinitely many derivations may have to tried.
❖ So if G does not generate w, the algorithm will never stop.
❖ So this problem appears to be **undecidable**.

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Why are CNFs useful?

Suppose you are given a CFG G as and a string w as input and you have to **write an algorithm that decides whether $\boldsymbol{G}$ generates $\boldsymbol{w}$**.

- Converting G first to a CNF alleviates this and **makes the problem decidable**.
- It limits the number of steps in derivations required to generate any $w \in L(G)$.
- If $w \in L(G)$, then a CFG in Chomsky Normal Form has **derivations of $\boldsymbol{2n-1}$ steps** for input strings w of length n (We will prove this shortly).

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A CFG in Chomsky Normal Form has derivations of $2n - 1$ steps for generating strings $w \in L(G)$ of length n.

Why are CNFs useful?

Suppose you are given a CFG G as and a string w as input and you have to **write an algorithm that decides whether $\boldsymbol{G}$ generates $\boldsymbol{w}$**.

1. Convert G to CNF.
2. List all derivations of $2n - 1$ steps, where $|w| = n$. (There are a finite number of these)
3. If ANY of these derivations generate w, output YES, otherwise output NO.

Chomsky Normal Form

A CFG G is in **CNF** if every rule of G is of the form

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where Var can be any variable, including the Start Variable, $Start\ Var$.

1) **A CFG in Chomsky Normal Form has derivations of $\boldsymbol{2n-1}$ steps for generating strings $\boldsymbol{w \in L(G)}$ of length $\boldsymbol{n}$.**

2) Any CFL can be generated by a CFG written in Chomsky Normal Form.

To prove 1) use induction!

Chomsky Normal Form

Prove that **a CFG in Chomsky Normal Form has derivations of $\mathbf{2n-1}$ steps for generating strings $\boldsymbol{w} \in \boldsymbol{L(G)}$ of length $\boldsymbol{n}$.**

Proof: Note that any CFG in CNF can be written as:

$$\boldsymbol{A \to BC} \qquad [B, C \text{ are not start variables}]$$
$$\boldsymbol{A \to a} \qquad [a \text{ is a terminal}]$$
$$\boldsymbol{S \to \epsilon} \qquad [S \text{ is the Start Variable}]$$

We will prove this by **induction.**

(Basic step) Let $|w| = 1$. Then **one** application of the second rule would suffice. So any derivation of w would need $2|w| - 1 = 1$ step.
(Inductive hypothesis) Assume the statement of the theorem to be true for any string of length at most k where $k \geq 1$. Now we shall show that it holds for any $w \in L(G)$ such that $|w| = k + 1$.

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Prove that **a CFG in Chomsky Normal Form has derivations of $2n - 1$ steps for generating strings $w \in L(G)$ of length n.**

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$$\begin{array}{ll} \boldsymbol{A \rightarrow BC} & [B, C \text{ are not start variables}] \\ \boldsymbol{A \rightarrow a} & [a \text{ is a terminal}] \\ \boldsymbol{S \rightarrow \epsilon} & [S \text{ is the Start Variable}] \end{array}$$

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(Inductive hypothesis) Assume the statement of the theorem to be true for any string of length at most k where $k \geq 1$. Now we shall show that it holds for any $w \in L(G)$ such that $|w| = k + 1$.

Since $|w| > 1$, any derivation will start from the rule $A \rightarrow BC$. So $\boldsymbol{w = xy}$**, where** $\boldsymbol{B \overset{*}{\Rightarrow} x, |x| > 0}$ **and** $\boldsymbol{C \overset{*}{\Rightarrow} y}, \boldsymbol{|y| > 0}$. But since $|x|, |y| \leq k$, and we have that by the inductive hypothesis: (i) number of steps in the derivation $\boldsymbol{B \overset{*}{\Rightarrow} x}$ **is** $\boldsymbol{2|x| - 1}$ and (ii) number of steps in the derivation $\boldsymbol{C \overset{*}{\Rightarrow} y}$ **is** $\boldsymbol{2|y| - 1}$. So the number of steps in the derivation of w is

$$\boldsymbol{1 + (2|x| - 1) + (2|y| - 1) = 2(|x| + |y|) - 1 = 2|w| - 1 = 2(k + 1) - 1}.$$

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1) A CFG in Chomsky Normal Form has derivations of $2n - 1$ steps for generating strings $w \in L(G)$ of length n.

2) **Any CFL can be generated by a CFG written in Chomsky Normal Form.**

Chomsky Normal Form

Any CFL can be generated by a CFG written in Chomsky Normal Form.

Proof: The proof is constructive. Suppose we have a CFG G with a set of rules. To convert G into CNF, we do the following:

1. Add a new start variable $S' \to S$

2. Remove ϵ rules of the form $A \to \epsilon$
   - Remove nullable symbols/rules

3. Remove unit (short) rules of the form $A \to B$
   - Remove useless symbols/rules

4. Remove long rules of the form $A \to u_1 u_2 \cdots u_k$
   - Remove useless symbols/rules

Thank You!