Applications

1) In combinatorics and graph theory.

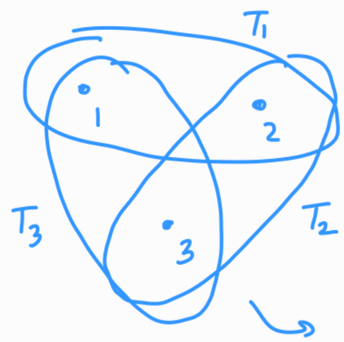
Let S be a set of elements and $T_1, T_2, \ldots, T_m \subseteq S$ s.t

$|T_i| = l \quad \forall i \in [1:m]$ | "Hypergraph"

$\underbrace{|T_i|}_{\text{Cardinality}}$

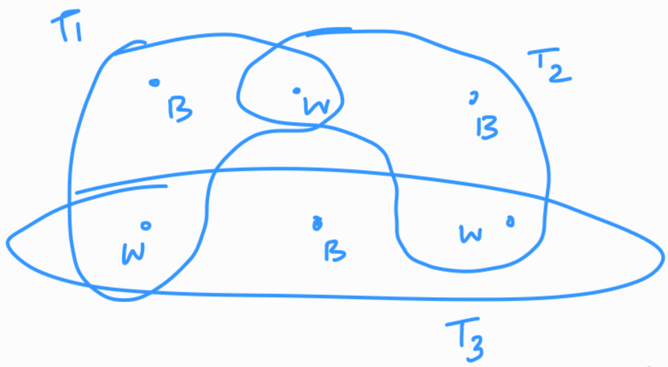
Q. Can we 2-colour S s.t no T_i is monochromatic i.e., each T_i should have both the colours ?

(Colouring func $C : S \to \{\text{Black}, \text{White}\}$).



$m = 3$, $2^{l-1} = 2$ $\Big\}$ Here $m > 2^{l-1}$

↳ Not 2 colourable.



↳ Each T_i has 2 colours.

$m = 3$, $2^{l-1} = 4$ $\Big\}$ Here $m < 2^{l-1}$

If we find one possible colouring s.t each T_i has 2 colours, then S is 2-colourable.

• **Th^m** : If $m < 2^{l-1}$, then there exists a 2-colouring such that no T_i is monochromatic $[\, T_1, T_2, T_3, \ldots, T_m \subseteq S$ and $|T_i| = l \,]$.
(Given)

(Implicit assumption : $l > 1$?)

Proof idea : Given $(\Omega, \mathcal{F}, P)$.

If $P(A) > 0$, then $\omega \in A$ (i.e., A is NOT empty).

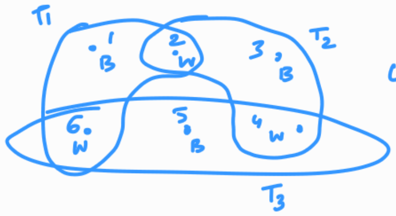
Proof : $S = \{x_1, x_2, \ldots, x_n\}$
$\quad\quad\quad B \quad W \quad W \quad\quad B$

Assign colours randomly and independently with each colour's probability $\frac{1}{2}$.

$\Omega = \{W, B\}^n$ $\quad\quad$ $A = \{\omega \in \Omega : \text{no } T_i \text{ is monochromatic}\}$

$P(\{W\}) = \frac{1}{2^n}$

Eg:



$\omega = BWBWBW$

$\omega \in \Omega$ i.e., ω is a colouring

This $\omega \in A$ $\because$ each T_i is not monochromatic

→ Now if we show $P(A) > 0$

Note " Strictly greater than sign ".

Let $E_i = \{\underset{\wedge}{}\ T_i \text{ is monochromatic}\}$. $\quad$ ($\omega \in \Omega$:)

Given a colouring, look at each T_i, if any of the T_i's is monochromatic then include that ω in corresponding E_i.

Eg:
$x_1 \quad x_2, \ldots\ldots \quad x_n$

$B \quad B \; BB \ldots B \ldots \longrightarrow \frac{2^{n-l}}{2^n}$
(or)
$W \quad \underbrace{W \; WW - W}_{|T_1| = l} - \longrightarrow \frac{2^{n-l}}{2^n}$

$$\Rightarrow P(E_i) = \frac{2^{n-l}}{2^n} + \frac{2^{n-l}}{2^n} = \frac{1}{2^{l-1}}$$

$$P\left(\bigcup_{i=1}^{m} E_i\right) \leq \sum_{i=1}^{m} P(E_i) = \frac{m}{2^{l-1}}$$

So if $m < 2^{l-1} \Rightarrow \sum_{i=1}^{m} P(E_i) < 1$

$\Rightarrow \quad P(\exists\, i \text{ s.t } T_i \text{ is monochromatic}) < 1$

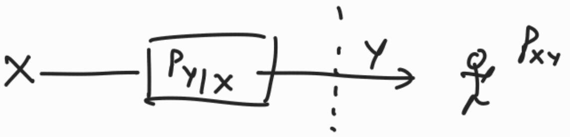
Complement.

$\Rightarrow \quad P(\underbrace{\text{no } T_i \text{ is monochromatic}}_{A}) > 0$

2) Estimation

Given RV X and PMF P_X and a conditional distri $P_{Y|X}$.

We are interested in estimating X knowing P_{XY} and Y.



"Mean squared error"

Estimate X.

Estimate of X $\leftarrow$ $\hat{X} = f(Y)$

Mean squared error : $E[(X - \hat{X})^2]$.

$= E[(X - f(Y))^2]$.

So, we want to find an estimate func f which minimises the mean squared error.

Theorem : The function $f(Y) = E[X \mid Y = y]$ minimises $E[(X - f(Y))^2]$

Proof : $E[(\underbrace{X - f(Y)}_{X'})^2]$

$= E[E[(X - f(Y))^2 \mid Y]]$

$= \sum_y P_Y(y) E[(X - f(y))^2 \mid Y = y]$

$E[E[X' \mid Y]] = E[X']$

$\phi(Y)$ Law of disjoint...

Here

$\phi(Y) = E[X' \mid Y = y]$

Exercise :

Quadratic in terms of $f(y)$.

So find minima of that expression.