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PRP A4

1. $F_X(x) = \begin{cases} 1 - \dfrac{a^3}{x^3} & x \geq a \\ 0 & x < a \end{cases}$

$\int f_X(x)\,dx = F_X(x)$ (by def^n)

$\Rightarrow f_X(x) = \dfrac{dF_X(x)}{dx}$ (Fundamental th^m of calculus)

$\Rightarrow f_X(x) = \begin{cases} \dfrac{3a^3}{x^4} & x \geq a \\ 0 & x \leq a \end{cases}$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{-\infty}^{a} x f_X(x)\,dx + \int_{a}^{\infty} x f_X(x)\,dx$$

$$= 0 + \int_a^{\infty} \cancel{x} \frac{3a^3}{\cancel{x^4}\, x^3}\,dx$$

$$= 3a^3 \int_a^{\infty} \frac{dx}{x^3}$$

$$= 3a^3 \left(-\frac{x^{-2}}{2} \right)_a^{\infty}$$

$$= \frac{3a^3}{2}\left[\frac{1}{a^2}\right]$$

$$= \boxed{\frac{3a}{2}} \text{ Ans}$$

$\text{var}(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = 0 + \int_a^{\infty} x^2 f_X(x)\,dx$$

$$= 3a^3 \int_a^{\infty} \frac{dx}{x^2} = 3a^3\left(\frac{1}{a}\right) = 3a^2$$

$$\text{var}(X) = 3a^2 - \frac{9a^2}{4} = \boxed{\frac{3a^2}{4}} \text{ Ans}$$

2. $P(A) = P(B) = \frac{1}{2}$ (fair coin-toss)

$f_{X|A}(x) = 1 \quad 0 \le x \le 1$
$f_{X|B}(x) = 3 \quad 0 \le x \le 1/3$

$P(A|X \le 1/4) = \dfrac{P(\{A\} \cup \{w | X(w) \le 1/4\})}{P(X \le 1/4)}$ {def^n of conditional prob.}

$= \dfrac{P(X \le 1/4 | A) \cdot P(A)}{P(X \le 1/4)}$

{We know $P(X \le 1/4 | A) = \dfrac{P(\{A\} \cup \{w | X(w) \le 1/4\})}{P(A)}$
Also from def^n of conditional prob.}

From Total Prob. th^m ($\because$ A & B are disjoint & cover entire sample space, forming its partion)

$P(X \le 1/4) = P(X \le 1/4 | A) P(A) + P(X \le 1/4 | B) P(B)$

$= \frac{1}{2}\left(F_{X|A}\left(\frac{1}{4}\right) + F_{X|B}\left(\frac{1}{4}\right)\right)$

We know $F_{X|E}(x') = \int_{-\infty}^{x'} f_{X|E}(x)\, dx$

$\Rightarrow F_{X|A}\left(\frac{1}{4}\right) = \int_{-\infty}^{1/4} f_{X|A}(x)\, dx$

$= \int_{-\infty}^{0} (0)\, dx + \int_{0}^{1/4} (1)\, dx = \underline{1/4}$

$F_{X|B}\left(\frac{1}{4}\right) = \int_{-\infty}^{1/4} f_{X|B}(x)\, dx$

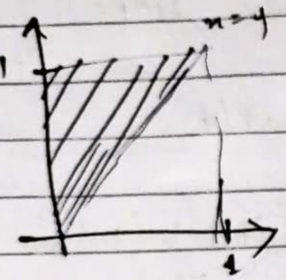
$= \int_{-\infty}^{0} (0)\, dx + \int_{0}^{1/4} (3)\, dx = \underline{3/4}$

$\therefore P(X \le 1/4) = \frac{1}{2}\left(\frac{1}{4} + \frac{3}{4}\right) = \underline{1/2}$

$\therefore P(A | X \le 1/4) = \dfrac{\frac{1}{2}(1/4)}{1/2} = \underline{\frac{1}{4}}$ Ans

3 $f_{XY}(x,y) = \begin{cases} k & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$

We can visualise it as follows (shaded region shows non-zero $f_{XY}(x,y)$)



$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

$= \int_{y=x}^{y=1} k\, dy$

$= k\, y\Big]_x^1$

$= k(1-x)$

for $0 \leq x < y < 1$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$

for $0 < x < y < 1$

$= \int_{x=0}^{x=y} k\, dx + 0$

$= k\, x\Big]_0^y = ky$

To summarise, $f_X(x) = \begin{cases} k(1-x) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$f_Y(y) = \begin{cases} ky & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

$f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$ (when $f_Y(y) \neq 0$)

~~Assuming $f_{X|Y}(x|y) = 0$ for $f_Y(y) = 0$~~ (~~is technically not defined~~)

$= \begin{cases} f_{XY}(x,y)/ky & 0<y<1 \\ \text{not defined} & \text{otherwise} \end{cases}$

Here $x \in \mathbb{R}$ but opening $f_{XY}(x,y)$ for x s.that $0<x<y$ is false $\underline{\underline{=0}}$

$\therefore f_{X|Y}(x|y) =$

$$= \begin{cases} \cancel{k}/\cancel{k}y = 1/y & 0 < x < y < 1 \\ 0 & x \in \mathbb{R} - (0,1) \\ 0 & 0 < y < 1,\ x \geq y \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} \quad [\text{if } f_X(x) \neq 0]$$

$$= \begin{cases} f_{XY}(x,y)/k(1-x) & 0 < x < 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

$y \in \mathbb{R}$ but $f_{XY}(x,y) = 0 \ \forall y \in \mathbb{R} - (0,1)$ or $y \leq x$

$= 1/1-x$

$$= \begin{cases} \cancel{k}/\cancel{k}(1-x) & 0 < x < y < 1 \\ 0 & y \in \mathbb{R} - (0,1) \\ 0 & 0 < x, y < 1,\ y \leq x \\ \text{undefined} & \text{oth} \end{cases}$$

$\therefore$ For $0 < x < y < 1$

$f_{X|Y}(x|y) = 1/y$

$f_{Y|X}(y|x) = 1/(1-x)$

4. $f_{XY}(x,y) = \begin{cases} 6xy & 0 \le x \le 1,\ 0 \le y \le \sqrt{x} \\ 0 & \text{otw} \end{cases}$

compute $\mathrm{var}(X \mid Y = y)$ for $y \in [0,1]$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{y^2}^{1} f_{XY}(x,y)\,dx + 0$$

$$= \begin{cases} \int_{x=y^2}^{x=1} 6xy\,dx & 0 \le y \le 1 \\ 0 & \end{cases}$$

$$= \begin{cases} \left[3y \left(x^2 \right) \right]_{y^2}^{1} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 3y(1-y^4) & 0 < y \le 1 \\ 0 & \text{otherwise} \end{cases}$$ ($y = 0$ also gives 0 in above)

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad (f_Y(y) \ne 0)$$

$$= \begin{cases} 6xy/(3y(1-y^4)) = 2x/(1-y^4) & 0 \le x \le 1,\ 0 < y \le \sqrt{x} \\ 0 & 0 < y \le 1,\ x \in \mathbb{R} - [0,1] \\ 0 & 0 < y \le 1,\ y \ge \sqrt{x} \end{cases}$$

$E[X \mid Y = y]$ for $0 < y \le 1$ (will treat $y = 0$ as a separate case)

$$= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \int_{y^2}^{1} x \frac{2x}{1-y^4}\,dx = \frac{2}{3(1-y^4)}(1-y^6)$$

$E[X^2 \mid Y = y]$ for $0 < y \le 1$

$$= \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\,dx = \int_{y^2}^{1} x^2 \frac{2x}{1-y^4}\,dx = \frac{1}{2(1-y^4)}(1-y^8)$$

~~(ii) For y = 0, $f_{X|Y} = 0$~~

$$Var(X|Y=y) = E[X^2|Y=y] - \left(E[X|Y=y]\right)^2$$

$$= \frac{1}{2}\left(\frac{1-y^8}{1-y^4}\right) - \frac{4}{9}\frac{(1-y^6)^2}{(1-y^4)^2}$$

We note that $y=0$ has no defined variance. The formula holds for $y=0^+$ (limit exists) but for $y=0$, $f_Y(y)=0 \Rightarrow$ undefined $f_{X|Y}(x,y)$

5(a) $P(U=V)=0$, s.that

given U, V are jointly continuous,

$$\Rightarrow F_{UV}(u,v) = \int_{x=-\infty}^{u}\int_{y=-\infty}^{v} f_{UV}(x,y)\,dy\,dx \quad \forall x,y \in \mathbb{R}$$

$$\Rightarrow P(U=V) = \iint_{\{(x,y):\,x=y\}} f_{UV}(x,y)\,dx\,dy \quad \{\text{outer integral}\}$$

This is equivalent to finding vol. beneath the area (inner integral) defined by the set $\{(x,y): x=y\}$ ($=S$ (sul))

$\because$ S represents a straight line $\Rightarrow A(S)=0$ & hence is a set of measure 0 & vol. beneath it is zero

$$\therefore \iint_{\{(x,y):\,x=y\}} f_{UV}(x,y)\,dx\,dy = 0 \Rightarrow P(U=V)=0$$

(b) No, there is no contradiction as the above result holds only for U, V being jointly continuous RVs. but, X, Y are not jointly continuous as we will prove next.

RTP $\nexists f_{XY}(u,v)$ s.that $F_{XY}(x,y) = \int_{v=0}^{y}\int_{u=0}^{x} f_{XY}(u,v)\,du\,dv$.

We know, $F_{XY}(x,y) = P(X \le x, Y \le y)$
$(\because X = Y)$
$= P(X \le x, X \le y) = P(X \le \min(x,y))$
$= F_X(\min(x,y))$
$= \int_0^{\min(x,y)} f_X(u)\,du$

$\because X$ is uniform RV $\Rightarrow f_X(x) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{otw.} \end{cases}$
$= \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otw.} \end{cases}$ here

$\therefore F_{XY}(x,y) = \int_0^{\min(x,y)} 1\,du$

$= \min(x,y) = \begin{cases} x & x \le y \\ y & y \le x \end{cases}$

Let us assume X & Y are jointly continuous.

$\Rightarrow f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$

$= \begin{cases} \frac{\partial^2}{\partial x \partial y} x & x \le y \\ \frac{\partial^2}{\partial x \partial y} y & y \le x \end{cases}$

$= \begin{cases} \partial 1/\partial y & x \le y \\ \partial 1/\partial x & y \le x \end{cases} = \begin{cases} 0 & x \le y \\ 0 & y \le x \end{cases}$

$= 0 \quad \forall\, x, y \in [0,1]$ (X is uniformly distributed over $[0,1]$ & $Y = X$)

$$P(\Omega) = P(0 \leq X \leq 1, 0 \leq Y \leq 1) = 1$$

$$\Rightarrow \int_0^1\int_0^1 f_{XY}(u,v)\, du\, dv = 1$$

Contradiction $\because f_{XY}(x,y) = 0 \ \forall\, x,y \in [0,1]$

Hence initial assumption was wrong

$\Rightarrow$ X & Y cannot be jointly continuous

$\Rightarrow$ hence there is no contradictory proofs in part (a) & (b) of this question $\because$ X & Y are not jointly continuous.

6. $F_Z(z) = P(Z \leq z)$

$= P(\max(X,Y) \leq z)$

$= P(\{X \leq z\} \cap \{Y \leq z\}) = F_{XY}(z,z)$

$= P(X \leq z).P(Y \leq z)$ ($\because$ X & Y are independent RVs)

We also know, if X, Y are independent

$F_{XY}(z,z) = F_X(z)\, F_Y(z)$ (proved in class)

$= (F_X(z))(F_X(z))$ ($\because$ common cdf F_X)

$= (F_X(z))^2$

$\because f_Z(z) = \dfrac{dF_Z(z)}{dz}$ (fundamental th^m of calculus)

$= 2F_X(z)\dfrac{dF_X(z)}{dz}$

$$\boxed{f_Z(z) = 2F_X(z)\, f_X(z)}$$

$$F_W(\omega) = P(W \le w)$$
$$= 1 - P(W > w) \quad \left(\begin{array}{l} W \le w \ \& \ W > w \text{ are partions of } \Omega \\ P(\Omega) = 1 \end{array} \right)$$
$$= 1 - P(\min(X, Y) > w)$$
$$= 1 - P((X > w) \cap (Y > w))$$
$$\cancel{1 - (1 - P(X \le w))}$$
$$= 1 - P(X > w) \cdot P(Y > w) \quad (X, Y : \text{independent RVs})$$

$$\int_w^\infty \int_w^\infty f_{XY}(x,y)\,dx\,dy = \int_w^\infty \int_w^\infty f_X(x) f_Y(y)\,dx\,dy = \int_w^\infty f_X(x)\,dx \int f_Y(y)\,dy = P(X > w)\,P(Y > w)$$

($X > w$ & $X \le w$ are partions of Ω, $P(\Omega) = 1$, similarly for Y)

$$= 1 - (1 - P(X \le w))(1 - P(Y \le w))$$
$$= 1 - (1 - F_X(\omega))(1 - F_X(\omega))$$
$$= 1 - (1 - F_X(\omega))^2$$

$$f_W(\omega) = \frac{dF_W(\omega)}{dw}$$
$$= -2(1 - F_X(\omega)) \frac{d(1 - F_X(\omega))}{dw}$$
$$= -2(1 - F_X(\omega))(-f_X(\omega))$$
$$= 2 f_X(\omega) (\cancel{F_X(\omega)}) (1 - F_X(\omega))$$

$$\therefore \boxed{f_W(\omega) = 2 f_X(\omega)(1 - F_X(\omega))}$$

If we define RVs Y_i s.that $Y_i = \frac{X_i}{\sum_i X_i}$, $i \in [1:n]$

$$\therefore \sum_{i=1}^m Y_i = \frac{\sum_{i=1}^m X_i}{\sum_{i=1}^n X_i}$$

$$\therefore \cancel{E[\sum_i Y_i]} \quad \therefore E\left[\frac{\sum_{i=1}^m X_i}{\sum_{i=1}^n X_i}\right] = E[\sum Y_i] = \sum E[Y_i]$$ (linearity of expectation)

$$E[Y_i] = \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n \text{ times}} \frac{x_i}{\sum_{k=1}^n x_k} f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)\, dx_n\, dx_{n-1} \cdots dx_1$$

($Y_i = f(X_1, X_2, \ldots X_n)$ & $\therefore$ this formula applies)

$$= \int_{x_1=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} \frac{x_i}{\sum_{k=1}^{n} x_k} f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)\, dx_n\, dx_{n-1} \cdots dx_1$$

$$= \int_{x_1=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} \frac{x_i}{\sum_{k=1}^{n} x_k} \prod_{p=1}^{n} f_X(x_p)\, dx_n\, dx_{n-1} \cdots dx_1$$

$\because$ common pdf f_X

Performing change of variable

$x_i \to x_j$, $x_j \to x_i$, $i \neq j$, $i, j \in [1:n]$

$$= \int_{x_1=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} \frac{x_j}{\sum x_k} \prod_{p=1}^{n} f_X(x_p)\, dx_n\, dx_{n-1} \cdots dx_1$$

(integrals order adjust themselves)

$= E[Y_j]$

$\therefore E[Y_i] = E[Y_j] \; \forall i, j \in [1:n], i \neq j$

$\therefore$ let $E[Y_1] = t = E[Y_2] = E[Y_3] = \cdots = E[Y_n]$

$$E\left[\frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i}\right] = \sum_{i=1}^{n} E(Y_i)$$

$\Rightarrow E[1] = nt$

$\Rightarrow 1 = nt \Rightarrow \underline{t = 1/n}$

$$\therefore E\left[\frac{\sum_{i=1}^{m} X_i}{\sum_{i=1}^{n} X_i}\right] = \sum_{i=1}^{m} E[Y_i]$$

$= mt$

$= \boxed{m/n}$ Ans

PTO

§ P: X, Y are independent RVs
Q: $f_{XY}(x,y) = g(x)\,h(y)$
RTP $P \Leftrightarrow Q$

If P then Q

X, Y are independent

$$f_{XY}(x,y) = f_X(x)\,f_Y(y)$$
$$= g(x)\,h(y)$$

[If $g(x) = f_X(x)$, $h(y) = f_Y(y)$]

$\therefore P \rightarrow Q$ (trivial)

$Q \rightarrow P$

$$f_{XY}(x,y) = g(x)\,h(y)$$
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$
$$= g(x)\int_{-\infty}^{\infty} h(y)\,dy$$

Similarly, $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

$$= h(y)\int_{-\infty}^{\infty} g(x)\,dx$$

$$\therefore f_X(x)\,f_Y(y) = g(x)\,h(y)\left(\int_{-\infty}^{\infty} h(y)\,dy\right)\left(\int_{-\infty}^{\infty} g(x)\,dx\right)$$
$$= g(x)\,h(y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,h(y)\,dy\,dx$$
$$= g(x)\,h(y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$$

$\because f_{XY}(x,y)$ is a valid pdf, it is normalisable
s. that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx = 1$ $(P(\Omega) = 1)$

$\therefore f_X(x)\,f_Y(y) = g(x)\,h(y)$
$\therefore f_{XY}(x,y) = f_X(x)\,f_Y(y)$ $(= g(x)h(y))$
$\Rightarrow X, Y$ independent $\therefore Q \rightarrow P$

$\because P \rightarrow Q$ & $Q \rightarrow P$ $\therefore P \Leftrightarrow Q$

$$f_{XY}(x,y) = 2e^{-x-y} \qquad 0 < x < y < \infty$$

$f_X(x)$ [for $x>0$] $= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

$= \int_{x}^{\infty} f_{XY}(x,y)\,dy$ [If $y < x$, $f_{XY}(x,y) = 0$]

$= 2e^{-x} \int_{x}^{\infty} e^{-y}\,dy$

$= 2e^{-x} \left. e^{-y} \right]_{\infty}^{x} = (2e^{-x})(e^{-x} - 0)$

$= 2e^{-2x} \qquad x > 0$

$f_Y(y)$ [For $y>0$] $= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

$= \int_{-\infty}^{0} f_{XY}(x,y)\,dx + \int_{0}^{y} f_{XY}(x,y)\,dx + \int_{y}^{\infty} f_{XY}(x,y)\,dx$

$= \int_{0}^{y} f_{XY}(x,y)\,dx$

$= \int_{0}^{y} 2e^{-x-y}\,dx$

$= 2e^{-y} \int_{0}^{y} e^{-x}\,dx = 2e^{-y}\left(\left. e^{-x} \right]_{y}^{0} \right)$

$= 2e^{-y}(1 - e^{-y}) \quad , y > 0$

$f_X(x)\,f_Y(y) = 4e^{-2x-y}(1-e^{-y}) \neq f_{XY}(x,y) \ \forall x,y$

(eg: If $x=1$, $y=2$ $\quad f_{XY}(1,2) = 2e^{-3}$

$f_X(1)\,f_Y(2) = 4e^{-4}(1-e^{-2})$

$f_{XY}(1,2) \neq f_X(1)\,f_Y(2)$)

$\because \exists\, x,y$ s.t. that

$f_X(x)\,f_Y(y) \neq f_{XY}(x,y) \Rightarrow X, Y$ are not independent