Lecture 20

(21 October 2024)

Theorem. Let X and Y be two jointly continuous random variables and let

$$Z = g_1(X, Y), \quad W = g_2(X, Y),$$

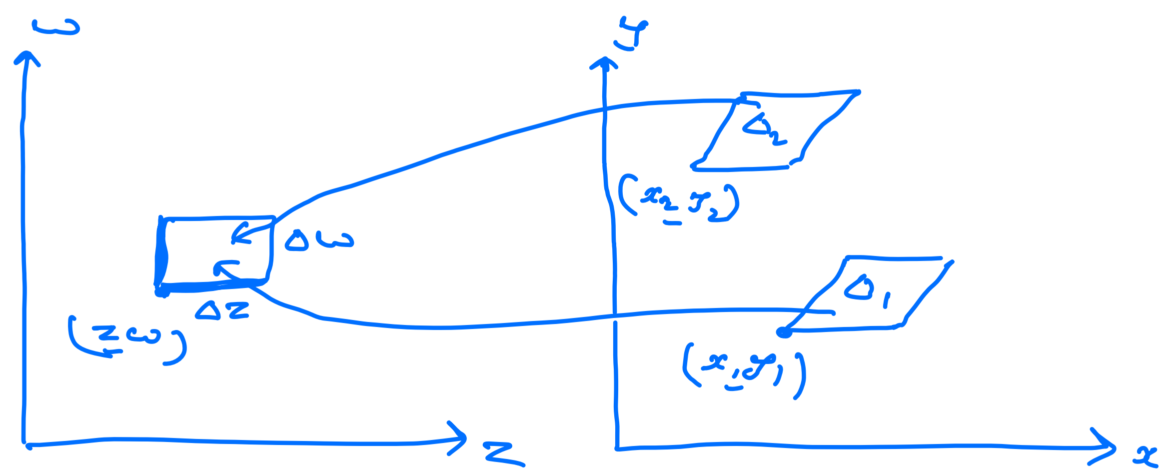
where g_1 and g_2 are continuous and differentiable functions. Then Z and W are jointly continuous with PDF

$$f_{ZW}(z, w) = \sum_{i=1}^{n} \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|},$$

when (x_i, y_i) $i \in [1:n]$ are the solutions of

$$g_1(x, y) = z, \quad g_2(x, y) = w.$$

Proof.



$$P(z < Z \le z+\Delta z,\ w < W \le w+\Delta w)$$

$$= f_{ZW}(z,w)\,\Delta z\,\Delta w$$

$$= \sum_{i=1}^{n} f_{XY}(x_i,y_i)\,|\Delta_i|$$

$$\Longrightarrow f_{ZW}(z,w) = \sum_{i=1}^{n} \frac{f_{XY}(x_i,y_i)}{\left(\frac{\Delta z\,\Delta w}{\Delta_i}\right)}$$

$$\text{as } \Delta z\,\Delta w \to 0 \quad = \sum_{i=1}^{n} \frac{f_{XY}(x_i,y_i)}{|J(x_i,y_i)|}.$$

<u>Exercise</u>. $Z = \max\{X,Y\}$ $W = \min\{X,Y\}$.

Find f_{ZW} in terms of f_{XY}.

We now study two transforms.

(1) Moment generating functions

(2) Characteristic functions

Moment Generating Functions (MGFs)

n^{th} moment of a RV X

$$E[X^n].$$

The main applications of MGFs (or in general transforms) are:

(i) They enable a convenient computation of moments

(ii) They can be used to solve problems involving the computation of the sums of random variables.

Definition. The MGF associated with a RV X is a function $M_X : \mathbb{R} \to [0, \infty)$ defined by

$$M_X(s) = E[e^{sX}].$$

The domain or region of convergence of M_X is the set $D_X = \{ s \in \mathbb{R} : M_X(s) < \infty \}$.

Discrete case:

$$M_X(s) = \sum_x e^{sx} p_X(x)$$

Continuous case:

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx$$

Example. Poisson random variable

$$p_X(k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$M_X(s) = E[e^{sX}]$$

$$= \sum_{k=0}^{\infty} e^{sk} e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^s}$$

$$= e^{\lambda(e^s - 1)}.$$

Example. Exponential RV

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

$$M_X(s) = \int_0^\infty e^{sx} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{x(s-\lambda)}\, dx$$

$$= \frac{\lambda}{\lambda - s} \quad \text{for } s < \lambda.$$

Theorem.

(i) Suppose $M_X(s)$ is finite for all $s \in [-\varepsilon, \varepsilon]$. Then $M_X(s)$ uniquely determines the CDF of X.

(ii) If X and Y are two RVs such that $M_X(s) = M_Y(s)$, $\forall s \in [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Then X and Y have the same CDF.

The proof of this theorem is beyond the scope of this course. However we will see an intuition for why this needs to be true in our discussion on characteristic functions.

Properties

(i) $M_X(0) = 1$ $\left(E[e^{0X}] = 1 \right)$

(ii) Moment generating property

Theorem: Suppose $M_X(s) < \infty$ for $s \in [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Then

$$\frac{d}{ds} M_X(s)\Big|_{s=0} = E[X],$$

$$\frac{d^n}{ds^n} M_X(s)\Big|_{s=0} = E[X^n].$$

Proof:

$$\frac{d}{ds} E[e^{sX}] = E\left[\frac{d}{ds} e^{sX}\right]$$

$$= E[X e^{sX}]$$

$$\frac{d^n}{ds^n} E[e^{sX}]\Big|_{s=0} = E[X^n e^{sX}]\Big|_{s=0}$$

$$= E[X^n].$$

(iii) $Y = aX + b$

$$M_Y(s) = e^{bs} M_X(as).$$

(iv) $Z = X + Y$ X and Y are independent

$$M_Z(s) = M_X(s) M_Y(s).$$

$$E[e^{sZ}] = E[e^{s(X+Y)}]$$

$$= E[e^{sX} \cdot e^{sY}]$$

$$= E[e^{sX}] \cdot E[e^{sY}]$$

$$= M_X(s) M_Y(s).$$

Example. Let $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, and X & Y are independent. Then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

For $N \sim N(0, 1)$

$$M_N(s) = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + sx} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-s)^2}{2}} e^{s^2/2} \, dx$$

$$= e^{s^2/2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-s)^2}{2}} \, dx}_{=1}$$

$$= e^{s^2/2}$$

$$\Rightarrow M_X(s) = e^{\mu_1 s + \sigma_1^2 s^2/2}.$$

Similarly $M_Y(s) = e^{\mu_2 s + \sigma_2^2 s^2/2}$.

$$M_Z(s) = M_X(s) M_Y(s) = e^{(\mu_1+\mu_2)s + (\sigma_1^2+\sigma_2^2)s^2/2}$$

$$\Rightarrow Z \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2).$$

(v) $Z = \sum_{i=1}^{N} X_i$, X_i are i.i.d. with MGF M_X

N is independent of X_i's with

MGF M_N.

$$M_Z(s) = E\left[e^{sZ}\right]$$

$$= E\left[e^{s\sum_{i=1}^{N} X_i}\right]$$

$$= E\left[E\left[e^{s\sum_{i=1}^{N} X_i} \mid N\right]\right]$$

$$= E\left[M_X(s)^N\right]$$

$$= E\left[e^{N\log M_X(s)}\right]$$

$$= M_N\left(\log M_X(s)\right).$$

<u>Characteristic</u> <u>Function</u>

$$\phi_X(t) = E\left[e^{itX}\right], \qquad i = \sqrt{-1}$$

For continuous case

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx$$

- Related to Fourier transform
- Characteristic function exist for all t.

$$|\phi_X(t)| = \left|\int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx\right|$$

$$\leq \int_{-\infty}^{\infty} |e^{itx}| f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

<u>Example</u>. Exponential RV.

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

$$\phi_X(t) = E[e^{itX}]$$

$$= \int_0^{\infty} e^{itx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - it)x}\, dx$$

$$= \frac{\lambda}{\lambda - it}$$

Inversion Theorem

If X is continuous with PDF f_X and characteristic function is ϕ_X, then

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi_X(t)\,dt$$

at every point x at which f_X is differentiable.

The proof is a consequence of inverse Fourier transform.

Properties.

(i) $\phi_X(0) = 1$

(ii) $$\left.\frac{d^n \phi_X(t)}{dt^n}\right|_{t=0} = i^n E[X^n]$$

$$\left.\frac{d^n \phi_X(t)}{dt^n}\right|_{t=0} = \left.\frac{d^n}{dt^n} E[e^{itX}]\right|_{t=0}$$

$$= \left. i^n E[X^n e^{itX}]\right|_{t=0}$$

$$= i^n E[X^n].$$

(iii) $y = ax + b$

$$\phi_y(t) = e^{ibt}\,\phi_x(at).$$

(iv) $z = x + y$ x and y are independent

$$\phi_z(t) = \phi_x(t)\,\phi_y(t).$$