Lecture 12

(12 September 2024)

Definition. A random variable X is called continuous random variable if its CDF can be expressed as

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du, \quad x \in \mathbb{R},$$

for some integrable function $f: \mathbb{R} \to [0, \infty)$ called the probability density function (PDF).

$$f(x) = F_X'(x)$$

[by Fundamental Theorem of Calculus]

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx.$$

The numerical value of $f(x)$ is not a probability. However we can think of $f(x)\Delta x$ as probability for infinitesimally small Δx since

$$P(x < X \le x+\Delta x) = \int_{x}^{x+\Delta x} f_X(x)\,dx$$

$$\approx f_X(x)\Delta x.$$

$$f_X(x) = \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}.$$

If B is a (sufficiently nice) subset of $\mathbb{R}$ (such as an interval or a countable union of intervals and so on) then

$$P(X \in B) = \int_B f_X(x)\,dx.$$

Theorem. If a continuous RV X has a PDF f_X, then

(a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

(b) $P(X=x) = 0$ for all $x \in \mathbb{R}$

Proof. (a) $\int_{-\infty}^{\infty} f_X(x)\,dx$

$= P(X \in (-\infty, \infty))$

$= P(\Omega) = 1.$

(b) $P(X=x) = \int_{x}^{x} f(u)\,du = 0.$

Expectation.

The expectation of a continuous RV X with PDF f_X is given by

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx.$$

Example. $f_X(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$

$$E[X] = \frac{2}{3}.$$

- If X is a continuous RV $g(X)$ is also a random variable. However $g(X)$ can be either a continuous RV or a discrete RV.

$Y = g(X) = X$ is continuous

If $g(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$, $Y = g(X)$ is discrete

Theorem. If x and $g(x)$ are continuous random variables then

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x)\,dx.$$

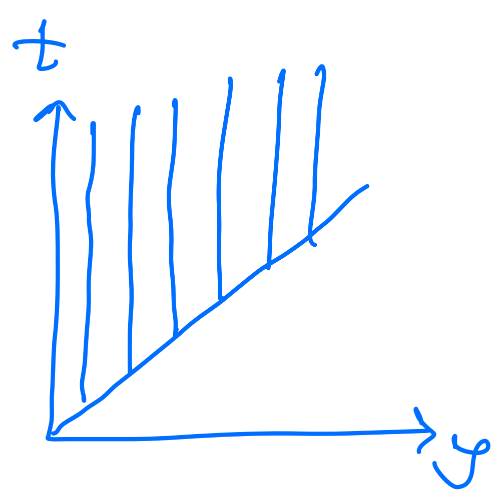
We first prove the following lemma.

Lemma. For a non-negative continuous random variable Y (i.e., f_Y s.t. $f_Y(y)=0$ when $y<0$),

$$E[Y] = \int_0^{\infty} P(Y>y)\,dy.$$

Proof. $\displaystyle\int_0^{\infty} P(Y>y)\,dy$

$$= \int_{y=0}^{\infty} \int_{t=y}^{\infty} f_Y(t)\,dt\,dy$$



$$= \int_{t=0}^{\infty} \int_{y=0}^{t} f_Y(t)\, dt\, dy$$

$$= \int_{t=0}^{\infty} f_Y(t) \left(\int_{y=0}^{t} dy \right) dt$$

$$= \int_{t=0}^{\infty} t\, f_Y(t)\, dt = E[Y].$$

We prove the above theorem assuming $g(X)$ is non-negative.

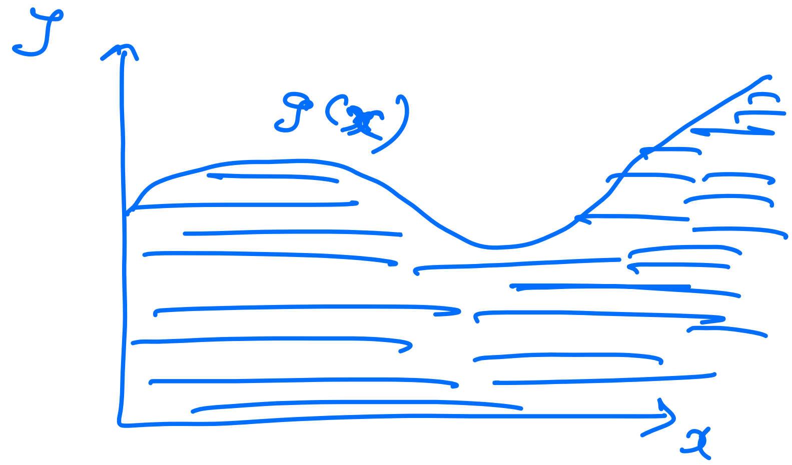
$$E[g(X)] = \int_{0}^{\infty} P(g(X) > y)\, dy$$

$$= \int_{y=0}^{\infty} \int_{x:\, g(x) > y} f_X(x)\, dx\, dy$$

$$\left[P(g(X) > y) = P(X \in B) = \int_{B} f_X(x)\, dx \right.$$

with $B = \{x : g(x) > y\}$]

$$= \int_{x=-\infty}^{\infty} \int_{y=0}^{g(x)} f_X(x)\,dx\,dy$$



$$= \int_{x=-\infty}^{\infty} f_X(x) \int_{y=0}^{g(x)} dy\,dx$$

$$= \int_{x=-\infty}^{\infty} g(x) f_X(x)\,dx,$$

__Exercise__. Complete the proof of the theorem for a general real-valued function g.

__Hint__. Show that

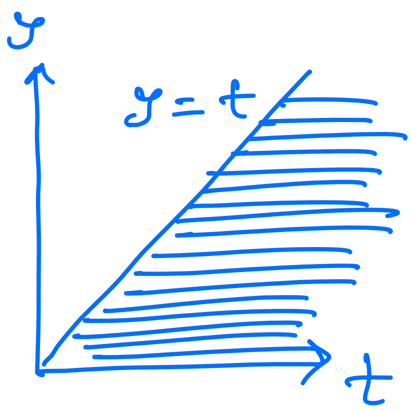
$$E[Y] = \int_0^{\infty} P(Y > y)\,dy - \int_0^{\infty} P(Y < -y)\,dy.$$

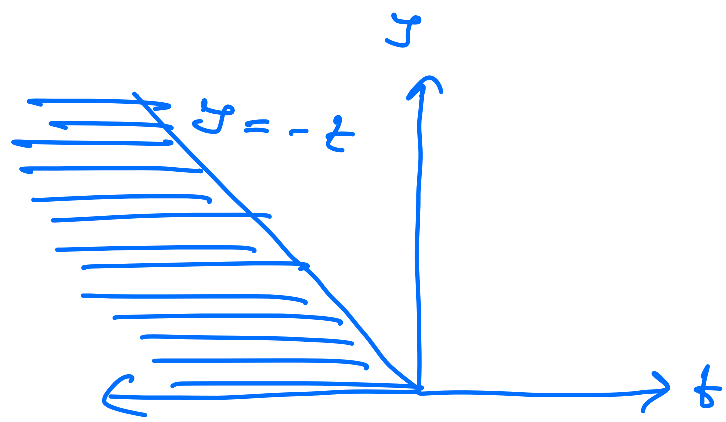
Proof of the above exercise follows.

$$\int_{0}^{\infty} P(Y>y)\,dy - \int_{0}^{\infty} P(Y<-y)\,dy$$

$$= \int_{y=0}^{\infty} \int_{t=y}^{\infty} f_Y(t)\,dt\,dy - \int_{y=0}^{\infty} \int_{t=-\infty}^{-y} f_Y(t)\,dt\,dy$$

$$= \int_{t=0}^{\infty} \int_{y=0}^{t} f_Y(t)\,dy\,dt - \int_{t=-\infty}^{0} \int_{y=0}^{-t} f_Y(t)\,dy\,dt$$





$$= \int_{t=0}^{\infty} t\, f_Y(t)\,dt + \int_{t=-\infty}^{\infty} t\, f_Y(t)\,dt$$

$$= E[Y].$$

$E[g(X)]$

$$= \int_0^\infty P(g(X) > y)\,dy - \int_0^\infty P(g(X) < -y)\,dy$$

$$= \int_{y=0}^\infty \int_{x:\, g(x) > y} f_X(x)\,dx\,dy$$

$$- \int_{y=0}^\infty \int_{x:\, g(x) < -y} f_X(x)\,dx\,dy$$

$$= \int_{x=-\infty}^\infty \int_{y:\, 0 \le y \le g(x)} f_X(x)\,dy\,dx$$

$$- \int_{x=-\infty}^\infty \int_{y:\, g(x) < -y \le 0} f_X(x)\,dy\,dx$$

$$= \int_{x=-\infty}^\infty g^+(x) f_X(x)\,dx + \int_{x=-\infty}^\infty g^-(x) f_X(x)\,dx$$

$$= \int_{x=-\infty}^\infty g(x) f_X(x)\,dx.$$

$$\left[\begin{array}{l} g^+(x) = \max\{0, g(x)\} \\ g^-(x) = \max\{0, -g(x)\} \end{array}\right]$$

Variance of x

$$Var(x) = E[(x - E[x])^2]$$

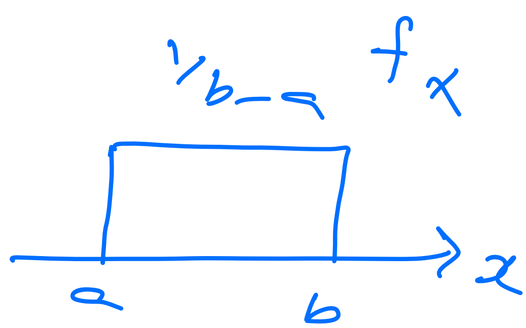
$$= \int_{-\infty}^{\infty} (x - E[x])^2 f_X(x)\, dx$$

$$= E[x^2] - E[x]^2.$$

<u>Examples of</u> <u>Continuous RVs</u>

<u>Uniform Random</u> <u>Variable</u>

$$f_X(x) = \begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{o.w.} \end{cases}$$



$$E[x] = \frac{a+b}{2}$$

$$Var(x) = E[x^2] - E[x]^2 = \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4}$$

$$= (b-a)^2/12$$

Exponential Random Variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

where λ is a positive parameter characterizing the PDF.

An exponential RV can be a good model for the amount of time until an incident of interest takes place, such as

- a message arriving at a computer
- an equipment breaking down, etc.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{0}^{\infty} x \lambda e^{-\lambda x}\,dx$$

$$= \left[x \int \lambda e^{-\lambda x} dx - \int 1 . \left(\int \lambda e^{-\lambda x} dx \right) dx \right]_0^\infty$$

$$= \left[-x e^{-\lambda x} + \int e^{-\lambda x} dx \right]_0^\infty$$

$$= 0 + \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty = 1/\lambda .$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x) dx$$

$$= \int_0^\infty \lambda x^2 e^{-\lambda x} dx$$

$$= \left[x^2 \int \lambda e^{-\lambda x} dx - \int 2x . e^{-\lambda x} dx \right]_0^\infty$$

$$= 0 + \frac{2 E[x]}{\lambda} = \frac{2}{\lambda^2}$$

$$\operatorname{var}(x) = \frac{2}{\lambda^2} - 1/\lambda^2 = 1/\lambda^2 .$$