Assignment 4

(MA6.102) Probability and Random Processes, Monsoon 2024

Release date: 30 September 2024, Due date: 7 October 2024

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. Find the PDF, the mean and the variance of the random variable X with the CDF

$$F_X(x) = \begin{cases} 1 - \frac{a^3}{x^3}, & \text{if } x \geq a, \\ 0, & \text{if } x < a, \end{cases}$$

where a is a positive constant.

Problem 2. One of the two wheels of fortune, A and B, is selected by the toss of a fair coin, and the wheel chosen is spun once to determine the value of a random variable X. The PDF of X given A is selected is $f_{X|A}(x) = 1$, $0 \leq x \leq 1$. The PDF of X given B is selected is $f_{X|B}(x) = 3$, $0 \leq x \leq \frac{1}{3}$. Find the probability that A was selected given that $X \leq \frac{1}{4}$.

Problem 3. Given

$$f_{XY}(x,y) = \begin{cases} k, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases} .$$

Determine the conditional PDFs $f_{X|Y}$ and $f_{Y|X}$.

Problem 4. Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} 6xy, & 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x} \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\text{Var}(X|Y = y)$, for $0 \leq y \leq 1$.

Problem 5. (a) If the random variables U and V are jointly continuous, show that $P(U = V) = 0$.
(b) Let X be uniformly distributed on $(0, 1)$, and let $Y = X$. Then X and Y are continuous, and $P(X = Y) = 1$. Is there a contradiction here?

Problem 6. Let X and Y be two independent random variables with common CDF F_X and PDF f_X. Find the PDFs f_Z and f_W, where $Z = \max\{X, Y\}$ and $W = \min\{X, Y\}$.

Problem 7. Let $X_1, X_2, , \dots, X_n$ be independent identically distributed random variables with common PDF f_X. Find

$$\mathbb{E}\left[\frac{\sum_{i=1}^{m} X_i}{\sum_{i=1}^{n} X_i}\right].$$

Problem 8. Show that X and Y are independent continuous random variables if and only if their joint probability density function f_{XY} factorizes as the product $f_{XY}(x, y) = g(x)h(y)$ of functions of the single variables x and y alone.

Problem 9. The joint PDF of two random variables X and Y is given by

$$f_{XY}(x, y) = 2\mathrm{e}^{-x-y},\ 0 < x < y < \infty.$$

Are they independent?