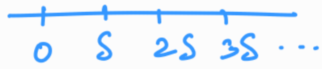
→ Suppose we're tossing a coin at time stamp - δ, 2δ, 3δ, 4δ, ...

Given p is the prob. of getting H in a single toss.



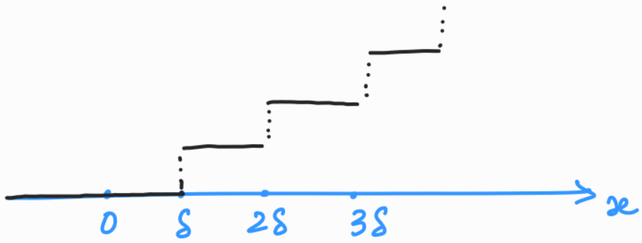
Let $X = k\delta$ where k: no. of tosses to get the first heads.

$$P(X = k\delta) = (1-p)^{k-1} p$$

$$F_X(n\delta) = 1 - (1-p)^n, \quad n \in \mathbb{N}.$$

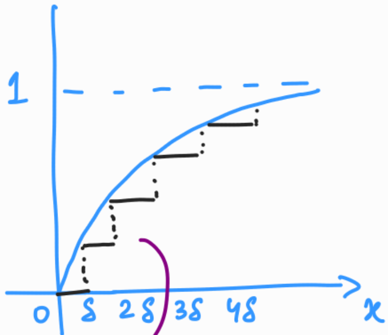
$F_X(x)$ where $x \in \mathbb{R}$? ⟶ Write this expression formally $\forall x \in \mathbb{R}$

Right conti func



→ $f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = 1 - e^{-\lambda x}, \quad x \geq 0.$$



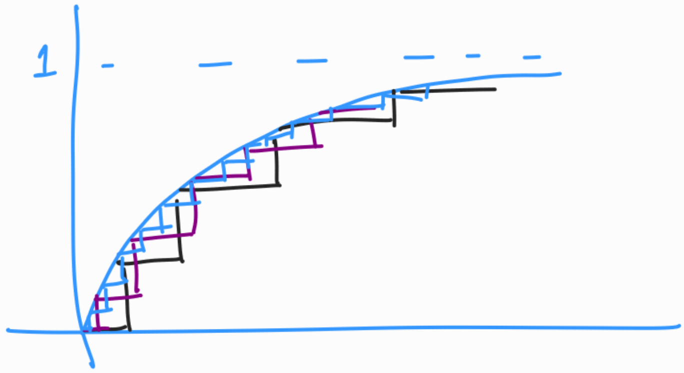
Graph similar to geo CDF

Intersection values : $e^{-\lambda n \delta} = (1-p)^n$

For $\delta > 0$, choose p as ⇒ We have p as a func of δ.

→ Starting with exponential RV. constructed geo RV for any δ.

→ If we decrease the value of δ, the plot of CDF of geo RV approaches exp. RV CDF plot.



⟹ As δ decreases,

geo RV ⟶ exp. RV

→ Exp. RV is essentially a geo RV where the time gap b/w 2 consecutive tosses is infinitesimally small.

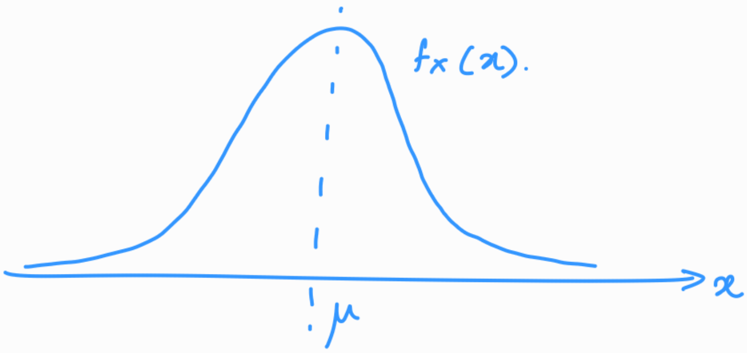
Formally, above can be stated as :

$$\lim_{\delta \to 0} \underbrace{F_X(x)}_{\text{Geo RV CDF}} = F_X^{\text{Exp}}(x)$$

→ Exercise.

• Gaussian RV (Normal RV).

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} , \quad x \in \mathbb{R}$$



→ $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ → Prove it.

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx.$$

$$t = \frac{x-\mu}{\sigma}.$$ (Change of variable).

↳ Whenever there is linear scaling of var. of interest

$$\Rightarrow dt = \frac{dx}{\sigma}$$

$$\Rightarrow I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

↳ Can't do integration by-parts of this. (Try it out).

Consider I^2 instead of I and if we show $I^2 = 1$ and I has to be $+1$

$$I^2 = \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt}_{I} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds.}_{I.}$$

because $f_X(x)$ is a non-negative quantity. So $\int f_x(x)$ would be positive.

So we just need to show $I^2 = 1$ and here we show $I = 1$.

Use Polar coordinate system

→ Sum of sq. in Cartesian → Convert to polar coord.

If we see scalar shift in vars of interest, change of var.

Use polar coord. change of var on $\int_{s=-\infty}^{\infty}\int_{t=-\infty}^{\infty} \frac{1}{2\pi} e^{-(s^2+t^2)/2} ds\,dt.$

$s = r\cos\theta, \quad t = r\sin\theta$

Exercise

$$I = \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{2\pi} e^{-r^2/2} r\,dr\,d\theta$$ → Solve this also. Show it's 1.

Mean of Gaussian : μ

Expectation of Gaussian : $E[X] = \int_{-\infty}^{\infty} x f_x(x) dx = \underline{\underline{\mu}}.$

↪ Use int. by parts

variance of Gaussian : $\underline{\underline{\sigma^2}}$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{\sigma^2}} dx$$

Change of var ($\because$ scalar shift of x) -

$$t = \frac{x-\mu}{\sigma}$$

$$\int_{-\infty}^{\infty} t^2 e^{-t^2/2}\, dt. \longrightarrow \text{Int. by parts.}$$

We know $\underline{\int t e^{-t^2/2}\, dt}$

So $t^2 e^{-t^2/2} = \underbrace{t\left(te^{-t^2/2}\right)}_{\text{split like this}}$

→ If X is a Gaussian / Normal RV, then we denote it as

$$X \sim N(\underbrace{\mu}_{\text{Mean}}, \underbrace{\sigma^2}_{\text{Variance.}})$$

• Properties of Gaussian RV

If $X \sim N(\mu, \sigma^2)$, then

(i) $Y = aX + b \sim N(a\mu + b,\ \sigma^2 a^2)$, $a \neq 0$

$X \sim f_X$

$Y = aX + b \sim f_Y.$

f_Y in terms of f_X ?

→ For conti. RV, PDF comes from its CDF. ←

For conti. RV, always start with CDF. Can't talk abt PDF directly.

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du \quad \& \quad f_X(x) = F_X'(x).$$

So first we find F_Y and then taking derivative gives f_Y.

$$F_Y(y) = P(Y \leq y)$$

$$= P(aX + b \leq y) = P(aX \leq y - b)$$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & \longrightarrow \text{When } a>0 \\ P\left(X \ge \frac{y-b}{a}\right) & \longrightarrow \text{When } a<0. \end{cases}$$

$$= \begin{cases} = F_X\left(\frac{y-b}{a}\right), & a>0 \\ = 1 - F_X\left(\frac{y-b}{a}\right), & a<0. \end{cases}$$

Notice this is X.

$$f_Y(y) = F_Y'(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Now $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2 a^2}} e^{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2}$ $\Rightarrow$ Y is also Normal.

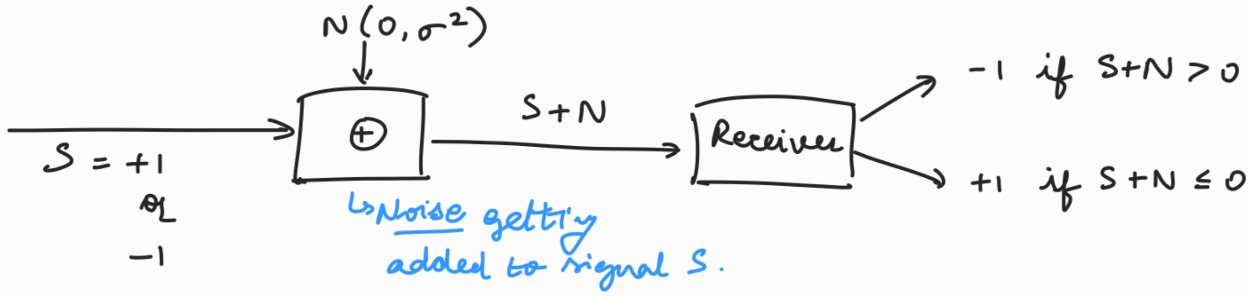
$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2 a^2}} e^{-\frac{(y-(a\mu+b))^2}{2a^2\sigma^2}}.$$

So Mean $= a\mu + b$, Var $= a^2\sigma^2$

- Standard Normal RV : $N(0,1) = Z$.

$$F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} \, dt = \Phi(z)$$

$\rightarrow$



Find the prob. of error when $S = -1$ is sent. : $P(N > 1)$