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PRP Assignment -2

1. Firstly, we notice that $\{X = x\}$ gives us a set of disjoint sets for all values of x, $x \in \mathbb{R}$.

Using the additivity ~~rule~~ axiom (of disjoint sets) for any set S of possible values of X, we have

$$P(X \in S) = \sum_{x \in S} P_X(x) \quad \text{--- (1)}$$

Now

$$P_Y(y) = P(\{Y = y\}) = P(\{f(X) = x\})$$

Let S be the set $\{x : f(x) = x\}$

$$\therefore P_Y(y) = P(X \in S) = \sum_{x \in S} P_X(x) \quad \text{(using (1))}$$

$$= \sum_{x : f(x) = x} P_X(x)$$

2. $E(X) = \sum_{k=0}^{\infty} k\, P(X=k)$ [we know]

Now, $k = \sum_{n=0}^{k-1} (1)$ {k times $1 = k$}

$$\Rightarrow E(X) = \sum_{k=0}^{\infty} \sum_{n=0}^{k-1} P(X=k)$$

$$= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} P(X=k)$$

$$= \sum_{n=0}^{\infty} P(X(k>n))$$

$$\boxed{E(X) = \sum_{n=0}^{\infty} P(X>n)}$$ (Proved)

Q3 $P_X(x) = \begin{cases} \dfrac{1}{b-a+1} & k = a, a+1, \ldots, b \\ 0 & \text{otherwise} \end{cases}$

$a < b$

Find $var(X) = E(X^2) - (E(X))^2$

$$E(x) = \sum_{k=0}^{\infty} k \, P_X(X=k)$$

$$\therefore E(X) = \sum_{k=a}^{b} k \cdot \frac{1}{(b-a+1)}$$

$$= \frac{1}{(b-a+1)} \sum_{k=a}^{b} k$$

$$= \frac{1}{(b-a+1)} \left[\frac{b(b+1)}{2} - \frac{(a)(a-1)}{2} \right]$$

$$= \frac{1}{2(b-a+1)} (b^2 + b - a^2 + a)$$

$$= \frac{\cancel{(b-a+1)}(a+b)}{2\cancel{(b-a+1)}}$$

$\because a \leq b \Rightarrow b-a+1 > 0$

$$= \frac{a+b}{2}$$

$$= \frac{\cancel{(b-a+1)}(a+b)}{2\cancel{(b-a+1)}} = \frac{a+b}{2}$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 P_X(X=k)$$

$$= \sum_{k=a}^{b} k^2 \frac{1}{(b-a+1)}$$

$$= \frac{1}{(b-a+1)}\left[\frac{b(b+1)(2b+1)}{6} - \frac{a(a+1)(2a+1)}{6}\right]$$

$$= \frac{(b(b+1)(2b+1) - a(a+1)(2a+1))}{6(b-a+1)}$$

$$(E[X])^2 = (a+b)^2/4$$

$$\therefore \text{var}(X) = E[X^2] - (E[X])^2$$

Subs. values

$$\Rightarrow \boxed{\text{Var}(X) = \frac{(b-a+1)^2 - 1}{12}}$$ Ans

Q4 let $F_1(x) = \frac{1}{2}(1 + \tanh(x))$

$$F_2(x) = \frac{1}{2}(1 + \tanh(x+1))$$

Now both are continuous, & strictly increasing functions, maps $\mathbb{R}$ to interval $(\Omega, 1)$

Now, $F_2(x) - F_1(x) = \tanh(x+1) - \tanh x$
> 0

$$\therefore F_2(x) > F_1(x)$$

Let's take RV X_1, X_2 such that

$X_1 = F_1^{-1}(x)$

$X_2 = F_2^{-1}(x)$

$X_1 = \tanh^{-1}(2u-1)$

$X_2 = \tanh^{-1}(2u-1) - 1$

let u be RV uniformly distributed on $(0,1)$

$X_1 = F_1^{-1}(u) = \tanh^{-1}(2u-1)$

$X_2 = F_2^{-1}(U) = \tanh^{-1}(2u-1) - 1$

$X_1 - X_2 = 1 \rightarrow X_1 > X_2$ (proved)

4] let $\Omega = \{(x,y) : x$ is heads/tails (denoted by H | T) & y is no. shown on dice$\}$

$\therefore \Omega = \{(H,1),(H,3),H(5),T(2),T(4),T(6)\}$

$X = \{y : (x,y) \in \Omega\}$

$\therefore X((H,1)) = 1, X((H,3)) = 3$, so on.

$E[X] = \sum_{k=0}^{n} k \, P_X(x)$

We notice

$P((H,1)) = P(H).P(1)$

$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = P((H,3)) = P((H,5))$
$= P((T,2)) = P((T,4)) = P((T,6))$

$\therefore P_X(x) = \begin{cases} 1/6 & \forall x \in [1:6] \\ 0 & \text{otherwise} \end{cases}$

$\therefore E[X] = \frac{1}{6} \sum_{k=1}^{6} k$

$= \frac{1}{\cancel{6}} \frac{(\cancel{6})(7)}{2} = \boxed{3.5}$ Ans

Q.6 X is a RV $R_x = \{a, a+1, \ldots, b\}$
$a < 0 < b$

find
PMF$(\max\{0, X\})$ & PMF$(\min\{0, X\})$

case I
let $X > 0 \Rightarrow \max\{0, X\} = X$, $\min\{0, X\} = 0$

$$P_X(\max\{0, X\}) = P_x(X) = P_x(X > 0)$$
$$= \frac{b}{b-a+1}$$

$$P_x(\min\{0, X\}) = P_x(X = 0) = \frac{1}{b-a+1}$$

case II For $x < 0$, $\max\{0, x\} = 0$
$\min\{0, x\} = X$

$$P_x(\max\{0, X\}) = P_x(x = 0) = 1/(b-a+1)$$
$$P_x(\min\{0, X\}) = P_x(x < 0) = a/(b-a+1)$$

case III For $x = 0$, $\max\{0, X\} = 0 = \min\{0, X\}$
$$P_x(\max\{0, X\}) = 1/(b-a+1)$$
$$= P_x(\min\{0, X\})$$