Practice Problem Set 1

(MA6.102) Probability and Random Processes, Monsoon 2024

Problem 1. Let $C_1, C_2, \ldots, C_n$ be events that form a partition of the sample space Ω. For two events A and B, suppose we know that

- A and B are conditionally independent given C_i, i.e., $P(A \cap B | C_i) = P(A|C_i)P(B|C_i)$, for all $i \in \{1, 2, \ldots, n\}$;
- B is independent of C_i, i.e., $P(B \cap C_i) = P(B)P(C_i)$, for all $i \in \{1, 2, \ldots, n\}$.

Are A and B independent events?

Problem 2. Let X and Y be independent random variables, each taking the values $+1$ or -1 with equal probability $\frac{1}{2}$, and let $Z = XY$.

(a) Are X, Y, and Z independent?
(b) Are X, Y, and Z pairwise independent?

Problem 3. Let X_1 and X_2 be two discrete independent random variables which are symmetric about 0; that is, X_i and $-X_i$ have exactly the same PMFs. Show that, for all x,

$$P(X_1 + X_2 \geq x) = P(X_1 + X_2 \leq -x).$$

Is the conclusion necessarily true without the assumption of independence?

Problem 4. A permutation on the numbers in $[1:n]$ can be represented as a function $\pi : [1:n] \to [1:n]$, where $\pi(i)$ is the position of i in the ordering given by the permutation. A fixed point of a permutation $\pi : [1:n] \to [1:n]$ is a value x for which $\pi(x) = x$. Let X be number of fixed points of a permutation chosen uniformly at random from all permutations. Find $\mathbb{E}[X]$.
Hint: Express X as a sum of indicator random variables.

Problem 5. Alvin shops for probability books for K hours, where K is a random variable that is equally likely to be $1, 2, 3,$ or 4. The number of books N that he buys is random and depends on how long he shops according to the conditional PMF $P_{N|K}(n|k) = \frac{1}{k}$, for $n = 1, 2, \ldots, k$.

(a) Find the joint PMF of K and N.

(b) Find the marginal PMF of N.

(c) Find the conditional PMF of K given that $N = 2$.

(d) Find the conditional mean and conditional variance of K, given that he bought at least 2 but no more than 3 books.

(e) The cost of each book is a random variable with mean Rs. 30. What is the expected value of his total expenditure?

Problem 6. Let $X_1, X_2, \ldots, X_n$ be independent discrete random variables and let $X = X_1 + X_2 + \cdots + X_n$. Suppose that each X_i is a geometric random variable with parameter p_i, and that $p_1, p_2, \ldots, p_n$ are chosen so that the mean of X is a given $\mu > 0$. Show that the variance of X is minimized if the p_i values are chosen to be all equal to $\frac{n}{\mu}$.

Problem 7. For two random variables X and Y, prove the triangle inequality:

$$\sqrt{\mathbb{E}[(X+Y)^2]} \leq \sqrt{\mathbb{E}[X^2]} + \sqrt{\mathbb{E}[Y^2]}.$$

[*Hint:* Use the Cauchy-Schwarz inequality.]

Problem 8. For a non-negative continuous random variable X, show that

$$\mathbb{E}[X^n] = \int_0^\infty n x^{n-1} P(X > x)\ dx.$$

Problem 9. Consider a random variable X with the following two-sided exponential PDF

$$f_X(x) = \begin{cases} p\lambda \mathsf{e}^{-\lambda x}, & \text{if } x \geq 0, \\ (1-p)\lambda \mathsf{e}^{\lambda x}, & \text{if } x < 0, \end{cases}$$

where λ and p are scalars with $\lambda > 0$ and $p \in [0,1]$. Find the mean and the variance of X.

Problem 10. A stick of length 1 is split at a point U that is uniformly distributed over $[0,1]$. Determine the expected length of the substick that contains the point p, $0 \leq p \leq 1$. Also, find the value of p that maximizes this expected length.

All the best for mid-semester examinations