Lecture 21

(24 October 2024)

Module 4 (Tail Bounds and Limit Theorems)

- Markov's Inequality
- Chebyshev's Inequality
- Chernoff Bound
- Convergence of Random Variables
- (Weak) Law of Large Numbers
- Central Limit Theorem

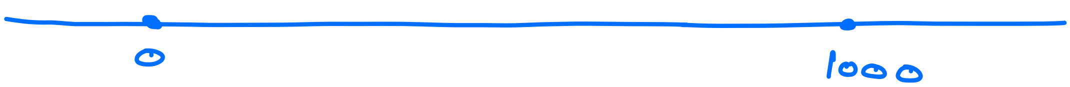
Suppose we want to compute $P(X \geq a)$. In some scenarios, it may be sufficient to have bounds on this probability instead of its exact value, e.g., when the distribution of X is unavailable or hard to compute. In such scenarios if we have exact values or bounds for the mean and variance of X, we can obtain meaningful bounds on the quantity of interest.

Markov's Inequality

If X is a non-negative random variable with $E[X] < \infty$, then

$$P(X \geq a) \leq \frac{E[X]}{a}, \quad \text{for all } a > 0.$$

Interpretation: "If $X \geq 0$ and $E[X]$ is small then the probability that X takes a large value must be small."



$$E[X] = \sum_x x\, P_X(x), \qquad P_X(1000) \rightarrow \text{small}$$

Proof:

$$X = X\left(\mathbb{1}\{X < a\} + \mathbb{1}\{X \geq a\}\right)$$

$$E[X] = E[X\,\mathbb{1}\{X < a\}] + E[X\,\mathbb{1}\{X \geq a\}]$$

$$\geq E[X\,\mathbb{1}\{X \geq a\}]$$

$$\geq a\,E[\mathbb{1}\{X \geq a\}]$$

$$= a P(X \geq a)$$

$$\Rightarrow \quad P(X \geq a) \leq \frac{E[X]}{a}.$$

Exercise. Can Markov's inequality hold with an equality? If so construct a distribution on X s.t. $P(X \geq a) = \frac{E[X]}{a}$.

Example. Let $X \sim \text{Binomial}(n, p)$. Using Markov's inequality, find an upper bound on $P(X \geq \alpha n)$ where $p < \alpha < 1$. Evaluate the bound for $p = 1/2$ and $\alpha = 3/4$.

$E[X] = np$.

$$P(X \geq \alpha n) \leq \frac{E[X]}{\alpha n} = \frac{\cancel{n}p}{\cancel{n}\alpha} = p/\alpha$$

$$= \frac{2}{3}$$

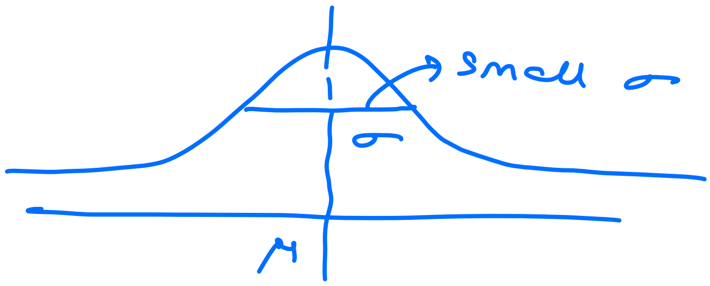
$$\therefore P(X \geq \alpha n) \leq \frac{2}{3}.$$

Chebyshev's Inequality

If X is a random variable with mean $\mu < \infty$ and variance $\sigma^2 < \infty$, then

$$P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}, \text{ for all } c>0.$$

Interpretation.



"If a random variable has small variance then the probability that it takes a value far from its mean is also small".

Recall that variance measures the spread of a RV X around its mean.

Proof. Let $Y = |X-\mu|$.

$$P(Y \geq c) = P(Y^2 \leq c^2)$$

$$\leq \frac{E[Y^2]}{c^2} = \frac{E[|X-\mu|^2]}{c^2} = \frac{\sigma^2}{c^2}.$$

(by Markov's inequality)

An alternative form of Chebyshev's Inequality:

$$P(|X-\mu| \geq k\sigma) \leq 1/k^2 .$$

<u>Example</u>. Let $X \sim \text{Binomial}(n, p)$. Using Chebyshev's inequality, find an upper bound on $P(X \geq \alpha n)$, where $p < \alpha < 1$. Evaluate for $p = 1/2$, $\alpha = 3/4$.

$$P(X \geq \alpha n) = P(X - np \geq n(\alpha - p))$$

$$\leq P(|X - np| \geq n(\alpha - p))$$

$$\leq \frac{Var(X)}{n^2(\alpha - p)^2}$$

$$= \frac{\cancel{n}p(1-p)}{n^{\cancel{2}}(\alpha - p)^2}$$

$$= 4/n .$$

Chernoff Bounds

Let X be a random variable with $M_X(s) = E[e^{sX}]$ for $s \in [-\varepsilon, \varepsilon]$. Then

$$P(X \geq a) \leq \inf_{s>0} \frac{E[e^{sX}]}{e^{as}},$$

$$P(X \leq a) \leq \inf_{s<0} \frac{E[e^{sX}]}{e^{as}}.$$

<u>Proof</u>. $P(X \geq a) = P(sX \geq sa)$

(for $s > 0$)

$$= P(e^{sX} \geq e^{sa})$$

$$\leq \frac{E[e^{sX}]}{e^{as}}$$

$$\Rightarrow P(X \geq a) \leq \inf_{s>0} \frac{E[e^{sX}]}{e^{as}}.$$

Similarly, $P(X \le a) = P(sX \ge as)$

(for $s < 0$)

$$= P(e^{sX} \ge as)$$

$$\le \frac{E[e^{sX}]}{e^{as}}$$

$$\Rightarrow P(X \le a) \le \inf_{s<0} \frac{E[e^{sX}]}{e^{as}}.$$

<u>Example</u>. Let $X \sim \text{Binomial}(n, p)$. Using Chernoff bound, give a bound on $P(X \ge \alpha n)$, where $p < \alpha < 1$. Evaluate the bound for $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

$$X = \sum_{i=1}^{n} X_i, \quad P_{X_i}(1) = p = 1 - P_{X_i}(0).$$

X_i are i.i.d.

(independent & identically distributed)

$$E[e^{sX}] = E\left[e^{s\sum_{i=1}^{n} X_i}\right]$$

$$= \prod_{i=1}^{n} E\left[e^{s x_i}\right]$$

$$= M_{x_1}(s)^n$$

$$= \left(pe^s + 1-p\right)^n.$$

$$P(X \geq \alpha n) \leq \inf_{s>0} \frac{\left(pe^s + 1-p\right)^n}{e^{n\alpha s}}$$

$$\frac{d}{ds}\left(e^{-n\alpha s}\left(pe^s + 1-p\right)^n\right) = 0$$

$$\Rightarrow e^s = \frac{\alpha(1-p)}{p(1-\alpha)}$$

$$\Rightarrow s = \log \frac{\alpha(1-p)}{p(1-\alpha)} > 0$$

since $\frac{1-p}{p} > \frac{1-\alpha}{\alpha}$ as $p < \alpha$.

Also check $\frac{d^2}{ds^2}\left(e^{-n\alpha s}\left(pe^s + 1-p\right)\right) \geq 0.$

We get

$$P(X \geq \alpha n) \leq \left(\frac{\alpha(1-p)}{p(1-\alpha)}\right)^{-n\alpha} \left(\frac{p\alpha(1-p)}{p(1-\alpha)} + 1-p\right)^{n}$$

$$= \left(\frac{\alpha(1-p)}{p(1-\alpha)}\right)^{-n\alpha} \left(\frac{1-p}{1-\alpha}\right)^{n}$$

$$= \left(\frac{1-p}{1-\alpha}\right)^{n(1-\alpha)} \cdot \left(\frac{p}{\alpha}\right)^{n\alpha}$$

$$= \left(\frac{1-1/2}{1-3/4}\right)^{n(1-3/4)} \left(\frac{2}{3}\right)^{3n/4}$$

$$= \frac{2^{n/4} \cdot 2^{3n/4}}{27^{n/4}} = \left(\frac{16}{27}\right)^{n/4}.$$

Comparison between Markov, Chebyshev and Chernoff bounds:

$$P(X \geq \alpha n) \leq \frac{2}{3} \quad [\text{Markov}]$$

$$P(X \geq \alpha n) \leq 4/n \quad [\text{Chebyshev}]$$

$$P(X \geq \alpha n) \leq \left(\frac{16}{27}\right)^{n/4} \quad [\text{Chernoff}]$$

The bound given by Markov is the weakest bound. It is a constant (does not depend on n).

Chebyshev's bound is stronger than Markov's. In particular note that $4/n \to 0$ as $n \to \infty$.

Chernoff's bound is the strongest bound. It goes to zero exponentially fast.

Exercise. Suppose X is a RV taking values in $[a,b]$. Obtain a bound on $P(|X-\mu| \geq c)$ using Chebyshev's inequality. In particular prove that

$$P(|X-\mu| \geq c) \leq \frac{(b-a)^2}{4c^2}.$$

Convergence of Random Variables

Recall the convergence of sequence of real numbers,

We say a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if for every $\varepsilon>0$ there exists $n_\varepsilon \in \mathbb{N}$ such that, for all $n \geq n_\varepsilon$ we have $|x_n - x| < \varepsilon$.

Convergence in Probability

Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables on some probability space $(\Omega, \mathcal{F}, P)$. We say $(X_n)_{n\in\mathbb{N}}$ converges to another RV X in probability if

$$P(|X_n - X| > \varepsilon) \to 0 \text{ as } n\to\infty, \text{ for all } \varepsilon > 0.$$

Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean μ and variance $\sigma^2 < \infty$. We have for every $\varepsilon > 0$

$$P\left(\left| \frac{\sum_{i=1}^{n} X_i}{n} - \mu \right| > \varepsilon \right) \to 0 \text{ as } n \to \infty,$$

(or)

$$\frac{\sum_{i=1}^{n} X_i}{n} \text{ converges to } \mu \text{ in probability.}$$

Proof. Let $M_n = \dfrac{\sum_{i=1}^{n} X_i}{n}$.

$$E[M_n] = \mu \qquad \text{Var}(M_n) = n \times \frac{1}{n^2} \text{Var}(X) = \frac{\sigma^2}{n}$$

By Chebyshev's inequality

$$P(|M_n - \mu| > \varepsilon) \leq \frac{\text{Var}(M_n)}{\varepsilon^2}$$

$$= \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty.$$