Question 1

Given: - X is an exponentially distributed random variable with parameter λ, meaning $X \sim \text{Exp}(\lambda)$. Its probability density function (PDF) is:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

Let: - $Y = \lfloor X \rfloor$: the integer part of X - $R = X - \lfloor X \rfloor$: the fractional part of X

1. PMF of $Y = \lfloor X \rfloor$

To find the PMF of Y, we need to find $P(Y = k)$ for $k = 0, 1, 2, \ldots$.

Since $Y = \lfloor X \rfloor = k$ means $k \leq X < k + 1$, we have:

$$P(Y = k) = P(k \leq X < k + 1)$$

Using the CDF of X, we can calculate this probability as follows:

$$P(Y = k) = P(k \leq X < k + 1) = F_X(k + 1) - F_X(k)$$

where $F_X(x)$ is the cumulative distribution function (CDF) of an exponential random variable:

$$F_X(x) = 1 - e^{-\lambda x}$$

Substituting, we get:

$$P(Y = k) = \left(1 - e^{-\lambda(k+1)}\right) - \left(1 - e^{-\lambda k}\right)$$

Simplifying, we obtain:

$$P(Y = k) = e^{-\lambda k} - e^{-\lambda(k+1)}$$

$$P(Y = k) = e^{-\lambda k}(1 - e^{-\lambda})$$

Thus, the PMF of Y is:

$$P(Y = k) = (1 - e^{-\lambda})e^{-\lambda k}, \quad k = 0, 1, 2, \ldots$$

2. PDF of $R = X - \lfloor X \rfloor$

Next, let's find the PDF of R, which is the fractional part of X.

For $r \in [0, 1)$, we want $F_R(r) = P(R \leq r)$:

$$F_R(r) = P(R \leq r) = P(X - \lfloor X \rfloor \leq r)$$

This event, $R \leq r$, happens whenever X is in a range where the fractional part of X is at most r. Mathematically, this is equivalent to saying:

$$F_R(r) = P(X \mod 1 \leq r)$$

To compute this, we sum over all possible integer values k for $\lfloor X \rfloor$:

$$F_R(r) = \sum_{k=0}^{\infty} P(k \leq X < k + r)$$

For each interval $[k, k+r)$, we use the exponential CDF:

$$P(k \leq X < k+r) = F_X(k+r) - F_X(k)$$

where $F_X(x) = 1 - e^{-\lambda x}$ is the CDF of X. Substituting, we get:

$$P(k \leq X < k+r) = \left(1 - e^{-\lambda(k+r)}\right) - \left(1 - e^{-\lambda k}\right)$$

$$= e^{-\lambda k} - e^{-\lambda(k+r)}$$

Thus,

$$F_R(r) = \sum_{k=0}^{\infty} \left(e^{-\lambda k} - e^{-\lambda(k+r)}\right)$$

This series simplifies using the geometric series formula, yielding:

$$F_R(r) = \frac{1 - e^{-\lambda r}}{1 - e^{-\lambda}}$$

To find the PDF $f_R(r)$, we differentiate $F_R(r)$ with respect to r:

$$f_R(r) = \frac{d}{dr} F_R(r) = \frac{\lambda e^{-\lambda r}}{1 - e^{-\lambda}}, \quad 0 \leq r < 1$$

Final Answer
- The PMF of $Y = \lfloor X \rfloor$ is:

$$P(Y = k) = (1 - e^{-\lambda}) e^{-\lambda k}, \quad k = 0, 1, 2, \ldots$$

- The PDF of $R = X - \lfloor X \rfloor$ is:

$$f_R(r) = \frac{\lambda e^{-\lambda r}}{1 - e^{-\lambda}}, \quad 0 \leq r < 1$$

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Date

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Problem 2

X and Y are two discrete random Variable

To prove

X and Y are independent if and only if
$F_{X,Y}(x,y) = F_X(x) F_Y(y)$

Assume X and Y are independent then

$$F_{X,Y}(x,y) = P_{X,Y}(X \leq x, Y \leq y)$$

$$= \sum_{a=-\infty}^{x} \sum_{b=-\infty}^{y} P_{X,Y}(X=a, Y=b)$$

since we assume X and Y are independent

$\therefore$

$$F_{X,Y}(x,y) = \sum_{a=-\infty}^{x} \sum_{b=\infty}^{y} P_X(X=a) P_Y(y=b)$$

$$= \sum_{a=-\infty}^{x} P_X(x=a) \sum_{b=\infty}^{y} P_Y(y=b)$$

$$= F_X(x) \cdot F_Y(y)$$

Now we prove other way round

i.e. If $F_{X,Y}(x,y) = F_X(x) F_Y(y)$

then X and Y are independent

$$f_{xy}(x,y) = f_x(x)\, f_y(y)$$

$$\sum_{a=\infty}^{x}\sum_{b=-\infty}^{y} P_{XY}(X=a, Y=b) = \sum_{a=-\infty}^{x} P_X(X=a) \sum_{b=-\infty}^{y} P_Y(Y=b)$$

$$= \sum_{a=-\infty}^{x}\sum_{b=-\infty}^{y} P_X(x=a)\, P_Y(y=b)$$

this equalty only hold if

$$P_{XY}(X=a, Y=b) = P_X(x=a)\, P_Y(y=b)$$

which means X and Y has to be independent

Question 6

Problem

Let X and Y be independent exponential random variables with rate λ. Define $Z = X + Y$ and $W = \frac{X}{Y}$. Find the joint distribution of Z and W, and show that they are independent.

Solution

Step 1: Joint PDF of X and Y

Since X and Y are independent exponential random variables with parameter λ, the joint probability density function (pdf) of X and Y is:

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) = \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} = \lambda^2 e^{-\lambda(x+y)}, \quad x, y \geq 0.$$

Step 2: Change of Variables

We define new variables $Z = X + Y$ and $W = \frac{X}{Y}$. We want to find the joint distribution of Z and W.

The inverse transformations are:

$$X = \frac{WZ}{1 + W}, \quad Y = \frac{Z}{1 + W}.$$

Next, we compute the Jacobian determinant of the transformation:

$$\frac{\partial X}{\partial Z} = \frac{W}{1 + W}, \quad \frac{\partial X}{\partial W} = \frac{Z}{(1 + W)^2},$$

$$\frac{\partial Y}{\partial Z} = \frac{1}{1 + W}, \quad \frac{\partial Y}{\partial W} = -\frac{Z}{(1 + W)^2}.$$

The Jacobian determinant is:

$$J = \left| \frac{W}{1 + W} \cdot \left(-\frac{Z}{(1 + W)^2} \right) - \frac{Z}{(1 + W)^2} \cdot \frac{1}{1 + W} \right| = \frac{Z}{(1 + W)^2}.$$

Step 3: Joint PDF of Z and W

Using the transformation, the joint pdf of Z and W is given by:

$$f_{Z,W}(z,w) = f_{X,Y}(x,y)\,|J|\,.$$

Substituting the joint pdf of X and Y and the Jacobian:

$$f_{Z,W}(z,w) = \lambda^2 e^{-\lambda z} \cdot \frac{z}{(1+w)^2}, \quad z \geq 0, w \geq 0.$$

Step 4: Independence of Z and W

Since the joint pdf factors as:

$$f_{Z,W}(z,w) = f_Z(z) f_W(w),$$

where $f_Z(z) = \lambda^2 z e^{-\lambda z}$ and $f_W(w) = \frac{1}{(1+w)^2}$, we conclude that Z and W are independent.

Q4. Let X be a continuous random variable with probability density function (PDF) $f_X(x)$, and let Y be a function of X defined as:

$$Y \triangleq \begin{cases} X & \text{if } X \geq 0, \\ X^2 & \text{if } X \leq 0. \end{cases}$$

Compute the PDF of Y in terms of $f_X(x)$.

Solution :

$$Y = \begin{cases} X & \text{if } X \leq 0, \\ X^2 & \text{if } X \geq 0. \end{cases}$$

$$\Rightarrow Y \geq 0.$$

$$\begin{aligned} f_Y(y) = \frac{d}{dy}P(Y < y) &= \frac{d}{dy}P(X \in [0, y] \cup X \in [-\sqrt{y}, 0]) \\ &= \frac{d}{dy}P(x \in [-\sqrt{y}, y]) \\ &= \frac{d}{dy}\left[F_X(y) - F_X(-\sqrt{y})\right] \\ &= f_X(y) + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}. \end{aligned}$$

Thus, the PDF of Y is

$$f_Y(y) = f_X(y) + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}.$$

Problem 5

Let X1, X2, and X3 be independent are uniformly distributed random variables on [0, 1]. Find the joint density function of $X = X_1X_2$ and Y = X_3^2 , and show that P (X $\geq$ Y) = $\frac{4}{9}$.

First let's find the cdf and hence pdf of X and Y.
$F_X(x) = P(X \leq x) = P(X_1X_2 \leq x) = P((X_1, X_2) \in \{(x_1, x_2) : x_1x_2 \leq x\})$
For x ≤ 0, $F_X(x) = 0$ since X_1, X_2 and uniformly distributed on [0,1]. For x ≥ 1, $F_X(x) = 1$. Thus, for x $\in$ (0,1),

$$\int\limits_{(x_1,x_2):x_1x_2\leq x} f_{X_1X_2}(x_1, x_2)dx_1dx_2 = \int\limits_0^1 \int\limits_0^{\frac{x}{x_1}} f_{X_1X_2}(x_1, x_2)dx_2dx_1 = \int\limits_0^1 \int\limits_0^{\frac{x}{x_1}} f_{X_1}(x_1)f_{X_2}(x_2)dx_2dx_1$$

$$[\because X_1, X_2 \quad are \quad independent.]$$

$$= \int\limits_0^x f_{X_1}(x_1)\int\limits_0^{\frac{x}{x_1}} f_{X_2}(x_2)dx_2dx_1 + \int\limits_x^1 f_{X_1}(x_1)\int\limits_0^{\frac{x}{x_1}} f_{X_2}(x_2)dx_2dx_1$$

$$= \int\limits_0^x f_{X_1}(x_1)\int\limits_0^1 (1)dx_2dx_1 + \int\limits_x^1 f_{X_1}(x_1)\int\limits_0^{\frac{x}{x_1}} (1)dx_2dx_1$$

$$[\because f_{X_2}(x_2) = 1 \forall x_2 \in [0, 1], f_{X_2}(x_2) = 0 \quad otherwise.]$$

$$= \int\limits_0^x f_{X_1}(x_1)(1)dx_1 + \int\limits_x^1 f_{X_1}(x_1)(\frac{x}{x_1})dx_1 = \int\limits_0^x (1)dx_1 + \int\limits_x^1 \frac{x}{x_1}dx_1$$

$$[\because f_{X_1}(x_1) = 1 \forall x_1 \in [0, 1], f_{X_1}(x_1) = 0 \quad otherwise.]$$

$$= x + xln(x_1)|_x^1 = x - xln(x).$$

Thus, $F_X(x) = x - xln(x) \ \forall x \in (0, 1)$.
$\implies f_X(x) = \frac{dF_X(x)}{dx} = 1 - ln(x) - 1 = -ln(x) \quad \forall x \in [0, 1]$, $f_X(x) = 0$ otherwise.
$F_Y(y) = P(Y \leq y) = P(X_3^2 \leq y) = P(X_3 \in \{x_3 : x_3^2 \leq y\})$
For y ≤ 0, $F_Y(y) = 0$ since X_3 is uniformly distributed on [0,1]. For y ≥ 1, $F_X(x) = 1$. Thus, for y $\in$ (0,1),

$$\int\limits_{x_3:x_3^2\leq y} f_{X_3}(x_3)dx_3 = \int\limits_0^{\sqrt{y}} f_{X_3}(x_3)dx_3 = \int\limits_0^{\sqrt{y}} (1)dx_3 = \sqrt{y}$$

$$[\because f_{X_3}(x_3) = 1 \quad \forall x_3 \in [0, 1], \quad f_{X_3}(x_3) = 0 \quad otherwise.]$$

Thus, $F_Y(y) = \sqrt{y} \ \forall y \in (0, 1)$.
$\implies f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2\sqrt{y}} \quad \forall y \in [0, 1]$, $f_Y(y) = 0$ otherwise.

Since X_1, X_2 and X_3 are independent, X and Y are also independent,
$\implies f_{XY}(x, y) = f_X(x)f_Y(y) = \frac{-ln(x)}{2\sqrt{y}} \ \forall \ x, y \in [0, 1]$ and 0 otherwise.

$$P(X \geq Y) = \int\limits_0^1 \int\limits_0^x f_{XY}(x, y)dydx = -\int\limits_0^1 ln(x)\int\limits_0^x \frac{1}{2\sqrt{y}}dydx = -\int\limits_0^1 ln(x)\sqrt{x}dx = \frac{4}{9}.$$