Q1]

(i) conditional independence of A and B given C_i.

$$P(A \cap B|C_i) = P(A|C_i)\,P(B|C_i) \quad \forall i \in \{1, 2, \dots n\}$$

(ii) Independence of B and C_i.

$$P(B \cap C_i) = P(B)\,P(C_i) \quad , \forall i \in \{1, 2, \dots n\}$$

We need to check for A & B independence

To prove :- $P(A \cap B) = P(A)\,P(B)$

By using total probability law

$$P(A \cap B) = \sum_{i=1}^{n} P(A \cap B \cap C_i)$$

$$= \sum_{i=1}^{n} P(C_i)\,P(A \cap B|C_i)$$

$$= \sum_{i=1}^{n} P(C_i)\,P(A|C_i)\,P(B|C_i)$$ using conditional independence condition

$$= \sum_{i=1}^{n} P(A|C_i)\,[P(C_i)\,P(B|C_i)]$$

$$= \cancel{P(B)} \cancel{\sum} \cancel{P(B}$$

Since B is independent of C_i

$$P(B|C_i) = P(B)$$

$$P(A \cap B) = \sum_{i=1}^{n} P(A|C_i)\,P(C_i)\,P(B)$$

$$= P(B) \sum_{i=1}^{n} P(A|C_i)\,P(C_i)$$

$$= P(B) \sum_{i=1}^{n} P(A \cap C_i)$$

$$= P(B)\, P(A)$$

$\therefore$ $P(A \cap B) = P(A)\, P(B)$ using the condn given

$\therefore$ A & B are independent

Q2]

X and Y are independent random variable

$$Z = XY$$

$$P(X=1) = P(X=-1) = 1/2$$

$$P(Y=1) = P(Y=-1) = 1/2$$

(a) To prove if X, Y and Z are independent

We need to prove

$$P(X=x, Y=y, Z=z) = P(X=x)\, P(Y=y)\, (Z=z)$$

$$\cancel{P(Z)}\ P(Z=1) = P(X=1)P(Y=1) + P(X=-1)P(Y=-1)$$

$$= 1/2 \times 1/2 + 1/2 \times 1/2$$

$$= 1/2$$

Similarly $P(Z=-1) = 1/2$

Let take $X = 1$ $Y = -1$ $Z = 1$

$P(X=1, Y=-1, Z=1) = 0$ cause $Z = XY$

$P(X=1)\, P(Y=1)\, P(Z=1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$\neq 0$

$\therefore$ X, Y and Z are not independent

value of Z dependents on values of X & Y

(b) X, Y, Z are pairwise independent

(i) X & Y.

its given X and Y are independent

and it can be easily proven that $P(X=x, Y=y)$

$= P(X=x)\, P(Y=y)$

(ii) $Z = XY$

to prove $P(X=x, Z=z) = P(X=x)\, P(Z=z)$

$P(X=1, Z=1) = \frac{1}{4}$

$P(X=1)\, P(Z=1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

it can be proven for all combination of x and z.

$\therefore$ X and Z are independent

(iii) Y and Z

$P(Y = y \text{ and } Z = z)$

=e 1 Let take an example

$P(Y=1, Z=-1) = 1/4$ $\qquad$ $y=1, x=1, z=1$

$P(Y=1)\,P(Z=-1/2) = 1/2 \times 1/2$

$= 1/4$

$P(Y=1, Z=-1) = P(Y=1)\,P(Z=-1)$

this can be proved for all values of Y and Z

Y and Z are also independent

Q3. Prove that $P(X_1 + X_2 \geq x) = P(X_1 + X_2 \leq -x)$ for Symmetric Random Variables

Given X_1 and X_2 be two discrete independent random variables that are symmetric about 0. This means X_i and $-X_i$ have the same PMFs. We want to show that:

$$P(X_1 + X_2 \geq x) = P(X_1 + X_2 \leq -x) \quad \text{for all } x.$$

Since X_1 and X_2 are symmetric about 0, their probability mass functions satisfy:

$$P(X_1 = x) = P(X_1 = -x) \quad \text{and} \quad P(X_2 = x) = P(X_2 = -x) \quad \text{for all } x.$$

The probability $P(X_1 + X_2 \geq x)$ can be written as:

$$P(X_1 + X_2 \geq x) = \sum_{x_1, x_2 : x_1 + x_2 \geq x} P(X_1 = x_1, X_2 = x_2).$$

Since X_1 and X_2 are independent, the joint probability is the product of the individual probabilities:

$$P(X_1 + X_2 \geq x) = \sum_{x_1, x_2 : x_1 + x_2 \geq x} P(X_1 = x_1) P(X_2 = x_2).$$

Now, consider the probability $P(X_1 + X_2 \leq -x)$:

$$P(X_1 + X_2 \leq -x) = \sum_{x_1, x_2 : x_1 + x_2 \leq -x} P(X_1 = x_1, X_2 = x_2).$$

By the symmetry of X_1 and X_2:

$$P(X_1 = x_1) = P(X_1 = -x_1) \quad \text{and} \quad P(X_2 = x_2) = P(X_2 = -x_2).$$

Thus,

$$P(X_1 + X_2 \leq -x) = \sum_{x_1, x_2 : x_1 + x_2 \leq -x} P(X_1 = -x_1) P(X_2 = -x_2).$$

Let's change variables to $y_1 = -x_1$ and $y_2 = -x_2$. Under this change, the condition $x_1 + x_2 \leq -x$ becomes $y_1 + y_2 \geq x$. This gives:

$$P(X_1 + X_2 \leq -x) = \sum_{y_1, y_2 : y_1 + y_2 \geq x} P(X_1 = y_1) P(X_2 = y_2).$$

This is exactly the expression for $P(X_1 + X_2 \geq x)$. Hence,

$$P(X_1 + X_2 \leq -x) = P(X_1 + X_2 \geq x).$$

Is Independence Necessary?

Yes, independence is a crucial assumption here. If X_1 and X_2 are not independent, the joint probability $P(X_1 = x_1, X_2 = x_2)$ cannot be factored into the product $P(X_1 = x_1)P(X_2 = x_2)$. Without independence, the symmetry of each individual random variable about 0 does not necessarily imply that their sum will have the same distribution properties. Therefore, the conclusion that $P(X_1+X_2 \geq x) = P(X_1+X_2 \leq -x)$ may not hold if X_1 and X_2 are dependent.

Q4. Expected Number of Fixed Points in a Random Permutation

Given a permutation $\pi : [1 : n] \to [1 : n]$, we want to find the expected number of fixed points $E[X]$, where a fixed point is a value x such that $\pi(x) = x$.

Let X be the total number of fixed points in a permutation π. We can express X as a sum of indicator random variables:

$$X = \sum_{i=1}^{n} I_i,$$

where I_i is an indicator random variable for whether i is a fixed point. Specifically, $I_i = 1$ if $\pi(i) = i$ and $I_i = 0$ otherwise.

Since each I_i is an indicator variable, its expected value is:

$$E[I_i] = P(\pi(i) = i).$$

In a uniformly random permutation, each number i has an equal probability of being mapped to any of the n positions. Thus, the probability that i is mapped to itself ($\pi(i) = i$) is:

$$P(\pi(i) = i) = \frac{1}{n}.$$

Since $X = \sum_{i=1}^{n} I_i$, and using the linearity of expectation, we get:

$$E[X] = E\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} E[I_i].$$

Substituting the value of $E[I_i]$ from the previous step:

$$E[X] = \sum_{i=1}^{n} \frac{1}{n} = n \times \frac{1}{n} = 1.$$

Conclusion

The expected number of fixed points $E[X]$ in a uniformly random permutation of $[1 : n]$ is:

$$E[X] = 1.$$

Problem 5 Solution

Given:

- K is equally likely to be 1, 2, 3, or 4, i.e., $P(K = k) = \frac{1}{4}$ for $k = 1, 2, 3, 4$.
- The conditional PMF of N given $K = k$ is $P_{N|K}(n|k) = \frac{1}{k}$ for $n = 1, 2, \ldots, k$.

(a) Joint PMF of K and N

The joint PMF $P_{K,N}(k, n)$ is given by:

$$P_{K,N}(k, n) = P(N = n \mid K = k) \cdot P(K = k).$$

So:

$$P_{K,N}(k, n) = \frac{1}{k} \cdot \frac{1}{4} = \frac{1}{4k}, \quad \text{for } n = 1, 2, \ldots, k \text{ and } k = 1, 2, 3, 4.$$

(b) Marginal PMF of N

To find the marginal PMF of N, sum over all possible values of K:

$$P_N(n) = \sum_{k=1}^{4} P_{K,N}(k, n).$$

For each n, this sum only includes terms where $n \leq k$. So:

$$P_N(1) = P_{K,N}(1, 1) + P_{K,N}(2, 1) + P_{K,N}(3, 1) + P_{K,N}(4, 1)$$

$$P_N(2) = P_{K,N}(2, 2) + P_{K,N}(3, 2) + P_{K,N}(4, 2)$$

$$P_N(3) = P_{K,N}(3, 3) + P_{K,N}(4, 3)$$

$$P_N(4) = P_{K,N}(4, 4)$$

Calculations:

$$P_N(1) = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} = \frac{48 + 24 + 16 + 12}{192} = \frac{100}{192} = \frac{25}{48}.$$

$$P_N(2) = \frac{1}{8} + \frac{1}{12} + \frac{1}{16} = \frac{24 + 16 + 12}{192} = \frac{52}{192} = \frac{13}{48}.$$

$$P_N(3) = \frac{1}{12} + \frac{1}{16} = \frac{16 + 12}{192} = \frac{28}{192} = \frac{7}{48}.$$

$$P_N(4) = \frac{1}{16} = \frac{12}{192} = \frac{3}{48}.$$

(c) Conditional PMF of K given $N = 2$

Using Bayes' rule:

$$P_{K|N}(k|2) = \frac{P_{K,N}(k,2)}{P_N(2)}.$$

For $K = 2, 3, 4$:

$$P_{K|N}(2|2) = \frac{\frac{1}{8}}{\frac{13}{48}} = \frac{1}{8} \times \frac{48}{13} = \frac{6}{13}.$$

$$P_{K|N}(3|2) = \frac{\frac{1}{12}}{\frac{13}{48}} = \frac{1}{12} \times \frac{48}{13} = \frac{4}{13}.$$

$$P_{K|N}(4|2) = \frac{\frac{1}{16}}{\frac{13}{48}} = \frac{1}{16} \times \frac{48}{13} = \frac{3}{13}.$$

(e) Expected Value of Total Expenditure

Each book costs Rs. 30 on average, and the total cost is the sum of the costs of N books:

$$E[\text{Total Cost}] = 30 \cdot E[N].$$

Where:

$$E[N] = 1 \cdot P_N(1) + 2 \cdot P_N(2) + 3 \cdot P_N(3) + 4 \cdot P_N(4).$$

Substituting values:

$$E[N] = 1 \cdot \frac{25}{48} + 2 \cdot \frac{13}{48} + 3 \cdot \frac{7}{48} + 4 \cdot \frac{3}{48} = \frac{25 + 26 + 21 + 12}{48} = \frac{84}{48} = \frac{7}{4}.$$

Thus, the expected expenditure:

$$E[\text{Total Cost}] = 30 \times \frac{7}{4} = 52.5.$$

Problem 6 solution

Let $X_1, X_2, \ldots, X_n$ be independent discrete random variables such that each X_i is a geometric random variable with parameter p_i. Let $X = X_1 + X_2 + \cdots + X_n$. The goal is to show that the variance of X is minimized if $p_1, p_2, \ldots, p_n$ are all equal to $\frac{n}{\mu}$, where $\mu > 0$ is the given mean of X.

Step 1: Properties of Geometric Random Variables

Let X_i be a geometric random variable with parameter p_i. The probability mass function of X_i is given by:

$$P(X_i = k) = (1 - p_i)^{k-1} p_i, \quad k = 1, 2, 3, \dots$$

The mean and variance of a geometric random variable X_i with parameter p_i are:

$$E[X_i] = \frac{1}{p_i}, \quad \text{and} \quad \mathrm{Var}(X_i) = \frac{1 - p_i}{p_i^2}.$$

Step 2: Mean and Variance of the Sum X

Since $X = X_1 + X_2 + \cdots + X_n$, the mean and variance of X are:

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = \sum_{i=1}^{n} \frac{1}{p_i},$$

$$\mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n) = \sum_{i=1}^{n} \frac{1 - p_i}{p_i^2}.$$

We are given that the mean of X is $\mu > 0$, so:

$$\sum_{i=1}^{n} \frac{1}{p_i} = \mu.$$

Step 3: Minimize the Variance Subject to the Constraint

To minimize the variance $\mathrm{Var}(X) = \sum_{i=1}^{n} \frac{1-p_i}{p_i^2}$ subject to the constraint $\sum_{i=1}^{n} \frac{1}{p_i} = \mu$, we use the method of perfect squares.

$$\sum \frac{1}{p_i^2} - \frac{1}{p_i} = \left(\frac{1}{p_i} - \frac{n}{\mu}\right)^2 + \left(\frac{2n}{\mu} - 1\right)\frac{1}{p_i} - \left(\frac{n}{\mu}\right)^2$$

Hence, it maximises when all $p_i = \frac{n}{\mu}$

Practice Set

Q7 To prove: $\sqrt{\mathbb{E}[(X+Y)^2]} \leq \sqrt{\mathbb{E}[X^2]} + \sqrt{\mathbb{E}[Y^2]}$

LHS: $\sqrt{\mathbb{E}[(X+Y)^2]} = \sqrt{\mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY]}$

We know from cauchy - Scharz:

$$-\sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]} \leq \mathbb{E}(XY) \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

$$\Rightarrow \sqrt{\mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY]} \leq \sqrt{\mathbb{E}(X^2) + \mathbb{E}(Y^2) + 2\sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}} = \sqrt{\left(\sqrt{\mathbb{E}[X^2]} + \sqrt{\mathbb{E}[Y^2]}\right)^2}$$

$$\leq \sqrt{|\mathbb{E}[X^2]|} + \sqrt{|\mathbb{E}[Y^2]|}$$

hence, proved.

#8

$X \to RV \leftarrow$ non negative & continous

$$\mathbb{E}[X^n] = \int_0^\infty n x^{n-1} P(X > x)\, dx \quad \leftarrow \text{To prove:}$$

Given RV Y (non-neg.)

$$\mathbb{E}[Y] = \int_0^\infty y f_Y(y)\, dy = \int_0^\infty P(Y > y)\, dy$$

Proof:

$$\int_0^\infty \boxed{P(Y>y)}\, dy = \int_{y=0}^\infty \int_{t=y}^\infty f_Y(t)\, dt\, dy = \int_{t=0}^\infty \int_{y=0}^t f_Y(t)\, dt\, dy = \int_{t=0}^\infty f_Y(t) \int_0^t \cancel{dy}\, dt = \int_{t=0}^\infty t f_Y(t)\, dt = \mathbb{E}[Y]$$

here let $X^n = Y$, as X is non neg. $\Rightarrow Y$ is non. neg. [$\because y = x^n \geq 0 \ \forall\ x \geq 0$]

$$\Rightarrow \mathbb{E}[X^n] = \mathbb{E}[Y] = \int_{y=0}^\infty P(Y > y)\, dy$$

$y = x^n$

$$\Rightarrow \mathbb{E}[X^n] = \mathbb{E}[Y] = \int_{x^n=0}^{x^n \to \infty} P(X^n > x^n)\, d(x^n) = \int_{x=0}^{x \to \infty} P(X > x)\, n x^{n-1}\, dx$$

[$\because$ as X is non. neg.] $X^n > x^n \equiv X > x$

$$= \int_0^\infty P(X > x)\, n x^{n-1}\, dx$$

Problem 9

Consider a random variable X with the following two-sided exponential PDF

$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ (1-p)\lambda e^{\lambda x}, & \text{if } x < 0, \end{cases}$$

where λ and p are scalars with $\lambda > 0$ and $p \in [0, 1]$. Find the mean and the variance of X.

Mean:

$$E[X] = \int_0^\infty xp\lambda e^{-\lambda x} dx + \int_{-\infty}^0 x(1-p)\lambda e^{\lambda x} dx$$

Evaluating the first integral using by-parts integration:

$$\int xp\lambda e^{-\lambda x} dx = \frac{xp\lambda e^{-\lambda x}}{-\lambda} - \int \frac{1 * p\lambda e^{-\lambda x}}{-\lambda}$$

$$\int xp\lambda e^{-\lambda x} dx = -xpe^{-\lambda x} - \frac{pe^{-\lambda x}}{\lambda}$$

Applying integral limits:

$$\int_0^\infty xp\lambda e^{-\lambda x} dx = [-xpe^{-\lambda x} - \frac{pe^{-\lambda x}}{\lambda}]_0^\infty = \frac{p}{\lambda}$$

Eavluating the second integral:

$$\int x(1-p)\lambda e^{\lambda x} dx = \frac{x(1-p)\lambda e^{\lambda x}}{\lambda} - \int (1-p)e^{\lambda x} = \frac{x(1-p)\lambda e^{\lambda x}}{\lambda} - \frac{(1-p)e^{\lambda x}}{\lambda}$$

Applying integral limits:

$$\int_{-\infty}^0 x(1-p)\lambda e^{\lambda x} dx = [\frac{x(1-p)\lambda e^{\lambda x}}{\lambda} - \frac{(1-p)e^{\lambda x}}{\lambda}]_{-\infty}^0 = \frac{(p-1)}{\lambda}$$

So for our distribution, we get

$$E[X] = \frac{(2p-1)}{\lambda}$$

For the variance:

You can similarly evaluate the integral:

Using by-parts, for the first term in the integral:

$$\int p\lambda x^2 e^{-\lambda x}\, dx = \frac{-x^2 e^{-\lambda x}}{\lambda} - \frac{\int -2xe^{-\lambda x}}{\lambda}, dx$$

$$\int p\lambda x^2 e^{-\lambda x}\, dx = -px^2 e^{-\lambda x} - \frac{2pxe^{-\lambda x}}{\lambda} - \frac{2pe^{-\lambda x}}{\lambda^2}$$

Now substituting our limits,$(0, \infty)$, we get:

$$\frac{2p}{\lambda^2}$$

Similarly, for the other term :

$$\frac{2(1-p)}{\lambda^2}$$

So our $E[X^2] = \frac{2}{\lambda^2}$

So for out variance we have

$$Var(X) = E[X^2] - E[X]^2 = \frac{1+4p-4p^2}{\lambda^2}$$

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$p \in [0,1]$ and U is uniformly distributed over $[0,1]$.

So $\int_0^1 f_U(u)\,du = \int_0^1 c\,du = c = 1$

So $f_U(u) = 1 \quad 0 \le u \le 1$.

For $u > p$, length of substick containing $p = u$
$u \le p$, length of substick containing $p = 1-u$.

$E[L] = \int_0^p (1-u) f_U(u)\,du + \int_p^1 u f_U(u)\,du$

$= \int_0^p f_U(u)\,du - \int_0^p u f_U(u)\,du + \int_p^1 u f_U(u)\,du$

$p - \frac{p^2}{2} + \frac{1}{2} - \frac{p^2}{2} = \frac{1}{2} + p - p^2 = \frac{1}{2} + p(1-p)$

For maximum expected length, $\frac{dE(l)}{dp} = 1 - 2p = 0 \Rightarrow p = \frac{1}{2}$

Thus expected length containing p is $\frac{1}{2} + p - p^2$ which is maximum for $p = \frac{1}{2}$, i.e. maximum expected length is

$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$