Properties of definite integral

1) $$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$

2) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

5) $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx & \text{, if } f(x) \text{ is even} \\ 0 & \text{, if } f(x) \text{ is odd} \end{cases}$$

6) $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx & \text{, if } f(2a-x)=f(x) \\ 0 & \text{, if } f(2a-x)=-f(x) \end{cases}$$

$$I = \int_0^{\pi} \frac{x \, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}$$

using property $$I = \int_0^{\pi} \frac{(\pi - x) \, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}$$

$$I = \int_0^{\pi} \frac{\pi \, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} - \int_0^{\pi} \frac{x \, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}$$

$$2I = \int_0^{\pi} \frac{\pi \, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}$$

$$\because f(\pi - x) = f(x)$$

using property no. 6

$$2I = 2\int_0^{\pi/2} \frac{\pi \, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} \quad \frac{*\sec^4 x}{*\sec^4 x}$$

$$I = \pi \int_0^{\pi/2} \frac{\sec^4 x}{(a^2\tan^2 x + b^2)^2} \, dx$$

$$I = \pi \int_0^{\pi/2} \frac{(1+\tan^2 x)\sec^2 x}{(a^2\tan^2 x + b^2)^2} \, dx$$

Ques.

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}} \quad \text{--- } (1)$$

using properties of definite integral

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}\, dx}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}} \quad \text{--- } (2)$$

$$2I = \int_0^{\pi/2} 1\, dx$$

$$\boxed{I = \frac{\pi}{4}}$$

Ques. $I = \int_0^1 \frac{\log x \, dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \log \frac{1}{2}$

Let $\boxed{x = \sin\theta}$ $dx = \cos\theta \, d\theta$

Lim: 0 to $\pi/2$

$$I = \int_0^{\pi/2} \frac{\log\sin\theta \, \cancel{\cos\theta} \, d\theta}{\cancel{\sqrt{1-\sin^2\theta}}}$$

$$I = \int_0^{\pi/2} \log\sin\theta \, d\theta \quad \text{—} (1)$$

using prop. of DI

$$I = \int_0^{\pi/2} \log\cos\theta \, d\theta \quad \text{—} (2)$$

Add (1) and (2)

$$2I = \int_0^{\pi/2} \log(\sin\theta\cos\theta) \, d\theta$$

$$2I = \int_0^{\pi/2} \log\sin 2\theta \, d\theta - \int_0^{\pi/2} \log 2 \, d\theta$$

Let $2\theta = t \Rightarrow d\theta = dt/2$

Limits: 0 to π

$$2I = \int_0^{\pi} \log\sin t \frac{dt}{2} - \log 2 . \frac{\pi}{2}$$

$(\because \log\sin(\pi - t) = \log\sin t)$

using property,

$$2I = 2\int_0^{\pi/2} \log\sin t \frac{dt}{2} - \log 2 . \frac{\pi}{2}$$

$\log 1/2$

$$I = \frac{\pi}{2} \log \frac{1}{2}$$

Let $a\tan x = b\tan y \implies \tan^2 x = b^2/a^2 \tan^2 y$

$a\sec^2 x\, dx = b\sec^2 y\, dy$

New Limits : 0 to $\pi/2$

$$I = \int_0^{\pi/2} \frac{(1+b^2/a^2 \tan^2 y) b/a \sec^2 y\, dy}{(a^2\tan^2 y + b^2)^2}$$

$$I = \frac{\pi}{a^3b^3}\int_0^{\pi/2} \frac{(a^2+b^2\tan^2 y)\cancel{\sec^2 y}\, dy}{\underbrace{(\tan^2 y + 1)^2}_{\sec^2 y * \cancel{\sec^2 y}}}$$

$$I = \frac{\pi}{a^3b^3}\int_0^{\pi/2} (a^2\cos^2 y + b^2\sin^2 y)\, dy$$

$$I = \int_0^1 \cot^{-1}(1-x+x^2)\,dx$$

$$I = \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx$$

$$I = \int_0^1 \tan^{-1}\left(\frac{x+1-x}{1-x(1-x)}\right) dx$$

$$I = \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$$

using properties

$$\int_0^1 \tan^{-1}x\,dx = \int_0^1 \tan^{-1}(1-x)\,dx$$

$$I = 2\int_0^1 \tan^{-1}x\,dx = \int_0^1 \underset{v}{1}.\overset{u}{\tan^{-1}x}\,dx$$

Integrating by parts:

$$I = \left[x.\tan^{-1}x - 1/2\ \log(1+x^2)\right]_0^1$$

$$I = \int_0^{\pi} \frac{x \, dx}{a^2 \sin^2 x + b^2 \cos^2 x}$$

using property $I = \int_0^{\pi} \frac{(\pi - x) \, dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

$$I = \int_0^{\pi} \frac{\pi \, dx}{a^2 \sin^2 x + b^2 \cos^2 x} - \int_0^{\pi} \frac{x \, dx}{a^2 \sin^2 x + b^2 \cos^2 x}$$

$$2I = \int_0^{\pi} \frac{\pi \, dx}{a^2 \sin^2 x + b^2 \cos^2 x}$$

$\because f(\pi - x) = f(x)$

using property no. 6

$$2I = 2\int_0^{\pi/2} \frac{\pi \, dx}{a^2 \sin^2 x + b^2 \cos^2 x} \quad \frac{* \sec^2 x}{* \sec^2 x}$$

$$I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 \tan^2 x + b^2} \, dx$$

$a \tan x = bt \Rightarrow a \sec^2 x \, dx = b \, dt$

$\lim : 0 \longrightarrow \infty$