# properties of beta gamma function:

(1) $\beta(m,n) = \beta(n,m)$

(2) $\Gamma 1 = 1$

(3) $\Gamma(n+1) = n\Gamma n$

(4) $\Gamma(n+1) = n!$ (If n is +ve integer)

(5) $\Gamma(1/2) = \sqrt{\pi}$

(6) $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

(7) $\Gamma m \, \Gamma(1-m) = \dfrac{\pi}{\sin m\pi}$

# Beta Gamma Function

$(m, n > 0)$

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

[Beta function]

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$$

[Gamma function]

(iv) $$\int_0^{\pi/4} \sin^4 x . \cos^2 x \, dx$$

$$\int_0^{\pi/4} (\sin^2 x) . \cos^2 x \, dx$$

$\sin^2 x \rightarrow (1-\cos 2x)/2$

$\cos^2 x \rightarrow (1+\cos 2x)/2$

$$\int_0^{\pi/4} (1-\cos 2x)^2 (1+\cos 2x) \, dx$$

Let $2x=t$

(v) $$\int_0^{\infty} e^{-x^4} x^2 dx * \int_0^{\infty} e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$$

Let $x^4 = t \Rightarrow 4x^3 dx = dt \Rightarrow dx = dt/4t^{3/4}$

$$\int_0^{\infty} e^{-t} t^{1/2} \frac{dt}{4t^{3/4}} * \int_0^{\infty} e^{-t} \frac{dt}{4t^{3/4}}$$

$$\frac{1}{16} \int_0^{\infty} e^{-t} t^{-1/4} dt * \int_0^{\infty} e^{-t} t^{-3/4} dt$$

$$\frac{1}{16} \Gamma(3/4) \Gamma(1/4) = \frac{1}{16} \frac{\pi}{\sin \frac{\pi}{4}}$$

Ex-14

(1) $\Gamma(7/2) = \Gamma(5/2+1) = 5/2\,\Gamma(5/2) = \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}$

$\Gamma(5/2) = \Gamma(3/2+1) = 3/2\,\Gamma(3/2)$

$\Gamma(3/2) = 1/2\,\Gamma(1/2) = 1/2\sqrt{\pi}$

(2) $$\frac{\Gamma(5/2)\,\Gamma(3/2)}{\Gamma(5)} = \frac{\frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{4!}$$

(3) $$\int_0^{\pi/2} \sin^4 x . \cos^2 x \, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma(4)}$$

$p, q > -1$

(8) $I = \int_0^{\pi/2} \sin^p x . \cos^q x \, dx = \dfrac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\, \Gamma\left(\frac{p+q+2}{2}\right)}$

$I = \int_0^{\pi/2} (\sin^2 x)^{p/2} . (\cos^2 x)^{q/2} dx$

Let $\sin^2 x = t \Rightarrow 2 \sin x . \cos x dx = dt \Rightarrow dx = \dfrac{dt}{2t^{1/2}(1-t)^{1/2}}$

Limits: 0 to 1

$I = \int_0^1 \dfrac{(t)^{p/2}(1-t)^{q/2}}{2t^{1/2}(1-t)^{1/2}} dt$

$I = \dfrac{1}{2} \int_0^1 (t)^{(p-1)/2} (1-t)^{(q-1)/2} dt$

$I = \dfrac{1}{2} \beta\left(\dfrac{p+1}{2}, \dfrac{q+1}{2}\right)$

$I = \dfrac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\, \Gamma\left(\frac{p+q+2}{2}\right)}$

Ques. Use beta gamma function to evaluate:

(i) $$\int_0^{\pi/4} (1-2\sin^2 x)^{3/2} \cos x\, dx$$

Hint: $\sqrt{2}\,\text{Sin}x = \text{Sin}\theta$

(ii) $$\int_0^{a} x^2 (a^2-x^2)^{3/2} dx$$

Let $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

new limits: 0 to $\pi/2$

$$a^6 \int_0^{\pi/2} \sin^2\theta.(1-\sin^2\theta)^{3/2} \cos\theta\, d\theta$$

$$a^6 \int_0^{\pi/2} \sin^5\theta.\cos\theta\, d\theta$$

(iii) $$\int_0^{2a} x^{9/2} (2a-x)^{-1/2} dx$$

Let $x = 2a\sin^2\theta \Rightarrow dx = 4a\,\text{Sin}\theta\,\text{Cos}\theta\, d\theta$

new limits: 0 to $\pi/2$

$$2^6 a^5 \int_0^{\pi/2} \sin^9\theta.(2a-2a\sin^2\theta)^{-1/2} \sin\theta\,\text{Cos}\theta\, d\theta$$

$$2^6 a^5 \int_0^{\pi/2} \sin^{10}\theta\, d\theta$$

(iv) $$\int_0^{\pi/6} \sin^2 6x.\cos^4 3x\, dx$$

let $3x = y$, $dx = dy/3$
limits: 0 to $\pi/2$

$$\int_0^{\pi/2} \sin^2 2y.\cos^4 y\, dy/3$$

$$\int_0^{\pi/2} 4\sin^2 y.\cos^6 y\, dy/3$$