1.3.2 Equation for forced oscillation and solution

For a system, executing forced oscillation, the force (F) acting on the system is balanced by restoring force (F_{res}), resistive or frictional force (F_{fric}), and external periodic force (F_{ext}) i.e. F = F_{res} + F_{fri} + F_{ext}

Thus, $m\frac{d^2y}{dt^2} = -\mu\frac{dy}{dt} - ky + F_0 \sin\omega''t$, where ω'' is the angular frequency and F_0 is the amplitude of the external periodic force.

i.e. $\frac{d^2y}{dt^2} + \frac{\mu}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F_0}{m}\sin\omega''t$ ----- (1.56)

which represents differential equation for forced oscillation.

Let the possible solution of equation (1.56) is $y = a\sin(\omega''t - \alpha)$ ----- (1.57)

So, $\frac{dy}{dt} = a\omega''\cos(\omega''t - \alpha)$ ----- (1.58)

And, $\frac{d^2y}{dt^2} = -a\omega''^2\sin(\omega''t - \alpha)(1.59)$

On putting the values of $y, \frac{dy}{dt}$, and $\frac{d^2y}{dt^2}$ from equation (!.57), (1.58), and (1.59) respectively in equation (1.56), we can write that:

$$-a\omega''^2\sin(\omega''t - \alpha) + \frac{\mu}{m}a\omega''\cos(\omega''t - \alpha) + \frac{k}{m}a\sin(\omega''t - \alpha) = \frac{F_0}{m}\sin\omega''t$$

i.e. $-a\omega''^2[\sin\omega''t.\cos\alpha - \cos\omega''t.\sin\alpha] + \frac{\mu}{m}a\omega''[\cos\omega''t.\cos\alpha + \sin\omega''t.\sin\alpha]$

$$+\frac{k}{m}a[\sin\omega''t.\cos\alpha - \cos\omega''t.\sin\alpha] - \frac{F_0}{m}\sin\omega''t = 0 \qquad \text{----- (1.60)}$$

When $\sin\omega''t = 1, \cos\omega''t = 0$ so that equation (1.60) reduces in the form:

$$-a\omega''^2\cos\alpha + \frac{\mu}{m}a\omega''\sin\alpha + \frac{k}{m}a\cos\alpha - \frac{F_0}{m} = 0 \qquad \text{----- (1.61)}$$

When $\cos\omega''t = 1, \ \sin\omega''t = 0$ so that equation (1.60) reduces in the form:

$$a\omega''^2\sin\alpha + \frac{\mu}{m}a\omega''\cos\alpha - \frac{k}{m}a\sin\alpha = 0 \qquad \text{----- (1.62)}$$

On dividing equation (1.62) by $\cos\alpha$ and simplifying, we have:

$$\tan\alpha\left[a\omega''^2 - k\frac{a}{m}\right] = -\frac{\mu}{m}a\omega''$$

i.e. $\tan\alpha\left[\frac{a}{m}\{m\omega''^2 - k\}\right] = -\frac{a}{m}\{\mu\omega''\}$

i.e. $\tan\alpha = \frac{\mu\omega''}{[k - m\omega''^2]} = \frac{A}{B}$ (let) ----- (1.63)

where $A = \mu\omega''$ and $B = [k - m\omega''^2]$ ----- (1.64)

From equation (1.63), we can write that

$\sin\alpha = \frac{A}{\sqrt{A^2+B^2}}$ and $\cos\alpha = \frac{B}{\sqrt{A^2+B^2}}$ ----- (1.65)

Again, on dividing equation (1.61) by $\cos\alpha$and simplifying, we have:

$$-a\omega''^2 + \frac{\mu}{m} a\omega'' \tan\alpha + \frac{k}{m} a - \frac{F_0}{m\cos\alpha} = 0$$

i.e. $\frac{a}{m}\left[\left\{k - m\omega''^2\right\} + \mu\omega'' tan\alpha\right] = \frac{F_0}{m\cos\alpha}$

i.e. $a\left[B + \frac{A^2}{B}\right] = \frac{F_0\sqrt{A^2+B^2}}{B}$, by using equation (1.63), (1.64) and (1.65)

i.e. $a = \frac{F_0}{\sqrt{A^2+B^2}}$ ----- (1.66)

On substituting the values of A and B from equation (1.64) in equation (1.66), we have

$$a = \frac{F_0}{\sqrt{\mu^2\omega''^2 + (k - m\omega''^2)^2}} \quad \text{----- (1.67)}$$

On substituting the values of 'a' from equation (1.66) in equation (1.57), we have

$$y = \frac{F_0}{\sqrt{\mu^2\omega''^2 + (k - m\omega''^2)^2}} \quad \sin(\omega'' t - \alpha) \quad \text{----- (1.68)}$$

But from the general solution for damped oscillation, where $F_0 = 0$, $y = [ae^{-bt}]\sin(\omega' t - \alpha)$. Thus, the general solution of forced oscillation will include both the particular solutions for damped and forced oscillation.

Hence, $y = [ae^{-bt}]\sin(\omega' t - \alpha) + \frac{F_0}{\sqrt{\mu^2\omega''^2 + (k - m\omega''^2)^2}} \quad \sin(\omega'' t - \alpha)$ ----- (1.69)

Also, by equation (1.63), we have

. $\tan\alpha = \frac{\mu\omega''}{[k - m\omega''^2]} = \frac{\mu\omega''}{m[\frac{k}{m} - \omega''^2]} = \frac{(\frac{\mu}{m})\omega''}{(\omega^2 - \omega''^2)}$ ----- (1.70)