Categories tannakiennes

P. Deligne

November 17, 2024

1 Introduction

In [6], N. Saavedra described certain categories equipped with a tensor product, the Tannakian Categories, as the categories of representation of gerbes (in particular: representations of a group-scheme). His presentation is incomplete (cf. [2].3.15). Our goal is to complete it. I was not able to write a short presentation giving only the missing arguments: many ideas of the article are in [6], due to Saavedra and, through him, to A. Grothendieck

The paragraphs 2 to 5 do not claim to be orignal. They gather results which, in paragraph 6, allows us to complete Saavedra's presentation. In paragraph 7, we show that in characteristic 0, a tensor category (1.2), whose every object has dimension (7.1) an integer ≥ 0 is Tannakian. In paragraph 8, we apply the methods of paragraph 6 and 7 to tensor categories which are not necessarily Tannakian. As an application (8.19), we describe tensor categories on k, say perfect, equipped with an exact $\otimes$-functor with values in a supervector spaces over k (1.4).

Paragraph 9 gives an application of the formalism of Tannakian categories to Picard-Vessiot theory. Let $(K.\partial)$ be a differntial field with the field of constants $K_0 := \{x|\partial x = 0\}$ algebraically closed of characteristic 0 and $y^n + a_1 y^{n-1} + \cdots + a_n = 0$ be a linear differntial equation of order n with coefficients in K. We see that the the existence of an extension (E, ∂) of (K, ∂), with the same field of constants, in which the equation admits n linearly independent solutions on the constants. This application is the result of discussions with D. Bertrand. The result is a version of E.R. Kolchin ([3 VI 6 prop. 13]). At the end of this introduction, after indicating some terminological conventions, we describe the main result 1.12 of paragraphs 2 to 6.

1.1 Terminology. *A ring (or an algebra) always means a ring (algebra) with a unit and morphisms send units to units.*

In the article , k will mean a commutative rings, often considered to be a field. We only consider the schemes over k. Often, we say schemes for schemes over k and morphism of schemes for morphism of schemes over k. We denote by $X \times Y$, the product over k, $X \times_{Spec\, k} Y$ and $Hom(X, Y)$ the be the set of morphisms of schemes over k of X to Y.

We identify a representable functor with the object it represents.

1.2. *Let k be a commutative field. In this article, we call simply a tensor category over k what in N. Saavedra [6] (resp. Deligne-Milne [2]) mean an abelian $\otimes$-category ACU (associative commutative unital) k-linear rigid, with $k \xrightarrow{\sim} End(1)$ (resp. an abelian tensor category, rigid, k-linear, with $k \xrightarrow{\sim} End(1)$). The axioms are recalled in 2.1. This is a k-linear abelian category $\mathcal{T}$ equipped with a functor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ with the constraints of associativity and commutativity for $\otimes$ (the functorial isomorphisms $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ and $X \otimes Y \xrightarrow{Y} \otimes X$) satisfying suitable axioms. Among the axioms is the existence of a unit object 1.*

1.3 Example. *The category Vect(k) of vector spaces of finite dimension over k, equipped with the tensor product evidently satisfies the constraints of associativity and commutativity.*

1.4 Example. *$\mathcal{T}$ is the category of vector spaces of finite dimension over k, with a $\mathbb{Z}/2\mathbb{Z}$-gradilng, $\otimes$ the tensor product, with the obvious constraint of associativity and commutativity being given by the Koszul's rule: $a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a$ for a and b homogeneous. This is the category of super vector spaces of finite dimension over k.*

1.5 Example. *Let G be an affine group scheme over k. We denote by $\mathcal{T}$, the category Rep(G) of linear representations of finite dimension of G over k, with $\otimes$ the tensor product of representations, with the usual constraints.*

When G is trivial, we recover example 1.3.

1.6. *A generalisation of example 1.5 will play an essential role for use. Before the generalisation, some preliminaries...*

Let S be a scheme over k. Recall (SGA 3 V1) that a k-groupoid acting on S is a scheme G over k equipped with source and target morphisms: $b, s : G \to S$ and a composition law $\circ : G \underset{{}^sS^b}{\times} G \to G$ which is a morphism of scheme over $S \times S$, such that for every scheme T over k, $S(T) := Hom(T, S)$, $G(T) := Hom(T, G)$, $s, b : G(T) \to S(T)$, $\circ$ defines a category (objects: $S(T)$, arrows: $G(T)$) for which all arrows are invertible. This can also be expressed in tems of diagrams: associativity expressed by the equality of the compositions

$$G \underset{{}^sS^b}{\times} G \underset{{}^sS^b}{\times} G \overset{\circ \times Id}{\underset{Id \times \circ}{\rightrightarrows}} G \underset{{}^sS^b}{\times} G \longrightarrow G,$$

the identity arrow is an $S \times S$ morphism $\epsilon : S \to G$ (S is an $S \times S$ schemes via the diagonal) such that the compositions

$$G = G \underset{{}^sS}{\times} S = S \underset{S^b}{\times} G \overset{\epsilon \times Id}{\underset{Id \times \epsilon}{\rightrightarrows}} G \underset{{}^sS^b}{\times} G \xrightarrow{\circ} G$$

are the identity morphism, the inverse by "-1" $: G \to G$ with s"-1" $=$ "-1"b,

b" -1" = " -1"s, and the following diagrams commute

$$
\begin{tikzcd}
G \arrow[r, "\text{"}-1\text{"}\times Id"] \arrow[d, "s"] & G \underset{sS^b}{\times} G \arrow[d, "\circ"] \\
S \arrow[r, "\epsilon"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "Id\times\text{"}-1\text{"}"] \arrow[d, "b"] & G \underset{sS^b}{\times} G \arrow[d, "\circ"] \\
S \arrow[r, "\epsilon"] & G.
\end{tikzcd}
$$

The terminology of SGA 3 V1 is that (S, G) is a (schemes over k)-groupoid.

For $u : T \to S$, the pullback of G over $u \times u : T \times T \to S \times S$ is a k-groupoid over T: the induced groupoid G_T.

A representation of G is a quasi-coherent sheaf V over S equipped with an action ρ of G, i.e. every given k-scheme T and every $g \in G(T)$, there is a homomorphism $\rho(g) : V_{s(g)} \to V_{b(g)}$ betwwen the two images of V by $s(g)$ and $b(g) : T \to S$. We require $\rho(g)$ to be compatible under base change $T' \to T$, $\rho(gh) = \rho(g)\rho(h)$ (for $s(g) = b(h)$) and the for every g, the identity automorphism $\epsilon(s)$ for $s \in S(T)$, $\rho(g)$ is the identity automorphism of V_s. Since G is a groupoid, $\rho(g)$ is an isomorphism. An action ρ is determined by $\rho(g)$ in the universal case $T = G$, $g = Id_G : G \to G$, i.e. for every morphism $u : s^*V \to b^*V$ between quasi-coherent sheaves on G. This morphism must satisfy:

1. *Over $G \underset{sS^b}{\times} G$, the inverse image of u under $\circ : G \underset{sS^b}{\times} G \to G$ is the composition $pr_1^*(u) \circ pr_2^*(u)$;*
2. *epsilon$^*(u)$ is identity.*

We say that the groupoid G is transitive over S (in the fpqc sence) if the morphism $(b, s) : G \to S \underset{k}{\times} S$ is a cover in the sence of fpqc, i.e. there exists a T faithfully flat quasi-compact over $S \times S$ with $Hom_{S\times S}(T, G) \neq \emptyset$. If G is transitive, then

1. *G is flat, so faithfully flat over $S \times S$ (3.6);*
2. *if G acts on quasi-coherent V and the fibers V_s are vector spaces of finite rank n over the residue field $k(s)$, so V is locally free of rank n(3.5).*

1.7 Example. *Let G be a k-groupoid acting transitively over a non-empty S over k. We take $\mathcal{T}$ to be the category of locally free sheaves of finite rank over S equipped with an action of G and for the tensor product functor $\otimes$ equipped with the obvious constraints. Notation: $Rep(S : G)$.*

When S is a point $S = Spec(k)$, we recover 1.5.

1.8 Remark. *Let G be a groupoid acting transitively on S, $u : T \to S$ and G_T is the induced groupoid (1.6). We verify (3.5) that if $T \neq \emptyset$, $u^* : Rep(S : G) \to Rep(T : G_T)$ is an equivalence of categories. In particular, if $S(k) \neq \emptyset$, so $x \in S(k)$ and G_x is an algebraic group over k over fixed x (the fiber of G over (x, x)), we have $Rep(S : G) \to Rep(G_x)$*

1.9. *Let $\mathcal{T}$ be a tensor category over k and S be a k-scheme. A fiber functor of $\mathcal{T}$ over S is a k-linear exact functor ω of $\mathcal{T}$ to the category of quasi-coherent sheaves on S, equipped with a natural isomorphism $\omega(X) \otimes \omega(Y) \xrightarrow{\sim} \omega(X \otimes Y)$ ACU, i.e. compatible with the constraints of associativity, commutativity and having a unit (2.7). The axioms imposed on $\mathcal{T}$ imply that ω takes values in locally free sheaves of finite rank (2.8). For $S = Spec(B)$, ω is identified by a functor of $\mathcal{T}$ to the category of finite-type projective B-modules. We again call ω a fiber functor over B.*

Let ω_1 and ω_2 be two fiber functors over S. A $\otimes$-isomorphism, or an isomorphism of fiber functors $u : \omega_1 \to \omega_2$ is an isomorphism of functors such that the diagram commutes

$$\begin{array}{ccc} \omega_1(X) \otimes \omega_1(Y) & \longrightarrow & \omega_1(X \otimes Y) \\ \downarrow{\scriptstyle u \otimes u} & & \downarrow{\scriptstyle u} \\ \omega_2(X) \otimes \omega_2(Y) & \longrightarrow & \omega_1(X \otimes Y) \end{array}$$

and such that $u : \omega_1(1) \to \omega_2(1)$ is the identity automorphism of $\mathcal{O}_S$.

1.10. *In [6], Saavedra claims with insuffcient demonstration (cf. [2] 3.15) that: (*) two fiber functors of $\mathcal{T}$ over $Spec(B)$ are locally isomorphic for the fpqc topology, i.e. there exists B' over B, faithfully flat, such that ω_1 and ω_2 after extension of scalars from B to B'.*

Our first goal is the show that the assertion (*) is true (with the additional necessary hypothesis, $k \xrightarrow{\sim} End(1)$ that Saavedra had forgotten) and justify all the results of [6].

1.11. *If ω_1 and ω_2 are two fiber functors over S, we denote by $\underline{Isom}_k^{\otimes}(\omega_1, \omega_2)$ the functor that takes T over $S : u : T \to S$ to the set of isomorphism of fiber functors $u^*\omega_1$ with $u^*\omega_2$. The functor is representable by an affine scheme over S. If ω_i is a fiber functor over S_i (i=1,2), we write*

$$\underline{Isom}_k^{\otimes}(\omega_2, \omega_1) := \underline{Isom}_{S_1 \times S_2}^{\otimes}(pr_2^*\omega_2, pr_1^*\omega_1)$$

For a fiber functor ω over S, we write $\underline{Aut}_S^{\otimes}(\omega) = \underline{Isom}_S^{\otimes}(\omega, \omega)$ and

$$\underline{Aut}_k^{\otimes}(\omega) = \underline{Isom}_k^{\otimes}(\omega, \omega)$$

Following conventions (1.1), we identity the functors with the schemes they represent. The scheme $\underline{Aut}_k^{\otimes}(\omega)$ is a k-groupoid acting over S. The target morphism b (resp. source s) is the composition of pr_1 (resp. pr_2) with the projections over $S \times S$. We prove in 1.13(b), 6.8, 6,14 and 6.15 the following result.

1.12 Theorem. *Let $\mathcal{T}$ be a tensor category over k and ω be a fiber functor of $\mathcal{T}$ over a k-scheme $S \neq \emptyset$.*

1. *The groupoid $\underline{Aut}_k^{\otimes}(\omega)$ is faithfully flat over $S \times S$;*

2. *ω induces an equivalence of $\mathcal{T}$ with the category $Rep(S : \underline{Aut}_k^{\otimes}(\omega))$ of representations of the groupoid $\underline{Aut}_k^{\otimes}(\omega)$*

Conversely, let G be a k-groupoid acting on affine $S \neq \emptyset$ and faithfully flat over $S \times S$ and ω is a fiber functor of $Rep(S : G)$ over S "forgetting the action of G." We have

3. $$G \xrightarrow{\sim} \underline{Aut}_k^{\otimes}(\omega)$$

The theorem provides a dictionary of tensor category over k equipped with a fiber functor over S and k-groupoids acting transitively over S and affine over $S \times S$.

1.13 Remark. 1. *If ω_1 and ω_2 are fiber functors over S_1 and S_2, there exists a disjoint union $T := S_1 \coprod S_2$ and a fiber funcotr ω, unique upto a unique isomorphism, equipped with isomorphisms $\omega|S_j = \omega_j$ (j=1,2). We apply 1.12.1 to ω. The pullback over $S_2 \times S_1$ of the scheme $\underline{Aut}_k^{\otimes}(\omega)$ to $T \times T$ is $\underline{Isom}_k^{\otimes}(\omega_1, \omega_2)$. According to 1.12.1, $\underline{Isom}_k^{\otimes}(\omega_1, \omega_2)$ is faithfully flat over $S_2 \times S_1$.*

For $S_1 = S_2 = S$, the restriction to the diagonal of $\underline{Isom}_k^{\otimes}(\omega_1, \omega_2)$ is $\underline{Isom}_S^{\otimes}(\omega_1, \omega_2)$. The S-scheme $\underline{Isom}_S^{\otimes}(\omega_1, \omega_2)$ is hence faithfully flat over S. This justifies 1.10(*).

2. *If 1.12 is true over affine S, then it is true in general. For the assertion 1, if S_i is an open affine covering of S, the pullback of $\underline{Aut}_k^{\otimes}(\omega)$ over $S_i \times S_j$ is $\underline{Isom}_k^{\otimes}(\omega|S_i, \omega|S_j)$ and, applying 1.13.1, we conclude that 1.12.1 applies to $S_i \coprod S_j$. For 2, observe that for U an non-empty affine open over S, we have by 1.8 $Rep(S : \underline{Aut}_k^{\otimes}(\omega)) \xrightarrow{\sim} Rep(U : \underline{Aut}_k^{\otimes}(\omega(U))$. We conclude by 1.12.2 for U. For 3. If U_1 and U_2 are non-empty open affine of S_x and that G_U is the induced groupoid over $U = U_1 \coprod U_2$, we have $Rep(S : G) \xrightarrow{\sim} Rep(U : G_U)$ and by 1.12.3 applied to U, the morphism $G \to \underline{Aut}_k(\omega)$ is an isomorphism by above on $U_1 \times U_2$.*

1.14. *Let G be a groupoid acting on $S = Spec(B)$. Suppose G is affine over $S \times S$, i.e. affine: $G = Spec(L)$. Since (b, s) makes G a scheme over $S \times S := S \underset{Spec(k)}{\times} S$ (1.1), L is a $B \underset{k}{\otimes} B$-module, i.e. a B, B-bimodule such that the two structures induce coinciding k-modules. We write to the left (resp. to the right) the structure of B-module defined by b (resp. s).*

The B, B-bimodule L is equipped with the following structure:

1. *L is a commutative $B \underset{k}{\otimes} B$-algebra, with the product*

$$p : L \underset{B \otimes B}{\otimes} L \to L.$$

2. *The law of composition $G \underset{{}_s S^b}{\times} G \to G$ corresponds to*

$$c : L \to L \underset{B}{\otimes} L$$

and identity $\epsilon : S \to G$ *to* $e : L \to B$.

Let M *be a* B*-module (= a quasi-coherent sheaf over* S*). An action of* G *over* M *is a morphism of* L*-modules*

$$L \underset{{}^sB}{\otimes} M \to M \underset{B^b}{\otimes} L$$

or, which again amounts to a morphism of B*-modules*

$$r : M \to M \underset{B^b}{\otimes} L$$

(to the right, the structure of B*-module defined by the right structure on* L*), with the compatibility of the composition to the neutral elements. The compatibility translates by 1.14.1, the equality of the following compositions*

$$M \xrightarrow{\ r\ } M \otimes L \overset{r\otimes L}{\underset{L\otimes c}{\rightrightarrows}} M \otimes L \otimes L,$$

and the equality of the composition $M \xrightarrow{r} M \otimes L \xrightarrow{M\otimes e} M$ *to the identity.*

We see that only structure (2) was used on L.

1.15. *Inspired by this remark, for every ring* B *not necessarily commutative, we define a co-algebra* L *be be a bimodule over* B *equipped with a bimodule morphism* $c : L \to L \underset{B}{\otimes} L$ *satisfying the axioms of coassociativity:* c *equalizes the double arrow* $(c \otimes 1, 1 \otimes c)$

$$L \xrightarrow{\ c\ } L \underset{B}{\otimes} L \overset{c\otimes 1}{\underset{1\otimes c}{\rightrightarrows}} L \underset{B}{\otimes} L \underset{B}{\otimes} L$$

and admits a counit $e : L \to B$*: a morphism of bimodules such the the compositions*

$$L \xrightarrow{\ c\ } L \otimes L \overset{e\otimes 1}{\underset{1\otimes e}{\rightrightarrows}} L$$

are identity. Note that if c *admits a counit, then it is unique. We will need only the case in which* B *is commutative, but not assuming it helps in not mixing up the left and the right.*

If B *is commutative and that the two structures of* B*-module over* L *coincide, we retrieve the co-algebra over* B*, hence the terminology. If* k *is a commutative ring and* B *is a* k*-algebra, we define a* k*-co-algebra* L *to be a co-algebra* L *such that the two structures of* k*-modules induced by the structures of* B*-modules coincide.*

A representation of L *is a right* B*-module* M *equipped with a coaction of* L*, i.e. a morphism of right* B*-modules* $r : M \to M \underset{B}{\otimes} L$ *satisfying (1.14.1) (1.14.2). If* L *is a flat left* B*-module, the category of representations of* L *is abelian and the forgetful functor of the coaction is exact.*

1.16. *Let $\mathcal{T}$ be a tensor category over k, $S = Spec(B)$ be an affine scheme over k and ω be a fiber functor of $\mathcal{T}$ over S. Imitating Saavedra, we begin by forgetting the tensor product and construct a k-co-algebra L acting on B such that ω factors through an equivalence of categories of $\mathcal{T}$ and the category of locally free sheaves of finite rank over S equipped with a coaction of L. The proof (6.1, 6.2) is an application of the theorem of Barr-Beck (4.1). It is similar the the theorem of faithfully flat descent SGA 1 VIII 1. The compatibility of ω with the tensor product equips L with a product and we verify that $G := Spec(L)$ is the groupoid $\underline{Aut}_k(\omega)$ acting on S (6.3 - 6.6).*

It remains to show that G is faithfully flat over $S \times S$. We construct a tensor category $\mathcal{T} \otimes \mathcal{T}$ with suitable properties - in particular that a fiber functor ω of $\mathcal{T}$ defines a fiber functor $\omega \times \omega$ of $\mathcal{T} \otimes \mathcal{T}$ over $S \times S$. The $B \otimes B$-module L will be faithfully flat as the image under $\omega \times \omega$ is an Ind-object containing 1 of $\mathcal{T} \otimes \mathcal{T}$.

We give two proofs of the existence of the tensor category $\mathcal{T} \otimes \mathcal{T}$. The first uses a theorem of passing to generic quotient (3.11) and the hypothesis $End(1) = k$ for seeing the if $\mathcal{T}$ is finitely $\otimes$-generated, there exists a fiber funtor ω_1 over the scheme S_1, which is the spectrum of a finite extension of k with $G_1 = \underline{Aut}_k^{\otimes}(\omega_1)$ faithfully flat over $S_1 \times S_1$. From the structure theorem $\mathcal{T} \sim Rep(S_1 : G_1)$ we deduce the existence of the required tensor category $\mathcal{T} \otimes \mathcal{T}$ (5.21). The second proof relies on a direct construction. It only applies when k is perfect.

2 Reminders and complements: tensor categories

Let k be a commutative field.

2.1. *The axioms of tensor categories over k (in our sense, see 1.2) are as follows.*

(2.1.1) The category $\mathcal{T}$ is equipped with a tensor product functor :make $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, satisfying the constraints of associativity and commutativity compatible for $\otimes$ ([4], [5] VII 7, where the terminology is "symmetric monoidal category", [6] I §1.2 or [2] §1 p.104) and there exists a unit object 1 ([6] I 1.3.2; the unit object is unique upto isomorphism; it is equipped with the constraint $X \otimes 1 \xrightarrow{\sim} X$ and $1 \otimes X \xrightarrow{\sim} X$).

These axioms allow us to define the product $\otimes$ over a finite family $(X_i)_{i \in I}$ of object of $\mathcal{T}$.

(2.1.2) For every X, there exists $X^\vee$ and morphism $ev : X \otimes X^\vee \to 1$ and $\delta : 1 \to X^\vee \otimes X$ such the the compositions

$$X \xrightarrow{X \otimes \delta} X \otimes X^\vee \otimes X \xrightarrow{ev \otimes X} X$$

$$X^\vee \xrightarrow{\delta \otimes X^\vee} X^\vee \otimes X \otimes X^\vee \xrightarrow{X^\vee \otimes ev} X^\vee$$

are identity.

(2.1.3) The category is abelian.

(2.1.4) *An isomorphism* $k \xrightarrow{\sim} End(1)$ *of* k *with the endomorphism ring of* 1 *is given.*

2.2. *Let* $\mathcal{M}$ *be a monoidal category ([5] VII 1), i.e. equipped with* $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$*, satisfying the constraints of associativity ([6 I 1.1.1]) and admitting a unit object 1. The functor sending* $\mathcal{M}$ *in the category* $\underline{Hom}(\mathcal{M}, \mathcal{M})$ *of functors of* $\mathcal{M}$ *to* $\mathcal{M}$: $X \mapsto$ *(the functor* $s_X : Z \mapsto X \otimes Z$*) is faithful. It admits a retraction* $s \mapsto s(1)$*. It is equipped with a natural isomorphism* $s_X \circ s_y \xrightarrow{\sim} s_{X \otimes Y} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$*. This isomorphism is* AU*: compatible with the constraints of associativity and admits* $s_1 \xrightarrow{\sim} Id_{\mathcal{M}}$*, compatible with the constraints of being a unit. After applying the functor* s*,*$X, X^\vee, ev$ *and* δ *as in (2.1.2), provide an adjunction of functors:* s_X *is left adjoint of* $s_{X^\vee}$*. Let* d *be the functor*

2.3 Proposition.

Proof. □

2.4.

2.5.

2.6 Proposition.

Proof. □

2.7.

2.8.

2.9.

2.10 Corollary.

Proof. □

2.11 Remark.

2.12.

2.13 Proposition.

Proof. □

2.14 Proposition.

Proof. □

2.15 Lemma.

Proof. □

2.16.

2.17 Corollary.

2.18.

2.19.

3 Reminders and complements: groupoids

3.1.

3.2.

3.3 Proposition.

Proof. ☐

3.4.

3.5.

3.6.

3.7 Proposition.

Proof. ☐

3.8 Corollary.

3.9 Corollary.

Proof. ☐

3.10.

3.11 Proposition.

Proof. ☐

4

4.1.

4.2.

4.3.

4.4 Proposition.

4.5 Proposition.

Proof. ☐

4.6 Remark.

4.7.

4.8 Example.

4.9.

4.10.

4.11.

4.12.

4.13 Proposition.

Proof. ☐

5 Tensor product of abelian categories

5.1.

5.2.

5.3 Proposition.

5.4 Corollary.

Proof. ☐

5.5 Proposition.

Proof. ☐

5.6.

5.7 Proposition.

Proof. ☐

5.8. *This section is missing*

5.9 Lemma.

Proof. ☐

5.10.

5.11 Proposition.

Proof. ☐

5.12.

5.13 Proposition.

5.14 Proposition.

Proof. ☐

5.15.

5.16.

5.17 Proposition.

Proof. ☐

5.18.

5.19.

5.20 Lemma.

Proof. ☐

5.21 Lemma.

Proof. ☐

6 The main theorem

6.1.

6.2 Proposition.

6.3.

6.4 Proposition.

Proof. □

6.5.

6.6 Proposition.

Proof. □

6.7.

6.8.

6.9 Lemma.

6.10.

6.11 Lemma.

Proof. □

6.12.

6.13 Lemma.

Proof. □

6.14.

6.15.

6.16.

6.17 Lemma.

6.18 Lemma.

Proof. □

6.19 Proposition.

Proof. □

6.20 Corollary.

Proof. □

6.21.