Title

Author

Contents

1 Introduction

For a fixed scheme S, let X be a scheme over S and G be a group scheme over S with the product map $\mu : G \times_S G \to G$, acting on X via $\sigma : G \times_S X \to X$. We shall call the pair $(\mathcal{F}, \phi_{\mathcal{F}})$ an equivariant sheaf, when $\mathcal{F}$ on X is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, and $\phi_{\mathcal{F}} : \sigma^*\mathcal{F} \to p_2^*\mathcal{F}$ is an isomorphism satisfying the cocycle condition

$$p_{23}^*\phi \circ (1_G \times_S \sigma)^*\phi = (m \times_S 1_X)^*\phi.$$

This cocycle condition is equivalent to saying that the following diagram commutes:

$$\begin{tikzcd}
& {[\sigma \circ (1_G \times \sigma)]^* \mathcal{F}} \arrow[dl, equal] \arrow[dr, "(1_G\times\sigma)^*\phi_{\mathcal{F}}"] & \\
{[\sigma \circ (\mu \times 1_X)]^* \mathcal{F}} \arrow[d, "(\mu\times 1_X)^*\phi_{\mathcal{F}}"] & & {[p_2 \circ (1_G \times \sigma)]^* \mathcal{F}} \arrow[d, equal] \\
{[p_2 \circ (\mu \times 1_X)]^* \mathcal{F}} \arrow[dr, equal] & & {[\sigma \circ (p_{23})]^* \mathcal{F}} \arrow[dl, "p_{23}^*\phi_{\mathcal{F}}"] \\
& {[p_2 \circ p_{23}]^* \mathcal{F}} &
\end{tikzcd}$$

A morphism $f : (\mathcal{F}, \phi_{\mathcal{F}}) \to (\mathcal{G}, \phi_{\mathcal{G}})$ between two equivariant sheafs is defined to be a morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules, such that the following diagram commutes

$$\begin{tikzcd}
\sigma^*\mathcal{F} \arrow[r, "\phi_{\mathcal{F}}"] \arrow[d, "\sigma^* f"] & p_2^*\mathcal{F} \arrow[d, "p_2^* f"] \\
\sigma^*\mathcal{G} \arrow[r, "\phi_{\mathcal{G}}"] & p_2^*\mathcal{G}
\end{tikzcd}$$

We see that a composition of a morphism of equivariant sheaf is well-defined, for if we have equivariant sheafs $(\mathcal{F}, \phi_{\mathcal{F}})$, $(\mathcal{G}, \phi_{\mathcal{G}})$ and $(\mathcal{H}, \phi_{\mathcal{H}})$ and morphisms $f : (\mathcal{F}, \phi_{\mathcal{F}}) \to (\mathcal{G}, \phi_{\mathcal{G}})$ and $g : (\mathcal{G}, \phi_{\mathcal{G}}) \to (\mathcal{H}, \phi_{\mathcal{H}})$ then the composition of the morphisms $g \circ f : (\mathcal{F}, \phi_{\mathcal{F}}) \to (\mathcal{H}, \phi_{\mathcal{H}})$ is also a morphisms of equivariant sheafs.

We see that $(\mathcal{F}, \phi_{\mathcal{F}})$ is a quasi-coherent sheaf if and only if $\mathcal{F}$ is an equivariant object in the fibered category of quasi-coherent sheafs. This means that under the setup $p : \mathcal{Q}coh/k \to Sch/k$ and $\mathcal{F} \in \mathcal{Q}coh(X)$ we have that $\mathcal{F}$ is a G-equivariant sheaf on X if and only if for every quasi-coherent sheaf $\mathcal{G}$ over $T \to \mathrm{Spec}(k)$, we have a naturalaction of $\mathrm{Hom}(T, G)$ on $\mathrm{Hom}(\mathcal{G}, \mathcal{F})$ such that

1. For every arrow $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ in $\mathcal{Q}coh/k$ mapping to a morphism of schemes $f : T_1 \to T_2$, the induced function $\phi^* : \mathrm{Hom}(T_2, \mathcal{F}) \to (T_1, \mathcal{F})$ is equivariant with respect to the homomorphism $\mathrm{Hom}(f, G) : \mathrm{Hom}(T_2, G) \to \mathrm{Hom}(T_1, G)$.

2. The function $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \to (T, X)$ induced by the functor p is $\mathrm{Hom}(T, G)$ equivariant.

A G-equivariant morphism $f : \mathcal{F} \to \mathcal{G}$ of quasi-coherent sheafs $\mathcal{F}$ and $\mathcal{G}$ is a morphism of sheafs, such that for each quasi-coherent quasi-coherent sheaf $\mathcal{V}$ over a k-scheme Y, the natural transformation of functors $\tilde{f}(\mathcal{V}) : \mathrm{Hom}(\mathcal{V}, \mathcal{F}) \Rightarrow \mathrm{Hom}(\mathcal{V}, \mathcal{G})$ induced by f is a a $\mathrm{Hom}(Y, G)$-equivariant map.

2 Equivariant Vector Bundles

Let k be a perfect field with algebraic closure $\bar{k}$ and let X be a complete connected reduced scheme over k. Denote by G, the constant profinite group $Gal(\bar{k}/k)$ over $\bar{k}$. Observe that G acts naturally on $\bar{X} := X \times_k \mathrm{Spec}(\bar{k})$. Denote the natural action of G on X by $\sigma : G \times_{\bar{k}} \bar{X} \to \bar{X}$. We study essentially finite vector bundles over $\bar{X}$ with G-equivariant structures.

2.1 Equivariant Vector Bundles

Let $\mathcal{C}$ be the full subcategory of the category of equivariant sheafs, whose objects are of the form (V, ϕ_V), where V is an essentially finite vector bundle. We shall show that $\mathcal{C}$ forms a Tannakian category.

Lemma 1. *Both $p_2 : G \times X \to X$ and $\sigma : G \times X \to X$ are flat.*

Proof. Since G is a limit of finite groups which are flat, we have that the structure map $G \to \mathrm{Spec}(\bar{k})$ is flat. So we have that the base change of this map by $\bar{X} \to \mathrm{Spec}(\bar{k})$ is flat, so $p_2 : G \times \bar{X} \to \bar{X}$ is flat.

Now we want to show that the group action σ is flat. For any finite Galois extension K of k, we set G_K to be the constant group $Gal(K/k)$ over $\mathrm{Spec}(K)$ and $X_K = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. We see that G_K is finite and flat over $\mathrm{Spec}(K)$. For a tower of Galois extensions $K' \supset K \supset k$, we have that

$$G_{K'} \to G_K \times_K \mathrm{Spec}(K') \xrightarrow{p_1} G_K.$$

Similarly we have a map $X_{K'} \to X_K$. We see that the limits of G_K and X_K over all finite Galois extensions K of k are G and $\bar{X}$ respectively. We similarly see that the limit of $G_K \times_K X_K$ over all finite Galois $K \supset k$ is $G \times_{\bar{k}} \bar{X}$. Now we have the natural action $\sigma_K : G_K \times_K X_K \to X_K$ of G_K on X_K. This action is flat and we have that this is a map of inverse systems whose limit is the map $\sigma : G \times \bar{X} \to \bar{X}$, so σ is flat. $\square$

Lemma 2. *For equivariant vector bundles (V_i, ϕ_{V_i}) and (W_i, ϕ_{W_i}) for $i = 1, 2$ and maps $f : (V_1, \phi_{V_1}) \to (V_2, \phi_{V_2})$ and $g : (W_1, \phi_{W_1}) \to (W_2, \phi_{W_2})$, $\mathrm{Hom}(V_1, \phi_{V_1})(W_1, \phi_{W_1})$ forms an abelian group and f and g induce a homomorphism of groups.*

Proof. Let h_1 and h_2 be two morphism from $\mathrm{Hom}((V_1, \phi_{V_1}), (W_1, \phi_{W_1}))$. Then $h_1 + h_2$ also defines an equivariant morphism from (V_1, ϕ_{V_1}) to (W_1, ϕ_{W_1}) as

$$\begin{aligned}(p_2^*(h_1 + h_2)) \circ \phi_{V_1} &= p_2^*(h_1) \circ \phi_{V_1} + p_2^*(h_2) \circ \phi_{V_1} \\ &= \phi_{V_2} \circ \sigma^*(h_1) + \phi_{V_2} \circ \sigma^*(h_2) \\ &= \phi_{V_2} \circ \sigma^*(h_1 + h_2).\end{aligned}$$

So we have that $\mathrm{Hom}((V_1, \phi_{V_1}), (V_2, \phi_{V_2}))$ is an abelian group. Now we want to show that f induces a homomorphism

$$\tilde{f} : \mathrm{Hom}((V_2, \phi_{V_2}), (W_1, \phi_{W_1})) \to \mathrm{Hom}((V_1, \phi_{V_1}), (W_1, \phi_{W_1})).$$

For $h \in \mathrm{Hom}((V_2, \phi_{V_2}), (W_1, \phi_{W_1}))$, we have that $\tilde{f}(h) = h \circ f$ so we have for $h_1, h_2 \in \mathrm{Hom}((V_2, \phi_{V_2}), (W_1, \phi_{W_1}))$,

$$\begin{aligned}\tilde{f}(h_1 + h_2) &= (h_1 + h_2) \circ f \\ &= h_1 \circ f + h_2 \circ f \\ &= \tilde{f}(h_1) + \tilde{f}(h_2)\end{aligned}$$

so $\tilde{f}$ is a homomorphism.

Similarly the map $\tilde{g}$ induced by g on the set of morphisms is also a homomorphism of abelian groups.

□

Lemma 3. *For two equivariant vector bundles (V_1, ϕ_{V_1}) and (V_2, ϕ_{V_2}) any equivariant morphism $f : (V_1, \phi_{V_1}) \to (V_2, \phi_{V_2})$ admits a kernel and a cokernel.*

Proof. From the hypothesis, we have a map $f : V_1 \to V_2$ of essentially finite vector bundles. Since the category of essentially finite vector bundles is Abelian, we have that f has a kernel $j_K : K \to V_1$ and a cokernel $q_C : V_2 \to C$. Since σ and p_2 are exact, we have the following diagram which commutes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \sigma^* K \arrow[r, "\sigma^* j_K"] \arrow[d, dotted, "\phi_K"] & \sigma^* V_1 \arrow[r, "\sigma^* f"] \arrow[d, "\phi_{V_1}"] & \sigma^* V_2 \arrow[r, "\sigma^* q_C"] \arrow[d, "\phi_{V_2}"] & \sigma^* C \arrow[r] \arrow[d, dotted, "\phi_C"] & 0 \arrow[d] \\
0 \arrow[r] & p_2^* K \arrow[r, "p_2^* j_K"] & p_2^* V_1 \arrow[r, "p_2^* f"] & p_2^* V_2 \arrow[r, "p_2^* q_C"] & p_2^* C \arrow[r] & 0
\end{tikzcd}$$

where ϕ_k is induced by the universal property of $p_2^* K$ being the kernel of $p_2^* f$, and ϕ_C is induced by the universal property of $\sigma_K^* C$ being the cokernel of $\sigma^* f$. By extending the rows on the left and right of the sequence by zeros, and using five lemma on the five columns to the left and the five columns to the right, we see that both ϕ_K and ϕ_C are isomorphisms.

Now we verify that both ϕ_K and ϕ_C indeed satisfy the hexagon. We have

$$\begin{aligned}&(1_G \times \sigma)^* j_K \circ p_{23}^* \phi_K \circ (1_G \times_S \sigma)^* \phi_K = \\ &p_{23}^* \phi_{V_1} \circ (1_G \times_S \sigma)^* \phi_{V_1} \circ (1_G \times \sigma)^* j_K = \\ &(m \times_S 1_X)^* \phi_{V1} \circ (1_G \times \sigma)^* j_K = \\ &(1_G \times \sigma)^* j_K \circ (m \times_S 1_X)^* \phi_{V1}.\end{aligned}$$

Since $(1_G \times \sigma)^* j_K$ is monic, we have that the cocycle condition for ϕ_K is satisfied. Thus we have That (K, ϕ_K) is an equivariant vector bundle. Similarly (C, ϕ_C) is an equivariant vector bundle. $\square$

So we see that $\mathcal{C}$ is an Abelian category. Now we shall show that we have a tensor product in $\mathcal{C}$, which makes it a Tannakian category

Define $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ by $(V, \phi_V) \otimes (W, \phi_W) = (V \otimes W, \phi_V \otimes \phi_W)$. To check that the functor is well defined, we have to check that $\phi_V \otimes \phi_W$ satisfies the cocycle condition:

$$\begin{aligned} p_{23}^*(\phi_V \otimes \phi_W) \circ (1_G \times_S \sigma)^*(\phi_V \otimes \phi_W) \\ (p_{23}^*\phi_V \circ p_{23}^*\phi_W) \otimes ((1_G \times_S \sigma)^*\phi_V \circ (1_G \times_S \sigma)^*\phi_W) \\ (p_{23}^*\phi_V \circ (1_G \times_S \sigma)^*\phi_V) \otimes (p_{23}^*\phi_W \circ (1_G \times_S \sigma)^*\phi_W) \\ = (m \times 1_X)^*\phi_V \otimes (m \times 1_X)^*\phi_W \\ = (m \times 1_X)^*\phi_V \otimes \phi_W. \end{aligned}$$

Thus $(V, \phi_V) \otimes (W, \phi_W) = (V \otimes W, \phi_V \otimes \phi_W)$ is a equivariant vector bundle.

Now we shall show that the sheaf hom $\mathrm{Hom}(V_1, V_2)$ can naturally be given a G-equivariant structure.

Lemma 4. *If (V_1, ϕ_{V_1}) and (V_2, ϕ_{V_2}) are G-equivariant sheafs, then the sheaf $\mathcal{H}om(V_1, V_2)$ has a natural G-equivariant structure.*

Proof. On $\mathcal{Q}coh/k$, we take the functor $F : \mathcal{Q}coh \to Sets$ defined, for a sheaf $\mathcal{V}$ over a scheme Y, by setting $F(\mathcal{V})$ to be the set of all pairs $(f, \tilde{f})$ where f is a morphism of k-schemes $f : Y \to X$, and $\tilde{f}$ is a homomorphism of $\Gamma(Y, \mathcal{O}_Y)$-modules $\Gamma(Y, F)$ to $\mathrm{Hom}(\Gamma(Y, V_1), \Gamma(Y, V_2))$ and for a morphism of sheaves $\tilde{g} : \mathcal{F} \to \mathcal{G}$ over the map $g : Y_1 \to Y_2$ of k-schemes, defining the map $F(\tilde{g}) : F(\mathcal{G}) \to F(\mathcal{F})$ by sending $(f, \tilde{f})$ to $(f \circ g, \tilde{f} \circ \tilde{g})$. We see that F defines a presheaf of sets on $\mathcal{Q}coh/k$, whose sheafification under the big Zariski topology is isomorphic to the sheaf $\mathrm{Hom}(-, \mathcal{H}om(V_1, V_2))$. Now we have an action of G on the preasheaf F, given by taking a $g \in \mathrm{Hom}(Y, G)$ and $(f, \tilde{f}) \in F(Y)$, and defining

$$g \cdot (f, \tilde{f}) = ((g \cdot f), (g \cdot \tilde{f} \cdot g^{-1})).$$

Thus G has an action on the sheaf $\mathrm{Hom}(-, \mathcal{H}om(V_1, V_2))$, giving us a G-equivariant structure on $\mathcal{H}om(V_1, V_2)$. $\square$

Lemma 5. *Ler $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ be G-equivariant sheafs on X. Then we have*

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{H}om(\mathcal{F}, \mathcal{G})).$$

Proof. From Yoneda lemma, we have that the set $\mathrm{Hom}_{\mathcal{C}}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G})$ is the same as the set of all G-equivariant natural transformations $F : \mathrm{Hom}(-, \mathcal{E} \otimes \mathcal{F}) \Rightarrow \mathrm{Hom}(-, \mathcal{G})$ of the set valued functors $\mathrm{Hom}(-, \mathcal{E} \otimes \mathcal{F})$ to $\mathrm{Hom}(-, \mathcal{G})$ on $\mathcal{Q}coh/k$. Similarly, $\mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$ is the same as the set of all G-equivariant natural transformations $F : \mathrm{Hom}(-, \mathcal{E}) \Rightarrow \mathrm{Hom}(-, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$ of the set valued functors $\mathrm{Hom}(-, \mathcal{E})$ to $\mathrm{Hom}(-, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$ on $\mathcal{Q}coh/k$.

Now we define preasheaf of sets $F_1 : \mathcal{Q}coh/k \to Sets$ by assigning to each quasi-coherent sheaf $\mathcal{V}$ over the k-scheme Y, the set of $(f, \tilde{f})$ such that $f : Y \to X$ is a morphism of k-schemes and $\tilde{f}$ is a $\Gamma(Y, \mathcal{O}_Y)$-module morphism of $\Gamma(Y, \mathcal{V})$ to $\Gamma(Y, f^*(\mathcal{E} \otimes \mathcal{F}))$. For a morphism $\tilde{g} : \mathcal{V}_1 \to \mathcal{V}_2$ over the morphism of schemes $g : Y_1 \to Y_2$, we set $F_1(g)$ to be the map which takes the pairs $(f, \tilde{f})$ and maps it to $(f \circ g, \tilde{f} \circ \tilde{g})$.

We define preasheaf of sets $F_2 : \mathcal{Q}coh/k \to Sets$ by assigning to each quasi-coherent sheaf $\mathcal{V}$ over the k-scheme Y, the set of $(f, \tilde{f})$ such that $f : Y \to X$ is a morphism of k-schemes and $\tilde{f}$ is a $\Gamma(Y, \mathcal{O}_Y)$-module morphism of $\Gamma(Y, \mathcal{V})$ to $\mathrm{Hom}(\Gamma(Y, f^*(\mathcal{F})), \Gamma(Y, f^*\mathcal{G}))$.

Now we have that G-equivariant natural transformations of F_1 to $\mathrm{Hom}(-, \mathcal{G})$ are isomorphic to the G-equivariant natural transformations of $\mathrm{Hom}(-, \mathcal{E})$ to F_2. Upon sheafification of both F_1 and F_2 under the big Zariski topology, we get the required isomorphism. $\square$

Let $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$ be a geometric point. We see that the fiber functor of $\mathcal{C}$ associated to $\bar{x}$ is an exact $\bar{k}$-linear faithful tensor functor and $\mathrm{End}(\mathcal{O}_{\bar{X}}) = \bar{k}$. Thus $\mathcal{C}$ along with $\bar{x}$ forms a Tannakian category with dual $\pi(\bar{x})$.