Proof (*a*), (*b*), and (*c*) are obvious, and (*d*) is an immediate consequence of the Schwarz inequality. By (*d*) we have

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) \\ &= \mathbf{x}\cdot\mathbf{x} + 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y} \\ &\le |\mathbf{x}|^2 + 2|\mathbf{x}|\,|\mathbf{y}| + |\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2, \end{aligned}$$

so that (*e*) is proved. Finally, (*f*) follows from (*e*) if we replace $\mathbf{x}$ by $\mathbf{x}-\mathbf{y}$ and $\mathbf{y}$ by $\mathbf{y}-\mathbf{z}$.

1.38 Remarks Theorem 1.37 (*a*), (*b*), and (*f*) will allow us (see Chap. 2) to regard R^k as a metric space.

R^1 (the set of all real numbers) is usually called the line, or the real line. Likewise, R^2 is called the plane, or the complex plane (compare Definitions 1.24 and 1.36). In these two cases the norm is just the absolute value of the corresponding real or complex number.

APPENDIX

Theorem 1.19 will be proved in this appendix by constructing R from Q. We shall divide the construction into several steps.

Step 1 The members of R will be certain subsets of Q, called *cuts*. A cut is, by definition, any set $\alpha \subset Q$ with the following three properties.

(I) α is not empty, and $\alpha \neq Q$.
(II) If $p \in \alpha$, $q \in Q$, and $q < p$, then $q \in \alpha$.
(III) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

The letters $p, q, r, \ldots$ will always denote rational numbers, and $\alpha, \beta, \gamma, \ldots$ will denote cuts.

Note that (III) simply says that α has no largest member; (II) implies two facts which will be used freely:

If $p \in \alpha$ and $q \notin \alpha$ then $p < q$.
If $r \notin \alpha$ and $r < s$ then $s \notin \alpha$.

Step 2 Define "$\alpha < \beta$" to mean: α is a proper subset of β.

Let us check that this meets the requirements of Definition 1.5.

If $\alpha < \beta$ and $\beta < \gamma$ it is clear that $\alpha < \gamma$. (A proper subset of a proper subset is a proper subset.) It is also clear that at most one of the three relations

$$\alpha < \beta, \qquad \alpha = \beta, \qquad \beta < \alpha$$

2.25 Examples In parts (*c*) and (*d*) of the preceding theorem, the finiteness of the collections is essential. For let G_n be the segment $\left(-\frac{1}{n}, \frac{1}{n}\right)$ $(n = 1, 2, 3, \ldots)$. Then G_n is an open subset of R^1. Put $G = \bigcap_{n=1}^{\infty} G_n$. Then G consists of a single point (namely, $x = 0$) and is therefore not an open subset of R^1.

Thus the intersection of an infinite collection of open sets need not be open. Similarly, the union of an infinite collection of closed sets need not be closed.

2.26 Definition If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\bar{E} = E \cup E'$.

2.27 Theorem *If X is a metric space and $E \subset X$, then*

(*a*) *$\bar{E}$ is closed,*
(*b*) *$E = \bar{E}$ if and only if E is closed,*
(*c*) *$\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.*

By (*a*) and (*c*), $\bar{E}$ is the *smallest* closed subset of X that contains E.

Proof
(*a*) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E. Hence p has a neighborhood which does not intersect E. The complement of $\bar{E}$ is therefore open. Hence $\bar{E}$ is closed.
(*b*) If $E = \bar{E}$, (*a*) implies that E is closed. If E is closed, then $E' \subset E$ [by Definitions 2.18(*d*) and 2.26], hence $\bar{E} = E$.
(*c*) If F is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus $F \supset \bar{E}$.

2.28 Theorem *Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.*

Compare this with the examples in Sec. 1.9.

Proof If $y \in E$ then $y \in \bar{E}$. Assume $y \notin E$. For every $h > 0$ there exists then a point $x \in E$ such that $y - h < x < y$, for otherwise $y - h$ would be an upper bound of E. Thus y is a limit point of E. Hence $y \in \bar{E}$.

2.29 Remark Suppose $E \subset Y \subset X$, where X is a metric space. To say that E is an open subset of X means that to each point $p \in E$ there is associated a positive number r such that the conditions $d(p, q) < r$, $q \in X$ imply that $q \in E$. But we have already observed (Sec. 2.16) that Y is also a metric space, so that our definitions may equally well be made within Y. To be quite explicit, let us say that E is *open relative to* Y if to each $p \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$. Example 2.21(*g*) showed that a set

Proof The monotonicity of the logarithmic function (which will be discussed in more detail in Chap. 8) implies that $\{\log n\}$ increases. Hence $\{1/n \log n\}$ decreases, and we can apply Theorem 3.27 to (10); this leads us to the series

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k(\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}, \tag{11}$$

and Theorem 3.29 follows from Theorem 3.28.

This procedure may evidently be continued. For instance,

$$\sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n} \tag{12}$$

diverges, whereas

$$\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^2} \tag{13}$$

converges.

We may now observe that the terms of the series (12) differ very little from those of (13). Still, one diverges, the other converges. If we continue the process which led us from Theorem 3.28 to Theorem 3.29, and then to (12) and (13), we get pairs of convergent and divergent series whose terms differ even less than those of (12) and (13). One might thus be led to the conjecture that there is a limiting situation of some sort, a "boundary" with all convergent series on one side, all divergent series on the other side—at least as far as series with monotonic coefficients are concerned. This notion of "boundary" is of course quite vague. The point we wish to make is this: No matter how we make this notion precise, the conjecture is false. Exercises 11(*b*) and 12(*b*) may serve as illustrations.

We do not wish to go any deeper into this aspect of convergence theory, and refer the reader to Knopp's "Theory and Application of Infinite Series," Chap. IX, particularly Sec. 41.

THE NUMBER e

3.30 Definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Here $n! = 1 \cdot 2 \cdot 3 \cdots n$ if $n \geq 1$, and $0! = 1$.

Put

$$\gamma_n = a_0 \beta_n + a_1\beta_{n-1} + \cdots + a_n \beta_0 .$$

We wish to show that $C_n \to AB$. Since $A_n B \to AB$, it suffices to show that

$$\lim_{n\to\infty} \gamma_n = 0. \tag{21}$$

Put

$$\alpha = \sum_{n=0}^{\infty} |a_n|.$$

[It is here that we use (*a*).] Let $\varepsilon > 0$ be given. By (*c*), $\beta_n \to 0$. Hence we can choose N such that $|\beta_n| \le \varepsilon$ for $n \ge N$, in which case

$$\begin{aligned}|\gamma_n| &\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1}a_{n-N-1} + \cdots + \beta_n a_0|\\ &\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon\alpha.\end{aligned}$$

Keeping N fixed, and letting $n \to \infty$, we get

$$\limsup_{n\to\infty} |\gamma_n| \le \varepsilon\alpha,$$

since $a_k \to 0$ as $k \to \infty$. Since ε is arbitrary, (21) follows.

Another question which may be asked is whether the series Σc_n, if convergent, must have the sum AB. Abel showed that the answer is in the affirmative.

3.51 Theorem *If the series* Σa_n, Σb_n, Σc_n *converge to* A, B, C, *and* $c_n = a_0 b_n + \cdots + a_n b_0$, *then* $C = AB$.

Here no assumption is made concerning absolute convergence. We shall give a simple proof (which depends on the continuity of power series) after Theorem 8.2.

REARRANGEMENTS

3.52 Definition Let $\{k_n\}$, $n = 1, 2, 3, \ldots,$ be a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from J onto J, in the notation of Definition 2.2). Putting

$$a'_n = a_{k_n} \qquad (n = 1, 2, 3, \ldots),$$

we say that $\Sigma a'_n$ is a *rearrangement* of Σa_n.

Corollary *If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.*

But f' may very well have discontinuities of the second kind.

L'HOSPITAL'S RULE

The following theorem is frequently useful in the evaluation of limits.

5.13 Theorem *Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose*

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a. \tag{13}$$

If

$$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a, \tag{14}$$

or if

$$g(x) \to +\infty \text{ as } x \to a, \tag{15}$$

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a. \tag{16}$$

The analogous statement is of course also true if $x \to b$, or if $g(x) \to -\infty$ in (15). Let us note that we now use the limit concept in the extended sense of Definition 4.33.

Proof We first consider the case in which $-\infty \leq A < +\infty$. Choose a real number q such that $A < q$, and then choose r such that $A < r < q$. By (13) there is a point $c \in (a, b)$ such that $a < x < c$ implies

$$\frac{f'(x)}{g'(x)} < r. \tag{17}$$

If $a < x < y < c$, then Theorem 5.9 shows that there is a point $t \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r. \tag{18}$$

Suppose (14) holds. Letting $x \to a$ in (18), we see that

$$\frac{f(y)}{g(y)} \leq r < q \qquad (a < y < c). \tag{19}$$

where $M = \sup|f(x)|$. Since $\alpha = \alpha_1 + \alpha_2$, it follows from (24) and (25) that

$$\left|\int_a^b f\,d\alpha - \sum_{i=1}^{N} c_n f(s_n)\right| \le M\varepsilon. \tag{26}$$

If we let $N \to \infty$, we obtain (23).

6.17 Theorem *Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$.*

Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case

$$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx. \tag{27}$$

Proof Let $\varepsilon > 0$ be given and apply Theorem 6.6 to α': There is a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P, \alpha') - L(P, \alpha') < \varepsilon. \tag{28}$$

The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$\Delta\alpha_i = \alpha'(t_i)\,\Delta x_i$$

for $i = 1, \ldots, n$. If $s_i \in [x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\,\Delta x_i < \varepsilon, \tag{29}$$

by (28) and Theorem 6.7(b). Put $M = \sup|f(x)|$. Since

$$\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\,\Delta x_i$$

it follows from (29) that

$$\left|\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\,\Delta x_i\right| \le M\varepsilon. \tag{30}$$

In particular,

$$\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i \le U(P, f\alpha') + M\varepsilon,$$

for all choices of $s_i \in [x_{i-1}, x_i]$, so that

$$U(P, f, \alpha) \le U(P, f\alpha') + M\varepsilon.$$

The same argument leads from (30) to

$$U(P, f\alpha') \le U(P, f, \alpha) + M\varepsilon.$$

Thus

$$|U(P, f, \alpha) - U(P, f\alpha')| \le M\varepsilon. \tag{31}$$

8.1 Theorem *Suppose the series*

$$\sum_{n=0}^{\infty} c_n x^n \tag{3}$$

converges for $|x| < R$, *and define*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (|x| < R). \tag{4}$$

Then (3) *converges uniformly on* $[-R+\varepsilon, R-\varepsilon]$, *no matter which* $\varepsilon > 0$ *is chosen. The function f is continuous and differentiable in* $(-R, R)$, *and*

$$f'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} \qquad (|x| < R). \tag{5}$$

Proof Let $\varepsilon > 0$ be given. For $|x| \le R - \varepsilon$, we have

$$|c_n x^n| \le |c_n(R-\varepsilon)^n|;$$

and since

$$\Sigma c_n(R-\varepsilon)^n$$

converges absolutely (every power series converges absolutely in the interior of its interval of convergence, by the root test), Theorem 7.10 shows the uniform convergence of (3) on $[-R+\varepsilon, R-\varepsilon]$.

Since $\sqrt[n]{n} \to 1$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \sqrt[n]{n|c_n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n|},$$

so that the series (4) and (5) have the same interval of convergence.

Since (5) is a power series, it converges uniformly in $[-R+\varepsilon, R-\varepsilon]$, for every $\varepsilon > 0$, and we can apply Theorem 7.17 (for series instead of sequences). It follows that (5) holds if $|x| \le R - \varepsilon$.

But, given any x such that $|x| < R$, we can find an $\varepsilon > 0$ such that $|x| < R - \varepsilon$. This shows that (5) holds for $|x| < R$.

Continuity of f follows from the existence of f' (Theorem 5.2).

Corollary *Under the hypotheses of Theorem* 8.1, *f has derivatives of all orders in* $(-R, R)$, *which are given by*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-k}. \tag{6}$$

In particular,

$$f^{(k)}(0) = k!c_k \qquad (k = 0, 1, 2, \ldots). \tag{7}$$

(Here $f^{(0)}$ means f, and $f^{(k)}$ is the kth derivative of f, for $k = 1, 2, 3, \ldots$).

8.16 Parseval's theorem *Suppose f and g are Riemann-integrable functions with period 2π, and*

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, \qquad g(x) \sim \sum_{-\infty}^{\infty} \gamma_n e^{inx}. \tag{82}$$

Then

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2 \, dx = 0, \tag{83}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx = \sum_{-\infty}^{\infty} c_n \bar{\gamma}_n, \tag{84}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2. \tag{85}$$

Proof Let us use the notation

$$\|h\|_2 = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(x)|^2 \, dx \right\}^{1/2}. \tag{86}$$

Let $\varepsilon > 0$ be given. Since $f \in \mathscr{R}$ and $f(\pi) = f(-\pi)$, the construction described in Exercise 12 of Chap. 6 yields a continuous 2π-periodic function h with

$$\|f - h\|_2 < \varepsilon. \tag{87}$$

By Theorem 8.15, there is a trigonometric polynomial P such that $|h(x) - P(x)| < \varepsilon$ for all x. Hence $\|h - P\|_2 < \varepsilon$. If P has degree N_0, Theorem 8.11 shows that

$$\|h - s_N(h)\|_2 \le \|h - P\|_2 < \varepsilon \tag{88}$$

for all $N \ge N_0$. By (72), with $h - f$ in place of f,

$$\|s_N(h) - s_N(f)\|_2 = \|s_N(h - f)\|_2 \le \|h - f\|_2 < \varepsilon. \tag{89}$$

Now the triangle inequality (Exercise 11, Chap. 6), combined with (87), (88), and (89), shows that

$$\|f - s_N(f)\|_2 < 3\varepsilon \qquad (N \ge N_0). \tag{90}$$

This proves (83). Next,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(f)\bar{g} \, dx = \sum_{-N}^{N} c_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx}\, \overline{g(x)} \, dx = \sum_{-N}^{N} c_n \bar{\gamma}_n, \tag{91}$$

and the Schwarz inequality shows that

$$\left| \int f\bar{g} - \int s_N(f)\bar{g} \right| \le \int |f - s_N(f)||g| \le \left\{ \int |f - s_N|^2 \int |g|^2 \right\}^{1/2}, \tag{92}$$

Here, as in Sec. 9.16, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ are the standard bases of R^n and R^m.

Proof Fix j. Since $\mathbf{f}$ is differentiable at $\mathbf{x}$,

$$\mathbf{f}(\mathbf{x} + t\mathbf{e}_j) - \mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})(t\mathbf{e}_j) + \mathbf{r}(t\mathbf{e}_j)$$

where $|\mathbf{r}(t\mathbf{e}_j)|/t \to 0$ as $t \to 0$. The linearity of $\mathbf{f}'(\mathbf{x})$ shows therefore that

$$\lim_{t\to 0} \frac{\mathbf{f}(\mathbf{x} + t\mathbf{e}_j) - \mathbf{f}(\mathbf{x})}{t} = \mathbf{f}'(\mathbf{x})\mathbf{e}_j. \tag{28}$$

If we now represent $\mathbf{f}$ in terms of its components, as in (24), then (28) becomes

$$\lim_{t\to 0} \sum_{i=1}^{m} \frac{f_i(\mathbf{x} + t\mathbf{e}_j) - f_i(\mathbf{x})}{t} \mathbf{u}_i = \mathbf{f}'(\mathbf{x})\mathbf{e}_j. \tag{29}$$

It follows that each quotient in this sum has a limit, as $t \to 0$ (see Theorem 4.10), so that each $(D_j f_i)(\mathbf{x})$ exists, and then (27) follows from (29).

Here are some consequences of Theorem 9.17:

Let $[\mathbf{f}'(\mathbf{x})]$ be the matrix that represents $\mathbf{f}'(\mathbf{x})$ with respect to our standard bases, as in Sec. 9.9.

Then $\mathbf{f}'(\mathbf{x})\mathbf{e}_j$ is the jth column vector of $[\mathbf{f}'(\mathbf{x})]$, and (27) shows therefore that the number $(D_j f_i)(\mathbf{x})$ occupies the spot in the ith row and jth column of $[\mathbf{f}'(\mathbf{x})]$. Thus

$$[\mathbf{f}'(\mathbf{x})] = \begin{bmatrix} (D_1 f_1)(\mathbf{x}) & \cdots & (D_n f_1)(\mathbf{x}) \\ \cdots & \cdots & \cdots \\ (D_1 f_m)(\mathbf{x}) & \cdots & (D_n f_m)(\mathbf{x}) \end{bmatrix}.$$

If $\mathbf{h} = \Sigma h_j \mathbf{e}_j$ is any vector in R^n, then (27) implies that

$$\mathbf{f}'(\mathbf{x})\mathbf{h} = \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} (D_j f_i)(\mathbf{x}) h_j \right\} \mathbf{u}_i. \tag{30}$$

9.18 Example Let γ be a differentiable mapping of the segment $(a, b) \subset R^1$ into an open set $E \subset R^n$, in other words, γ is a differentiable curve in E. Let f be a real-valued differentiable function with domain E. Thus f is a differentiable mapping of E into R^1. Define

$$g(t) = f(\gamma(t)) \quad (a < t < b). \tag{31}$$

The chain rule asserts then that

$$g'(t) = f'(\gamma(t))\gamma'(t) \qquad (a < t < b). \tag{32}$$

9.35 Theorem *If $[A]$ and $[B]$ are n by n matrices, then*

$$\det([B][A]) = \det[B]\det[A].$$

Proof If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the columns of $[A]$, define

$$\Delta_B(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \Delta_B[A] = \det([B][A]). \tag{85}$$

The columns of $[B][A]$ are the vectors $B\mathbf{x}_1, \ldots, B\mathbf{x}_n$. Thus

$$\Delta_B(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \det(B\mathbf{x}_1, \ldots, B\mathbf{x}_n). \tag{86}$$

By (86) and Theorem 9.34, Δ_B also has properties 9.34 (*b*) to (*d*). By (*b*) and (84),

$$\Delta_B[A] = \Delta_B\Big(\sum_i a(i, 1)\mathbf{e}_i, \mathbf{x}_2, \ldots, \mathbf{x}_n\Big) = \sum_i a(i, 1)\,\Delta_B(\mathbf{e}_i, \mathbf{x}_2, \ldots, \mathbf{x}_n).$$

Repeating this process with $\mathbf{x}_2, \ldots, \mathbf{x}_n$, we obtain

$$\Delta_B[A] = \sum a(i_1, 1)a(i_2, 2)\cdots a(i_n, n)\,\Delta_B(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}), \tag{87}$$

the sum being extended over all ordered n-tuples $(i_1, \ldots, i_n)$ with $1 \le i_r \le n$. By (*c*) and (*d*),

$$\Delta_B(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}) = t(i_1, \ldots, i_n)\,\Delta_B(\mathbf{e}_1, \ldots, \mathbf{e}_n), \tag{88}$$

where $t = 1, 0$, or -1, and since $[B][I] = [B]$, (85) shows that

$$\Delta_B(\mathbf{e}_1, \ldots, \mathbf{e}_n) = \det[B]. \tag{89}$$

Substituting (89) and (88) into (87), we obtain

$$\det([B][A]) = \Big\{\sum a(i_1, 1)\cdots a(i_n, n)t(i_1, \ldots, i_n)\Big\}\det[B],$$

for all n by n matrices $[A]$ and $[B]$. Taking $B = I$, we see that the above sum in braces is $\det[A]$. This proves the theorem.

9.36 Theorem *A linear operator A on R^n is invertible if and only if* $\det[A] \neq 0$.

Proof If A is invertible, Theorem 9.35 shows that

$$\det[A]\det[A^{-1}] = \det[AA^{-1}] = \det[I] = 1,$$

so that $\det[A] \neq 0$.

If A is not invertible, the columns $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of $[A]$ are dependent (Theorem 9.5); hence there is one, say, $\mathbf{x}_k$, such that

$$\mathbf{x}_k + \sum_{j \neq k} c_j \mathbf{x}_j = 0 \tag{90}$$

for certain scalars c_j. By 9.34 (*b*) and (*d*), $\mathbf{x}_k$ can be replaced by $\mathbf{x}_k + c_j\mathbf{x}_j$ without altering the determinant, if $j \neq k$. Repeating, we see that $\mathbf{x}_k$ can