



A numerical method to estimate the parameters of the CEV model implied by American option prices: Evidence from NYSE

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ABSTRACT

We develop a highly efficient procedure to forecast the parameters of the constant elasticity of variance (CEV) model implied by American options. In particular, first of all, the American option prices predicted by the CEV model are calculated using an accurate and fast finite difference scheme. Then, the parameters of the CEV model are obtained by minimizing the distance between theoretical and empirical option prices, which yields an optimization problem that is solved using an ad-hoc numerical procedure. The proposed approach, which turns out to be very efficient from the computational standpoint, is used to test the goodness-of-fit of the CEV model in predicting the prices of American options traded on the NYSE. The results obtained reveal that the CEV model does not provide a very good agreement with real market data and yields only a marginal improvement over the more popular Black–Scholes model.

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1. Introduction

One of the most popular models for pricing financial derivatives is the Black–Scholes model [7], which is based on the assumption that asset prices follow a log-normal diffusion process with constant volatility. This yields a considerable amount of mathematical tractability and allows us to price European vanilla options using simple analytical expressions, the so-called Black–Scholes formulae. Nevertheless, as revealed by several empirical studies, the probability distribution of log-returns is far from being normal, and thus the Black–Scholes model has been further improved in order to properly take into account this fact.

In particular, [15] and [16] have developed the so-called constant elasticity of variance (CEV) model, according to which the volatility is specified as a function of the price of the options' underlying asset. The CEV model is still nowadays quite popular among researchers and practition-

ers (among the more recent works let us recall those by [24,32,39,48,49,53,54,57–59,61]) as it offers the following advantages: first of all, the volatility is specified as a simple function of the asset price, without introducing any additional stochastic process; second, the leverage effect, i.e. the inverse relationship that is frequently observed between prices and volatility, can be taken into account.

However, in the technical literature there is not an unanimous consensus about the performances of the CEV model. In fact, according to [6,13,14,27,42,46,50,51], the CEV model allows one to obtain a particularly accurate description of asset prices; on the contrary, other empirical studies [3,20,22,34] show that the CEV model does not yield a substantial improvement over the log-normal model. In the present manuscript, in order to shed light on this subject, we test the performances of the CEV model in describing American option prices. Note that some of the works cited above [6,13,14,20,34,42,46,51] are concerned with the pricing of options, but only of European type. Nevertheless, when evaluating the goodness-of-fit of the CEV model in describing realized option prices, it is important to consider American options (rather than European options), as the majority of the options traded on the markets are of this type.

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Fitting the CEV model to American options is a challenging problem, as the prices of American options cannot be calculated using an exact analytical formula; thus, we have to employ an approximate solution, but such an approximate solution must be accurate and fast to compute, because, in order to obtain a reliable estimation of the implied model parameters (for example by least-squares fitting), American option prices need to be calculated several times and with sufficient precision.

In the present paper, an accurate and fast procedure to forecast the parameters of the CEV model implied by American options is developed as follows. First of all, the American option prices predicted by the CEV model are computed using a highly efficient Richardson-extrapolated finite differences scheme; then, the parameters of the CEV model are computed as the ones that minimize the difference between theoretical and empirical option prices. This second step leads us to an optimization problem in two decision variables, which is solved using an ad-hoc modification of the numerical algorithm that has been proposed in [10]. Such an approach turns out to be particularly suitable as it does not require the use of derivatives, which are not very simple to obtain in our numerical model (see Section 3).

In summary, the calibration procedure developed in this paper reduces the complexity of the problem to a large extent, so that, as shown by practical experiments reported in Section 5, the parameters of the CEV model implied by American options can be computed extremely quickly.

The proposed approach is used to assess the capability of the CEV model in predicting the prices of American options on fifty equities traded on the NYSE. The results obtained reveal that the CEV model does not provide a very good agreement with market data, as there are forty equities (among the fifty considered) for which the relative difference (RM-SRE) between predicted and realized option prices is greater than 10%. Moreover, we also find that the CEV model yields only a marginal improvement over the more popular Black-Scholes model (see Section 5).

This paper is organized as follows: in Section 2 we introduce the problem of pricing American options under the CEV model; in Section 3 we describe the finite difference scheme used to evaluate the American option prices; in Section 4 we show the parameter estimation method; in Section 5 we present and discuss the results obtained; in Section 6 some conclusions are drawn.

2. Pricing American options under the constant elasticity of variance model

According to the constant elasticity of variance (CEV) model (see [15,16,42]), the price of an asset, which we denote $S(t)$, satisfies (under the dividend-adjusted risk neutral measure) the stochastic differential equation:

$$dS(t) = (r - q)S(t)dt + \delta S^{\frac{\beta}{2}}(t)dW(t), \quad (1)$$

where r and q are the (constant) interest rate and the (constant) dividend yield, respectively and δ and β are positive constants. It immediately follows that the above process is an extension of the well-known geometric Brownian motion to the case where the volatility of the asset, which we denote

σ , is price dependent and equal to:

$$\sigma(S) = \delta S^{\frac{\beta}{2}-1}. \quad (2)$$

In particular, when $\beta = 2$ the volatility (2) is constant and the CEV model (1) reduces to the geometric Brownian motion on which the famous Black-Scholes model stands [7]. By constraint, if $\beta < 2$ price and volatility are inversely proportional (leverage effect), whereas if $\beta > 2$ they are directly proportional.

The price of an American option on an underlying asset described by the CEV model can be computed as follows:

American Call option

Let $C(S, t)$ denote the price of an American Call option on an underlying asset described by (1), with maturity T and strike price E . As is well-known (see [56]), if q is strictly positive, there is a certain value $B(t)$, which is commonly referred to as optimal exercise boundary, such that for $S \geq B(t)$ the option is exercised and we have:

$$C(S, t) = S - E. \quad (3)$$

Instead, for $0 \leq t < T$, the American option price satisfies the following partial differential problem:

$$\begin{aligned} \frac{\partial C(S, t)}{\partial t} + \frac{1}{2} \delta^2 S^\beta \frac{\partial^2 C(S, t)}{\partial S^2} + (r - q) \frac{\partial C(S, t)}{\partial S} \\ - rC(S, t) = 0, \end{aligned} \quad (4)$$

$$C(0, t) = 0, \quad \left. \frac{\partial C(S, t)}{\partial S} \right|_{S=B(t)} = 1, \quad (5)$$

$$C(S, 0) = \max[S - E, 0]. \quad (6)$$

Moreover, the optimal exercise boundary is implicitly defined by the equation:

$$C(B(t), t) = B(t) - E. \quad (7)$$

It is worth recalling that if the dividend yield q is equal to zero, then it is never optimal to exercise the American option prior to maturity and thus the American option price $C(S, t)$ is equal to the price of an European option with same strike and maturity (see, for example, [17]).

American Put option

Let $P(S, t)$ denote the price of an American Put option on an underlying asset described by (1), with maturity T and strike price E . In analogy with the case of the Call option, there is a certain value $B(t)$, which is commonly referred to as optimal exercise boundary, such that for $S \leq B(t)$ the option is exercised and we have:

$$P(S, t) = E - S. \quad (8)$$

Instead, for $S > B(t)$, the American option price satisfies the following partial differential problem:

$$\begin{aligned} \frac{\partial P(S, t)}{\partial t} + \frac{1}{2} \delta^2 S^\beta \frac{\partial^2 P(S, t)}{\partial S^2} + (r - q) \frac{\partial P(S, t)}{\partial S} \\ - rP(S, t) = 0, \end{aligned} \quad (9)$$

$$P(0, t) = E, \quad \left. \frac{\partial P(S, t)}{\partial S} \right|_{S=B(t)} = -1, \quad (10)$$

$$P(S, 0) = \max[E - S, 0]. \quad (11)$$

Moreover, the optimal exercise boundary is implicitly defined by the equation:

$$P(B(t), t) = E - B(t). \quad (12)$$

3. Numerical approximation of the American option price

Problem (4)–(7) and problem (9)–(12) do not have exact closed-form solutions and thus some approximation is mandatory. In the technical literature, a large variety of numerical methods for option pricing, or, more in general, for solving partial differential equations of parabolic type, have been proposed (see, e.g., [2,4,11,12,18,21,36,37,44,45,52–54,60]). In this paper, we employ a finite difference scheme based on space and time extrapolation, which is described in the following. For the sake of brevity, in showing the proposed numerical approach we only consider the case of American Call options (problem (4)–(7)), but the case of Put options (problem (9)–(12)) is perfectly analogous.

First of all, in place of problem (4)–(7), we consider a time discretization of it that amounts to computing the American option price by Richardson extrapolation of the prices of Bermudan options; let us recall that a Bermudan option is an option that can be exercised only at a discrete set of dates. As shown in [2,4,11,12,23,31,41], such an approach is very efficient from the computational standpoint.

We also observe that problem (4)–(7) could also be solved using the method developed by Brennan and Schwartz in [9], which is based on a linear complementarity approach and has the same (linear) computational complexity as the Bermudan approximation (see also [35] for a thorough theoretical investigation of the Brennan–Schwartz algorithm). Thus, it would be interesting to perform a numerical comparison between the Bermudan approximation and the method by Brennan and Schwartz, which, to the best of our knowledge, has not been done yet. This is left as a future work.

In the interval $[0, T]$ let us consider $N_t + 1$ equally spaced time levels t_0, t_1, \dots, t_{N_t} such that $t_k = k\Delta t$, $k = 0, 1, \dots, N_t$, where $\Delta t = \frac{T}{N_t}$. Let $C_{\Delta t}(S, t)$ denote the price of a Bermudan option with maturity T and exercise dates t_0, t_1, \dots, t_{N_t} . The Bermudan option price $C_{\Delta t}(S, t)$ is obtained using the following recursion procedure. First of all set $k = N_t$, and define:

$$\Psi(S, t_k) = \phi(S). \quad (13)$$

Then, for $t \in [t_{k-1}, t_k]$, solve the partial differential problem:

$$\frac{\partial C_{\Delta t}(S, t)}{\partial t} + \frac{1}{2} \delta^2 S^2 \frac{\partial^2 C_{\Delta t}(S, t)}{\partial S^2} + (r - q) \frac{\partial C_{\Delta t}(S, t)}{\partial S} - r C_{\Delta t}(S, t) = 0, \quad (14)$$

$$C_{\Delta t}(0, t) = 0, \quad C_{\Delta t}(S, t) \sim S - K, \text{ as } S \rightarrow +\infty, \quad (15)$$

$$C_{\Delta t}(S, t_k) = \Psi(S, t_k). \quad (16)$$

Then set:

$$\Psi(S, t_{k-1}) = \max[C_{\Delta t}(S, t_{k-1}), \phi(S)]. \quad (17)$$

Finally, update the counter:

$$k := k - 1, \quad (18)$$

and repeat the cycle (14)–(18) until $k = 0$.

Note that problem (14)–(17) amounts to performing a time discretization of the early exercise opportunity by imposing the constraint (17) only at the dates t_0, t_1, \dots, t_{N_t} .

Finally, it remains to solve the partial differential problem (14)–(17). To this aim, first of all, we approximate the time derivative in (14) through a single step of the implicit Euler finite difference scheme (see [52]). Precisely, let $C_{\Delta t}^k(S)$ and $\Psi_{\Delta t}^k(S)$ denote approximate values of $C_{\Delta t}(S, t_k)$ and $\Psi_{\Delta t}(S, t_k)$, respectively. Problem (14)–(17) is discretized in time as follows:

$$\begin{aligned} \frac{\Psi_{\Delta t}^k(S) - C_{\Delta t}^{k-1}(S)}{\Delta t} + \frac{\partial C_{\Delta t}^{k-1}(S, t)}{\partial t} + \frac{1}{2} \delta^2 S^2 \frac{\partial^2 C_{\Delta t}^{k-1}(S, t)}{\partial S^2} \\ + (r - q) \frac{\partial C_{\Delta t}^{k-1}(S, t)}{\partial S} - r C_{\Delta t}^{k-1}(S, t) = 0, \end{aligned} \quad (19)$$

$$C_{\Delta t}^{k-1}(0) = 0, \quad C_{\Delta t}^{k-1}(S) \sim S - K, \text{ as } S \rightarrow +\infty. \quad (20)$$

Then, we discretize problem (19) and (20) in the S variable. To this aim, the infinite spatial domain $[0, +\infty)$ is replaced with a bounded one $[0, S_{\max}]$, where S_{\max} is chosen large enough such that the truncation error is negligible (following [53] we set $S_{\max} = 2K$). Moreover, let us consider $N_S + 1$ equally spaced points S_0, S_1, \dots, S_{N_S} such that $S_j = j\Delta S$, $j = 0, 1, 2, \dots, N_S$, where $\Delta S = \frac{S_{\max}}{N_S}$, and let $C_{\Delta S, \Delta t}^{k-1, j}$ denote an approximate value of $C_{\Delta t}^{k-1}(S_j)$, $j = 0, 1, 2, \dots, N_S$. For the sake of computational accuracy, N_S is chosen such that one of the points S_0, S_1, \dots, S_{N_S} coincides with the strike price K (see, e.g., [52]). This can be easily accomplished as follows: since we use $S_{\max} = 2K$, it is enough to set $N_S = 2p$, where p can be any positive integer. Eq. (19) is discretized in space using the (centered) three-point finite difference scheme (see, e.g., [52]):

$$\begin{aligned} \frac{\Psi_{\Delta t}^{k, j} - C_{\Delta S, \Delta t}^{k-1, j}}{\Delta t} + (r - q) S_j \frac{C_{\Delta S, \Delta t}^{k-1, j+1} - C_{\Delta S, \Delta t}^{k-1, j-1}}{\Delta S} \\ + \frac{1}{2} \delta^2 S_j^2 \frac{C_{\Delta S, \Delta t}^{k-1, j+1} - 2C_{\Delta S, \Delta t}^{k-1, j} + C_{\Delta S, \Delta t}^{k-1, j-1}}{\Delta S^2} - r C_{\Delta S, \Delta t}^{k-1, j} = 0, \end{aligned} \quad (21)$$

The boundary conditions (20) are simply imposed as follows:

$$C_{\Delta S, \Delta t}^{k-1, 0} = 0, \quad C_{\Delta S, \Delta t}^{k-1, N_S} = S_{\max} - K. \quad (22)$$

Relations (21) and (22) constitute a tridiagonal system of linear equations in the unknowns $C_{\Delta S, \Delta t}^{k-1, 0}, C_{\Delta S, \Delta t}^{k-1, 1}, \dots, C_{\Delta S, \Delta t}^{k-1, N_S}$, which is solved very quickly by means of the well known Thomas algorithm (see [55]). According to the recursion (14)–(18), the numerical scheme (21) and (22) is used at every time step t_k , $k = N_t, N_t - 1, \dots, 1$, so that we can obtain the values $C_{\Delta S, \Delta t}^{k, j}$ for every $k \in \{0, 1, \dots, N_t\}$ and $j \in \{0, 1, \dots, N_S\}$.

The accuracy of the approximation $C_{\Delta S, \Delta t}^{k, j}$ increases as the number of the time intervals N_S and N_t increase, or, equivalently, as ΔS and Δt tend to zero. Precisely, the difference

between $C_{\Delta S, \Delta t}^{k,j}$ and $C(S_j, t_k)$ contains leading terms that grow like $O(\Delta S^2)$ and $O(\Delta t)$ as $\Delta S \rightarrow 0$ and $\Delta t \rightarrow 0$, $k = 0, 1, \dots, N_t$ and $j = 0, 1, \dots, N_S$ (see [1,2,31]). Therefore, it is possible to remove the $O(\Delta S^2)$ and $O(\Delta t)$ error terms using the Richardson extrapolation procedure (see, e.g., [26]):

$$C_{\Delta S, \Delta t, \text{extr}}^{k,j} = \frac{4C_{\Delta S/2, \Delta t/4}^{k,j} - C_{\Delta S, \Delta t}^{k,j}}{3}, \quad k = 0, 1, \dots, N_t, \\ j = 0, 1, \dots, N_S, \quad (23)$$

where $C_{\Delta S/2, \Delta t/4}^{m,\ell}$, $m = 0, 1, \dots, 4N_t$, $\ell = 0, 1, \dots, 2N_S$, denote the approximate option prices obtained using a finite difference mesh of $2N_S + 1$ spatial nodes and $4N_t + 1$ time levels (i.e. with spacings $\Delta S/2$ and $\Delta t/4$).

The values $C_{\Delta S, \Delta t, \text{extr}}^{k,j}$, $k = 0, 1, \dots, N_t$, $j = 0, 1, \dots, N_S$, are very accurate approximations of the true American option prices. In particular, the discretization error tends to zero faster than ΔS^2 and Δt as ΔS and Δt tend to zero.

Finally, we note that the numerical method described above provides an approximation of the American option price only at the nodes S_0, S_1, \dots, S_{N_S} . Then, for values of S that do not coincide with the mesh nodes, the American option price is obtained by cubic interpolation of the option prices at the four nodes closest to S (which is extremely quick and does not compromise the $o(\Delta S^2)$ accuracy of the spatial extrapolation scheme).

4. The estimation method

Let $V_1, V_2, \dots, V_{N_{op}}$ denote the realized prices of N_{op} American options of both Call and Put type written on the same underlying asset and having different strikes and maturities. Moreover, let $V_1^{AP}(\beta, \delta), V_2^{AP}(\beta, \delta), \dots, V_{N_{op}}^{AP}(\beta, \delta)$ denote the corresponding values of the prices predicted by the CEV model (computed using the method described in Section 3). Note that, as explicitly indicated, the quantities $V_1^{AP}, V_2^{AP}, \dots, V_{N_{op}}^{AP}$ depend on the model parameters β and δ .

The parameters β and δ are estimated by minimizing the root-mean-square relative error (RMSRE) defined as follows:

$$(\beta^*, \delta^*) = \arg \min_{\beta, \delta} \text{RMSRE}(\beta, \delta), \quad (24)$$

where

$$\text{RMSRE}(\beta, \delta) = \sqrt{\frac{1}{N_{op}} \sum_{i=1}^{N_{op}} \left(\frac{V_i - V_i^{AP}(\beta, \delta)}{V_i} \right)^2}. \quad (25)$$

Now, to solve the above optimization problem we could think to apply a numerical algorithm that makes use of the derivatives of the function $\text{RMSRE}(\beta, \delta)$ with respect to β and δ . However, computing these derivatives poses some difficulty. In fact, we could obtain partial differential equations for the derivatives of $\text{RMSRE}(\beta, \delta)$ by differentiating (21) with respect to β and δ , but then we would not know how to impose the American constraint (17). Alternatively, we could compute the derivatives of $\text{RMSRE}(\beta, \delta)$ by finite difference approximation. That is, for example, we could approximate $\frac{\partial}{\partial \beta}(\text{RMSRE}(\beta, \delta))$ and $\frac{\partial}{\partial \delta}(\text{RMSRE}(\beta, \delta))$ with $\frac{1}{\Delta \beta}(\text{RMSRE}(\beta + \Delta \beta, \delta) - \text{RMSRE}(\beta, \delta))$ and $\frac{1}{\Delta \delta}(\text{RMSRE}(\beta, \delta + \Delta \delta) - \text{RMSRE}(\beta, \delta))$, respectively, where $\Delta \beta$ and $\Delta \delta$

are some (small) finite variations. Nevertheless, this is computationally costly, as two further evaluations of the function RMSRE are required. Moreover, $\text{RMSRE}(\beta, \delta)$, $\text{RMSRE}(\beta + \Delta \beta, \delta)$ and $\text{RMSRE}(\beta, \delta + \Delta \delta)$ are affected by numerical error (as they are obtained by finite difference approximation), and thus, once we divide by the small quantities $\Delta \beta$ and $\Delta \delta$, we could obtain values of $\frac{\partial}{\partial \beta}(\text{RMSRE}(\beta, \delta))$ and $\frac{\partial}{\partial \delta}(\text{RMSRE}(\beta, \delta))$ which are fairly inaccurate.

Therefore, the optimization problem (24) is solved using the Brent's numerical algorithm, which is very simple to implement and does not require us to evaluate the derivatives of $\text{RMSRE}(\beta, \delta)$ with respect to β and δ . For a detailed description of such an approach, the interested reader is directly referred to [10]. Here, we simply observe that the Brent's method is designed for solving one-dimensional optimization problems (see [10]). Therefore, we employ a variant of this algorithm, according to which problem (24) is split into two nested optimization problems:

$$\beta^* = \arg \min_{\beta} \text{RMSRE}(\beta, \tilde{\delta}(\beta)), \quad (26)$$

where

$$\tilde{\delta}(\beta) = \arg \min_{\delta} \text{RMSRE}(\beta, \delta). \quad (27)$$

That is we minimize RMSRE as a function of the only parameter β , which can be done by considering the parameter δ as a function of β , namely the function $\tilde{\delta}$ defined as in (27).

Now, the optimization problems (26) and (27) involve only one decision variable each and thus they can be conveniently solved using the Brent's method. Finally, once that the optimal value of β^* is obtained as the solution of problem (26), the optimal value of δ^* is simply computed as $\delta^* = \tilde{\delta}(\beta^*)$, i.e. as the solution of the optimization problem (27).

5. Results

We estimate the parameters of the CEV model for 50 equities of the NYSE. We consider American options of both Call and Put type with various different strikes and maturities. In particular, for each one of the 50 equities we consider 24 option prices (12 Call and 12 Put) observed on 24 July 2015, so that in (25) we have $N_{op} = 24$. Moreover, each of the 50 sets of 12 Call options and of the 50 sets of 12 Put options is chosen as follows: we have four different maturities (which are nearly equally spaced from 3 months to 24 months), and for each of these four maturities, we have three different strike prices, namely one out-of-the-money option, one at-the-money option and one in-the-money option. In addition, according to a rather standard procedure (see, e.g., [20,25,33,43]) for every equity and every option we consider (recorded) daily closing prices.

Following a common approach (see, e.g., [19,28,29]), for each equity the (continuous) dividend yield q is determined by imposing the equivalence between the discounted actual value of the continuous dividend flow and the discounted actual value of the discrete dividends that have been paid in the time period from 1 January 2014 to 24 July 2015. In addition, the interest rate r is estimated as the 12 month T-bill rate on 24 July 2015. However, we have performed several numerical experiments (which we do not report in the paper to save

Table 1
Calibration of the CEV model.

Equity	δ^*	β^*	RMSRE	Err_{δ^*}	Err_{β^*}	CPUTime (s)
Agilent Technologies Inc (A)	0.7347	1.422	0.1452	7.90×10^{-4}	2.97×10^{-4}	13.4
Alcoa Inc (AA)	0.7564	1.3543	0.0503	1.00×10^{-4}	6.55×10^{-5}	14.5
AbbVie Inc (ABBV)	3.9721	0.6397	0.1404	6.61×10^{-3}	4.97×10^{-3}	14.2
Abbott Laboratories (ABT)	1.7293	0.8746	0.0797	8.98×10^{-3}	5.26×10^{-3}	26.1
Aecom (ACM)	0.4790	1.8351	0.1752	3.33×10^{-4}	1.07×10^{-4}	21.1
Accenture PLC (ACN)	0.2383	1.8663	0.2115	4.46×10^{-3}	1.03×10^{-3}	23.5
Aetna Inc (AET)	18.7246	0.1718	0.1328	2.95×10^{-3}	7.28×10^{-3}	18.2
Alaska Air Group Inc (ALK)	27.4035	-0.1002	0.1274	2.70×10^{-4}	1.28×10^{-3}	19.4
Abercrombie & Fitch Co (ANF)	4.7022	0.3809	0.2172	1.62×10^{-4}	2.72×10^{-4}	19.1
Apache Corporation (APA)	5.4997	0.5544	0.1823	1.29×10^{-3}	1.24×10^{-3}	19.1
Anadarko Petroleum Corp (APC)	0.3510	1.9222	0.0644	1.89×10^{-4}	4.20×10^{-5}	24.8
Alibaba Group Holding Ltd (BABA)	4.6854	0.7635	0.0562	4.17×10^{-4}	2.48×10^{-4}	23.1
BB&T Corp (BBT)	1.5343	0.8408	0.1146	2.54×10^{-3}	1.67×10^{-3}	18.3
BHP Billiton Ltd (BHP)	21.7996	-0.3277	0.1220	1.31×10^{-3}	2.31×10^{-3}	18.8
Blackstone Group LP (BX)	1.0288	1.2056	0.1290	5.69×10^{-3}	2.63×10^{-3}	25.4
Chicago Bridge & Iron Company (CBI)	2.6384	1.0724	0.1816	1.15×10^{-4}	5.10×10^{-5}	16.9
CBS Corp (CBS)	4.9242	0.5331	0.1069	1.11×10^{-3}	1.08×10^{-3}	18.2
Campbell Soup Co (CPB)	1.1847	1.0605	0.1721	1.56×10^{-3}	7.73×10^{-4}	16.7
Computer Sciences Corp (CSC)	6.8367	0.4236	0.1978	7.81×10^{-4}	9.44×10^{-4}	13.7
Cablevision Systems Corp (CVC)	0.7549	1.5188	0.2812	2.79×10^{-4}	8.53×10^{-5}	24.5
3D Systems Corp (DDD)	3.8728	0.5382	0.0983	1.55×10^{-4}	2.06×10^{-4}	12.6
Dean Foods Co (DF)	2.0027	0.7444	0.2147	3.01×10^{-4}	2.95×10^{-4}	16.7
Ensco PLC (ESV)	0.9214	1.5739	0.1791	2.07×10^{-5}	3.27×10^{-6}	15.0
Fiat Chrysler Automobiles NV (FCAU)	0.5163	1.7571	0.0889	2.54×10^{-4}	1.04×10^{-4}	19.8
Freeport-McMoRan Inc (FCX)	1.1601	1.5055	0.1896	8.07×10^{-5}	4.81×10^{-5}	12.1
FedEx Corp (FDX)	0.6530	1.5357	0.1297	4.36×10^{-3}	1.13×10^{-3}	18.5
Goldcorp Inc (GG)	0.8079	1.6590	0.0990	9.39×10^{-5}	5.88×10^{-5}	16.3
Corning Inc (GLW)	1.5326	0.8403	0.0596	1.21×10^{-3}	9.96×10^{-4}	21.1
GameStop Corp (GME)	8.4952	0.3403	0.0938	3.10×10^{-4}	5.83×10^{-4}	18.5
Genworth Financial Inc (GNW)	0.6528	1.6155	0.1862	8.72×10^{-6}	2.85×10^{-5}	12.0
Gap Inc (GPS)	4.9154	0.3125	0.1986	4.20×10^{-3}	7.70×10^{-3}	18.7
HCA Holdings Inc (HCA)	6.4777	0.5467	0.1277	1.37×10^{-3}	1.12×10^{-3}	16.3
Hecla Mining Co (HL)	0.6706	1.7544	0.1992	1.05×10^{-5}	1.46×10^{-5}	15.9
Hewlett-Packard Co (HPQ)	0.3793	1.7947	0.1076	1.43×10^{-3}	4.64×10^{-4}	15.4
Hertz Global Holdings Inc (HTZ)	4.6502	0.3043	0.2281	7.54×10^{-5}	1.87×10^{-4}	19.1
Lennar Corp (LEN)	1.7792	1.043	0.1888	2.51×10^{-3}	1.25×10^{-3}	19.6
Altria Group Inc (MO)	1.4415	0.9618	0.2822	2.75×10^{-4}	9.03×10^{-5}	19.4
Marathon Oil Corp (MRO)	1.2118	1.1258	0.1306	1.14×10^{-3}	6.94×10^{-4}	13.6
ArcelorMittal (MT)	2.5185	0.2262	0.2486	1.78×10^{-5}	6.21×10^{-5}	17.2
Newmont Mining Corp (NEM)	1.8091	1.0029	0.0800	4.07×10^{-4}	2.60×10^{-4}	18.2
National Oilwell Varco Inc (NOV)	0.5279	1.8325	0.3164	2.19×10^{-4}	5.86×10^{-5}	16.8
Petroleo Brasileiro SA Petrobras (PBR)	2.2631	0.5677	0.1760	2.11×10^{-5}	4.17×10^{-5}	14.0
QEP Resources Inc (QEP)	3.7705	0.1535	0.5941	1.90×10^{-4}	1.10×10^{-3}	25.3
Banco Santander SA (SAN)	0.4894	1.4749	0.3153	5.97×10^{-4}	4.44×10^{-4}	16.3
Seadrill Ltd (SDRL)	1.2589	1.3731	0.1568	1.11×10^{-4}	1.27×10^{-4}	15.6
Silver Wheaton Corp (SLW)	1.4311	1.1962	0.1681	1.37×10^{-4}	9.41×10^{-5}	14.0
Suncor Energy Inc (SU)	0.5292	1.6667	0.0903	3.92×10^{-4}	1.70×10^{-4}	17.8
UnitedHealth Group Inc (UNH)	1.5459	1.1632	0.1543	2.65×10^{-3}	9.42×10^{-4}	22.1
United Technologies Corp (UTX)	2.9349	0.7745	0.1371	9.16×10^{-3}	5.03×10^{-3}	22.4
Whiting Petroleum Corp (WLL)	1.8072	1.2180	0.3002	4.44×10^{-4}	2.19×10^{-4}	14.3

space) where we have tried to vary q and r (in an economically reasonable range of values), and we have found that the parameters have only a very small effect on the calibration of the CEV model. Finally, as far as the discretization parameters N_S and N_t are concerned, we set $N_S = 80$ and $N_t = 80$.

The (relative) error on δ^* and β^* due to the finite difference approximation is computed as follows:

$$Err_{\delta^*} = \frac{|\delta_{exact}^* - \delta^*|}{\delta_{exact}^*}, \quad Err_{\beta^*} = \frac{|\beta_{exact}^* - \beta^*|}{\beta_{exact}^*}, \quad (28)$$

where δ_{exact}^* and β_{exact}^* denote the values of δ^* and β^* which we would obtain if the option prices were computed using an exact analytical solution. Since δ_{exact}^* and β_{exact}^* are not available, an accurate estimation of them is obtained by numerical

approximation of problems (4)–(7) and (9)–(12). In particular, we employ the same finite difference method described in Section 3 and use a mesh with a very large number of nodes in both the S and the t directions ($N_S = 160$, $N_t = 240$).

The results obtained are shown in Table 1. These figures also report the computer times required in order to obtain the parameters δ^* and β^* (the simulations are performed on a personal computer with an Intel Core i7-2600 3.40 GHz and the software codes are written in Matlab 7.0). As we can observe, for all the 50 equities considered, the estimated values of β^* are smaller than 2, which indicates an inverse proportionality between prices and volatilities (leverage effect).

We may also note that the proposed calibration procedure yields an accurate estimation of the parameters of the

Table 2

Improvement of the CEV model over the Black–Scholes model.

Equity	ϵ
Agilent Technologies Inc (A)	4.60×10^{-3}
Alcoa Inc (AA)	8.46×10^{-2}
AbbVie Inc (ABBV)	6.97×10^{-2}
Abbott Laboratories (ABT)	1.64×10^{-2}
Aecom (ACM)	1.78×10^{-4}
Accenture PLC (ACN)	5.64×10^{-4}
Aetna Inc (AET)	1.06×10^{-2}
Alaska Air Group Inc (ALK)	7.03×10^{-2}
Abercrombie & Fitch Co (ANF)	1.36×10^{-2}
Apache Corporation (APA)	5.98×10^{-3}
Anadarko Petroleum Corp (APC)	5.43×10^{-5}
Alibaba Group Holding Ltd (BABA)	1.60×10^{-2}
BB&T Corp (BBT)	3.52×10^{-3}
BHP Billiton Ltd (BHP)	6.19×10^{-2}
Blackstone Group LP (BX)	3.27×10^{-3}
Chicago Bridge & Iron Company (CBI)	2.33×10^{-3}
CBS Corp (CBS)	1.18×10^{-2}
Campbell Soup Co (CPB)	3.31×10^{-2}
Computer Sciences Corp (CSC)	4.65×10^{-2}
Cablevision Systems Corp (CVC)	6.17×10^{-4}
3D Systems Corp (DDD)	6.54×10^{-2}
Dean Foods Co (DF)	1.77×10^{-2}
EnSCO PLC (ESV)	1.04×10^{-3}
Fiat Chrysler Automobiles NV (FCAU)	1.92×10^{-3}
Freeport-McMoRan Inc (FCX)	4.98×10^{-3}
FedEx Corp (FDX)	9.11×10^{-3}
Goldcorp Inc (GG)	1.22×10^{-2}
Corning Inc (GLW)	7.28×10^{-2}
GameStop Corp (GME)	1.76×10^{-2}
Genworth Financial Inc (GNW)	6.68×10^{-3}
Gap Inc (GPS)	2.21×10^{-3}
HCA Holdings Inc (HCA)	5.99×10^{-3}
Hecla Mining Co (HL)	1.98×10^{-3}
Hewlett-Packard Co (HPQ)	3.46×10^{-4}
Hertz Global Holdings Inc (HTZ)	1.80×10^{-2}
Lennar Corp (LEN)	5.22×10^{-3}
Altria Group Inc (MO)	8.94×10^{-3}
Marathon Oil Corp (MRO)	7.76×10^{-3}
ArcelorMittal (MT)	5.26×10^{-2}
Newmont Mining Corp (NEM)	3.87×10^{-2}
National Oilwell Varco Inc (NOV)	2.28×10^{-4}
Petroleo Brasileiro SA Petrobras (PBR)	9.23×10^{-2}
QEP Resources Inc (QEP)	1.04×10^{-2}
Banco Santander SA (SAN)	3.98×10^{-3}
Seadrill Ltd (SDRL)	9.92×10^{-3}
Silver Wheaton Corp (SLW)	1.18×10^{-2}
Suncor Energy Inc (SU)	1.94×10^{-3}
UnitedHealth Group Inc (UNH)	3.31×10^{-3}
United Technologies Corp (UTX)	9.69×10^{-3}
Whiting Petroleum Corp (WLL)	1.38×10^{-3}

CEV model, as the parameters δ^* and β^* can be computed with (relative) errors that are never greater than 8.98×10^{-3} and 7.70×10^{-3} respectively. Moreover, the parameter estimation method is also very efficient from the computational standpoint. In fact, for each one of the 50 equities considered, the parameters δ^* and β^* can be obtained in a time smaller than 27 s. This is rather satisfactory, especially if we think that for each equity we are taking into account 24 options, whose prices must be computed several times. Finally, to better demonstrate the performance of the proposed calibration approach, we report that for each of the 50 equities the number of evaluations of the function *RMSRE* (see (25)) required by the Brent's algorithm is in the range [65, 143] (the number

of function evaluations averaged over the whole set of equities is approximately equal to 99).

Finally, for the sake of completeness, we shall observe that we have no theoretical guarantee that the Brent's method finds a global minimum (rather than a local one) of the function *RMSRE*. Nevertheless, we have checked empirically (by plotting *RMSRE* as a function of β and δ) that a global minimum is actually obtained for each of the 50 equities. On the other hand, we shall also observe that in the technical literature no optimization method exists which is guaranteed to reach a global minimum for any optimization problem in a reasonable time (see, for example, [38,40,47]).

Moreover, let us evaluate the empirical performances of the CEV model. Overall, it does not provide a very good agreement with real market data, as for 40 equities the relative difference (*RMSRE*) between predicted and realized option prices is greater than 10%.

Furthermore, we can also compare the performance of the CEV model with those of the Black–Scholes model in describing American option prices. First of all, let us observe that for the Black–Scholes model, since $\beta = 2$, we have only to estimate the parameter δ , which is thus computed as $\tilde{\delta}(2)$, where again $\tilde{\delta}$ is given in (27). Then, the improvement that the CEV model brings to the Black–Scholes model can be measured by the following indicator:

$$\epsilon = \frac{RMSRE(2, \tilde{\delta}(2)) - RMSRE(\beta^*, \delta^*)}{RMSRE(2, \tilde{\delta}(2))}. \quad (29)$$

The values of ϵ obtained are reported in Table 2. As we can observe, the CEV model provides only a marginal improvement over the Black–Scholes model, in fact, for each one of the 50 equities considered ϵ is of the order 10^{-2} or even smaller.

6. Conclusions

A highly efficient procedure to forecast the parameters of the constant elasticity of variance (CEV) model implied by American options is developed. In particular, first of all, the values of the American option prices predicted by the CEV model are calculated using an accurate and fast finite difference scheme. Then, the parameters of the CEV model are obtained by minimizing the distance between theoretical and empirical option prices, which yields an optimization problem that is solved using a suitable variant of the Brent's algorithm. The proposed approach is used to assess the performances of the CEV model in describing the prices of American options on fifty equities traded on the NYSE. The results obtained reveal that the overall calibration procedure turns out to be very efficient from the computational standpoint. In fact, the parameters of the CEV model can be obtained with a (relative) error that is never greater than 8.98×10^{-3} in a time smaller than 27 s. However, the CEV model does not provide a very good agreement with real market data (for forty among the fifty equities considered the *RMSRE* difference between predicted and realized option prices is greater than 10%) and yields only a marginal improvement over the more popular Black–Scholes model. This is presumably due to the fact that the CEV model is too simplified, and, actually, the volatility cannot be specified as a function of market prices only. Thus, a more sophisticated

model such as a stochastic volatility model (see, e.g., [5,8,30]) would be preferable.

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