A Problem in Enumerating Extreme Points

Katta G. Murty
Department of Industrial and Operations Engineering
University of Michigan
Ann Arbor, MI 48109-2117, USA
Phone: 734-763-3513, fax: 734-764-3451
e-mail:katta_murty@umich.edu

October 2002

Abstract

We describe the central problem in developing an efficient algorithm for enumerating the extreme points of a convex polytope specified by linear constraints, and discuss a conjecture for its solution.

Key words Convex polytopes, enumeration of extreme points, adjacency, segments, central problem, strategy for its solution.

1 Introduction

We consider the problem of enumerating all the extreme points (also called vertices) of a convex polytope specified by a system of linear constraints.

For any matrix H, we denote by $H_{i.}, H_{.j}$ its ith row vector, jth column vector respectively. We will use the abbreviation "LP" for "linear program". Superscript T indicates transposition, i.e., for $y \in R^n$, y^T is its transpose.

If $\{a^1, \ldots, a^t\}$ is a set of points in R^n, we denote its convex hull by $< a^1, \ldots, a^t >$. The affine rank of $\{a^1, \ldots, a^t\}$ is defined to be the rank of $\{a^2 - a^1, \ldots, a^t - a^1\}$; it is the dimension of the affine space of $\{a^1, \ldots, a^t\}$.

A general system of linear constraints may consist of linear equations, inequalities and/or bounds on variables; when its set of feasible solutions is bounded (which is the case that we deal with in this paper), such a general system can be transformed into a system of the form (1) given below by simple transformations that preserve one-to-one correspondence between faces of the original and transformed systems. So, we consider the convex polytope K which is the set of feasible solutions of

$$Ax = b, \qquad x \geq 0 \tag{1}$$

where A is a matrix of order $m \times (n+m)$ and rank m. We assume that A, b are integer and that K is nonempty, bounded, and of dimension $n + m - m = n$. System (1) is said to be in **standard form**.

For each $j = 1$ to $n+m$, without any loss of generality we assume that the set of feasible solutions of

$$\begin{aligned} Ax &= b \\ x_j &= 0 \\ x_i &\geq 0, \qquad i = 1, \ldots, j-1, j+1, \ldots, n+m. \end{aligned}$$

is a facet, denoted by F_j of K. Since the equation "$x_j = 0$" defines the facetal hyperplane containing the facet F_j, we will define "x_j" to be the **facetal constraint function (FCF)** corresponding to F_j.

Let $x_B = (x_{n+1}, \ldots, x_{n+m})$ be a feasible basic vector for (1). Let $x_N = (x_1, \ldots, x_n)$. Then (x_B, x_N) is a basic , nonbasic partition of the variables in the vector x. The important property that holds in this (basic, nonbasic) partition is that B = corresponding basis = the $m \times m$ matrix consisting of column vectors of basic variables x_B in (1), is nonsingular and hence invertible. Let N be the $m \times n$ matrix consisting of column vectors of nonbasic variables. Then, after rearranging variables, $Ax = b$ becomes $Bx_B + Nx_N = b$. So, (1) is equivalent to $x_B + B^{-1}Nx_N = B^{-1}b, \quad x_B, x_N \geq 0$. Eliminating the basic variables, K can be represented by

$$\begin{aligned} B^{-1}Nx_N &\leq B^{-1}b \\ x_N &\geq 0 \end{aligned} \tag{2}$$

in the space of the independent variables x_N in which it is of full dimension.

2 Historical Note

Extreme points of convex polyhedra were brought to prominence by George Dantzig when he introduced the simplex method for solving linear programs in 1947 [3]. This work of his is perhaps the most fundamental contribution in the annals of the study of convex polyhedra. Until then work on them remained the domain of abstract thinkers and their imagination. His work brought a new computational dimension to the study of convex polyhedra [4].

Until Dantzig's work, visualization was only possible for 2-dimensional polyhedra. The techniques that originated in Dantzig's simplex method made it possible to compute any portion of a convex polyhedron that we want to see and visualize. One of the most important outcomes of Dantzig's work for mathematics is to bring about the possibility of clear visualization to convex polyhedra of dimension ≥ 3.

Ever since his work in 1947, the problem of enumerating all the extreme points of a convex polytope specified by linear constraints has been recognized as an important one.

There are several algorithms for this problem already, most of them are based on enumerating the feasible bases for (1). Public software programs implementing these algorithms are also available. But in all these algorithms, after computing r extreme points, the effort needed to compute the next one grows exponentially with r in the worst case.

An algorithm for this problem is said to be an efficient algorithm, or polynomial time algorithm, if it satisfies the following properties:

1. It should obtain the extreme points of K sequentially one after the other in a list.

2. When the list of known extreme points of K, $\{d^1, \ldots, d^r\}$, contains r extreme points, the effort needed in the algorithm to check whether it contains all the extreme points of K, and if it does not, to generate a new extreme point, should be bounded above by a polynomial in r and the size of system (1).

The goal of developing an efficient algorithm for this problem is more of a mathematical challenge, than a practical one. We will address this mathematical challenge.

3 Relationship Between the Problems of Enumerating Extreme Points, and Facets

Consider a convex polytope P of dimension n in R^n. There are two different ways of representing it algebraically. These are:

1. Convex hull representation: Let $\{x^1, \dots, x^\ell\}$ be the set of all extreme points of P. Then $P \; = \; < x^1, \dots, x^\ell >$, the convex hull of its set of extreme points.

2. Halfspace intersection representation: P can be represented as the intersection of a finite number of halfspaces of R^n.

These two different representations lead to two different enumeration problems associated with P. They are:

Extreme point enumeration problem: When P is given in halfspace intersection representation (i.e., as the set of feasible solutions of a system of linear inequalities), enumerate all the extreme points of P.

Facet enumeration problem: When P is given as the convex hull of its extreme points, enumerate all the facetal inequalities to represent it as the set of feasible solutions of a system of linear inequalities.

In the theory of convex polytopes, these two problems are known as the duals of each other. An algorithm for one of these problems can be used to solve the other problem directly. To show this, we denote by:

Procedure A: Any algorithm for enumerating the extreme points of P when it is specified as the set of feasible solutions of a system of linear inequalities.

Procedure B: Any algorithm for enumerating all the facetal inequalities to represent P when it is given as the convex hull of its extreme points.

How to enumerate all the facetal inequalities to represent P when the set of its extreme points $\{x^1, \dots, x^\ell\}$ is given, using Procedure A

We assume that each x^t is an extreme point of $P \; = \; < x^1, \dots, x^\ell >$ of full dimension n. Let $\bar{x} = (x^1 + \ldots + x^\ell)/\ell$, an interior point of P.

Take the transformation $x = y + \bar{x}$, let $y^t = x^t - \bar{x}$ for $t = 1$ to ℓ. Then $P \; = \; < x^1, \dots, x^\ell > \; = \; < y^1, \dots, y^\ell > + \bar{x}$. Also 0 is an interior point of

$< y^1, \dots, y^\ell >$.

Let $q = (q_1, \dots, q_n)$ be a row vector of variables in R^n. Use Procedure A to generate all the extreme points of the polytope in q-space defined by the inequalities $qy^t \leq 1$, $t = 1$ to ℓ. Let $\{q^1, \dots, q^M\}$ be the set of extreme points of this polytope. Then

$$< y^1, \dots, y^\ell > \; = \; \{y : \quad q^s y \leq 1, \quad s = 1, ..., M\}$$

So

$$P = \; < x^1, \dots, x^\ell > \; = \; \{x : \quad q^s x \leq 1 + q^s \bar{x}, \quad s = 1, ..., M\}$$

with each inequality being a facetal inequality for K.

How to enumerate all the extreme points of $P = \{x : Qx \leq h\}$ using Procedure B

Let Q be of order $m \times n$. Assume that each inequality in the representation of P corresponds to a facet of P, and that the dimension of P is n. Find an interior point $\bar{x}$ of P by using the Low dimension step in Murty & Chung [7] (also discussed in the next section) to generate extreme points of P until the convex hull of generated extreme points is of full dimension. Take the transformation $x = y + \bar{x}$. Then $P = \{y : \quad Qy \leq h - Q\bar{x}\} + \bar{x}$. Also, since $\bar{x}$ is an interior point of P, $h - Q\bar{x} > 0$. For each $i = 1$ to m, define $a^i = Q_{i.}(1/(h_i - Q_{i.}\bar{x}))$ where $Q_{i.}$ is the i-th row vector of Q. Therefore $P = \{y : \quad a^i y \leq 1, \quad i = 1, ..., m\} + \bar{x}$.

Use Procedure B to generate a linear inequality representation of $< a^1, \dots, a^m >$. Suppose it is $\{a : \quad ad^r \leq 1, \quad r = 1, ..., L\}$.

Then $\{d^1, \dots, d^L\}$ is the set of extreme points of $\{y : \quad a^i y \leq 1, \; i = 1, ..., m\}$.

So, the set of extreme points of P is $\{d^1 + \bar{x}, \dots, d^L + \bar{x}\}$.

Here we will study the extreme point enumeration problem.

4 Segments and Their Properties

Murty and Chung [7] introduced the concept of a *segment* of a polytope and used it to enumerate its extreme points and faces. They defined segments of various orders ranging from 1 to (-2 + the dimension of the polytope). Here we will only use segments of order 1, and hence we will refer to segments of order 1 as segments.

Definition 3.1: (Segment): A *segment* of a convex polytope K of dimension n is the convex hull, Ω, of a subset of its extreme points satisfying: **(i)** its dimension is the same as that of K, n, **(ii)** adjacency of extreme points on Ω coincides with that on K (i.e., every edge of Ω is also an edge of K).

In [7] the following two steps, LD and NS-1, have been developed for our problem (these names for the steps are new, they have not been used in [7]).

LD (Low Dimension Step): Let $\Gamma = < d^1, \ldots, d^r >$ be the convex hull of the present list of known extreme points of K. If dimension(Γ) $<$ dimension(K), find the equation for a hyperplane containing Γ, in the space of the variables x_N in which K has full dimension. To do this, for $k = 1$ to r, let (d^k_B, d^k_N) be the partition of the vector d^k according to (x_B, x_N) partition selected above for the variables x.

The dimension of $\{d^1, \ldots, d^r\}$ is the rank of the set of vectors $\{d^k_N - d^1_N : k = 2, \ldots, r\}$.

If this dimension is $< n$, the following homogeneous system of linear equations in variables f_N (written as a row vector)

$$f_N(d^k_N - d^1_N) = 0, \quad k = 2, \ldots, r$$

has a nonzero solution $f_N \neq 0$, find it. Let $\beta = f_N d^1_N$. Then all the extreme points $d^1, \ldots, d^r$ lie on the hyperplane represented by the equation $f_N x_N = \beta$ in the space of the independent variables x_N.

Now solve the two LPs, minimize $f_N x_N$, and maximize $f_N x_N$ subject to (1). One or both of these LPs will have as an optimum extreme point a point not in the current list. Call it d^{r+1}, add it to the list.

NS-1 (Non-Segment-1 Step): Let $\Gamma = < d^1, \ldots, d^r >$ be the convex hull of the present list of known extreme points of K satisfying dimension(Γ) $=$ dimension(K).

Let $p = (p_1, \ldots, p_{n+m})^T, q = (q_1, \ldots, q_{n+m})^T$ be a pair of extreme points of **K**. They are:

adjacent on **K**	iff rank of $\{A_{.j} : j$ such that either p_j or q_j or both are $> 0\}$ is -1 + (its cardinality)
nonadjacent on **K**	iff rank of $\{A_{.j} : j$ such that either p_j or q_j or both are $> 0\}$ is < -1 + (its cardinality).

p, q are adjacent on $\Gamma = < d^1, \ldots, d^r >$ iff the following system in variables $c = (c_1, \ldots, c_{n+m})$ has a feasible solution, which can be checked by solving an LP.

$$\begin{aligned} c(p-q) &= 0 \\ c(p-d^k) &> 0, \quad \text{for all } k \text{ such that } d^k \neq p \text{ or } q. \end{aligned} \tag{3}$$

If a pair of extreme points of Γ; p, q say; which are adjacent on Γ, are not adjacent on K, then from the vector c satisfying (3); we get the supporting hyperplane $cx = cp$ of Γ that contains only extreme points p, q but no other expreme point of Γ, which is not a supporting hyperplane for K.

Now solve the LP: maximize cx over K to get an extreme point optimum solution. That extreme point may be a new extreme point of K not in the present list, but it could also be p itself. In the latter case p, q are the only extreme points of K known at present on the face S of K determined by

$$\begin{aligned} Ax &= b \\ cx &= \gamma \\ x &\geq 0 \end{aligned}$$

Therefor by applying the LD step on S a new extreme point of K on S not in the present list can be determined in this case.

The application of NS-1 can be continued until at some stage the convex hull of the present list becomes a segment of K. To continue the enumeration then, some new steps are needed. One of those is LDF-1 given below.

LDF-1 (Low Dimension Facet Intersection Step): Let $\Gamma = < d^1, \ldots, d^r >$ be the convex hull of the present list of known extreme points of K which is a segment of K. If there is a facet F of K satisfying dimension$(F \cap \Gamma) <$ dimension(F), apply the LD step on F and obtain a new extreme point of K on F and add it to the list.

The application of Steps NS-1, LDF-1 can again be continued until at some stage the conditions in the following definition hold for the extreme points in the present list.

Definition 3.2 (Mukka): A *mukka* of a convex polytope K is the convex hull of a subset of extreme points of K which is a segment of K that has a full dimensional intersection with every facet of K.

When the convex hull of the present list of extreme points of K becomes a mukka of K, to continue the enumeration some new steps are needed. These are discussed in the next section.

5 The Central Problem

Let $\Gamma \ = \ < d^1, \ldots, d^r >$ be the convex hull of the present list of known extreme points of K which is a mukka of K. A facet F of K is said to be

- a **complete facet** of K for mukka Γ if $F \cap \Gamma = F$
- an **incomplete facet** of K for mukka Γ if $F \cap \Gamma \neq F$.

The central problem in enumerating extreme points: Given a mukka Γ of K, develop a criterion for identifying any incomplete facets of K for Γ efficiently.

I have the following conjecture for this central problem.

Conjecture: Let K be the convex polytope defined by (1), and let $\Gamma = < d^1, \ldots, d^r >$ be a mukka of K. Consider a facet of K, say F_1 defined by the FCF x_1. For $j = 2$ to $n+m$, the FCF of the facet F_j of K is x_j, and let

$[\alpha_j, \beta_j]$ be the interval of values of x_j, the FCF of the facet F_j, on F_1 (i.e., $\alpha_j = \min \{x_j : x \in F_1\}$; and $\beta_j = \max \{x_j : x \in F_1\}$),

P_j, Q_j be the optimal faces for $\min \{x_j : x \in F_1\}$; and $\max \{x_j : x \in F_1\}$ respectively,

$[\gamma_j, \delta_j]$ be the interval of values of x_j on $F_1 \cap \Gamma$ (extreme points of $F_1 \cap \Gamma$ are known, so this can be computed in $O(r)$ time),

R_j, S_j be the optimal faces for $\min \{x_j : x \in F_1 \cap \Gamma\}$; and $\max \{x_j : x \in F_1 \cap \Gamma\}$ respectively, these sets are determined in terms of their extreme points.

Then F_1 is an incomplete facet of K for Γ iff

either $[\gamma_j, \delta_j]$ is a proper subset of $[\alpha_j, \beta_j]$ for some $j = 2$ to $n+m$,

or $[\gamma_j, \delta_j] = [\alpha_j, \beta_j]$ for all $j = 2$ to $n+m$, but either dimension$(R_j) <$ dimension(P_j), or dimension$(S_j) <$ dimension(Q_j) for some j.

If this conjecture is true, it provides an efficient criterion for identifying all the incomplete facets of K for mukka Γ. If F_t is an incomplete facet of K for mukka Γ, then of course $F_t \cap \Gamma$ is a segment of F_t but it may not be a mukka. If $F_t \cap \Gamma$ is not a mukka of F_t, then the enumeration can be continued by applying the LDF-1 step on F_t.

If $F_t \cap \Gamma$ is a mukka of F_t, then by applying the conjecture on F_t and continuing in the same way, a face G of F_t satisfying dimension$(G \cap \Gamma) <$ dimension(G) can be identified. Then the enumeration can be continued by applying the LD step on G.

Example: Consider the polytope $P \subset R^4$ defined by the constraints

$$a_i(x) = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 \leq 1, \quad i = 1, \ldots, 15$$

where $\{(a_{i1}, a_{i2}, a_{i3}, a_{i4}) : i = 1 \text{ to } 15\} = \{(1, 1, 1, -1), (1, 1, -1, 1), (1, 1, -1, -1), (1, , -1, 1, 1), (1, -1, 1, -1), (1, -1, -1, 1), (1, -1, -1, -1), (-1, 1, 1, 1), (-1, 1, 1, -1), (-1, 1, -1, 1), (-1, 1, -1, -1), (-1, -1, 1, 1), (-1, -1, 1, -1), (-1, -1, -1, 1), (-1, -1, -1, -1)\}$. Here P is defined by a general system of linear constraints, which is not in standard form.

Let I denote the unit matrix of order 4. It can be verified that $\Omega =$ convex hull of $\{I_{.j}, -I_{.j} : j = 1 \text{ to } 4\}$ is a mukka of P.

Let F_t be the facet of P on the facetal hyperplane defined by $a_t(x) = 1$, for $t = 1$ to 15. $a_t(x)$ is the FTF corresponding to F_t for $t = 1$ to 15.

Consider the problem of maximizing $a_2(x)$, the FTF of F_2, on F_1 and on $F_1 \cap \Omega =< I_{.1}, I_{.2}, I_{.3}, -I_{.4} >$. The optimum objective value is 1 in both problems, but the dimension of the optimum face on F_1 is 2; while on $F_1 \cap \Omega$ it is only 1. This shows that F_1 is an incomplete facet of P for Ω. A similar argument shows that the facets F_4 and F_8 are also incomplete facets of P for Ω.

6 References

[1] A. Y. Alfakih, T. Yi, K. G. Murty, "Facets of an Assignment Problem with a $0-1$ Side Constraint", *Journal of Combinatorial Optimization*, 4(2000)365-388.
[2] D. Avis and K. Fukuda, "A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra", *Discrete and Computational Geometry* 8(1992)295-313.
[3] G. B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, 1963.
[4] B. Grünbaum, *Covex Polytopes*, Wiley Interscience, NY, 1967.
[5] K.G. Murty, "Adjacency on convex polyhedra", *SIAM Review* 13, no. 3(1971)377-386.
[6] K.G. Murty, *Linear Programming* (Wiley, NY, 1983).
[7] K.G. Murty, S. J. Chung, "Segments in Enumerating Facets", *Mathematical Programming* 70(1995)27-45.
[8] J.S. Provan, "Efficient enumeration of the vertices of polyhedra associated with network LPs", *Mathematical Programming* 63(1994)47-64.