

Homework #7

April 11, 2024

Name: Ravi Raghavan

Extension: No

Problem 1. *Voronoi description of halfspace.* Let a and b be distinct points in \mathbb{R}^n . Show that the set of all points that are closer (in Euclidean norm) to a than b , i.e., $\{x : \|x - a\|_2 \leq \|x - b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

Solution. We can see that the following two sets are equivalent:

$$S_1 = S_2 \text{ where } S_1 = \{x : \|x - a\|_2 \leq \|x - b\|_2\} \text{ and } S_2 = \{x : \|x - a\|_2^2 \leq \|x - b\|_2^2\}$$

Let's work with S_2 since it will be a lot easier

$$\begin{aligned} S_2 &= \{x : \|x - a\|_2^2 \leq \|x - b\|_2^2\} \\ S_2 &= \{x : \|x\|_2^2 - 2\langle x, a \rangle + \|a\|_2^2 \leq \|x\|_2^2 - 2\langle x, b \rangle + \|b\|_2^2\} \\ S_2 &= \{x : -2\langle x, a \rangle + \|a\|_2^2 \leq -2\langle x, b \rangle + \|b\|_2^2\} \\ S_2 &= \{x : 2\langle x, b - a \rangle \leq \|b\|_2^2 - \|a\|_2^2\} \\ S_2 &= \{x : 2(b - a)^T x \leq \|b\|_2^2 - \|a\|_2^2\} \\ S_2 &= \{x : (b - a)^T x \leq 0.5(\|b\|_2^2 - \|a\|_2^2)\} \end{aligned}$$

This is a closed half-space

Problem 2. Which of the following sets S are polyhedra? If possible, express S in the form $S = \{x | Ax \preceq b, Fx = g\}$

(b) Yes S is a polyhedra.

Let $M_1 = [a_1, a_2, \dots, a_n] \in \mathbb{R}^{1 \times n}$ and let $M_2 = [a_1^2, a_2^2, \dots, a_n^2] \in \mathbb{R}^{1 \times n}$

Let F be the vertical concatenation of 1^T , M_1 , and M_2 . Let g be $[1, b_1, b_2]^T$

Let $A = -I$

We can express S , via compact notation, as $S = \{x | Ax \preceq 0, Fx = g\}$

(c) S is NOT a Polyhedra

Problem 3. *Hyperbolic sets.* Show that the *hyperbolic set* is $\{x \in \mathbb{R}_+^2 : x_1 x_2 \geq 1\}$ is convex. As a generalization, show that $\{x \in \mathbb{R}_+^n : \prod_{i=1}^n x_i \geq 1\}$ is convex. *Hint.* If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1 - \theta)b$

Solution. Let S be $\{x \in \mathbb{R}_+^n : \prod_{i=1}^n x_i \geq 1\}$. Let's have two vectors j and k that are in S . Let the elements of j be j_1, j_2, \dots, j_n . Let the elements of k be k_1, k_2, \dots, k_n

We want to prove that $\theta j + (1 - \theta)k \in S$.

Since we know that $j \in S$ and $k \in S$, we can state the following:

- $j_1 \geq 0, j_2 \geq 0, \dots, j_n \geq 0$
- $j_1 j_2 j_3 \dots j_n \geq 1$
- $k_1 \geq 0, k_2 \geq 0, \dots, k_n \geq 0$
- $k_1 k_2 k_3 \dots k_n \geq 1$

For $i \in [1, n]$, $j_i, k_i \geq 0$ and $0 \leq \theta \leq 1$, we can see that $0 \leq j_i^\theta k_i^{1-\theta} \leq \theta j_i + (1 - \theta)k_i$.

We can also see that:

$$\prod_{i=1}^n j_i^\theta k_i^{1-\theta} = (\prod_{i=1}^n j_i)^\theta (\prod_{i=1}^n k_i)^{1-\theta}$$

Since $j_1 j_2 j_3 \dots j_n \geq 1$ and $k_1 k_2 k_3 \dots k_n \geq 1$, we can say that:

$$\prod_{i=1}^n j_i^\theta k_i^{1-\theta} = (\prod_{i=1}^n j_i)^\theta (\prod_{i=1}^n k_i)^{1-\theta} \geq 1$$

Since $j_i^\theta k_i^{1-\theta} \leq \theta j_i + (1 - \theta)k_i$,

$$1 \leq \prod_{i=1}^n j_i^\theta k_i^{1-\theta} \leq \prod_{i=1}^n (\theta j_i + (1 - \theta)k_i)$$

We have shown that $\theta j_i + (1 - \theta)k_i \in S$ and that S is a convex set!

Since we have proved the generalized case, we can say that $\{x \in \mathbb{R}_+^2 : x_1 x_2 \geq 1\}$ is convex as well

Problem 4. Problem 2.16:

Show that if S_1 and S_2 are convex sets in \mathbb{R}^{m+n} , then so is their partial sum $S = \{(x, y_1 + y_2) : x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}$

Solution. Let's say that we have two points in S , namely $(x_1, y_{11} + y_{12})$ and $(x_2, y_{21} + y_{22})$. To prove that S is convex, we need to show that $\theta(x_1, y_{11} + y_{12}) + (1 - \theta)(x_2, y_{21} + y_{22})$ is in S .

Based on the definition of S , we can see the following:

- $(x_1, y_{11}) \in S_1$
- $(x_1, y_{12}) \in S_2$
- $(x_2, y_{21}) \in S_1$
- $(x_2, y_{22}) \in S_2$

Since S_1 and S_2 are convex, based on the definition of convex sets, we can see that:

- $\theta(x_1, y_{11}) + (1 - \theta)(x_2, y_{21}) \in S_1$
 $(\theta x_1 + (1 - \theta)x_2, \theta y_{11} + (1 - \theta)y_{21}) \in S_1$
- $\theta(x_1, y_{12}) + (1 - \theta)(x_2, y_{22}) \in S_2$
 $(\theta x_1 + (1 - \theta)x_2, \theta y_{12} + (1 - \theta)y_{22}) \in S_2$

By definition of Set S , we can see that:

$$(\theta x_1 + (1 - \theta)x_2, \theta y_{11} + (1 - \theta)y_{21} + \theta y_{12} + (1 - \theta)y_{22}) \in S$$

$$(\theta x_1 + (1 - \theta)x_2, \theta(y_{11} + y_{12}) + (1 - \theta)(y_{21} + y_{22})) \in S$$

$$\theta(x_1, y_{11} + y_{12}) + (1 - \theta)(x_2, y_{21} + y_{22}) \text{ is in } S.$$

Problem 5 (Problem 2.19(a)). *Linear-fractional functions and convex sets.* Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be the linear-fractional function

$$f(x) = (Ax + b)/(c^T x + d), \quad \text{dom } f = \{x | c^T x + d > 0\} \quad (1)$$

In this problem, we study the inverse image of a convex set C under f , i.e.,

$$f^{-1}(C) = \{x \in \text{dom } f : f(x) \in C\} \quad (2)$$

For each of the following sets $C \subseteq \mathbb{R}^n$, give a simple description of $f^{-1}(C)$

Solution. Let's look at the halfspace $C = \{y : g^T y \leq h\}$ (with $g \neq 0$).

$$f^{-1}(C) = \{x : g^T((Ax + b)/(c^T x + d)) \leq h, c^T x + d > 0\}$$

Since $c^T x + d > 0$

$$f^{-1}(C) = \{x : g^T(Ax + b) \leq h(c^T x + d), c^T x + d > 0\}$$

$$f^{-1}(C) = \{x : g^T Ax + g^T b \leq hc^T x + hd, c^T x + d > 0\}$$

$$f^{-1}(C) = \{x : (g^T A - hc^T)x \leq hd - g^T b, c^T x + d > 0\}$$

$$f^{-1}(C) = \{x : (A^T g - ch^T)^T x \leq hd - g^T b, c^T x + d > 0\}$$

Let's call a new vector $p^T = (A^T g - ch^T)^T$ and $q = hd - g^T b$

$$f^{-1}(C) = \{x : p^T x \leq q, c^T x + d > 0\}$$

We can see that $f^{-1}(C)$ is just the intersection of a halfspace and the domain of f !

Problem 6 (Problem 3.17). Suppose $p < 1, p \neq 0$. Show that the function

$$f(x) = \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \quad (3)$$

with $\text{dom } f = \mathbb{R}_{++}^n$ is concave. This includes as special cases $f(x) = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ and the *harmonic mean* $f(x) = (\sum_{i=1}^n \frac{1}{x_i})^{-1}$. *Hint.* Adapt the proofs for the log-sum-exp function and the geometric mean in 3.1.5

Solution. Gradient: $\frac{\partial f}{\partial x_k} = (\sum_{i=1}^n x_i^p)^{\frac{1}{p}-1} x_k^{p-1}$

Jacobian:

$$\frac{\partial^2 f}{\partial^2 x_k} = (1 - p) \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-2} [x_k^{2p-2} - x_k^{p-2} \sum_{i=1}^n x_i^p]$$

$$\frac{\partial^2 f}{\partial x_k \partial x_l} = (1 - p) \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-2} x_l^{p-1} x_k^{p-1}$$

It is clear that we can express $\nabla^2 f(x)$ as follows:

$$\nabla^2 f(x) = (1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2} [qq^T - (\sum_{i=1}^n x_i^p) \text{diag}(x_1^{p-2}, x_2^{p-2}, \dots, x_n^{p-2})] \text{ where } q_i = x_i^{p-1}$$

For concavity, we need $v^T \nabla^2 f(x) v \leq 0$ to hold true for all $x \in \mathbb{R}_{++}^n$

$$v^T \nabla^2 f(x) v = (1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2} [v^T qq^T v - (\sum_{i=1}^n x_i^p) v^T \text{diag}(x_1^{p-2}, x_2^{p-2}, \dots, x_n^{p-2}) v]$$

$$v^T \nabla^2 f(x) v = (1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2} [(\sum_{i=1}^n q_i v_i)^2 - \sum_{i=1}^n x_i^p (\sum_{i=1}^n x_i^{p-2} v_i)]$$

$$v^T \nabla^2 f(x) v = (1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2} [(\sum_{i=1}^n x_i x_i^{p-2} v_i)^2 - \sum_{i=1}^n x_i^p (\sum_{i=1}^n x_i^{p-2} v_i)]$$

The Cauchy Schwartz Inequality tell us that $(a^T a)(b^T b) \geq (a^T b)^2$

If we set $a_i = x_i^{\frac{p}{2}}$ and we set $b_i = x_i^{\frac{p-2}{2}} v_i$, substituting this into the Cauchy Schwartz inequality allows us to see that

$$(\sum_{i=1}^n x_i^p)(\sum_{i=1}^n x_i^{p-2} v_i) \geq (\sum_{i=1}^n x_i^{p-1} v_i)^2 = (\sum_{i=1}^n q_i v_i)^2$$

By extension, we can see that

$$(\sum_{i=1}^n x_i^p) v^T \text{diag}(x_1^{p-2}, x_2^{p-2}, \dots, x_n^{p-2}) v \geq v^T qq^T v$$

$$[v^T qq^T v - (\sum_{i=1}^n x_i^p) v^T \text{diag}(x_1^{p-2}, x_2^{p-2}, \dots, x_n^{p-2}) v] \leq 0$$

Since $p < 1$ and $x_i \in \mathbb{R}_{++}$, $(1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2}$ is a positive constant. Hence, $v^T \nabla^2 f(x) v \leq 0$ which means that we have proven concavity!

Problem 7. Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following

- (a) $f(X) = \text{tr}(X^{-1})$ is convex on $\text{dom} f = \mathbb{S}_{++}^n$

Solution. To verify convexity, we can consider an arbitrary line given by $X = Z + tV$ where $Z \in \mathbb{S}_{++}^n$ and $V \in \mathbb{S}^n$

$$\begin{aligned} g(t) &= f(Z + tV) = \text{tr}((Z + tV)^{-1}) \\ &= \text{tr}((Z^{\frac{1}{2}}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})Z^{\frac{1}{2}})^{-1}) \\ &= \text{tr}((Z^{-\frac{1}{2}}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})^{-1}Z^{\frac{1}{2}})) \end{aligned}$$

It is a well known fact that $\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$

Hence, we can continue our simplification

$$\begin{aligned} &= \text{tr}((Z^{-\frac{1}{2}}Z^{\frac{1}{2}}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})^{-1})) \\ &= \text{tr}((Z^{-1}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})^{-1})) \end{aligned}$$

Let us represent $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}}$ as an Eigenvalue Decomposition of QDQ^T

$$\begin{aligned} &= \text{tr}((Z^{-1}(I + tQDQ^T)^{-1})) \\ &= \text{tr}((Z^{-1}(QIQ^T + tQDQ^T)^{-1})) \\ &= \text{tr}((Z^{-1}(Q(I + tD)Q^T)^{-1})) \\ &= \text{tr}((Z^{-1}Q(I + tD)^{-1}Q^T)) \end{aligned}$$

Again, we can use the cyclic property of the trace:
 $= \text{tr}((Q^T Z^{-1} Q (I + tD)^{-1}))$

$$= \sum_{i=1}^n (Q^T Z^{-1} Q)_{ii} (1 + t\lambda_i)^{-1}$$

Note: λ_i are eigenvalues of $Z^{-\frac{1}{2}} V Z^{-\frac{1}{2}}$

Since Z is a positive definite, symmetric matrix, we know that Z^{-1} is a positive definite, symmetric matrix as well. Hence, $(Q^T Z^{-1} Q)_{ii}$ is always positive.

Let's now look at $(1 + t\lambda_i)^{-1}$. Let's take the second derivative with respect to t . This second derivative is equal to $\frac{2\lambda_i^2}{(1+t\lambda_i)^3}$. The numerator is clearly positive.

We know that $Z + tV \in \mathbb{S}_{++}^n$

This means that $Z^{\frac{1}{2}} (I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}}) Z^{\frac{1}{2}} \in \mathbb{S}_{++}^n$

This means that $(I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}}) \in \mathbb{S}_{++}^n$ must be True

This means that $1 + t\lambda_i > 0$ must be true for all i !. Hence, the denominator of $\frac{2\lambda_i^2}{(1+t\lambda_i)^3}$ is positive as well.

We have shown that the second derivative of $(1 + t\lambda_i)^{-1}$ is positive over the values of t that are in the domain of g which means that $(1 + t\lambda_i)^{-1}$ is a convex function over the domain of g .

$\sum_{i=1}^n (Q^T Z^{-1} Q)_{ii} (1 + t\lambda_i)^{-1}$ is a non-negative weighted sum of convex functions which is convex!

We have completed the proof!

Problem 8. *Nonnegative weighted sums and integrals*

- (a) Show that $f(x) = \sum_{i=1}^r \alpha_i x_{[i]}$ is a convex function of x , where $\alpha_1 \geq \alpha_2 \geq \dots \alpha_r \geq 0$, and $x_{[i]}$ denotes the i th largest component of x . (You can use the fact that $f(x) = \sum_{i=1}^r x_{[i]}$ is convex on \mathbb{R}^n)

Solution. $f(x) = \alpha_r (\sum_{i=1}^r x_{[i]}) + (\alpha_{r-1} - \alpha_r) (\sum_{i=1}^{r-1} x_{[i]}) + (\alpha_{r-2} - \alpha_{r-1}) (\sum_{i=1}^{r-2} x_{[i]}) + \dots + (\alpha_1 - \alpha_2) (x_{[1]})$

Since $\alpha_i > \alpha_{i+1}$, $\alpha_i - \alpha_{i+1} > 0$.

We know that a non-negative combination of convex functions is convex. Hence, we have completed the proof

- (b) Let $T(x, w)$ denote the trigonometric polynomial

$$T(x, w) = x_1 + x_2 \cos w + x_3 \cos 2w + \dots + x_n \cos (n-1)w \quad (4)$$

Show that the function

$$f(x) = - \int_0^{2\pi} \log T(x, w) dw \quad (5)$$

is convex on $\{x \in \mathbb{R}^n : T(x, w) > 0, 0 \leq w \leq 2\pi\}$

Solution. Let $g(x, w) = -\log T(x, w) = -\log(x_1 + x_2 \cos w + x_3 \cos 2w + \dots + x_n \cos (n-1)w)$

Let's prove that $g(x, w)$ is convex in x when we fix w

$$h(t) = g(z + tv, w) = -\log T(z + tv, w) = -\log(z_1 + tv_1 + (\cos w)(z_2 + tv_2) + (\cos 2w)(z_3 + tv_3) + \cdots + (\cos(n-1)w)(z_n + tv_n))$$

$$h'(t) = -\frac{v_1 + v_2 \cos w + v_3 \cos 2w + \cdots + v_n \cos(n-1)w}{T(z + tv, w)}$$

$$h''(t) = \frac{(v_1 + v_2 \cos w + v_3 \cos 2w + \cdots + v_n \cos(n-1)w)^2}{T(z + tv, w)^2} \geq 0$$

This means that $g(x, w)$ is convex for a fixed w

Hence, $f(x) = \int_0^{2\pi} g(x, w) dw$ is like an infinite non-negative weighted sum. Hence, $f(x)$ is convex!
