ECE 509: Convex Optimization **Rutgers: Spring 2024**

Homework #2

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Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with dom f being closed. Prove that the function is closed.

Solution. Let's analyze any given sublevel set of the function f. Given any constant $\alpha \in \mathbb{R}$, a sublevel set is defined as

$S = \{x \in dom f : f(x) \leq \alpha\}$

Our goal is to prove that any such sub-level set is closed.

Let's say we have a convergent sequence $H = (x_n)_{n \in \mathbb{N}}$ which lies in the sublevel set S and we know that this sequence H converges to c.

Since H lies within S and S is a subset of the domain of f, we know that H lies within the domain of f.

Since the domain of f is a closed set and H lies within the domain of f, we know that c lies within the domain of f.

Since the function is continuous, H lies within the domain of f, and c lies within the domain of f, we know that the sequence $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(c)$.

Since H lies within S, we know that $(f(x_n))_{n \in \mathbb{N}}$ is always $\leq \alpha$.

Limit Property:
Given two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$, if $a_n \leq b_n$ for all n, then $\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n$.

Usage of Limit Property in Proof:
If we consider a_n to be $f(x_n)$ and consider b_n to be a constant sequence where the value is always α, we can see the following:

$f(x_n) \leq \alpha \to \lim_{n \to \infty} f(x_n) \leq \lim_{n \to \infty} \alpha$

$f(x_n) \leq \alpha \to f(c) \leq \alpha$

Since we can say that $f(c) \leq \alpha$ and we know that c is in the domain of f, we know that c is within S.

Hence, since H lies within S and H converges to c and c is in S, we have proved that S is a closed set.

By proving that any such sub-level set is closed, we have showed that the function f is closed!

Problem 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with dom f being open. Prove that the function is closed if and only if for every sequence $x_i \in dom f$ such that $\lim_i x_i = x \in bd\ \ dom f$, we have $\lim_{i\to\infty} f(x_i) = \infty$

Solution. Proof

Step 1: "if" direction
We know that the function f is continuous and the domain f is open. Let's analyze any given sublevel set of the function f. Given any constant $\alpha \in \mathbb{R}$, a sublevel set is defined as $S = \{x \in dom f : f(x) \leq \alpha\}$

Our goal is to prove that any such sub-level set is closed when for every sequence $x_i \in dom f$ such that $\lim_i x_i = x \in bd\ \ dom f$, we have $\lim_{i\to\infty} f(x_i) = \infty$. Let's say we have a convergent sequence $H = (x_n)_{n\in\mathbb{N}}$ which lies in the sublevel set S and we know that this sequence H converges to c.

Since H lies within S and S is a subset of the domain of f, we know that H lies within the domain of f.

Since H lies within the domain of f and the domain of f is an OPEN set, there are usually a few cases for where c can be. We know that limit points of convergent sequences in an open set either lie within the interior of the set or on the boundary of the set. Hence, the first case is that c is on the Boundary of the domain. The second case is that c is within the domain.

However, if we look at the first case, this fact jumps out at us: for every sequence $x_i \in dom f$ such that $\lim_i x_i = x \in bd\ \ dom f$, we have $\lim_{i\to\infty} f(x_i) = \infty$. This would mean that H CANNOT possibly be within S since we need $f(x_i) \leq \alpha$ to ALWAYS be true. This would then violate our assumption at the beginning that H lies in the sublevel set S. Hence, we know that only the second case must be true which means that c is in the domain of f.

<u>Further Analysis for why c CANNOT BE ON THE BOUNDARY OF THE DOMAIN f</u>
We know that we have a convergent sequence $H = (x_n)_{n\in\mathbb{N}}$ which lies in the sublevel set S and we know that this sequence H converges to c. Let's look at this closer. Given two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, if $a_n \leq b_n$ for all n, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$. Since H lies within the domain of f, we know that we can have $f(x_n)_{n\in\mathbb{N}}$ corresponding to each element of H.

If we consider a_n to be $f(x_n)$ and consider b_n to be a constant sequence where the value is always α, we can see the following:

$f(x_n) \leq \alpha \to \lim_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} \alpha$

$f(x_n) \leq \alpha \to \lim_{n\to\infty} f(x_n) \leq \alpha$

Since H lies within the sublevel set S, we know that $\lim_{n\to\infty} f(x_n) \leq \alpha$. Hence, this would mean that it is IMPOSSIBLE for $\lim_{n\to\infty} f(x_n) = \infty$ to be true.

Case: c lies within the domain of f
Since the function is continuous, H lies within the domain of f, and c lies within the domain of f, we know that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.

Since H lies within S, we know that $(f(x_n))_{n\in\mathbb{N}}$ is always $\leq \alpha$.

Limit Property:
Given two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, if $a_n \leq b_n$ for all n, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Usage of Limit Property in Proof:

If we consider a_n to be $f(x_n)$ and consider b_n to be a constant sequence where the value is always α, we can see the following:

$f(x_n) \leq \alpha \rightarrow \lim_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} \alpha$

$f(x_n) \leq \alpha \rightarrow f(c) \leq \alpha$

Since we can say that $f(c) \leq \alpha$ and we know that c is in the domain of f, we know that c is within S.

Hence, since H lies within S and H converges to c and c is in S, we have proved that S is a closed set.

By proving that any such sub-level set is closed, we have showed that the function f is closed!

Step 2: "only if" direction
We know that the function f is continuous and closed and the domain f is open. Let's analyze any given sublevel set of the function f. Given any constant $\alpha \in \mathbb{R}$, a sublevel set is defined as $S = \{x \in dom f : f(x) \leq \alpha\}$. Due to the fact that the function f is closed, we know that any such sublevel set is CLOSED. The definition of a closed set is that the limit point of every CONVERGENT sequence is within that set.

Let us have a sequence. We can describe this sequence as $x_i \in dom f$ such that $\lim_i x_i = x \in bd\ dom f$. We know that the sequence is contained within the domain of f, f is continuous along its domain, and f is closed. Hence, based on the general definition of limits, we can see that there are several cases for what we can observe for $\lim_{i\to\infty} f(x_i)$. This limit can have a finite value, the limit can have an infinite value(i.e. $f(x_i)$ is unbounded and, thus, diverges), or the limit may not even exist(i.e. sequence $f(x_i)$ is bounded but does NOT have a limit. This, for example, happens with oscillating functions). To prove that this limit must have an infinite value, I will show that it CANNOT have a finite value and I will show that, for the case where the limit does not exist, it CANNOT occur.

Case: The Limit is a finite value(i.e. $\lim_{i\to\infty} f(x_i) = c$ where $c \in \mathbb{R}$) .

Since our sequence is defined as $x_i \in dom f$ such that $\lim_i x_i = x \in bd\ dom f$, we know that $f(x_i)$ is defined at every point along the sequence. Hence, in this case, the sequence $f(x_i)$ would be a CONVERGENT SEQUENCE.

Given the case where $\lim_{i\to\infty} f(x_i) = c$ where $c \in \mathbb{R}$, the sequence($x_i \in dom f$ such that $\lim_i x_i = x \in bd\ dom f$), based on the definition of a sublevel set, would be in the sublevel set S_M defined as such: $S_M = \{x \in dom f : f(x) \leq M\}$.

The reason is that convergent sequences are bounded. So, if we take M to be the "maximum" value that can be attained in the sequence $f(x_i)$, we can say that the entire sequence($x_i \in dom f$ such that $\lim_i x_i = x \in bd\ dom f$) will be in the sublevel set $S_M = \{x \in dom f : f(x) \leq M\}$.

In this case, the limit point of the sequence($x_i \in dom f$ such that $\lim_i x_i = x \in bd\ dom f$) would NOT be in S since the domain of f is open. This would mean that S is an open set which CONTRADICTS the fact that we know that S HAS to be closed.

Case: $f(x_i)$ is bounded but the limit does NOT exist. In other words, the Limit of the sequence $f(x_i)$ does

NOT exist but $f(x_i)$ is bounded.
If the limit of the sequence $f(x_i)$ does NOT exist and $f(x_i)$ is bounded, we know that the sequence $f(x_i)$ oscillates between several values. In this case, we can say that the sequence $f(x_i)$ is bounded by some M where $M \in \mathbb{R}$.

So, if we take M to be the "maximum" value that can be attained in the sequence $f(x_i)$, we can say that the entire sequence($x_i \in dom f$ such that $\lim_i x_i = x \in bd\ \ dom f$) will be in the sublevel set $S_M = \{x \in dom f : f(x) \leq M\}$. Hence, in this case, we can see that the sequence($x_i \in dom f$ such that $\lim_i x_i = x \in bd\ \ dom f$), based on the definition of a sublevel set, would be in S_M.

In this case, the limit point of the sequence($x_i \in dom f$ such that $\lim_i x_i = x \in bd\ \ dom f$) would NOT be in S_M since the domain of f is open. This would mean that S is an open set which CONTRADICTS the fact that we know that S HAS to be closed.

We have shown that $\lim_{i\to\infty} f(x_i) = \infty$

This completes the proof!

Problem 3. Let $C \in \mathbb{R}^n$ be a convex set, with $x_1, ..., x_k \in C$, and let $\theta_1, ..., \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$. (The definition of convexity is that this holds for k = 2; you must show it for arbitrary k.)

Solution. Proof by Induction

Base Case: (k = 2)
This is true by definition of convex set

Inductive Step:
Let's say this holds true for k. We must show that this holds true for $k+1$

We know that $\theta_1 x_1 + ... + \theta_k x_k \in C$ given that $\theta_i \geq 0$, $\theta_1, ..., \theta_k \in \mathbb{R}$, and $\theta_1 + ... + \theta_k = 1$

Let's introduce $\theta_{k+1} \in \mathbb{R}$ such that $\theta_i \geq 0$ and $\theta_i \in \mathbb{R}$, for $1 \leq i \leq k+1$, and let's also say that $\theta_1 + ... + \theta_k + \theta_{k+1} = 1$. It is evident to see that $\theta_i \leq 1$, for $1 \leq i \leq k+1$, must be true as well.

We want to prove that $\theta_1 x_1 + ... + \theta_k x_k + \theta_{k+1} x_{k+1} \in C$

Let's do some re-writing of mathematical expressions!

$\theta_1 x_1 + ... + \theta_k x_k + \theta_{k+1} x_{k+1}$
$(\theta_1 x_1 + ... + \theta_k x_k) + \theta_{k+1} x_{k+1}$

$(1 - \theta_{k+1})(\frac{\theta_1}{1-\theta_{k+1}} x_1 + ... + \frac{\theta_k}{1-\theta_{k+1}} x_k) + \theta_{k+1} x_{k+1}$

Since $\theta_1 + ... + \theta_k + \theta_{k+1} = 1$, we know that $\theta_1 + ... + \theta_k = 1 - \theta_{k+1}$. Hence, $\frac{\theta_1}{1-\theta_{k+1}} + ... + \frac{\theta_k}{1-\theta_{k+1}} = 1$

Since $\theta_{k+1} \in \mathbb{R}$ and $\theta_{k+1} \geq 0$, we can see that $1 - \theta_{k+1} \leq 1$ and $1 - \theta_{k+1} \in \mathbb{R}$. Since $\theta_{k+1} \leq 1$, $1 - \theta_{k+1} \geq 0$

Since $\theta_i \geq 0$, $1 - \theta_{k+1} \geq 0$, and $\theta_i \in \mathbb{R}$, for $1 \leq i \leq k+1$, we can see that $\frac{\theta_i}{1-\theta_{k+1}} \geq 0$ and $\frac{\theta_i}{1-\theta_{k+1}} \in \mathbb{R}$

Since, $\frac{\theta_1}{1-\theta_{k+1}} + ... + \frac{\theta_k}{1-\theta_{k+1}} = 1$, $\frac{\theta_i}{1-\theta_{k+1}} \geq 0$, and $\frac{\theta_i}{1-\theta_{k+1}} \in \mathbb{R}$ for $1 \leq i \leq k+1$, we can say that $\frac{\theta_1}{1-\theta_{k+1}}x_1 + ... + \frac{\theta_k}{1-\theta_{k+1}}x_k \in C$

Now, let's represent $\frac{\theta_1}{1-\theta_{k+1}}x_1 + ... + \frac{\theta_k}{1-\theta_{k+1}}x_k = x_p$ where, as we have already proven, $x_p \in C$

Our goal is to now prove that $(1 - \theta_{k+1})(x_p) + \theta_{k+1}x_{k+1}$. However, based on the base case(k = 2) which is true by the definition of convexity, we know that this must be true!

This completes the proof that $\theta_1 x_1 + ... + \theta_k x_k + \theta_{k+1}x_{k+1} \in C$

Problem 4. Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

Solution. Proof
Step 1: Show that a set is convex if and only if its intersection with any line is convex

- Part 1: The "If" Part
  We know that the intersection of the set and any line is convex. Our goal is to show that the set itself is convex.

  Let us start with a set C. Let us also denote an arbitrary line as L. Let us denote two distinct points m and n that are in $C \cap L$. Since we know that $C \cap L$ is convex, we can state the following:

  $\alpha m + (1 - \alpha)n \in C \cap L$ where $\alpha \in [0, 1]$

  This leads us to the following:

  $\alpha m + (1 - \alpha)n \in C$ where $\alpha \in [0, 1]$
  $\alpha m + (1 - \alpha)n \in L$ where $\alpha \in [0, 1]$

  Hence, by definition, C is convex as well!

- Part 2: The "Only If" Part
  We know that the set is convex. Our goal is to prove that the intersection of the set and any line is convex as well.

  Let us start with a convex set C. Let us also denote an arbitrary line as L.

  Let us denote two distinct points m and n that are in $C \cap L$. For $C \cap L$ to be convex, we need the following to be true:

$\alpha m + (1-\alpha)n \in C \cap L$ where $\alpha \in [0,1]$

Since C is convex, we know that $\alpha m + (1-\alpha)n \in C$ where $\alpha \in [0,1]$

Since m and n are in $C \cap L$, they must be points on L. We can define L as $tm + (1-t)n$ where $t \in \mathbb{R}$. It is easy to see that $\alpha m + (1-\alpha)n \in L$ where $\alpha \in [0,1]$

Since $\alpha m + (1-\alpha)n \in L$ where $\alpha \in [0,1]$ AND $\alpha m + (1-\alpha)n \in C$ where $\alpha \in [0,1]$, we can conclude by saying that:
$\alpha m + (1-\alpha)n \in C \cap L$ where $\alpha \in [0,1]$

Hence, $C \cap L$ is convex as well!

Step 2: Show that a set is affine if and only if its intersection with any line is affine.

- Part 1: The "If" Part
  We know that the intersection of the set and any line is affine. Our goal is to show that the set itself is affine. Let us start with a set C. Let us also denote an arbitrary line as L. Let us denote two distinct points m and n that are in $C \cap L$. Since we know that $C \cap L$ is affine, we can state the following:

  $\alpha m + (1-\alpha)n \in C \cap L$ where $\alpha \in \mathbb{R}$

  This leads us to the following:

  $\alpha m + (1-\alpha)n \in C$ where $\alpha \in \mathbb{R}$
  $\alpha m + (1-\alpha)n \in L$ where $\alpha \in \mathbb{R}$

  Hence, by definition, C is affine as well!

- Part 2: The "Only If" Part
  We know that the set is affine. Our goal is to prove that the intersection of the set and any line is affine as well. Let us start with an affine set C. Let us also denote an arbitrary line as L.

  Let us denote two distinct points m and n that are in $C \cap L$. For $C \cap L$ to be affine, we need the following to be true:

  $\alpha m + (1-\alpha)n \in C \cap L$ where $\alpha \in \mathbb{R}$

  Since C is affine, we know that
  $\alpha m + (1-\alpha)n \in C$ where $\alpha \in \mathbb{R}$

  Since m and n are in $C \cap L$, they must be points on L. We can define L as $tm + (1-t)n$ where $t \in \mathbb{R}$. It is easy to see that $\alpha m + (1-\alpha)n \in L$ where $\alpha \in \mathbb{R}$

  Since $\alpha m + (1-\alpha)n \in L$ where $\alpha \in \mathbb{R}$ AND $\alpha m + (1-\alpha)n \in C$ where $\alpha \in \mathbb{R}$, we can conclude by saying that:
  $\alpha m + (1-\alpha)n \in C \cap L$ where $\alpha \in \mathbb{R}$

  Hence, $C \cap L$ is affine as well!