**ECE 509: Convex Optimization** **Rutgers: Spring 2024**

# Homework #1

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*Name: Ravi Raghavan* *Extension: No*

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**Problem 1.** *Device sizing.* In a device sizing problem the goal is to minimize power consumption subject to the total area not exceeding 50, as well as some timing and manufacturing constraints. Four candidate designs meet the timing and manufacturing constraints, and have power and area listed in the table below.

| Design | Power | Area |
|---|---|---|
| A | 10 | 50 |
| B | 8 | 55 |
| C | 10 | 45 |
| D | 11 | 50 |

Table 1: Table

**Solution.** (a): **False**
(b): **False.** We can clearly see that Design C and Design A are both feasible designs due to the fact that the total areas for both designs don't exceed 50. In this optimization problem, the "objective function" is the power consumption value. As the table depicts, Design C and Design A achieve the same objective value(i.e. Power is 10 in both designs). Hence, since Design C and A are both feasible and both achieve the same objective value, Design C and Design A are equally as good.
(c): **True**

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**Problem 2.** *Some famous inequalities.* The Cauchy-Schwarz states that

$$|a^T b| \leq \|a\|_2 \|b\|_2$$

for all vectors $a, b \in \mathbb{R}^n$

(a) Prove the Cauchy-Schwartz Inequality
*Hint:* A simple proof is as follows. With a and b fixed, consider the function $g(t) = \|a + tb\|_2^2$ of the scalar variable $t$. This function is nonnegative for all t. Find an expression for $inf_t g(t)$ (the minimum value of g), and show that the Cauchy-Schwarz inequality follows from the fact that $inf_t g(t) \geq 0$.

**Solution.** Let us fix two vectors, $a$ and $b$ which are both in $\mathbb{R}^n$. We will now consider the function $g(t) = \|a + tb\|_2^2$

$g(t) = \langle (a + tb), (a + tb) \rangle$

$g(t) = \langle a, (a + tb) \rangle + \langle tb, (a + tb) \rangle$

$g(t) = \langle a, a \rangle + \langle a, tb \rangle + \langle tb, a \rangle + \langle tb, tb \rangle$

$g(t) = \|a\|_2^2 + \langle a, tb\rangle + \langle tb, a\rangle + \|tb\|_2^2$

$g(t) = \|a\|_2^2 + \langle a, tb\rangle + \langle tb, a\rangle + t^2\|b\|_2^2$

$g(t) = \|a\|_2^2 + t\langle a, b\rangle + t\langle b, a\rangle + t^2\|b\|_2^2$

$g(t) = \|a\|_2^2 + t\langle a, b\rangle + t\langle a, b\rangle + t^2\|b\|_2^2$

$g(t) = \|a\|_2^2 + 2t\langle a, b\rangle + t^2\|b\|_2^2$

Conveniently, $g(t)$ is a function of one variable. So taking first and second order derviatives should be straightforward: $g'(t) = 2\langle a, b\rangle + 2t\|b\|_2^2$ and $g''(t) = 2\|b\|_2^2$
Since the $L_2$ norm of a vector is always non-negative, we know that $2\|b\|_2^2 \geq 0$ and $g''(t) \geq 0$ at all times. Since $g''(t) \geq 0$ at all times, we know that the graph of $g(t)$ is a quadratic function that opens upwards. It is clear to see that solving for $g'(t) = 0$ gives us the minimum of $g(t)$.

$g'(t) = 2\langle a, b\rangle + 2t\|b\|_2^2 = 0$
$t = \frac{-\langle a,b\rangle}{\|b\|_2^2}$

Substitute this value of $t$ into $g(t)$ to solve for the minimum value of $g(t)$

$g(\frac{-\langle a,b\rangle}{\|b\|_2^2}) = \|a\|_2^2 + 2(\frac{-\langle a,b\rangle}{\|b\|_2^2})\langle a, b\rangle + (\frac{-\langle a,b\rangle}{\|b\|_2^2})^2\|b\|_2^2$

$g(\frac{-\langle a,b\rangle}{\|b\|_2^2}) = \|a\|_2^2 - 2\frac{(\langle a,b\rangle)^2}{\|b\|_2^2} + (\frac{-\langle a,b\rangle}{\|b\|_2^2})^2\|b\|_2^2$

$g(\frac{-\langle a,b\rangle}{\|b\|_2^2}) = \|a\|_2^2 - 2\frac{(\langle a,b\rangle)^2}{\|b\|_2^2} + \frac{(\langle a,b\rangle)^2}{\|b\|_2^2}$

$g(\frac{-\langle a,b\rangle}{\|b\|_2^2}) = \|a\|_2^2 - \frac{(\langle a,b\rangle)^2}{\|b\|_2^2}$

$g(\frac{-\langle a,b\rangle}{\|b\|_2^2}) = \frac{\|a\|_2^2\|b\|_2^2 - (\langle a,b\rangle)^2}{\|b\|_2^2}$

Since $g(t) = \|a + tb\|_2^2$, we know that the minimum of $g(t) \geq 0$. Hence, we can say that

$\frac{\|a\|_2^2\|b\|_2^2 - (\langle a,b\rangle)^2}{\|b\|_2^2} \geq 0$

$\|a\|_2^2\|b\|_2^2 - (\langle a, b\rangle)^2 \geq 0$

$\|a\|_2^2\|b\|_2^2 \geq (\langle a, b\rangle)^2$

$(\langle a, b\rangle)^2 \leq \|a\|_2^2\|b\|_2^2$

Take the square root of both sides: $|\langle a, b\rangle| \leq \|a\|_2\|b\|_2$

Using the fact that $\langle a, b\rangle = a^T b$

$|a^T b| \leq \|a\|_2\|b\|_2$

We have completed the proof!

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(b) The 1-norm of a vector $x$ is defined as $\|x\|_1 = \sum_{k=1}^{n} |x_k|$. Use the Cauchy Schwarz inequality to show that

$$\|x\|_1 \leq \sqrt{n}\|x\|_2$$

for all x

**Solution.** We already have $x = [x_0, x_1, x_2, ..., x_{n-1}]^T \in \mathbb{R}^n$

Let us define $a = [1, 1, 1, ......, 1]^T \in \mathbb{R}^n$ and $b = [|x_0|, |x_1|, |x_2|, ..., |x_{n-1}|]^T \in \mathbb{R}^n$

According to the Cauchy-Schwartz Inequality
$|a^T b| \leq \|a\|_2\|b\|_2$

This can be simplified as we know that $\|a\|_2 = \sqrt{n}$
$|a^T b| \leq \sqrt{n}\|b\|_2$

We can conduct another simplification as we know that $a^T b = \langle a, b\rangle = |x_0| + |x_1| + |x_2| + .... + |x_{n-1}|$
$||x_0| + |x_1| + |x_2| + .... + |x_{n-1}|| \leq \sqrt{n}\|b\|_2$

Since absolute values are always $\geq 0$, we know that the sum of absolute values is always $\geq 0$ as well. Hence, we can further simplify
$|x_0| + |x_1| + |x_2| + .... + |x_{n-1}| \leq \sqrt{n}\|b\|_2$

Since $\|x\|_1 = \sum_{k=1}^{n} |x_k|$, we can further simplify:

$\|x\|_1 \leq \sqrt{n}\|b\|_2$

$\|b\|_2 = \sqrt{(|x_0|^2 + |x_1|^2 + .... + |x_{n-1}|^2)}$

$\|b\|_2 = \sqrt{(x_0^2 + x_1^2 + .... + x_{n-1}^2)}$

It is clear to see that $\|x\|_2 = \|b\|_2$. Since we already know that $\|x\|_1 \leq \sqrt{n}\|b\|_2$, we can further state that:

$\|x\|_1 \leq \sqrt{n}\|x\|_2$

We have completed the proof!

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(c) The *harmonic mean* of a positive vector $x \in \mathbb{R}^n_{++}$ is defined as

$$(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k})^{-1}$$

Use the Cauchy-Schwarz inequality to show that the arithmetic mean $\frac{(\sum_k x_k)}{n}$ of a positive n-vector is greater than or equal to its harmonic mean.

**Solution.** We already have $x = [x_0, x_1, x_2, ..., x_{n-1}]^T \in \mathbb{R}^n$

Let us define $a = [\sqrt{x_0}, \sqrt{x_1}, ..., \sqrt{x_{n-1}}]^T$ and $b = [\frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{x_1}}, ..., \frac{1}{\sqrt{x_{n-1}}}]^T$

By Cauchy-Schwarz Inequality, we know that:
$|a^T b| \leq \|a\|_2 \|b\|_2$

Substituting the values of $a$ and $b$ gives us
$|n| \leq \sqrt{(x_0 + x_1 + ... + x_{n-1})}\sqrt{(\frac{1}{x_0} + \frac{1}{x_1} + ... + \frac{1}{x_{n-1}})}$

Square both sides

$n^2 \leq (x_0 + x_1 + ... + x_{n-1})(\frac{1}{x_0} + \frac{1}{x_1} + ... + \frac{1}{x_{n-1}})$

Divide both sides by $(\frac{1}{x_0} + \frac{1}{x_1} + ... + \frac{1}{x_{n-1}})$

$n^2((\frac{1}{x_0} + \frac{1}{x_1} + ... + \frac{1}{x_{n-1}}))^{-1} \leq (x_0 + x_1 + ... + x_{n-1})$

Divide both sides by $n$
$n((\frac{1}{x_0} + \frac{1}{x_1} + ... + \frac{1}{x_{n-1}}))^{-1} \leq \frac{1}{n}(x_0 + x_1 + ... + x_{n-1})$

$(\frac{1}{n}(\frac{1}{x_0} + \frac{1}{x_1} + ... + \frac{1}{x_{n-1}}))^{-1} \leq \frac{1}{n}(x_0 + x_1 + ... + x_{n-1})$

$(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k})^{-1} \leq \frac{(\sum_k x_k)}{n}$

$\frac{(\sum_k x_k)}{n} \geq (\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k})^{-1}$

We have completed the proof