

## Homework #7

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Extension: No

**Problem 1.** *Voronoi description of halfspace.* Let  $a$  and  $b$  be distinct points in  $\mathbb{R}^n$ . Show that the set of all points that are closer (in Euclidean norm) to  $a$  than  $b$ , i.e.,  $\{x : \|x - a\|_2 \leq \|x - b\|_2\}$ , is a halfspace. Describe it explicitly as an inequality of the form  $c^T x \leq d$ . Draw a picture.

**Solution.** We can see that the following two sets are equivalent:

$$S_1 = S_2 \text{ where } S_1 = \{x : \|x - a\|_2 \leq \|x - b\|_2\} \text{ and } S_2 = \{x : \|x - a\|_2^2 \leq \|x - b\|_2^2\}$$

Let's work with  $S_2$  since it will be a lot easier

$$\begin{aligned} S_2 &= \{x : \|x - a\|_2^2 \leq \|x - b\|_2^2\} \\ S_2 &= \{x : \|x\|_2^2 - 2 \langle x, a \rangle + \|a\|_2^2 \leq \|x\|_2^2 - 2 \langle x, b \rangle + \|b\|_2^2\} \\ S_2 &= \{x : -2 \langle x, a \rangle + \|a\|_2^2 \leq -2 \langle x, b \rangle + \|b\|_2^2\} \\ S_2 &= \{x : 2 \langle x, b - a \rangle \leq \|b\|_2^2 - \|a\|_2^2\} \\ S_2 &= \{x : 2(b - a)^T x \leq \|b\|_2^2 - \|a\|_2^2\} \\ S_2 &= \{x : (b - a)^T x \leq 0.5(\|b\|_2^2 - \|a\|_2^2)\} \end{aligned}$$

This is a closed half-space

**Problem 2.** Which of the following sets  $S$  are polyhedra? If possible, express  $S$  in the form  $S = \{x | Ax \preceq b, Fx = g\}$

(b) Yes  $S$  is a polyhedra.

Let  $M_1 = [a_1, a_2, \dots, a_n] \in \mathbb{R}^{1 \times n}$  and let  $M_2 = [a_1^2, a_2^2, \dots, a_n^2] \in \mathbb{R}^{1 \times n}$

Let  $F$  be the vertical concatenation of  $1^T$ ,  $M_1$ , and  $M_2$ . Let  $g$  be  $[1, b_1, b_2]^T$

Let  $A = -I$

We can express  $S$ , via compact notation, as  $S = \{x | Ax \preceq 0, Fx = g\}$

(c)  $S$  is NOT a Polyhedra

**Problem 3.** *Hyperbolic sets.* Show that the *hyperbolic set* is  $\{x \in \mathbb{R}_+^2 : x_1 x_2 \geq 1\}$  is convex. As a generalization, show that  $\{x \in \mathbb{R}_+^n : \prod_{i=1}^n x_i \geq 1\}$  is convex. *Hint.* If  $a, b \geq 0$  and  $0 \leq \theta \leq 1$ , then  $a^\theta b^{1-\theta} \leq \theta a + (1 - \theta)b$

**Solution.** Let  $S$  be  $\{x \in \mathbb{R}_+^n : \prod_{i=1}^n x_i \geq 1\}$ . Let's have two vectors  $j$  and  $k$  that are in  $S$ . Let the elements of  $j$  be  $j_1, j_2, \dots, j_n$ . Let the elements of  $k$  be  $k_1, k_2, \dots, k_n$

We want to prove that  $\theta j + (1 - \theta)k \in S$ .

Since we know that  $j \in S$  and  $k \in S$ , we can state the following:

- $j_1 \geq 0, j_2 \geq 0, \dots, j_n \geq 0$
- $j_1 j_2 j_3 \dots j_n \geq 1$
- $k_1 \geq 0, k_2 \geq 0, \dots, k_n \geq 0$
- $k_1 k_2 k_3 \dots k_n \geq 1$

For  $i \in [1, n]$ ,  $j_i, k_i \geq 0$  and  $0 \leq \theta \leq 1$ , we can see that  $0 \leq j_i^\theta k_i^{1-\theta} \leq \theta j_i + (1 - \theta)k_i$ .

We can also see that:

$$\prod_{i=1}^n j_i^\theta k_i^{1-\theta} = (\prod_{i=1}^n j_i)^\theta (\prod_{i=1}^n k_i)^{1-\theta}$$

Since  $j_1 j_2 j_3 \dots j_n \geq 1$  and  $k_1 k_2 k_3 \dots k_n \geq 1$ , we can say that:

$$\prod_{i=1}^n j_i^\theta k_i^{1-\theta} = (\prod_{i=1}^n j_i)^\theta (\prod_{i=1}^n k_i)^{1-\theta} \geq 1$$

Since  $j_i^\theta k_i^{1-\theta} \leq \theta j_i + (1 - \theta)k_i$ ,

$$1 \leq \prod_{i=1}^n j_i^\theta k_i^{1-\theta} \leq \prod_{i=1}^n (\theta j_i + (1 - \theta)k_i)$$

We have shown that  $\theta j_i + (1 - \theta)k_i \in S$  and that  $S$  is a convex set!

Since we have proved the generalized case, we can say that  $\{x \in \mathbb{R}_+^2 : x_1 x_2 \geq 1\}$  is convex as well

**Problem 4.** Problem 2.16:

Show that if  $S_1$  and  $S_2$  are convex sets in  $\mathbb{R}^{m+n}$ , then so is their partial sum  $S = \{(x, y_1 + y_2) : x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}$

**Solution.** Let's say that we have two points in  $S$ , namely  $(x_1, y_{11} + y_{12})$  and  $(x_2, y_{21} + y_{22})$ . To prove that  $S$  is convex, we need to show that  $\theta(x_1, y_{11} + y_{12}) + (1 - \theta)(x_2, y_{21} + y_{22})$  is in  $S$ .

Based on the definition of  $S$ , we can see the following:

- $(x_1, y_{11}) \in S_1$
- $(x_1, y_{12}) \in S_2$
- $(x_2, y_{21}) \in S_1$
- $(x_2, y_{22}) \in S_2$

Since  $S_1$  and  $S_2$  are convex, based on the definition of convex sets, we can see that:

- $\theta(x_1, y_{11}) + (1 - \theta)(x_2, y_{21}) \in S_1$   
 $(\theta x_1 + (1 - \theta)x_2, \theta y_{11} + (1 - \theta)y_{21}) \in S_1$
- $\theta(x_1, y_{12}) + (1 - \theta)(x_2, y_{22}) \in S_2$   
 $(\theta x_1 + (1 - \theta)x_2, \theta y_{12} + (1 - \theta)y_{22}) \in S_2$

By definition of Set  $S$ , we can see that:

$$(\theta x_1 + (1 - \theta)x_2, \theta y_{11} + (1 - \theta)y_{21} + \theta y_{12} + (1 - \theta)y_{22}) \in S$$

$$(\theta x_1 + (1 - \theta)x_2, \theta(y_{11} + y_{12}) + (1 - \theta)(y_{21} + y_{22})) \in S$$

$$\theta(x_1, y_{11} + y_{12}) + (1 - \theta)(x_2, y_{21} + y_{22}) \text{ is in } S.$$

**Problem 5** (Problem 2.19(a)). *Linear-fractional functions and convex sets.* Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be the linear-fractional function

$$f(x) = (Ax + b)/(c^T x + d), \quad \text{dom } f = \{x | c^T x + d > 0\} \quad (1)$$

In this problem, we study the inverse image of a convex set  $C$  under  $f$ , i.e.,

$$f^{-1}(C) = \{x \in \text{dom } f : f(x) \in C\} \quad (2)$$

For each of the following sets  $C \subseteq \mathbb{R}^n$ , give a simple description of  $f^{-1}(C)$

**Solution.** Let's look at the halfspace  $C = \{y : g^T y \leq h\}$  (with  $g \neq 0$ ).

$$f^{-1}(C) = \{x : g^T((Ax + b)/(c^T x + d)) \leq h, c^T x + d > 0\}$$

Since  $c^T x + d > 0$

$$f^{-1}(C) = \{x : g^T(Ax + b) \leq h(c^T x + d), c^T x + d > 0\}$$

$$f^{-1}(C) = \{x : g^T Ax + g^T b \leq hc^T x + hd, c^T x + d > 0\}$$

$$f^{-1}(C) = \{x : (g^T A - hc^T)x \leq hd - g^T b, c^T x + d > 0\}$$

$$f^{-1}(C) = \{x : (A^T g - ch^T)^T x \leq hd - g^T b, c^T x + d > 0\}$$

Let's call a new vector  $p^T = (A^T g - ch^T)^T$  and  $q = hd - g^T b$

$$f^{-1}(C) = \{x : p^T x \leq q, c^T x + d > 0\}$$

We can see that  $f^{-1}(C)$  is just the intersection of a halfspace and the domain of  $f$ !

**Problem 6** (Problem 3.17). Suppose  $p < 1, p \neq 0$ . Show that the function

$$f(x) = \left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \quad (3)$$

with  $\text{dom } f = \mathbb{R}_{++}^n$  is concave. This includes as special cases  $f(x) = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$  and the *harmonic mean*  $f(x) = (\sum_{i=1}^n \frac{1}{x_i})^{-1}$ . *Hint.* Adapt the proofs for the log-sum-exp function and the geometric mean in 3.1.5

**Solution.** Gradient:  $\frac{\partial f}{\partial x_k} = (\sum_{i=1}^n x_i^p)^{\frac{1}{p}-1} x_k^{p-1}$

Jacobian:

$$\frac{\partial^2 f}{\partial^2 x_k} = (1 - p) \left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-2} [x_k^{2p-2} - x_k^{p-2} \sum_{i=1}^n x_i^p]$$

$$\frac{\partial^2 f}{\partial x_k \partial x_l} = (1 - p) \left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}-2} x_l^{p-1} x_k^{p-1}$$

It is clear that we can express  $\nabla^2 f(x)$  as follows:

$$\nabla^2 f(x) = (1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2} [qq^T - (\sum_{i=1}^n x_i^p) \text{diag}(x_1^{p-2}, x_2^{p-2}, \dots, x_n^{p-2})] \text{ where } q_i = x_i^{p-1}$$

For concavity, we need  $v^T \nabla^2 f(x) v \leq 0$  to hold true for all  $x \in \mathbb{R}_{++}^n$

$$v^T \nabla^2 f(x) v = (1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2} [v^T qq^T v - (\sum_{i=1}^n x_i^p) v^T \text{diag}(x_1^{p-2}, x_2^{p-2}, \dots, x_n^{p-2}) v]$$

$$v^T \nabla^2 f(x) v = (1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2} [(\sum_{i=1}^n q_i v_i)^2 - \sum_{i=1}^n x_i^p (\sum_{i=1}^n x_i^{p-2} v_i)]$$

$$v^T \nabla^2 f(x) v = (1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2} [(\sum_{i=1}^n x_i x_i^{p-2} v_i)^2 - \sum_{i=1}^n x_i^p (\sum_{i=1}^n x_i^{p-2} v_i)]$$

The Cauchy Schwartz Inequality tell us that  $(a^T a)(b^T b) \geq (a^T b)^2$

If we set  $a_i = x_i^{\frac{p}{2}}$  and we set  $b_i = x_i^{\frac{p-2}{2}} v_i$ , substituting this into the Cauchy Schwartz inequality allows us to see that

$$(\sum_{i=1}^n x_i^p)(\sum_{i=1}^n x_i^{p-2} v_i) \geq (\sum_{i=1}^n x_i^{p-1} v_i)^2 = (\sum_{i=1}^n q_i v_i)^2$$

By extension, we can see that

$$(\sum_{i=1}^n x_i^p) v^T \text{diag}(x_1^{p-2}, x_2^{p-2}, \dots, x_n^{p-2}) v \geq v^T qq^T v$$

$$[v^T qq^T v - (\sum_{i=1}^n x_i^p) v^T \text{diag}(x_1^{p-2}, x_2^{p-2}, \dots, x_n^{p-2}) v] \leq 0$$

Since  $p < 1$  and  $x_i \in \mathbb{R}_{++}$ ,  $(1-p)(\sum_{i=1}^n x_i^p)^{\frac{1}{p}-2}$  is a positive constant. Hence,  $v^T \nabla^2 f(x) v \leq 0$  which means that we have proven concavity!

**Problem 7.** Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following

- (a)  $f(X) = \text{tr}(X^{-1})$  is convex on  $\text{dom} f = \mathbb{S}_{++}^n$

**Solution.** To verify convexity, we can consider an arbitrary line given by  $X = Z + tV$  where  $Z \in \mathbb{S}_{++}^n$  and  $V \in \mathbb{S}^n$

$$\begin{aligned} g(t) &= f(Z + tV) = \text{tr}((Z + tV)^{-1}) \\ &= \text{tr}((Z^{\frac{1}{2}}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})Z^{\frac{1}{2}})^{-1}) \\ &= \text{tr}((Z^{-\frac{1}{2}}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})^{-1}Z^{\frac{1}{2}})) \end{aligned}$$

It is a well known fact that  $\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$

Hence, we can continue our simplification

$$\begin{aligned} &= \text{tr}((Z^{-\frac{1}{2}}Z^{\frac{1}{2}}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})^{-1})) \\ &= \text{tr}((Z^{-1}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})^{-1})) \end{aligned}$$

Let us represent  $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}}$  as an Eigenvalue Decomposition of  $QDQ^T$

$$\begin{aligned} &= \text{tr}((Z^{-1}(I + tQDQ^T)^{-1})) \\ &= \text{tr}((Z^{-1}(QIQ^T + tQDQ^T)^{-1})) \\ &= \text{tr}((Z^{-1}(Q(I + tD)Q^T)^{-1})) \\ &= \text{tr}((Z^{-1}Q(I + tD)^{-1}Q^T)) \end{aligned}$$

Again, we can use the cyclic property of the trace:  
 $= \text{tr}((Q^T Z^{-1} Q (I + tD)^{-1}))$

$$= \sum_{i=1}^n (Q^T Z^{-1} Q)_{ii} (1 + t\lambda_i)^{-1}$$

Note:  $\lambda_i$  are eigenvalues of  $Z^{-\frac{1}{2}} V Z^{-\frac{1}{2}}$

Since  $Z$  is a positive definite, symmetric matrix, we know that  $Z^{-1}$  is a positive definite, symmetric matrix as well. Hence,  $(Q^T Z^{-1} Q)_{ii}$  is always positive.

Let's now look at  $(1 + t\lambda_i)^{-1}$ . Let's take the second derivative with respect to  $t$ . This second derivative is equal to  $\frac{2\lambda_i^2}{(1+t\lambda_i)^3}$ . The numerator is clearly positive.

We know that  $Z + tV \in \mathbb{S}_{++}^n$

This means that  $Z^{\frac{1}{2}} (I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}}) Z^{\frac{1}{2}} \in \mathbb{S}_{++}^n$

This means that  $(I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}}) \in \mathbb{S}_{++}^n$  must be True

This means that  $1 + t\lambda_i > 0$  must be true for all  $i$ !. Hence, the denominator of  $\frac{2\lambda_i^2}{(1+t\lambda_i)^3}$  is positive as well.

We have shown that the second derivative of  $(1 + t\lambda_i)^{-1}$  is positive over the values of  $t$  that are in the domain of  $g$  which means that  $(1 + t\lambda_i)^{-1}$  is a convex function over the domain of  $g$ .

$\sum_{i=1}^n (Q^T Z^{-1} Q)_{ii} (1 + t\lambda_i)^{-1}$  is a non-negative weighted sum of convex functions which is convex!

We have completed the proof!

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