Homework #1
Randomized and Big Data Algorithms, Fall 16
600.464

Due on: Tuesday, October 4th, 5 p.m.

Where to submit: On blackboard, under student assessment
Late submissions: will NOT be accepted

Format: Please start each problem on a new page.
Please type your answers.

September 19, 2016

Problem 1 (7 points)

Let $[n] = \{1, 2, \ldots, n\}$ and $g : [n] \mapsto \{-1, 1\}$ be a function and let $\epsilon < 0.1$. We say that g is ϵ-good for $S \subseteq [n]$ if $|\sum_{i \in S} g(i)| \leq n^{1-\epsilon}$. Give a randomized algorithm that finds g such that for a set S the probability that g is not ϵ-good is at most $\frac{1}{n^{1-2\epsilon}}$. Note that your algorithm does not know the sets S ahead of time (otherwise it is trivial). Give a randomized algorithm that finds g such that for fixed sets $S_1, \ldots, S_t$ with $t = n^\epsilon$ g is ϵ-good for all sets with probability $1 - o(1)$. Note that your algorithm does not know the sets $S, S_1, \ldots, S_t$ ahead of time.

Hint: use Chebyshev inequality and the union bound.

1 Problem 2 (1 point)

Let G be an undirected weighted graph with non-negative weights. In class we have learned the randomized algorithm for the MAX-CUT problem such that the expected value of the resulting cut is at least $0.5OPT$ where OPT is the value of a maximum cut in G. Design a randomized algorithm that runs in polynomial time and finds, with probability at least 0.99, a cut in G of size X such that $X \geq 0.49OPT$.

Hint: Use the Markov inequality.

2 Problem 3 (2 points)

Explain why $ZPP \supseteq RP \cap coRP$. An informal (but correct) argument will be sufficient.

3 Problem 4 (1 point)

In class we have learned the randomized algorithm for the MIN-CUT problem that finds a minimum cut with probability at least $\frac{2}{n(n-1)}$ where n is the number of nodes in the graph, $n = |V|$. The algorithm can be implemented in time $O(n^2)$. Using the aforementioned algorithm design another (simple) algorithm that finds a minimum cut with probability at least $1 - \frac{1}{n^{10}}$. What is the running time of your algorithm?

4 Problem 5 (1 point)

Let H be a class of all functions from M to N. Is it true that H is a 2-universal family? Prove your answer. Is it a good idea to use H for applications? Please explain your answer.

Problem 6 (2 points)

In class we have learned how to build a family H of 2-universal hash functions $h : M \mapsto N$ where $|M| = m, |N| = n$. Let $n > 10^{10}$ be a sufficiently large parameter. Can you construct a family of 2-universal hash functions H such that $|H| \leq 10$? Give an example of such family and prove its 2-universality. Otherwise prove that no such family exists.

5 Problem 7 (1 points)

Design a randomized algorithm that finds, in polynomial time, **all** min-cuts with probability at least 0.99. Can you derive an upper bound on the number of different min-cuts?