600.464 Randomized and Big Data Algorithms
Homework #2 Answers

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Problem 1 (3 points)

Alice and Bob play checkers often. Alice is a better player. so the probability that she wins any given game is 0.6, independent of all other games. They decide to play a tournament of n games. Bound the probability that Alice loses the tournament using a Chernoff bound.

Answer:

If Alice loses, she wins less than half the games. Let X be the number of games Alice wins, we want to bound $P(X < n/2)$.

Problem 2 (3 points)

a. In an election with two candidates using paper ballots, each vote is independently misrecorded with probability p = 0.02. Use a Chernoff bound to bound the probability that more than 4% of the votes are misrecorded in an election of 1,000,000 ballots.

b. Assume that a misrecorded ballot always counts as a vote for the other candidate. Suppose that candidate A received 51O,000 votes and that candidate B received 490,000 votes. Use Chernoff bounds to bound the probability that candidate B wins the election owing to misrecorded ballots. Specifically, let X be the number of votes for candidate A that are misrecorded and let Y be the number of votes for candidate B that are misrecorded. Bound $Pr((X > k) \cap (Y < l))$ for suitable choices of k and e.

Answer:

Problem 3 (3 points)

Recall that a function f is said to be *convex* if for any $x1$, $x2$ and for $0 \leq \lambda \leq 1$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

a. Let Z be a random variable that takes on a (finite) set of values in the interval [0,1], and let $p = E[Z]$. Define the Bernoulli random variable X by $Pr(X=1) = p$ and $Pr(X=0) = 1-p$, show that $E[f(Z)] \leq E[f(X)]$ for any convex function f.

b. Use the fact that $f(x) = e^{tx}$ is convex for any $t \geq 0$ to obtain a Chernoff-like bound for Z baesd on a Chernoff bound for X.

Answer:

Problem 4 (3 points)

Let $X_0 = 0$ and for $j \geq 0$ let X_{j+1} be chosen uniformly over the real interval $[X_j, 1]$. Show that, for $k \geq 0$, the sequence

$$Y_k = 2^k(1 - X_k)$$

is a martingale.

Answer:

Problem 5 (3 points)

Consider a random graph from $G_{n,N}$, where $N = cn$ for some constant $c > 0$. Let X be the expected number of isolated verticies (i.e., verticies of degree 0).

a. Determine $E[X]$

b. Show that:

$$P(|X - E[X]| \geq 2\lambda\sqrt{cn}) \leq 2e^{-\lambda^2/2}$$

Answer: