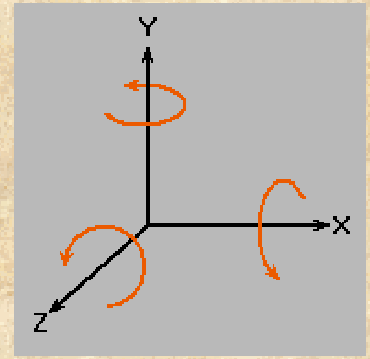
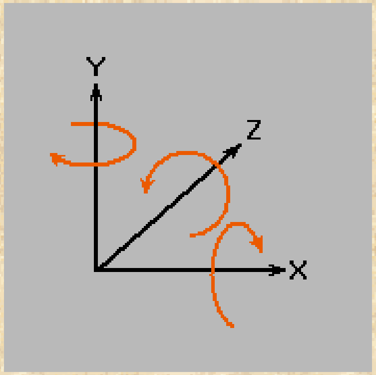
Three-Dimensional Graphics

- **Use of a right-handed coordinate system (consistent with math)**
- **Left-handed suitable to screens.**
- **To transform from right to left, negate the z values.**



Right Handed Space



Left Handed Space

Homogeneous representation of 3D point:

$|x\ y\ z\ 1|^T$
(w=1 for a 3D point)

Transformations will be represented by 4x4 matrices.

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation

$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scale

$$Sh = \begin{bmatrix} 1 & sh_x^y & sh_x^z & 0 \\ sh_y^x & 1 & sh_y^z & 0 \\ sh_z^x & sh_z^y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$P' = P \cdot Sh$

$$X' = X + Sh_x^y Y + Sh_x^z Z$$

$$Y' = Sh_y^x X + Y + sh_y^z Z$$

$$Z' = Sh_z^x X + Sh_z^y Y + Z$$

Shearing

Reflection (xy plane)

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

yz plane

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

xz plane

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrices along:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

X-axis

$$\begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Y-axis

$$\begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Z-axis

Why is the sign reversed in one case ?

Rotation About an Arbitrary Axis in Space

Assume we want to perform a rotation about an axis in space passing through the point (x0, y0, z0) with direction cosines (cx, cy, cz) by θ degrees.

1) First of all, translate by: - (x0, y0, z0) = |T|.

2) Next, we rotate the axis into one of the principle axes, let's pick, Z (|Rx|, |Ry|).

R_θ

$M = R_T^{-1} R_y^{-1} R_x^{-1} R_\theta R_y R_x R_T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

3) We rotate next by θ degrees in Z (|Rz(θ)|).

4) Then we undo the rotations to align the axis.

5) We undo the translation: translate by (x0, y0, z0)

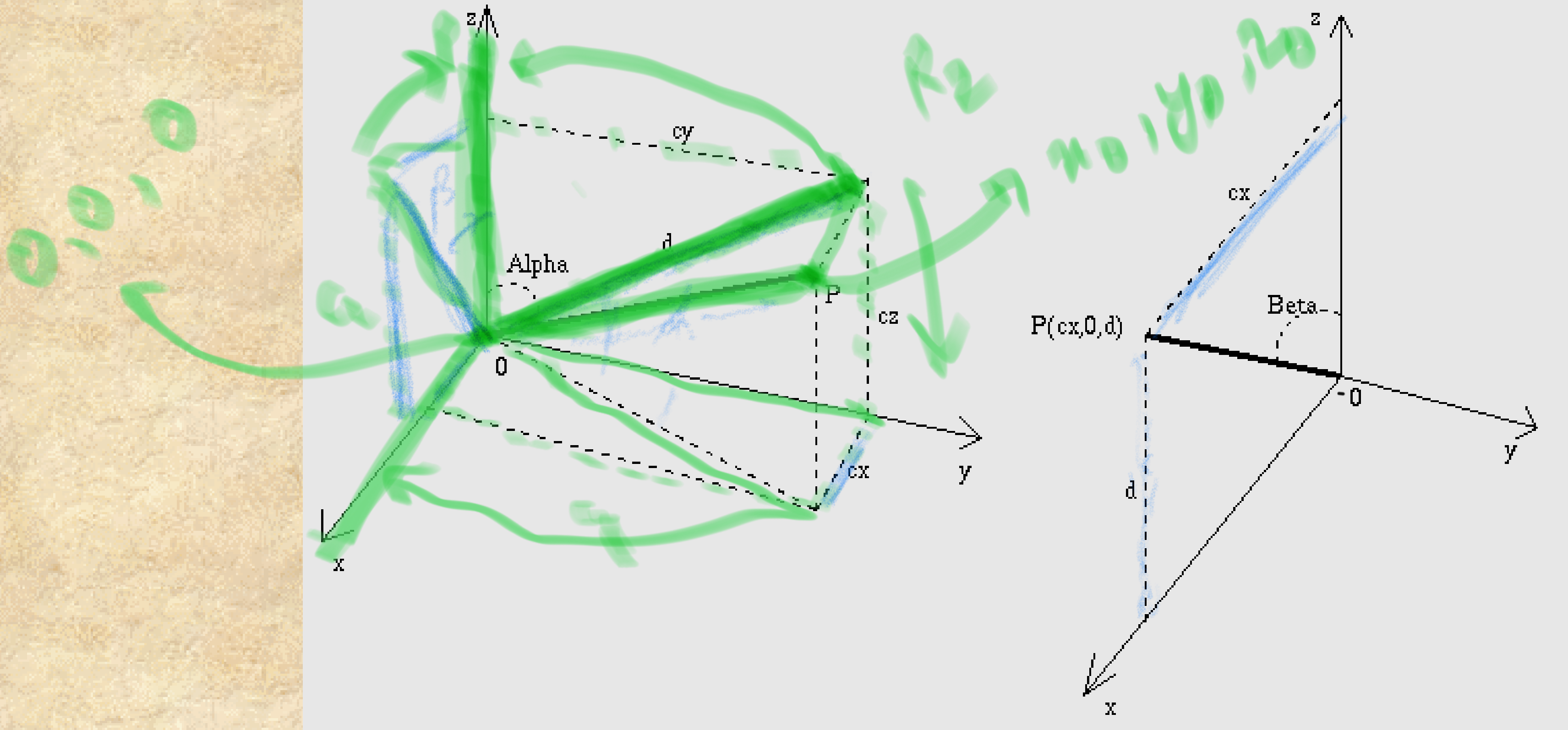
$x' = Mx$

The tricky part is (2) above.

This is going to take 2 rotations,

1 about x (to place the axis in the xz plane) and

1 about y (to place the result coincident with the z axis).



(a) (b)

Rotation about x by α:
How do we determine α?

Project the unit vector, along OP, into the yz plane as shown below.
The y and z components are c_y and c_z, the direction cosines of the unit vector along the arbitrary axis. It can be seen from the diagram above, that :

$$d = \mathrm{sqrt}(c_y^2 + c_z^2),$$

therefore

$$\cos(\alpha) = c_z/d$$
$$\sin(\alpha) = c_y/d$$

Rotation by β about y:

How do we determine β? Similar to above:

Determine the angle β to rotate the result into the Z axis:
The x component is c_x and the z component is d.

$\cos(\beta) = d = d/(\text{length of the unit vector})$
$\sin(\beta) = c_x = c_x/(\text{length of the unit vector})$.

Final Transformation:

Mx

$$M = |T|^{-1}\ |R_x|^{-1}\ |R_y|^{-1}\ |R_z|\ |R_y|\ |R_x|\ |T|$$

If you are given 2 points instead, you can calculate the direction cosines as follows:

$$V = |\ (x1 - x0)\ \ (y1 - y0)\ \ (z1 - z0)\ |^T$$

$c_x = (x1 - x0)/\ |V|$
$c_y = (y1 - y0)/\ |V|$
$c_z = (z1 - z0)/\ |V|$, **where $|V|$ is the length of the vector V.**

Spaces

Object Space

definition of objects. Also called Modeling space.

World Space

where the scene and viewing specification is made

***Eyespace* (Normalized Viewing Space)**

where eye point (COP) is at the origin looking down the Z axis.

3D Image Space

A 3D Perspected space.
Dimensions: -1:1 in x & y, 0:1 in Z.
Where Image space hidden surface algorithms work.

Screen Space (2D)

Coordinates 0:width, 0:height

Projections

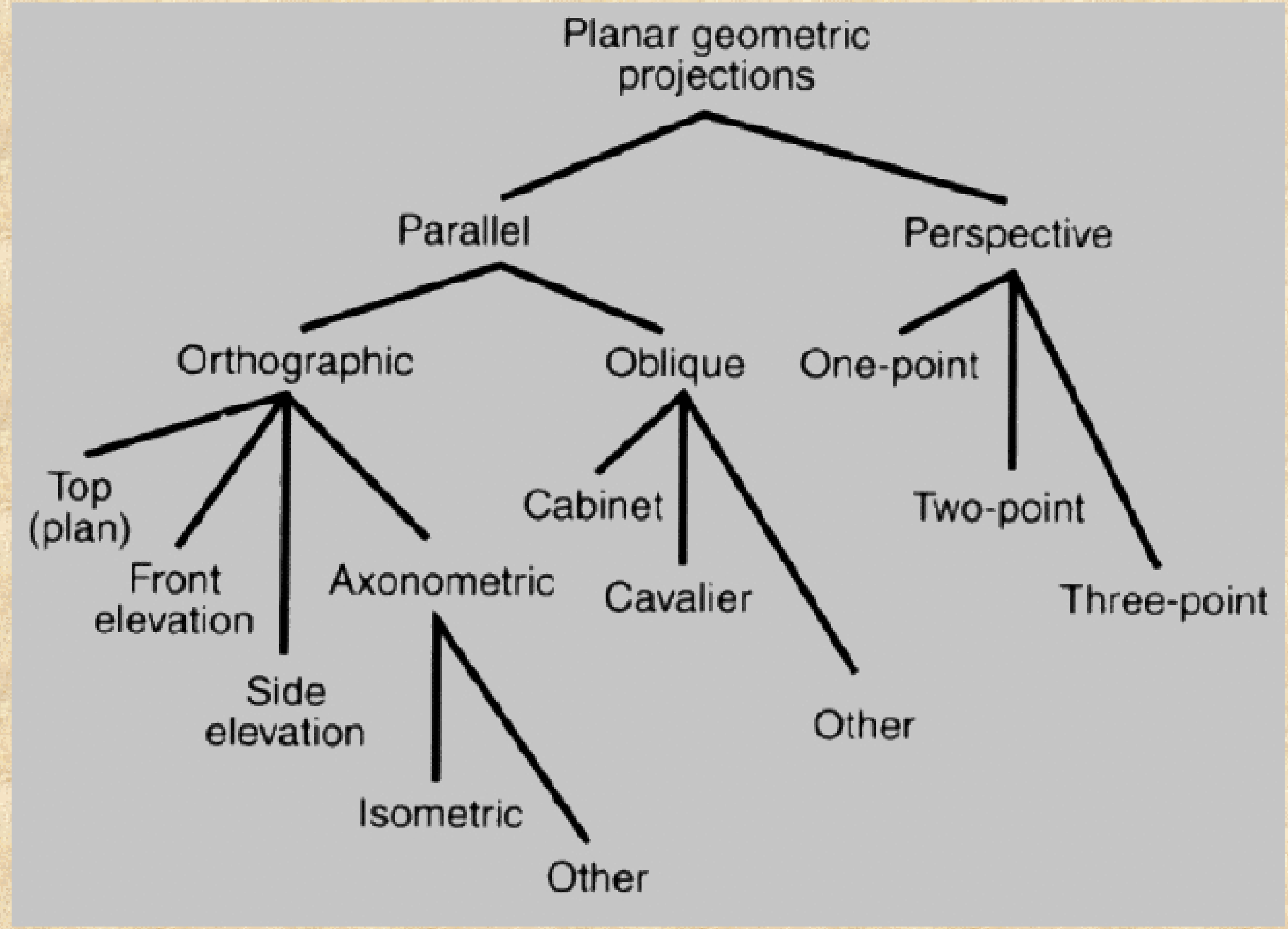
We will look at several planar geometric 3D to 2D projection:

-Parallel Projections
Orthographic
Oblique

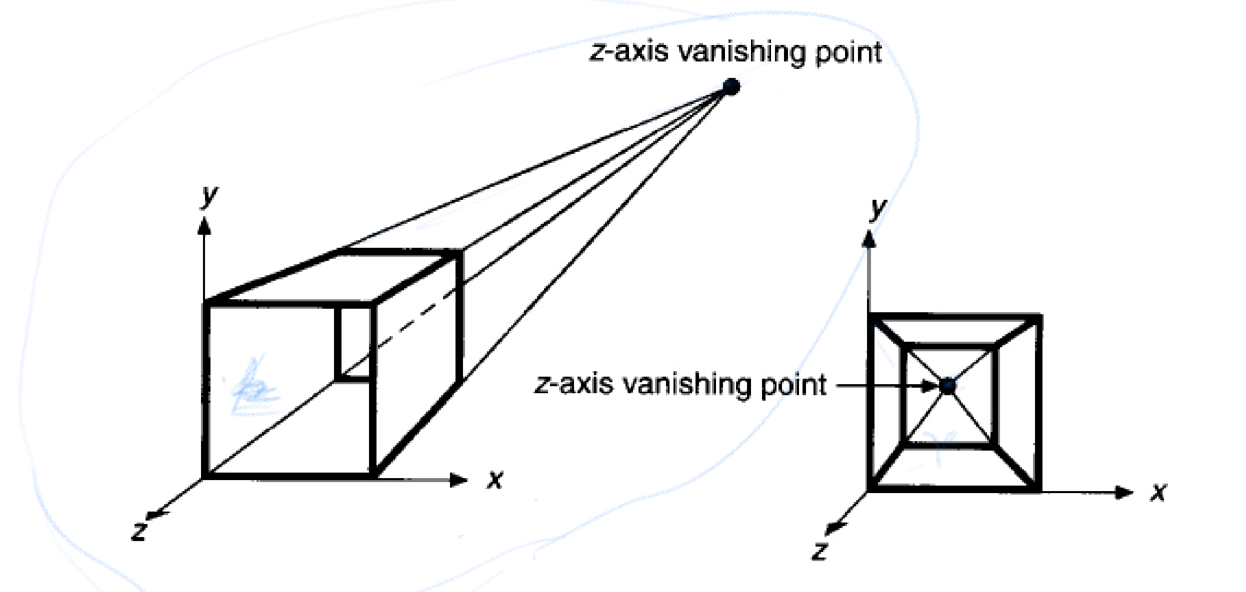
-Perspective

Projection of a 3D object is defined by straight projection rays (projectors) emanating from the center of projection (COP) passing through each point of the object and intersecting the projection plane.

Classification of Geometric Projections



Perspective Projections

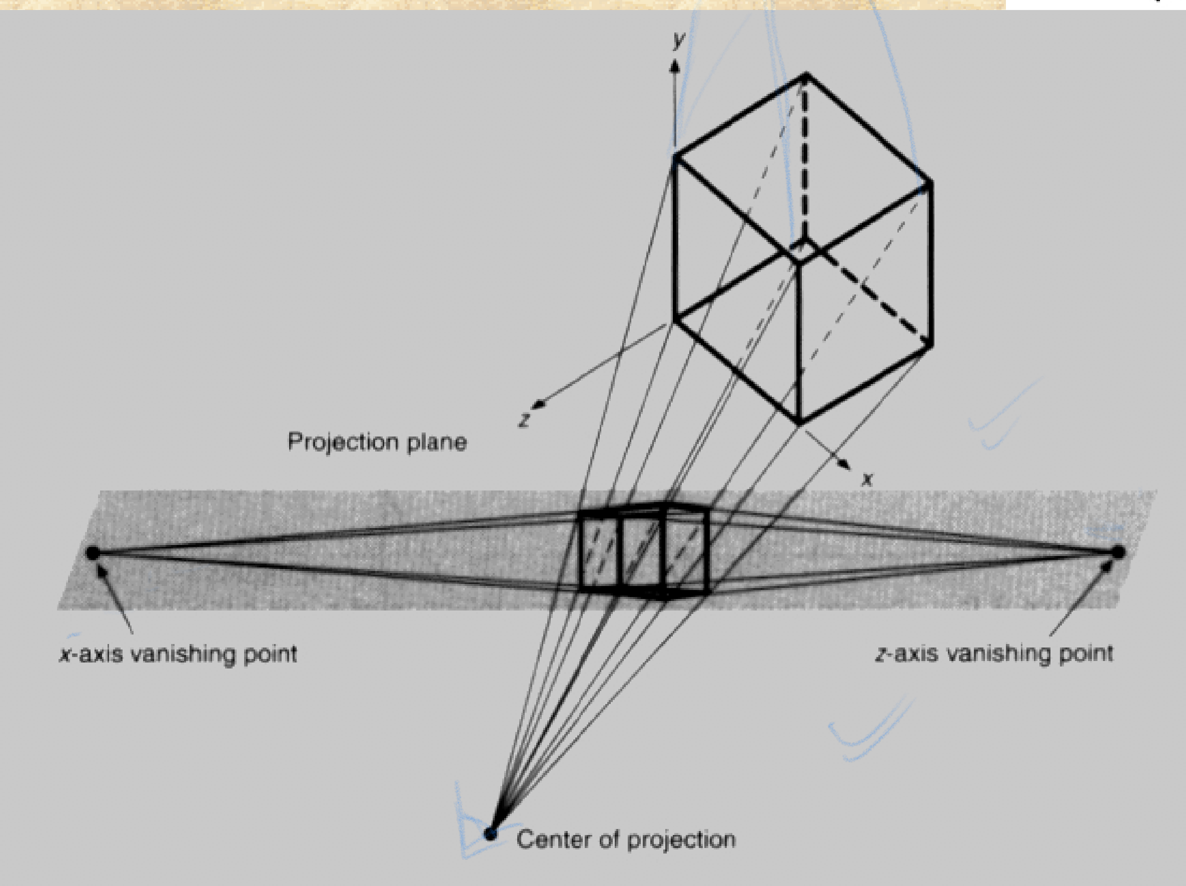


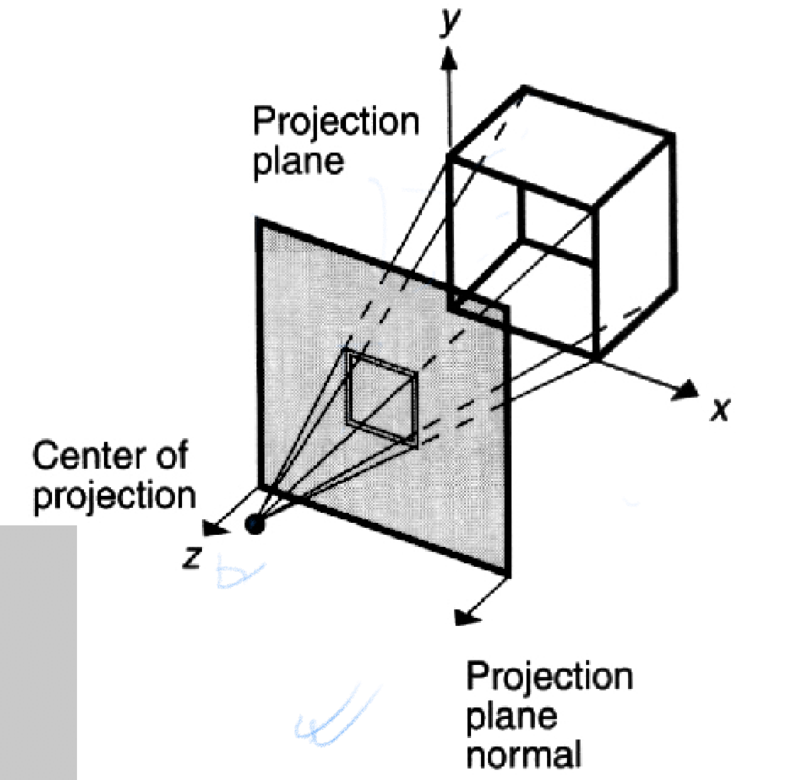
Distance from COP to projection plane is finite. The projectors are not parallel & we specify a center of projection.

Center of Projection is also called the Perspective Reference Point
COP = PRP

Perspective foreshortening: the size of the perspective projection of the object varies inversely with the distance of the object from the center of projection.

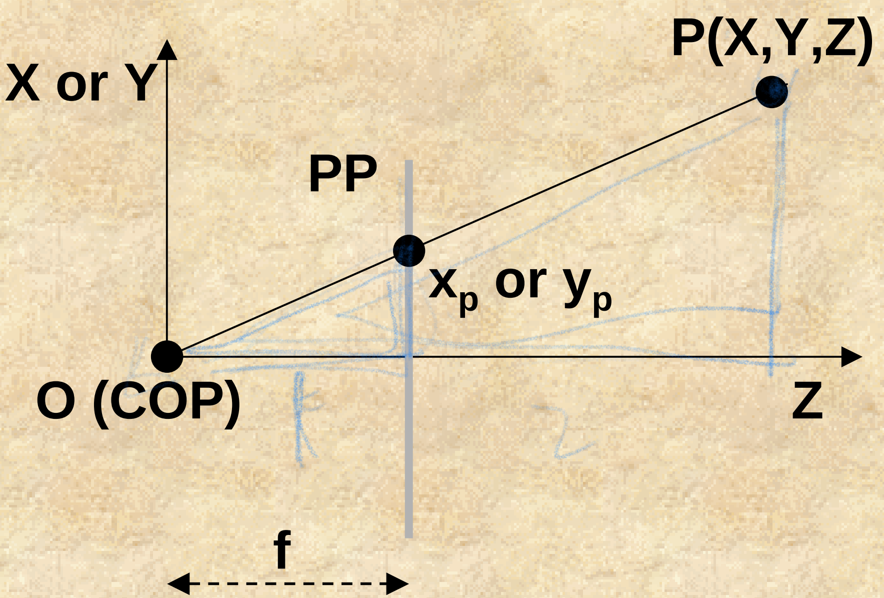
Vanishing Point: The perspective projections of any set of parallel lines that are not parallel to the projection plane converge to a vanishing point.

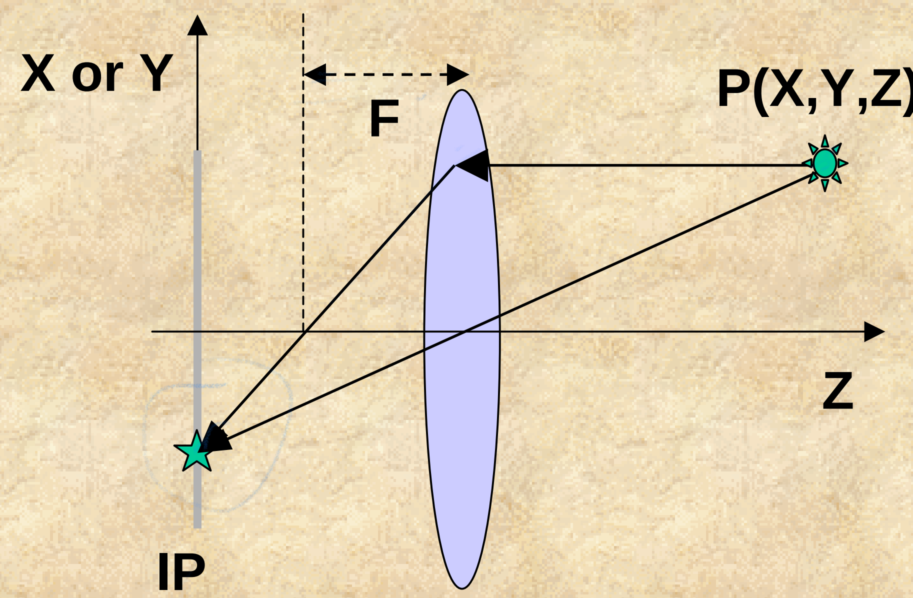

y
z
x
Projection plane
x-axis vanishing point
z-axis vanishing point
Center of projection

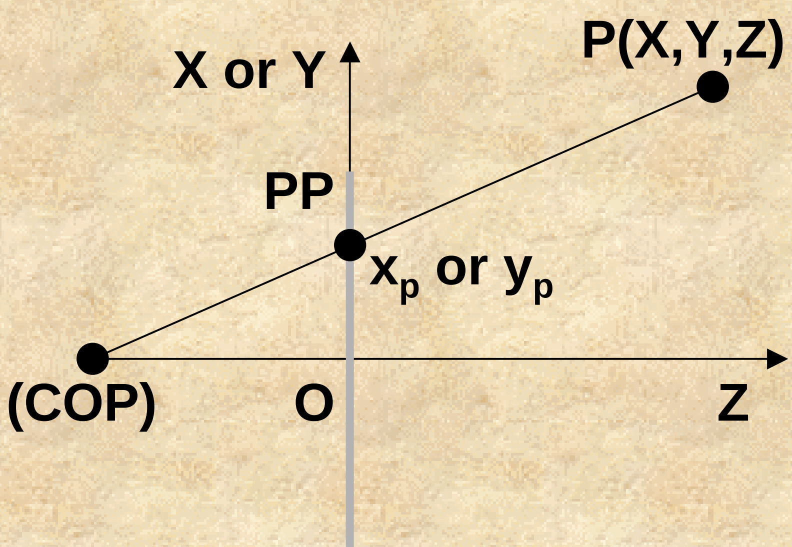

y
Projection plane
x
Center of projection
z
Projection plane normal

Perspective geometry and Camera Models

$\frac{x_p}{f} = \frac{Y}{}$







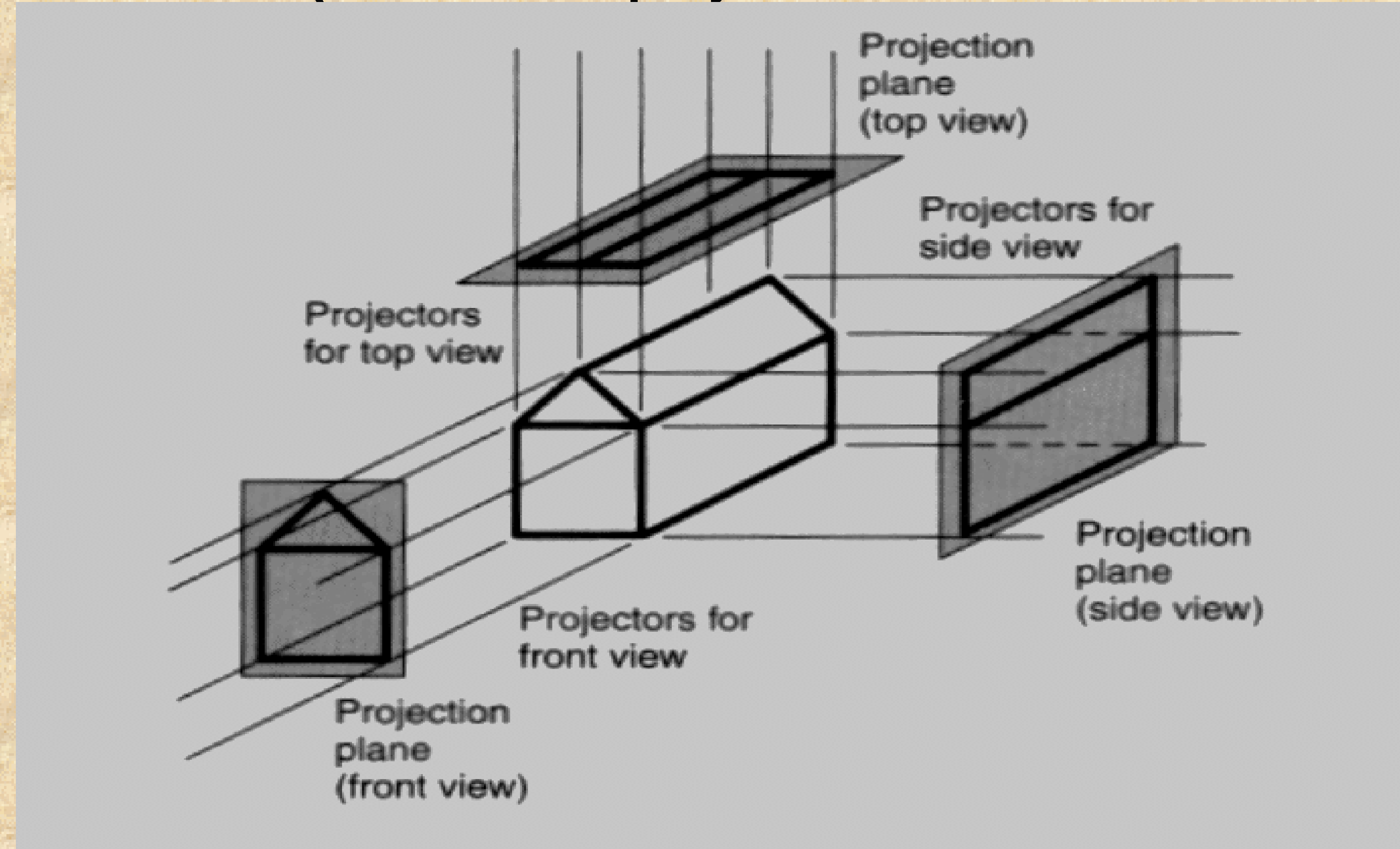
Parallel Projection

Distance from COP to projection plane is infinite.

Therefore, the projectors are parallel lines & we specify a direction of projection (DOP)

Orthographic: the direction of projection and the normal to the projection plane are the same. (direction of projection is normal to the projection plane).

Example of Orthographic Projection:

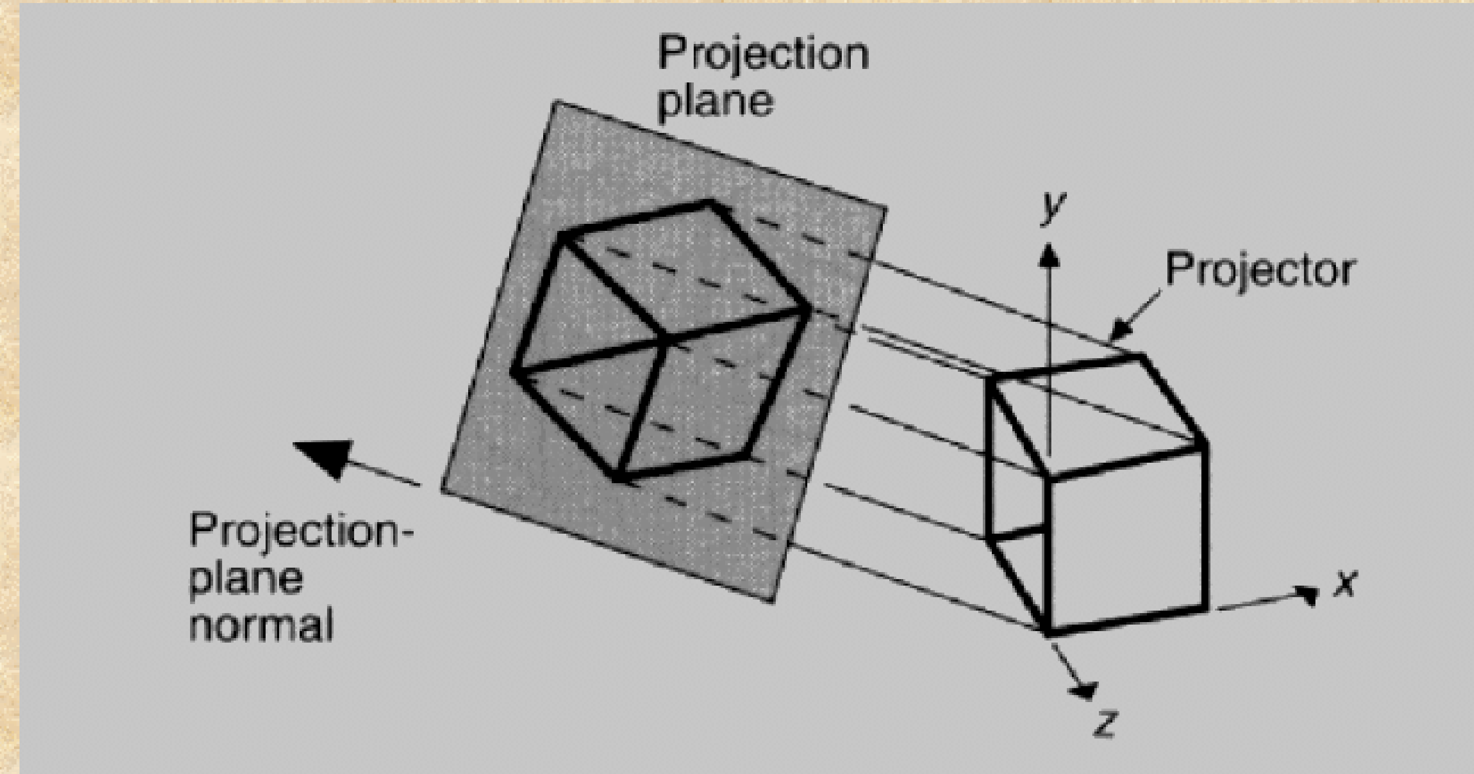


<u>Axometric orthographic projections</u> **use planes of projection that are not normal to a principal axis (they therefore show multiple face of an object.)**

Isometric projection: projection plane normal makes equal angles with each principle axis. DOP Vector: [1 1 1].

All 3 axis are equally foreshortened allowing measurements along the axes to be made with the same scale.

Example Isometric Projection:



Oblique projections : **projection plane normal and the direction of projection differ.**

Plane of projection is normal to a principle axis

Projectors are not normal to the projection plane

Example Oblique Projection