Digital Image Processing

Ch-4 Filtering in the Frequency Domain

Image Enhancement in freq. Domain

→ From spatial domain to freq. domain
then processed

→ Inverse transform is applied to bring
back into spatial domain

→ Filters → smoothening & Sharpening
→ removing high & low freq.

→ Change → whole image

Types of filters

a) Low Pass filters
↳ smoothen

b) high pass filters
↳ sharpen

Ideal
Butterworth
Gaussian

Fourier Transform :

- relation b/w spatial & freq. domain
→ Image enhancement in freq. domain

(1) Discrete FT

$$x(n) \overset{DFT}{\longleftrightarrow} X(k)$$

Spatial domain   Freq. domain

$N \to$ no of samples

$$\therefore X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \; ; \; 0 \le k \le N-1$$

$$X(k) \overset{IDFT}{\longleftrightarrow} x(n)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn} \; ; \; 0 \le n \le N-1$$

Fourier Spectrum

$$|F(u)| = [R^2(u) + I^2(u)]^{1/2}$$

Phase angle $\phi(u) = \tan^{-1}[I(u)/R(u)]$

Power spectrum $P(u) = |F(u)|^2 = R^2(u) + I^2(u)$

2D Discrete Fourier Transform [2D DFT]

$$f(x,y) \xleftrightarrow{\text{2D DFT}} F(u,v)$$

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

$$F(u,v) \xleftrightarrow{\text{2D IDFT}} f(x,y)$$

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

u & v $\rightarrow$ transform or freq. variables

x & y $\rightarrow$ spatial or image variables

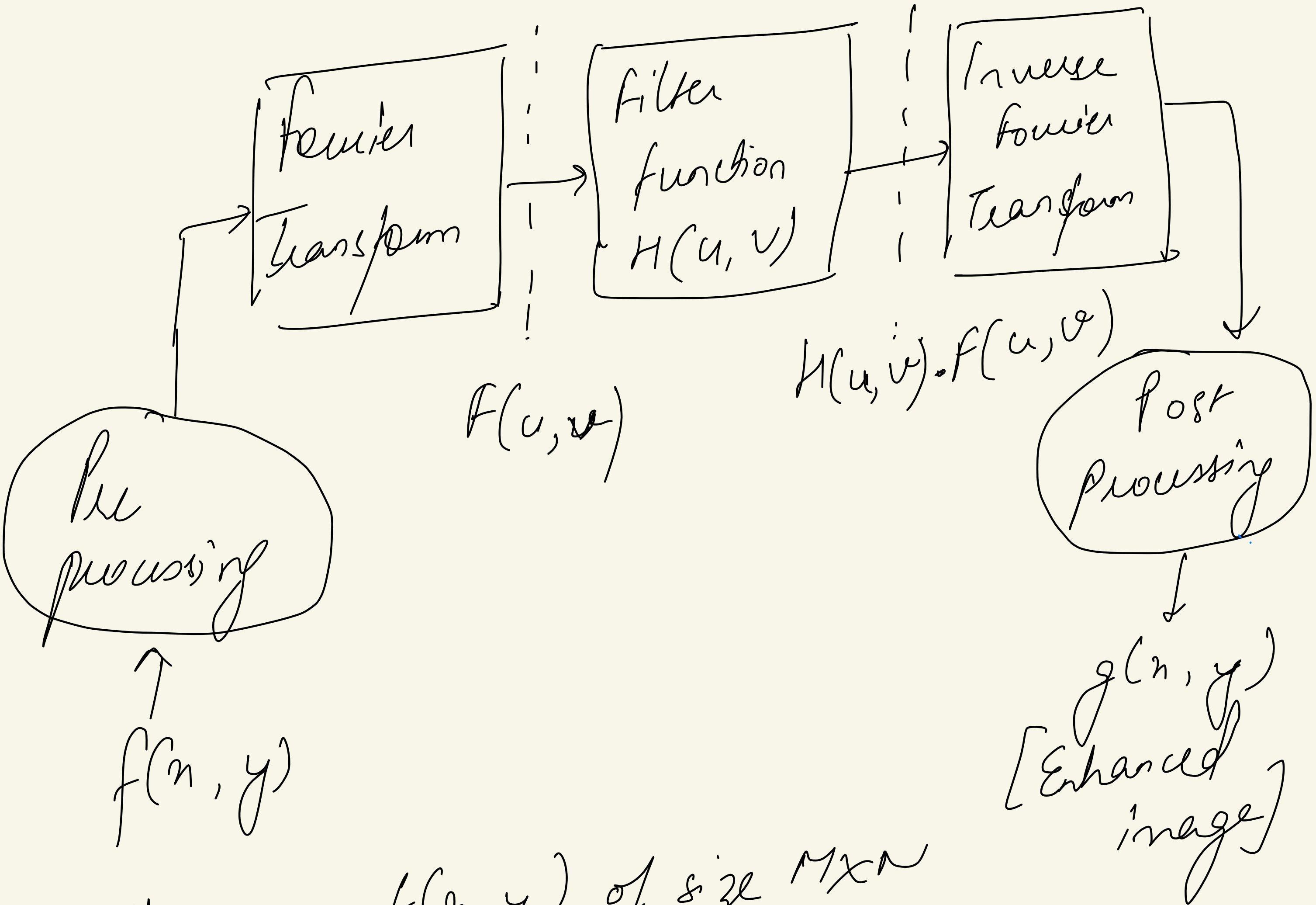
Fourier Spectrum

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

Phase angle $\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$

Power spectrum $P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$

Steps for filtering in freq. domain :



1) Image $f(x,y)$ of size $M \times N$

2) $f(x,y)\,(-1)^{x+y}$

3) $F(u,v) \rightarrow$ F.T of $f(x,y)$

4) $H(u,v) \rightarrow$ filter in freq. domain

$$G(u,v) = H(u,v) \cdot F(u,v)$$

5) $g(x,y) = \mathcal{F}^{-1}[G(u,v)](-1)^{x+y}$

Image Smoothening & Sharpening using freq. domain filters:

a) Lowpass filter

→ Smoothen

i) Ideal lowpass filter

$$H(u,v) = \begin{cases} 1 \;; & D(u,v) \le D_0 \\ 0 \;; & D(u,v) > D_0 \end{cases}$$

$D_0 \Rightarrow$ non -ve quantity

$D(u,v) \rightarrow$ distance from pt. (u,v)

$f(x,y) \rightarrow M \times N$ $\quad D(u,v) = \left[u - \frac{M}{2}\right]^2 + \left[v - \frac{N}{2}\right]^2$

ii) Butterworth LPF

$$H(u,v) = \frac{1}{1 + \left[D(u,v)/D_0\right]^2}$$

Advantage : useful in defining the edges

iii) Gaussian LPF

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

removes low freq. noise

b) High pass filter

→ Sharpen

i) Ideal Highpass filter

$$H(u,v) = \begin{cases} 0 \;; & D(u,v) \le D_0 \\ 1 \;; & D(u,v) > D_0 \end{cases}$$

Disadvantage : blurred edges

ii) Butterworth HPF

$$H(u,v) = \frac{1}{1 + \left[D_0/D(u,v)\right]^2}$$

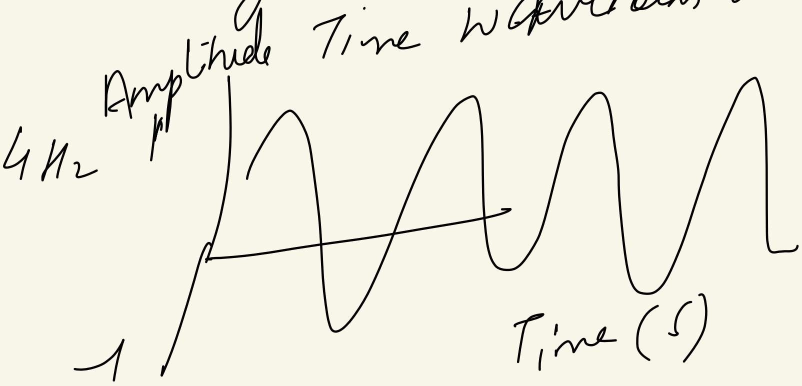
iii) Gaussian HPF

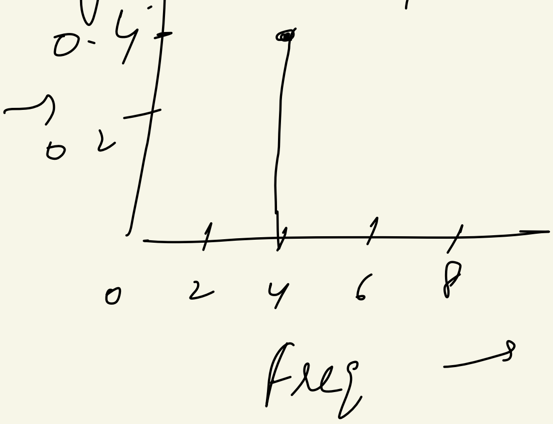
$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

removes high freq. noise

What is FT?

Converting time domain signal into freq domain signal





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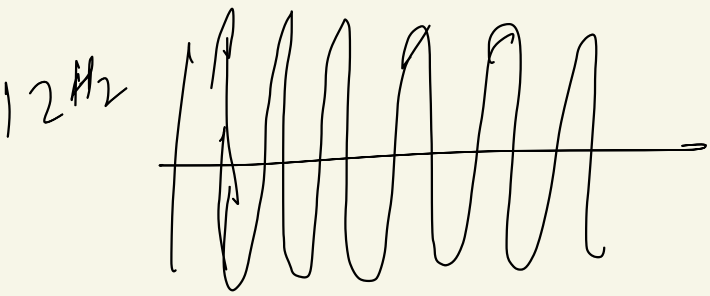

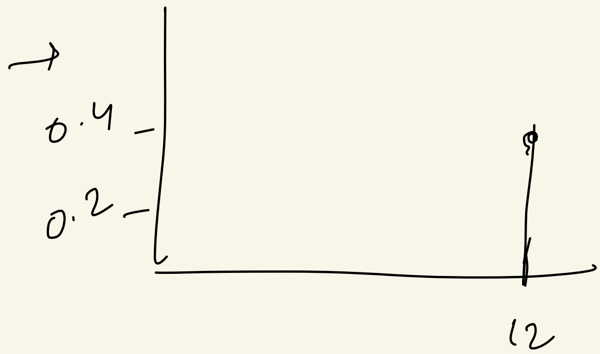

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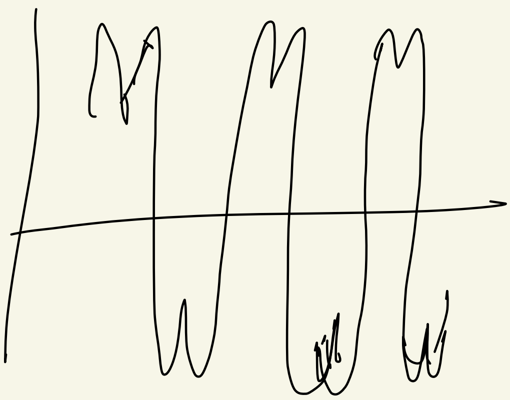

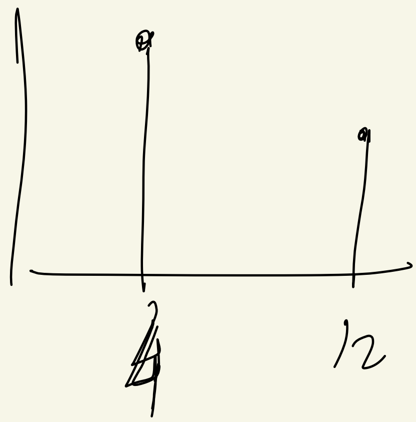

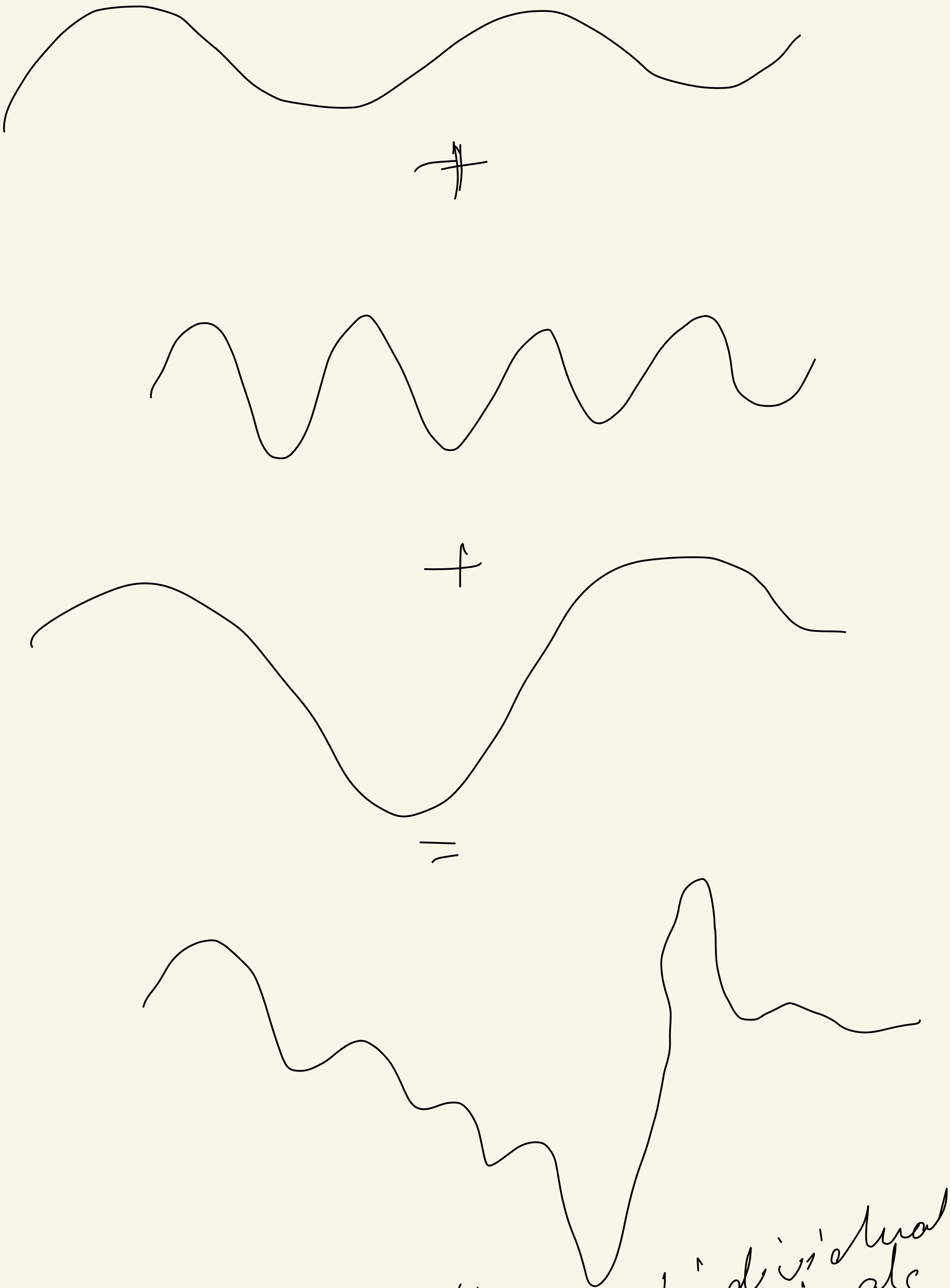

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What freq bands / individual signals this has?

In time domain we can't predict

F.T. is an important image processing tool which is used to decompose an image into its sine & cosine components.

As we are only concerned with digital images, we will restrict to DFT.

For image size $N \times N$

2D DFT:

$$F(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{N}+\frac{vy}{N}\right)}$$

IDFT:

$$f(x,y) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} F(u,v) e^{j2\pi\left(\frac{ux}{N}+\frac{vy}{N}\right)}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

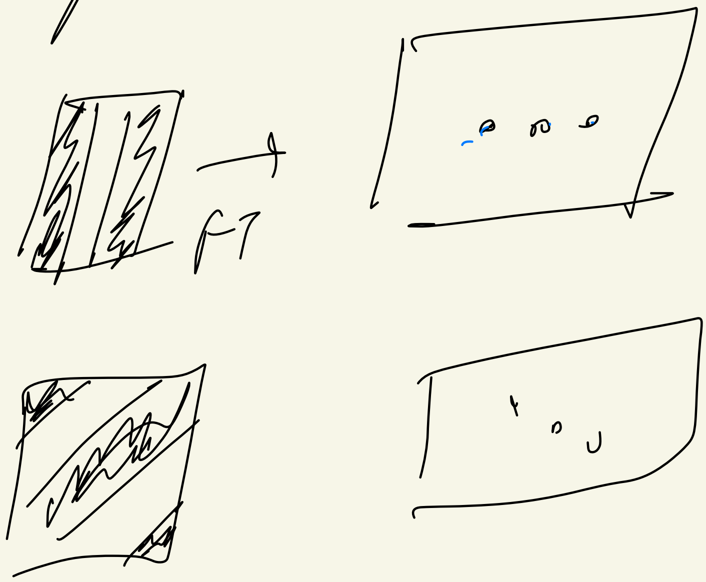
$f(x,y) \rightarrow$ spatial domain

Exponential term is the basis fn.

The basis fns. are sine & cosine waves with increasing frequencies

The response of the FT to periodic patterns in the spatial domain images can be seen very easily in the following artificial images:-

```
I = imread("1.png");
I = M×N×3 uint8
im = fft2(I);
figure, imshow(im);
```



Ques 2D DFT of 4x4 gray scale image

$$f(x,y) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$F(u,v) = \text{kernel} \times f(x,y) \times \text{kernel}^T$$

DFT basis ~~fn~~ (kernel) for $N=4$ is

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}_{k}$$

$$f(x,y) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F(u,v) = K \cdot f(x,y) \cdot K^{T}$$

$$= \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

=

Q $f(n) = \{1, 0, 0, 1\}$
IDFT

$$F[k] = \sum_{n=0}^{N-1} f(n)\, e^{-j2\pi kn/N}$$
where $k = 0, 1, \ldots N-1$

$0 \rightarrow$ Black
$255 \rightarrow$ white

Q. DFT $f(n) = \{1, 0, 0, 1\}$

$$F[k] = \sum_{n=0}^{N-1} f(n)\, e^{-j2\pi kn/N}$$

where $k = 0, 1, \dots N-1$

$N = 4$

$$F[k] = \sum_{n=0}^{3} f(n)\, e^{-j2\pi kn/4}$$

$$= f(0)e^{0} + f(1)e^{-j2\pi k/4} + f(2)e^{-j\pi k} + f(3)e^{-j3\pi k/2}$$

$$= 1 + 0 + 0 + e^{-j3\pi k/2}$$

$$= 1 + e^{-j3\pi k/2}$$

$k=0 \quad F[0] = 1 + e^{0} = 1+1 = 2$

$\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$
$0 - j \times -1$
$= +j$

$k=1 \quad F[1] = 1 + e^{-j3\pi/2} = 1 + j$

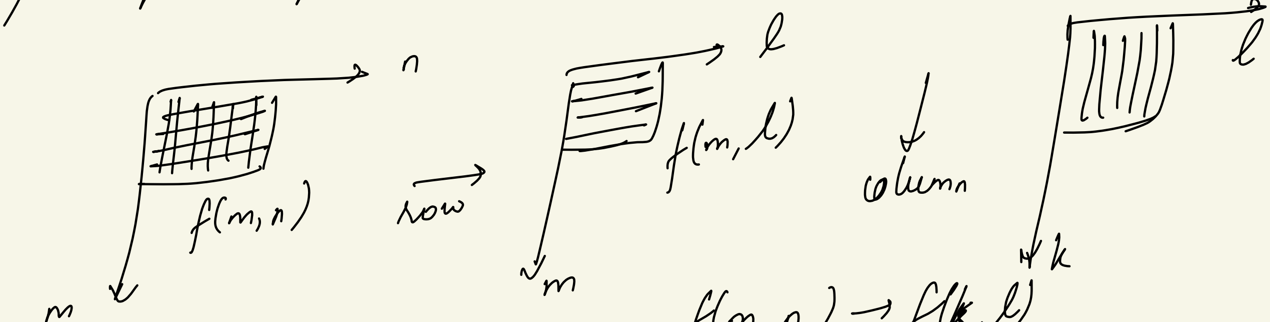
$k=2 \quad F[2] = 1 + e^{-j3\pi} = 1 - 1 = 0$

$k=3 \quad F[3] = 1 + e^{-j9\pi/2} = 1 - j$

$$f[k] = \{\ , 1+j, 0, 1-j\}$$

Properties 2D DFT

1) Seperable property: Successive 1D operation on row & column



$$f(m,n) \to F(k,l)$$

$$F(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) \cdot e^{-j\frac{2\pi}{N}mk} \cdot e^{-j\frac{2\pi}{N}nl}$$

$$= \sum_{m=0}^{N-1} \underbrace{\left(\sum_{n=0}^{N-1} f(m,n) \cdot e^{-j\frac{2\pi}{N}nl} \right)} \cdot e^{-j\frac{2\pi}{N}mk}$$

$$= \sum_{m=0}^{N-1} f(m,l) \cdot e^{-j\frac{2\pi}{N}mk}$$

$$= F(k,l)$$

2) Spatial shift property:

$$f(m,n) \longrightarrow f(m-m_0, n)$$

$m - m_0$

$$f(m-m_0, n) \xrightarrow{DFT} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl}$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m-m_0, n)\, e^{-j\frac{2\pi}{N}(m-m_0)k} \cdot e^{-j\frac{2\pi}{N}m_0k} \cdot e^{-j\frac{2\pi}{N}nl}$$

$$= e^{-j\frac{2\pi}{N} m_0 k} \left(\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m-m_0, n) \cdot e^{-j\frac{2\pi}{N} nl} \right) e^{-j\frac{2\pi}{N}(m-m_0)k}$$

$$= e^{-j\frac{2\pi}{N} m_0 k} \sum_{m=0}^{N-1} f(m-m_0, l)\, e^{-j\frac{2\pi}{N}(m-m_0)k}$$

$$= \underbrace{e^{-j\frac{2\pi}{N} m_0 k}}_{\text{Direction of shift}} f(k,l)$$

3) Periodicity $\quad f(k,l) = f(k+pN,\ l+qN)$

$$f(k+pN, l+qN) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N} m(k+pN)} \cdot e^{-j\frac{2\pi}{N} n(l+qN)}$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) \cdot e^{-j\frac{2\pi}{N} mk} \cdot e^{-j\frac{2\pi}{\cancel{N}} mp\cancel{N}} \cdot e^{-j\frac{2\pi}{N} nl} \cdot e^{-j\frac{2\pi}{\cancel{N}} nq\cancel{N}}$$

$$= \underbrace{e^{-j2\pi mp} \cdot e^{-j2\pi nq}}_{=1} \cdot \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N} mk}\, e^{-j\frac{2\pi}{N} nl}$$

$$= \sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N} nl} \right) e^{-j\frac{2\pi}{N} mk}$$

$$= \sum_{m=0}^{N-1} f(m,l) \cdot e^{-j\frac{2\pi}{N} mk} = f(k,l)$$

4) Scaling → to increase / decrease size of image

expansion in 1 domain ≈ compression in other domain

$$f(ax, by) \rightarrow \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right)$$

5) Convolution property → Convolution in spatial domain = Multiplication in freq. domain

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

$$f(x,y) * g(x,y) = F(u,v) \cdot G(u,v)$$

6) Multiplication by exponential in spatial domain leads to freq. shift

$$f(x,y) \cdot e^{j\pi \frac{x w_0}{\cancel{x}}} \cdot e^{j2\pi \frac{x v_0}{X}}$$

$$= F\left[(w - w_0), (v, v_0)\right]$$

7) Rotation $\quad f(x,y) = f(r\cos\theta, y\sin\theta)$

$$\downarrow DFT$$

$$F\left[R\cos\phi, R\sin\phi\right]$$

$$DFT\left(r\cos(\theta+\theta_0), \cancel{r}\sin(\theta+\theta_0)\right) \rightarrow F\left[R\cos(\phi+\phi_0), R\sin(\phi+\phi_0)\right]$$

i.e if image is rotated by specific angle, spectrum is rotated by same angle