Transformation of a Random Variable (RV)

Consider a (continuous) RV X with probability density function (PDF) $p(X)$.

Consider a transformed variable $Y := g(X)$, where $g(\cdot)$ is an **increasing** function.

(Right now, we consider only the special case of monotonic functions).

• What is the PDF $q(Y)$ of Y ?

Consider probability "mass" of X in the interval (a, b) getting mapped to the probability "mass" of Y in the interval $(g(a), g(b))$. In other words, consider the events $\{x : x \in (a, b)\}$ and $\{y : y \in (g(a), g(b))\}$

Because we assumed increasing $g(\cdot)$, mass conservation holds, i.e., $P(g(a) < Y < g(b)) = P(a < X < b)$

Now, $P(g(a) < Y < g(b)) := \int_{g(a)}^{g(b)} q(y)dy$

Also, $P(a < X < b) := \int_a^b p(x)dx$

Write the above integral in terms of y, using the known relationship $y = g(x)$

We have, $x = g^{-1}(y)$

$dx = \left(\frac{d}{dy} g^{-1}(y)\right) dy$

Then, $P(a < X < b) = \int_{g(a)}^{g(b)} p(g^{-1}(y)) \left(\frac{d}{dy} g^{-1}(y)\right) dy$

This mass conservation holds for *every interval* (a, b), however small it may be.

Thus, $q(y) = p(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$, for all y

If $g(\cdot)$ is increasing, then (i) $a < b \implies g(a) < g(b)$ and (ii) the derivative $\frac{d}{dy} g^{-1}(y)$ is non-negative. So, the above formula holds good.

If $g(\cdot)$ is decreasing, then (i) $a < b \implies g(a) > g(b)$ and (ii) the derivative $\frac{d}{dy} g^{-1}(y)$ is negative. In this case, we can negate the derivative and switch the upper and lower limits to retain the same analysis.

For convenience, to handle both cases above, we write $q(y) = p(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right|$, and then take the integral limits to go from a smaller number to a larger number.
We need to take the absolute value because the function $q(y)$ cannot be negative, it being a PDF.

• Classic Example 1: Consider a RV $X \sim U(0, 1)$ (generated by the C/C++ rand() function). Consider the transformation $Y = (-1/\lambda) \log(X)$, where $\lambda > 0$. What is $q(Y)$?

$y = -(1/\lambda) \log(x) \implies x = \exp(-\lambda y)$. This is the $g^{-1}(\cdot)$ function.

$\left|\frac{d}{dy} g^{-1}(y)\right| = \lambda \exp(-\lambda y)$

So, $q(y) = p(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| = \lambda \exp(-\lambda y)$

Thus, the PDF of Y is the exponential PDF with parameter λ, i.e., mean $= 1/\lambda$

• Classic Example 2: Consider a RV $X \sim U(-a/2, a/2)$. Consider $Y = \exp(X)$. What is $q(Y)$?

$y = \exp(x) \implies x = \log(y)$. This is the $g^{-1}(\cdot)$ function.

$\left| \frac{d}{dy} g^{-1}(y) \right| = 1/y$

So, $q(y) = p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = (1/a)(1/y)$

Thus, the PDF of Y has form $q(y) = 1/(ay)$ for $y \in (\exp(-a/2), \exp(a/2))$

• Classic Example 3: Consider a RV $X \sim G(0, 1)$ (standard Normal PDF). Consider $Y = aX$ with $a > 0$. What is $q(Y)$?

$$
\begin{aligned}
& y := ax \implies x = y/a \implies g^{-1}(y) = y/a \\
& \left| \frac{d}{dy} g^{-1}(y) \right| = 1/a \\
& q(y) := p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = p\left(\frac{y}{a}\right) \frac{1}{a} = \frac{1}{a\sqrt{2\pi}} \exp\left(-\frac{y^2}{2a^2}\right)
\end{aligned}
$$

Thus, $p(Y)$ is also a Gaussian with σ^2 scaled by a factor of a^2

• Classic Example 4: Consider a RV $X \sim G(0, a^2)$. Consider $Y = b + X$. What is $q(Y)$?

$$
\begin{aligned}
& y := b + x \implies x = y - b \implies g^{-1}(y) = y - b \\
& \left| \frac{d}{dy} g^{-1}(y) \right| = 1 \\
& q(y) := p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = p(y - b) \cdot 1 = \frac{1}{a\sqrt{2\pi}} \exp\left(-\frac{(y-b)^2}{2a^2}\right)
\end{aligned}
$$

Thus, $p(Y)$ is also a Gaussian with μ translated by b

• Example 5: Consider a PDF $P(X)$ as follows:

$P(x) := 0$ for $x \leq -1$
$P(x) := 0.5$ for $x \in (-1, 0)$
$P(x) := 0.5 \exp(-x)$ for $x \geq 0$

Consider a transformation function $y := g(x) := x^2$

What is PDF $q(y)$ of Y ?

Transformation function:

$$
\begin{aligned}
& y := x^2 \implies x = \pm\sqrt{y} \implies g^{-1}(y) = \pm\sqrt{y} \\
& \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{2\sqrt{y}}
\end{aligned}
$$

Case 1: $x \in (-1, 0)$. In this case, $g(\cdot)$ is a *decreasing* function. Mass conservation applies.

For $y \in (0,1)$: $q_1(y) := p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = (0.5) \frac{1}{2\sqrt{y}} = \frac{1}{4\sqrt{y}}$

Case 2: $x \geq 0$. In this case, $g(\cdot)$ is a *increasing* function. Mass conservation applies.

$$\text{For } y \geq 0 : q_2(y) := p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = (0.5 \exp(-\sqrt{y})) \frac{1}{2\sqrt{y}} = \frac{\exp(-\sqrt{y})}{4\sqrt{y}}$$

Desired PDF $q(y) = q_1(y) + q_2(y)$

In the region $y \in (0, 1)$, the probability mass comes from Case 1 as well as Case 2.

Thus,
(i) for $y \in (0, 1)$, PDF $q(y) = \frac{1}{4\sqrt{y}}(1 + \exp(-\sqrt{y}))$
(ii) for $y \geq 1$, PDF $q(y) = \frac{\exp(-\sqrt{y})}{4\sqrt{y}}$

Note the step discontinuity at $y = 1$, where the left limit $= \frac{1+\exp(-1)}{4}$ and the right limit $= \frac{\exp(-1)}{4}$

• Classic Example 6: Let $X \sim G(0, 1)$. Let $Y := X^2$. Then, what is $P(Y)$, defined as the chi-square PDF ?

Transformation function:

$$y := x^2 \implies x = \pm\sqrt{y} \implies g^{-1}(y) = \pm\sqrt{y}$$
$$\left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{2\sqrt{y}}$$

Case 1: $x \leq 0$. In this case, $g(\cdot)$ is a *decreasing* function. Mass conservation applies.

$$\text{For } y \geq 0 : q_1(y) := p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{\exp(-0.5(\sqrt{y})^2)}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} = \frac{\exp(-0.5y)}{2\sqrt{y2\pi}}$$

Case 2: $x > 0$. In this case, $g(\cdot)$ is a *increasing* function. Mass conservation applies.

$$\text{For } y > 0 : q_2(y) := \frac{\exp(-0.5y)}{2\sqrt{y2\pi}}$$

Desired chi-square PDF is $q(y) = q_1(y) + q_2(y) = (1/\sqrt{y2\pi}) \exp(-0.5y)$

• Classic Example 7: Let X have a Gamma PDF $P(x) = \text{Gamma}(x|\alpha, \beta) = (\beta^\alpha/\Gamma(\alpha))x^{\alpha-1} \exp(-\beta x)$, where $\alpha > 0, \beta > 0, x > 0$, and $\Gamma(\cdot)$ is the well-known gamma function defined for all complex numbers with real part positive.

Consider the transformation $Y := 1/X$

What is the PDF of Y ?

Transformation function:

$$y := 1/x \implies x = 1/y \implies g^{-1}(y) = 1/y$$
$$\left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{y^2} \text{for } y > 0$$

For $x > 0$, $g(\cdot)$ is a *decreasing* function. Mass conservation applies.

$$\text{For } y \geq 0 : q_1(y) := p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = (\beta^\alpha/\Gamma(\alpha)) y^{1-\alpha} \exp(-\beta/y) \frac{1}{y^2} = (\beta^\alpha/\Gamma(\alpha)) y^{-\alpha-1} \exp(-\beta/y)$$

This is called the inverse-Gamma PDF.