CS 215 : Data Analysis and Interpretation

(Instructor : Suyash P. Awate)

Quiz (Maximum Marks 40; Closed Book)

Date: 9 Nov 2017. Time: 11:05 am – 12:30 pm

Roll Number: ____________________

Name: ____________________

For all questions, if you feel that some information is missing, make justifiable assumptions, state them clearly, and answer the question.

Relevant Formulae

• Univariate Gaussian: $P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$

• Fourth central moment for the univariate Gaussian above is $3\sigma^4$
• Multivariate Gaussian: $P(x) = \frac{1}{(2\pi)^{d/2}|C|^{0.5}} \exp(-0.5(x-\mu)^\top C^{-1}(x-\mu))$

• Product of two univariate Gaussians: $G(z;\mu_1,\sigma_1^2)G(z;\mu_2,\sigma_2^2) \propto G(z;\mu_3,\sigma_3^2)$
where $\mu_3 = \frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}$ and $\sigma_3^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$

• Exponential distribution: $P(x;\lambda) = \lambda\exp(-\lambda x); \forall x > 0$

• Gamma distribution: $P(x;\alpha,\beta) = \frac{\beta^\alpha x^{\alpha-1}\exp(-\beta x)}{\Gamma(\alpha)}$

• Gamma function: $\Gamma(z) = \int_0^\infty x^{z-1}\exp(-x)dx$ for real-valued z. When z is integer valued, then $\Gamma(z) = (z-1)!$, where ! denotes factorial.

• Fisher information $I(\theta_{\text{true}}) := E_{P(X|\theta_{\text{true}})}[\left(\frac{\partial}{\partial\theta}\log P(X|\theta)\Big|_{\theta_{\text{true}}}\right)^2]$

---

1. [20 points] Derive the Jeffreys prior, to its most simplified form (using simple polynomial or exponential functions only), for the following cases:

   • (5 points) Mean μ for the univariate Gaussian probability density function (PDF), when variance is known.

   See `https://en.wikipedia.org/wiki/Jeffreys_prior#Gaussian_distribution_with_mean_para`

   • (7 points) Standard deviation σ for the univariate Gaussian probability density function (PDF), when mean is known.

See https://en.wikipedia.org/wiki/Jeffreys_prior#Gaussian_distribution_with_standard_deviation_parameter

---

• (8 points) If you reparametrize the univariate Gaussian PDF by substituting $\theta := \log \sigma^2$, then find the Jeffreys prior when the mean is known.

See https://en.wikipedia.org/wiki/Jeffreys_prior#Gaussian_distribution_with_standard_deviation_parameter

---

2. [15 points] The entropy $H(X)$ of a random variable X is a measure of the spread of the distribution of the random variable, defined as $H(X) := E_{P(X)}[\log P(X)]$.

   • (3 points) Derive the entropy of the Bernoulli random variable as a function of the associated parameter $\theta \in [0, 1]$. Find the parameter value θ for which the entropy is maximized.

See https://en.wikipedia.org/wiki/Binary_entropy_function

• (5 points) Derive the entropy of the univariate Gaussian random variable as a function with parameters μ and σ^2. Find the parameter values for which the entropy is maximum.

See http://web.ntpu.edu.tw/~phwang/teaching/2012s/IT/slides/chap08.pdf

---

• (7 points) Derive the entropy of the multivariate Gaussian random variable as a function with parameters μ and C. Find the parameter values for which the entropy is maximum.

See http://web.ntpu.edu.tw/~phwang/teaching/2012s/IT/slides/chap08.pdf

---

3. [5 points] Given a dataset $\{x_i\}_{i=1}^N$, where each x_i is known to be drawn independently from a D-variate Gaussian PDF with mean μ and covariance C, give a step-by-step algorithm to:

   • (1 point) Compute the estimate of the mean.

   • (1 point) Compute the estimate of the covariance.

   • (3 points) Compute the principal modes / directions of variation and the variances along those modes / directions.

Please see class notes, or any of the reference books.
