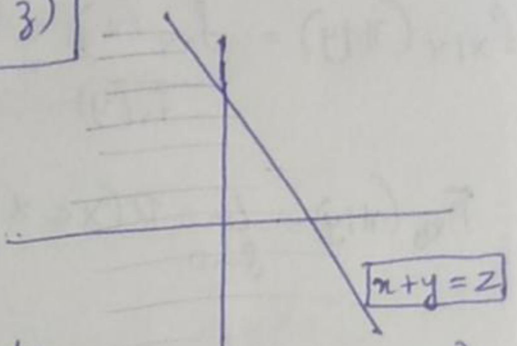
1. (a) Given pdf's $f_X(x)$, $f_Y(y)$, $f_{XY}(x,y)$

$$Z = X + Y$$

$$f_Z(z) = F_Z'(z) ; \boxed{F_Z(z) = P(Z \le z)}$$



Limits $x \in (-\infty, z-y)$
$y \in (-\infty, \infty)$

$$F_Z(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$
$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_{XY}(x,y)\,dx\,dy$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} \left[\frac{d}{dz} \int_{-\infty}^{z-y} f_{XY}(x,y)\,dx \right] dy$$

Using Leiblitz principle:

$$P(x) = \int_{a(x)}^{b(x)} q(x,y)\,dy$$

$$P'(x) = q(x, b(x))\,b'(x) - q(x, a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}(q(x,y))\,dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z-y, y) + 0 + 0$$

$$= \int_{-\infty}^{\infty} f_{XY}(z-y, y)\,dy$$

(a) if X, Y are independent then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\,dy$$

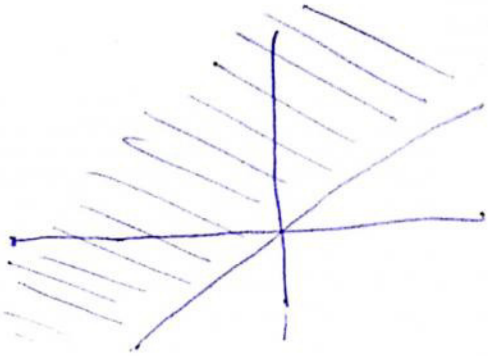
using Convolution

$$\boxed{f_Z(z) = f_X(z) \circledast f_Y(z)}$$

(b) for $P(x \le Y)$

$$P[(x,y)\in C] = \iint_{(x,y)\in C} f_{XY}(x,y)\,dx\,dy$$

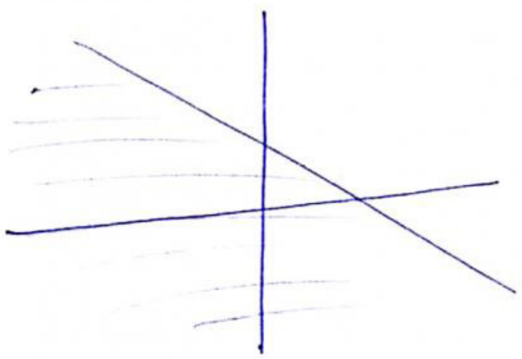
$$P(X<y) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{y} f_{XY}(x,y)\,dx\right]dy$$



if X and y are independent

$$= \int_{-\infty}^{\infty} f_Y(y)\,dy \int_{-\infty}^{y} f_X(x)\,dx$$

→ Submitted one more assume that $X \le y$ is continuation of $Z = X+y$ hence taking intersection of both



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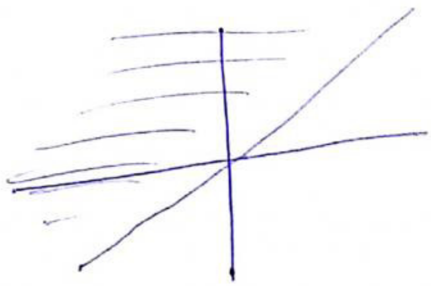

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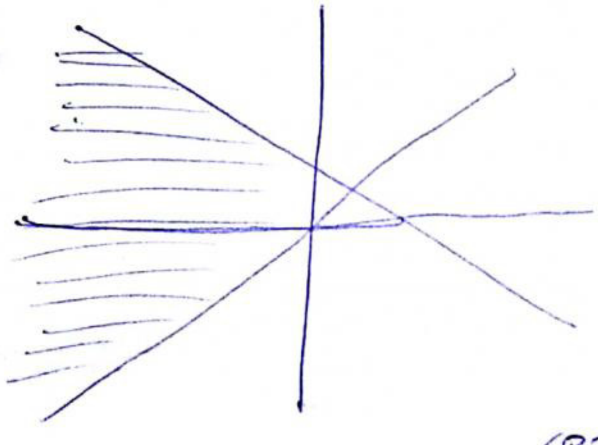
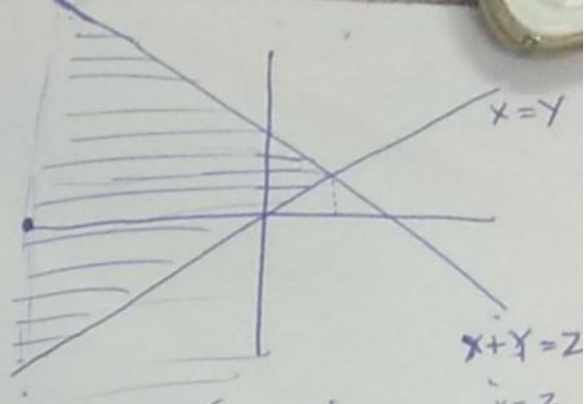


(PTO)

(b) if $P(X \le y)$ Change of Limits



$$F_Z(z) = P(Z \le z)$$
$$= P(X+Y \le z)$$
$$= \int_{x}^{z-x}\left[\int_{-\infty}^{z/2} f_{XY}(x,y)\,dx\right]dy$$

$X = (-\infty, \frac{z}{2})$

$y = (x, z-x)$

$$f_Z(z) = F_Z'(z) = \left[\int_{-\infty}^{z/2} \frac{d}{dz}\left[\int_{x}^{z-x} f(x,y)\,dy\right]\right]dx$$

using Leibnitz

$$= \int_{-\infty}^{z/2} f_{XY}(x, z-x)\,dx$$

if X and Y are independent

$$= \int_{0}^{z/2} [f_X(x)][f_Y(z-x)]\,dx$$

2)

Given $X_1, X_2, \ldots X_n$ are independent identically distributed random Variables

For $Y_1 = \max\{X_1, X_2 \ldots X_n\}$:-

cdf : $F_{Y_1}(y) = P(Y \leq y) \Rightarrow F_{Y_1}(y) = P(Y_1 \leq y)$

$P(Y_1 \leq y) = P(X_1 \leq y, X_2 \leq y \ldots X_n \leq y)$ (as $Y_1 = \max\{X_1, X_2 \ldots X_n\}$ if $Y_1 \leq y$ then each element is less than or equal to y)

$\Rightarrow P(Y_1 \leq y) = P(X_1 \leq y) \cdot P(X_2 \leq y) \ldots P(X_n \leq y)$ (as they are independent)

$\Rightarrow P(Y_1 \leq y) = F_{X}(y) \cdot F_X(y) \cdots F_X(y) = (F_X(y))^n$

$\therefore$ put x inplace of y

$\Rightarrow P(Y_1 \leq x) = (F_X(x))^n$

$\therefore$ cdf $F_{Y_1}(x) = P(Y_1 \leq x) = (F_X(x))^n$

Pdf $f_{Y_1}(x) = F'_{Y_1}(x) = n(F_X(x))^{n-1} F'_X(x)$

Pdf $f_{Y_1}(x) = n(F_X(x))^{n-1} f_X(x)$

For $Y_2 = \min\{X_1, X_2 \ldots X_n\}$:-

cdf. $F_{Y_2}(x) = P_{Y_2}(Y_2 \leq x)$

$$P(Y_2 \geq x) = P(X_1 \geq x, X_2 \geq x, X_3 \geq x \ldots, X_n \geq x)$$

{ $|||^{ly}$ reason for Y_1 }

$$\Rightarrow P(Y_2 \geq x) = P(X_1 \geq x) \cdot P(X_2 \geq x) \ldots P(X_n \geq x)$$

(as they are independent)

$$\Rightarrow P(Y_2 \geq x) = \underline{(1 - F_X(x))^n}$$ (as $P(X_1 \geq x) = 1 - P(X_1 \leq x) = 1 - F_X(x)$)

$$\Rightarrow P(Y_2 \leq x) = 1 - P(Y_2 \geq x) = 1 - (1 - F_X(x))^n$$

$$\therefore \text{cdf } F_{Y_2}(x) = P(Y_2 \leq x) = 1 - (1 - F_X(x))^n$$

pdf $\cancel{P(x)}$ $f_{Y_2}(x) = F'_{Y_2}(x) = n(1 - F_X(x))^{n-1}(-F'_X(x))$

$$= -n(1 - F_X(x))^{n-1} f_X(x)$$

$$\therefore \underline{f_{Y_2}(x) = -n f_X(x)(1 - F_X(x))^{n-1}}$$

3) Given mean $= \mu$, Variance $= \sigma^2$

if $\tau > 0$:- $x - \mu \geq \tau$

add $b\ (>0)$ on both sides

$$\Rightarrow x - \mu + b \geq \tau + b$$

$\therefore$ $P(x - \mu \geq \tau)$ same as $P(x - \mu + b \geq \tau + b)$

consider $E((x-\mu+b)^2) = E((x-\mu)^2 + b^2 + 2b(x-\mu))$

$$= E((x-\mu)^2) + b^2 + 2bE(x-\mu)$$

$$= \sigma^2 + b^2 + 2b \cdot 0 \quad (\text{as } E((x-\mu)^2) = \sigma^2)$$

$$\therefore E((x-\mu+b)^2) = \sigma^2 + b^2$$

also $E((x-\mu+b)^2) = \int_{-\infty}^{\infty} (x-\mu+b)^2 \, P(x = x_i)\,dx$ (by def'n)

$$\sigma^2 + b^2 = \int_{-\infty}^{\tau+\mu} (x-\mu+b)^2 P(x=x_i)\,dx + \int_{\tau+\mu}^{\infty} (x-\mu+b)^2 P(x = x_i)\,dx$$

$$\sigma^2 + b^2 \geq \int_{\tau+\mu}^{\infty} (x-\mu+b)^2 P(x = x_i)\,dx$$

$$\sigma^2 + b^2 \geq (\tau+b)^2 \, P(x - \mu \geq \tau)$$ (as for $x \geq \tau + \mu$, $x - \mu + b \geq \tau + b$ $(\tau, b > 0)$)

(also $\int_{\tau+\mu}^{\infty} P(x=x_i)\,dx = P(x-\mu \geq \tau)$)

$$\Rightarrow P(x - \mu \geq \tau) \leq \frac{\sigma^2 + b^2}{(\tau+b)^2} = f(x)$$

$\hookrightarrow$ $\forall\, b > 0$

$\therefore$ $P(x - \mu \geq \tau)$ must less than minimum value of $f(x)$

To find minimum value of $f(x)$:

$$\frac{d}{dx} f(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\sigma^2+b^2}{(\tau+b)^2}\right) = 0$$

$$\Rightarrow \cancel{(\sigma^2+b^2)}(2b)(\tau+b)^2 = (\sigma^2+b^2)(2)\cancel{(\tau+b)}$$

$$\tau b + b^2 = \sigma^2 + b^2$$

$$\Rightarrow b = \frac{\sigma^2}{\tau}$$

$\therefore$ minimum value of $f(x)$ is at $b = \frac{\sigma^2}{\tau}$

$$\Rightarrow \text{min value} = \frac{\sigma^2 + \left(\frac{\sigma^2}{\tau}\right)^2}{\left(\tau + \frac{\sigma^2}{\tau}\right)^2} = \underline{\frac{\sigma^2}{\sigma^2+\tau^2}}$$

$$\therefore \quad P(x-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

<u>if $\tau < 0$</u> :- consider $P(x-\mu \leq \tau)$

add "$-b$" $(b>0)$ on both sides

$$\Rightarrow \underline{P(x-\mu-b \leq \tau - b)}$$

llly to case I $(\tau > 0)$ consider

$$E((x-\mu-b)^2) = E((x-\mu)^2 + b^2 - 2b(x-\mu))$$

$$= E((x-\mu)^2) + b^2 - 2b\,E(x-\mu)$$

$$= \underline{\sigma^2 + b^2}$$

we know $E((x-\mu-b)^2) = \int_{-\infty}^{\infty} (x-\mu-b)^2 P(x=x_i)dx$

$$\sigma^2 + b^2 = \int_{-\infty}^{\mu+\tau} (x-\mu-b)^2 P(x=x_i)dx + \int_{\mu+\tau}^{\infty} (x-\mu-b)^2 P(x=x_i)dx$$

$$\geq \int_{-\infty}^{\mu+\tau} (x-\mu-b)^2 P(x=x_i)dx$$

$$\Rightarrow \sigma^2+b^2 \geq (x-b)^2 P(x-\mu \leq \tau) \quad \Big(\text{as } \tau < 0,$$ value of $(x-\mu-b)^2$ increases as x decreases from $\mu+\tau$ $\Big)$ ($b>0$)

$$\Rightarrow \sigma^2+b^2 \geq (\tau-b)^2 P(x-\mu \leq \tau)$$

$$\Rightarrow P(x-\mu \leq \tau) \leq \underbrace{\frac{\sigma^2+b^2}{(\tau-b)^2}}_{=g(x)} \quad \forall b<0$$

Illy in case ① $P(x-\mu \leq \tau)$ is less than min. value of $g(x)$

min. value of $g(x)$ Illy to case ① we get

when $\cancel{b=}\ \frac{\sigma^2}{\tau}$ $\frac{d}{dx} g(x) = 0$

$$\Rightarrow (\tau-b)^2 (2b) = 2(\tau-b)(\sigma^2+b^2)(-1)$$

$$\tau b - \cancel{b^2} = -\sigma^2 - \cancel{b^2}$$

$$\Rightarrow b = \frac{-\sigma^2}{\tau}$$

Substitute b in $g(x)$ we get $g(x)_{min} = \frac{\sigma^2}{\sigma^2+\tau^2}$

$$\Rightarrow P(x-\mu \leq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2} \quad (\text{for } \tau<0)$$

$$\Rightarrow 1 - P(x-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\Rightarrow P(x-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2} \quad (\text{for } \tau<0)$$

$\therefore$ for $\tau<0$, $P(x-\mu \geq \tau) \geq 1 - \dfrac{\sigma^2}{\sigma^2+\tau^2}$

4) $\phi_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} P(x=x_i)\,dx = \text{MGF}$

<u>Take $\phi_x(t)$</u>: $\phi_x(t) = \int_{-\infty}^{\infty} e^{tx} P(X=x_i)\,dx$

$$= \int_{-\infty}^{x} e^{tx} P(X=x_i)\,dx + \int_{x}^{\infty} e^{tx} P(X=x_i)\,dx \rightarrow (1)$$

$$\geq \int_{x}^{\infty} e^{tx} P(X=x_i)\,dx \longrightarrow (2)$$

$$\geq e^{tx} \int_{x}^{\infty} P(X=x_i)\,dx \quad (\text{as } t>0 \downarrow \; e^{tx} \geq e^{tx} \; \forall x \geq x)$$

$$\Rightarrow \geq e^{tx} (P(X \geq x))$$

$$\Rightarrow P(X \geq x) \leq e^{-tx} \phi_x(t) \text{ for } t>0$$

<u>if $t<0$</u>: <u>In eq (1) above</u>

$$\cancel{\phi_x(t) \geq \int_{x}^{\infty} e^{tx} P(x=x_i)\,dx}$$

$$\phi_x(t) = \int_{-\infty}^{x} e^{tx} P(x=x_i)\,dx + \int_{x}^{\infty} e^{tx} P(x=x_i)\,dx$$

$$\phi_x(t) \geq \int_{-\infty}^{x} e^{tx} P(x=x_i)\,dx$$

$$\geq e^{tx} \int_{-\infty}^{x} P(x=x_i)\,dx \quad (\text{as } t<0,\; e^{tx} \geq e^{tx} \; \forall x \leq x)$$

$$\geq e^{tx} P(X \leq x)$$

$$\Rightarrow P(X \leq x) \leq e^{-tx} \phi_x(t) \text{ for } t<0$$

4) Given $X = X_1 + X_2 + \cdots X_n$

where $X_1, X_2 \ldots X_n$ are independent bernoulli random Variables

Now for any $t \geq 0$, $\delta > 0$ we have

$$P\{(x > (1+\delta)\mu\} = P\{e^{tx} > e^{t(1+\delta)\mu}\}$$

By markev's Inequality,

we have

$$P\{e^{tx} > e^{t(1+\delta)\mu}\} \leq \frac{E(e^{tx})}{e^{t(1+\delta)\mu}}$$

$$\Rightarrow P(X > x\} \leq \frac{\phi_x(t)}{e^{t(1+\delta)\mu}}$$

$$\phi_x(t) = E(e^{tx}) = E[e^{t(x_1+x_2+\cdots x_n)}]$$

$$= E[e^{tx_1}] . E[e^{tx_2}] \ldots E[e^{tx_n}]$$

[They are independent]

Given, $E(X_i) = P_i$

we know that, $E[e^{tx}] = 1 - E[x] + E[x]e^t$

so, $E[e^{tx_1}] = 1 - P_1 + P_1 e^t = \cancel{e^t(P_1-1)}\ P_1(e^t-1)+1$

$\vdots$

$E[e^{tx_n}] = 1 - P_n + P_n e^t = P_n(e^t-1)+1$

so, $E[e^{tx}] = (P_1(e^t-1)+1) \ldots (P_n(e^t-1)+1)$

$$E(e^{tx}) \leq e^{P_1(e^t-1)} . e^{P_2(e^t-1)} \ldots e^{P_n(e^t-1)}$$

$(\because 1+x \leq e^x)$

$$E(e^{tx}) \leq e^{(e^t-1)(\sum_i P_i)} = e^{\mu(e^t-1)}$$

$$\therefore \boxed{P\{X > (1+\delta)\mu\} \leq \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}}}$$

5) In second method :-

probability that each person does not have disease is $(1-p)$

As all persons independently have disease with probability 'p'.

probability that no one have disease is $\underline{(1-p)^k}$ (as they are independent. $P((\text{person}_1 \text{ not have disease}) \cap (P_2 \text{ not have disease}) \cap \cdots (P_n \text{ not have disease})) = P(P_1 \text{ not have disease}) \times \cdots P(P_n \text{ not have disease})$

$= (1-p)\times(1-p) \cdots (1-p)$

$= (1-p)^k$)

$\therefore$ Probability that test on mixture shows negative is $(1-p)^k$

$\Rightarrow$ probability that test on mixture shows positive is $1-(1-p)^k$

$\therefore$ Expected number of tests $= (1-p)^k \times 1 + (k+1)\left(1-(1-p)^k\right)$

$= (1-p)^k + (k+1) - (k+1)(1-p)^k$

$= \underline{(k+1) - k(1-p)^k}$

$\therefore$ Expected number of tests in 2nd method is

$\underline{(k+1) - k(1-p)^k}$

Let it be '$f(k)$' $= (k+1) - k(1-p)^k$

for what values of P, Expected value of k in 2nd case is Smaller than in 1st case

$\Rightarrow$ $f(k) < k$ (expected value in 1st case)

as always k tests performed

$\Rightarrow (k+1) - k(1-P)^{k} \leq k$

$\Rightarrow 1 \leq k(1-P)^{k}$

$\Rightarrow (1-P)^{k} \geq 1/k$

$\Rightarrow (1-P) \geq \left(\frac{1}{k}\right)^{1/k}$

$\Rightarrow P \leq 1 - \left(\frac{1}{k}\right)^{1/k}$

$\therefore$ for $P \leq 1 - \left(\frac{1}{k}\right)^{1/k}$ expected value of k for 2nd method is less than 1st method.

Given $k \in [2, 25]$

$P \leq 1 - \left(\frac{1}{k}\right)^{1/k}$

$\therefore$ P is less than minimum value of $1 - \left(\frac{1}{k}\right)^{1/k}$ in $[2, 25]$

Let us take $y = \left(\frac{1}{k}\right)^{1/k}$

$\ln y = \frac{1}{k} \ln\left(\frac{1}{k}\right) = -\frac{1}{k} \ln k$

$\frac{1}{y} \times \frac{dy}{dk} = -\frac{1}{k^2} + \frac{1}{k^2} \ln k = \frac{1}{k^2} (\ln k - 1)$

$\therefore \frac{dy}{dk} = \left(\frac{1}{k}\right)^{1/k} \left(\frac{1}{k^2}\right) (\ln k - 1)$

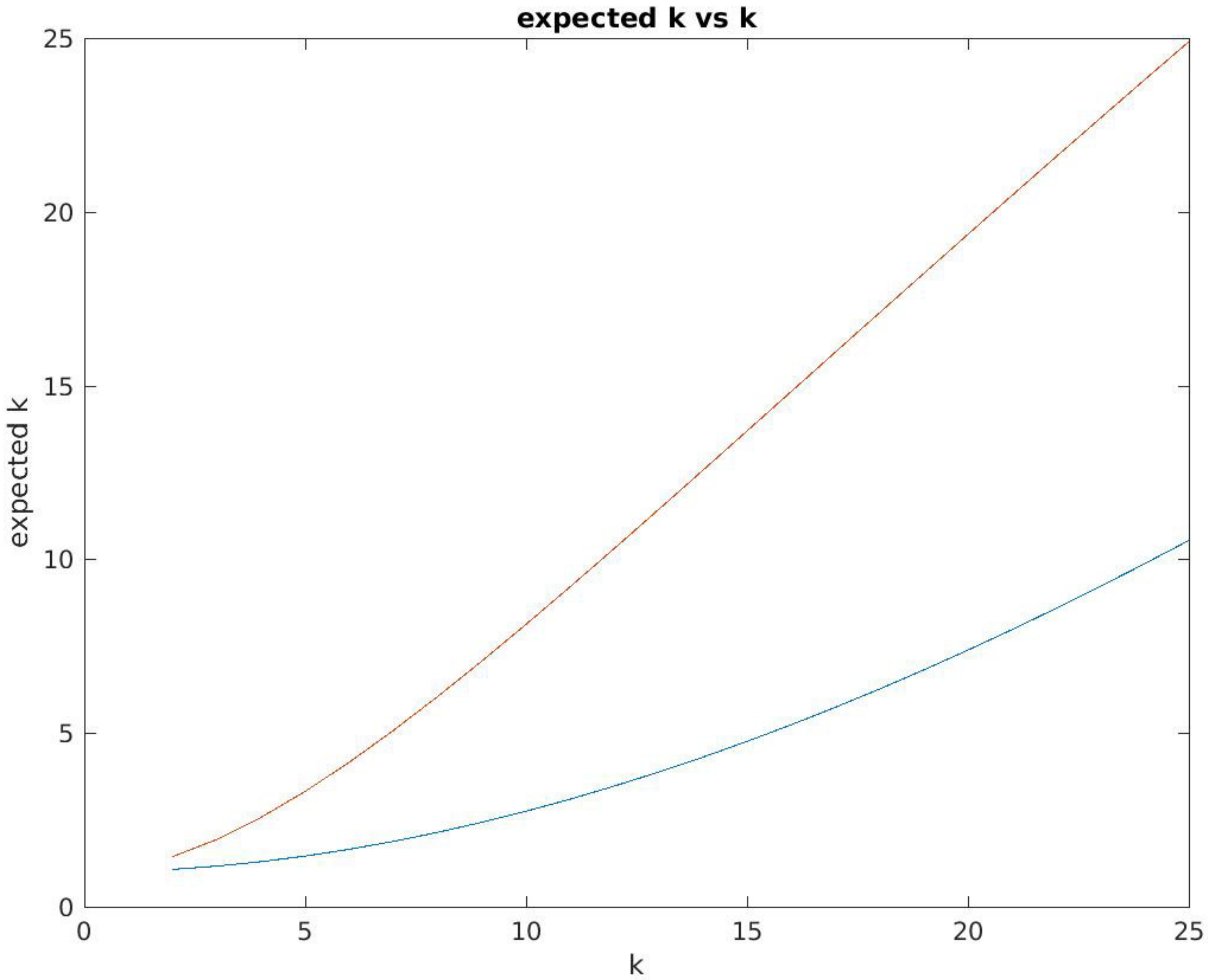
$\therefore$ for $k \in (e, \infty)$ $\frac{dy}{dk} > 0$

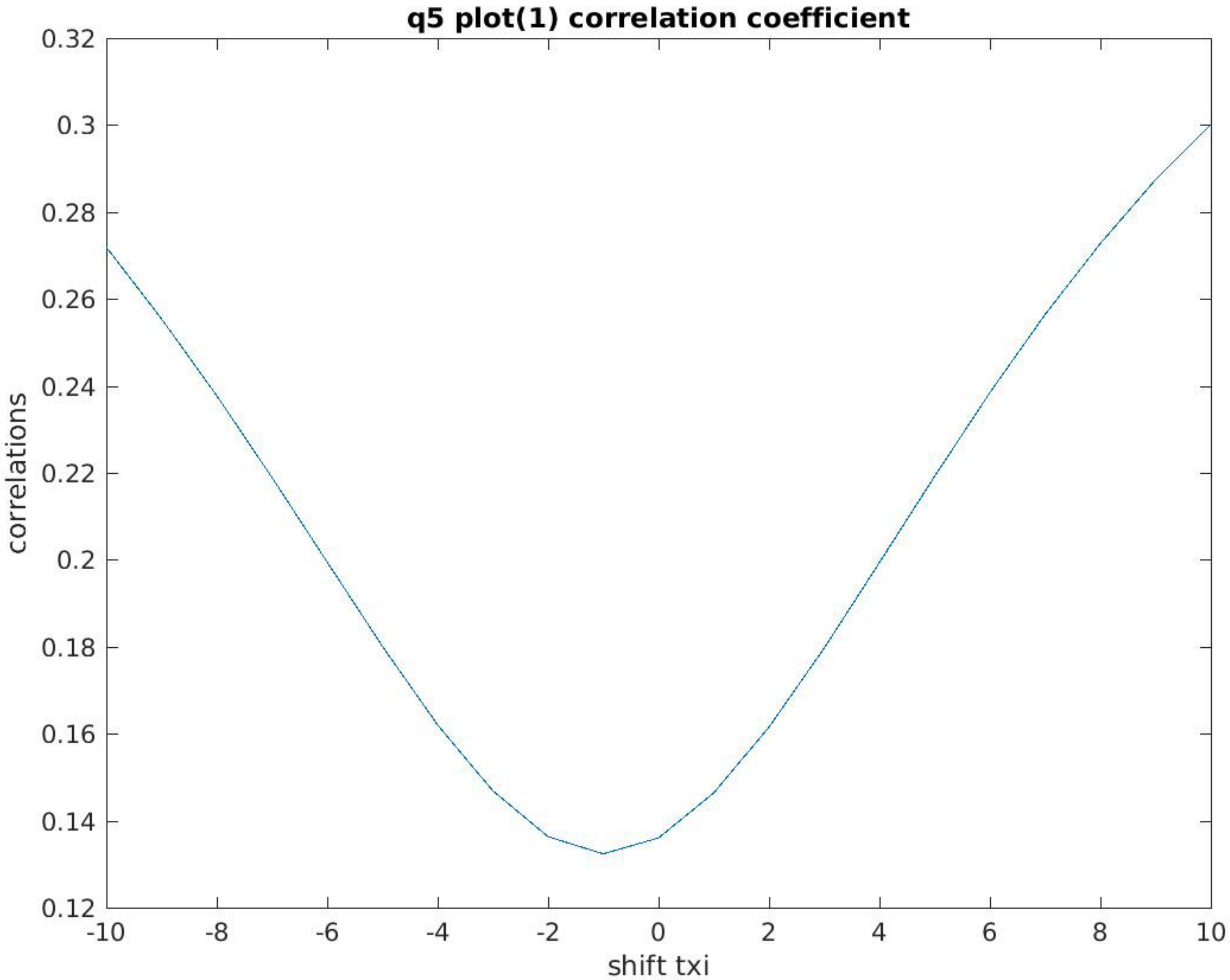
$\therefore$ y is increasing function

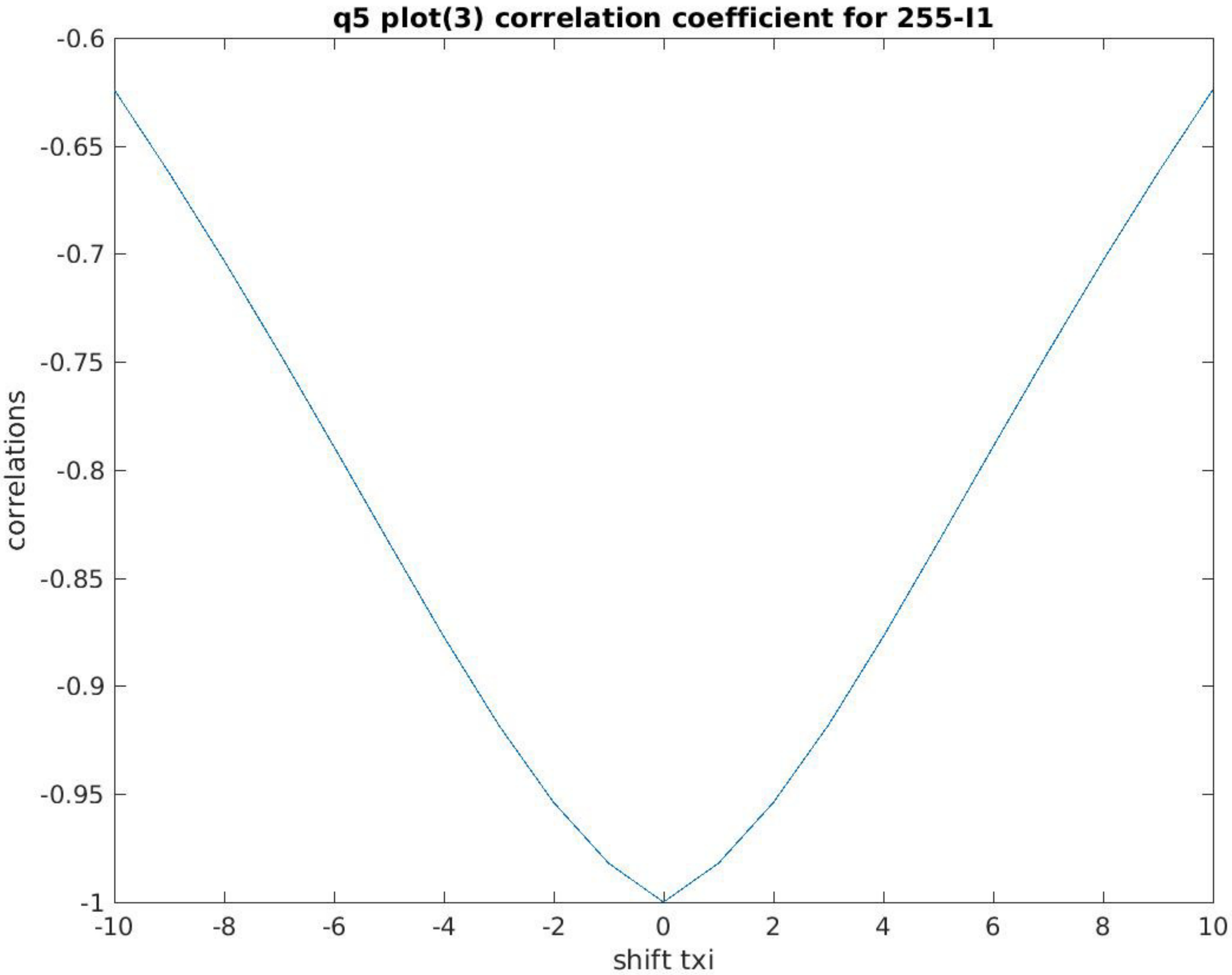
maximum value of y is when $k = 25$

$\therefore P \leq 1 - \left(\frac{1}{k}\right)^{1/k} \leq 1 - \left(\frac{1}{25}\right)^{1/25}$

$\therefore$ for graph of f(k) Versus 'k' take '2' random values of 'p' which satisfies $p < 1 - \left(\frac{1}{25}\right)^{1/25}$


expected k vs k
expected k
k
0
5
10
15
20
25
0
5
10
15
20
25


q5 plot(1) correlation coefficient
correlations
shift txi
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
-10
-8
-6
-4
-2
0
2
4
6
8
10


q5 plot(3) correlation coefficient for 255-I1
correlations
shift txi
-0.6
-0.65
-0.7
-0.75
-0.8
-0.85
-0.9
-0.95
-1
-10
-8
-6
-4
-2
0
2
4
6
8
10