problem 1 :-

Bivariate random Variable $X = (X_1, X_2)$

X_1, X_2 independent and uniform distributions on $(-1,1)$

$X_1 \sim U(-1,1)$ $X_2 \sim U(-1,1)$

(a) Let x_1 denotes x-axis, $X_2 \rightarrow$ Y-axis

probability that X takes values inside circle is

$$P(X_1^2 + X_2^2 \leq 1)$$

~~Let~~ $\therefore$ $f_{X_1}(x) = 1/2$, $-1 \leq x \leq 1$

$= 0$ otherwise

$f_{X_2}(x) = 1/2$, $-1 \leq x \leq 1$

$= 0$, otherwise.

Let $Y = X_1^2 + X_2^2$

$$P(X_1^2 + X_2^2 \leq 1) = P(-\sqrt{1-X_2^2} \leq X_1 \leq \sqrt{1-X_2^2})$$

$$F_Y(y) = \int_{-1}^{y} \int_{-\sqrt{1-X_2^2}}^{\sqrt{1-X_2^2}} f_{X_1X_2}(x_1,x_2)\,dx_1\,dx_2$$

$$F_Y(1) = P(X_1^2 + X_2^2 \leq 1) = \int_{-1}^{1} \int_{-\sqrt{1-X_2^2}}^{\sqrt{1-X_1^2}} f_{X_1X_2}(x_1,x_2)\,dx_1\,dx_2$$

Since X_1 and X_2 are independent

$$P(X_1^2 + X_2^2 \leq 1) = \int_{-1}^{1} \int_{-\sqrt{1-X_2^2}}^{\sqrt{1-X_2^2}} f_{X_1}(x_1)\, f_{X_2}(x_2)\,dx_1\,dx_2$$

$$= \frac{1}{4} \cdot \int_{-1}^{1} 2\sqrt{1-X_2^2}\,dx_2$$ ($\int_{-1}^{1} \sqrt{1-x^2}\,dx$ denotes area of circle of radius 1)

$$= \frac{\pi}{4}$$

b) For this we take 'N' points inside 2×2 square

by $y=1, y=-1; x=1$ & $x=-1$

find no. of points lie inside circle of radius 1 is

given by $x_1^2+x_2^2=1$

No. of points lying inside circle denoted by n

$$\frac{n}{N} = \frac{\text{Area of Circle}}{\text{Area of square}}$$

from (a),

$$\frac{\pi}{4} = \frac{n}{N}$$

$$\Rightarrow \pi = \frac{4n}{N}$$

c) From matlab,

N	estimate of π
10^1	2.00
10^2	3.24
10^3	3.28
10^4	3.1316
10^5	3.140510
10^6	3.141868
10^7	3.141377
10^8	3.141560
10^9	3.141644

d) Let 'Y' be random variable takes 1 if

selected point from square inside circle otherwise '0'

$$P(Y=1) = \frac{\pi}{4} \quad ; \quad P(Y=0) = 1-\frac{\pi}{4}$$

Y is bernoulli r.v with $p = \frac{\pi}{4}$

$$\text{Var}(Y) = \frac{\pi}{4}\left(1-\frac{\pi}{4}\right)$$

Q(N) be r.v which denotes estimate of π

from sample size 'N'

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Q(1) be '4' times estimate of Y

Since we are doing for 'N', Q(N) will be $\frac{4}{N}$ times sum of estimates of Y i.e

$$Q(N) = \frac{4}{N}\sum_{i=1}^{N} y_i$$

$$Var(Q(N)) = Var\left(\frac{4}{N}\sum_{i=1}^{N} y_i\right)$$

$$= \frac{16}{N^2} Var\left(\sum_{i=1}^{N} y_i\right)$$

Because they are independent we can write

$$Var(Q(N)) = \frac{16}{N^2}\sum_{i=1}^{N} Var(y_i)$$

$$= \frac{16}{N^2}\sum_{i=1}^{N}\left(\frac{\pi}{4}\right)\left(1-\frac{\pi}{4}\right)$$

$$= \frac{1}{N}\pi(4-\pi)$$

$$\Rightarrow \sigma(Q(N)) = \sqrt{\frac{\pi(4-\pi)}{N}}$$

→ for large N, central limit theorm can be applied to distribution Q(N), distribution (Q(N) − π) will approach a normal distribution with mean '0' and standard deviation derived from (3)

→ 95% of confidence for a Gaussian of mean 'μ' and deviation 'σ' is (μ − 2σ, μ + 2σ)

$$P(-2\sigma \le Q(N) - \pi \le 2\sigma) \approx 0.95$$

$$\Rightarrow 2\sqrt{\frac{\pi(4-\pi)}{N}} \approx 0.01$$

$$\Rightarrow N \simeq 107870$$

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