Dynamic Local Average Treatment Effects

Ravi B. Sojitra

I am presenting joint work with Vasilis Syrgkanis.

Paper's QR Code

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Treatment dynamics & noncompliance are rife!

In panel data settings, assignment often depends on prior states, and there is noncompliance.

Example Setting	Education	Marketing	Medical Treatment
Outcome (Y)	Student SAT score	Customer spend	Patient tumor size
Treatment (D)	Enroll in advanced course	Redeem discount	Continue strong drug
Dynamic encouragement (Z)	Offer advanced course if grades were high last year	Offer discount if customer added-to-cart yesterday	Encourage status quo if toxicity is acceptable
Treatment Noncompliance (D≠Z)	Students may not enroll in advanced classes	Customers may not redeem offered discounts	Physician shifts to weaker drug
Unobserved D-Y confounder (U)	Student has limited time (e.g. provides childcare to siblings)	Customer already owns the specific product	Patient puts more weight on quality than quantity of life

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Imbens & Angrist ('94) gives ID for T=1 LATEs.

The LATE is identified because it equals an estimable function of the observational distribution.

$$\mathbb{E}[Y(1) - Y(0) \mid D(1) - D(0) = 1] = \frac{\mathbb{E}[Y \mid Z = 1] - \mathbb{E}[Y \mid Z = 0]}{\mathbb{E}[D \mid Z = 1] - \mathbb{E}[D \mid Z = 0]}$$

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Effect of Z on D for everyone

What is the result for >1 time period settings?

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$$\Big\downarrow$$

$$\mathbb{E}\left[Y(d_1, d_2) - Y(0, 0) \mid ???\right]$$

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$$\Downarrow$$

$$\mathbb{E}\left[Y(d_1, d_2) - Y(0,0) \mid ???\right]$$

???

1. **What about the >1 time period setting?**
2. **Are the assumptions/conditions reasonable?**
3. **Do the result have the same interpretation?**

Noncompliance in Dynamic Treatment Regimes

Marketing Example

Each day of Thanksgiving Week, non-members who visited checkout yesterday, did not buy, and visit the checkout page today are offered free membership for the week if they buy.

Noncompliance in Dynamic Treatment Regimes

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Each day of Thanksgiving Week, $\underbrace{\text{non-members}}_{\text{Target population}}$ who visited checkout yesterday, did not buy, and visit the checkout page today are offered free membership for the week if they buy.

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Target population

Purchase Intent

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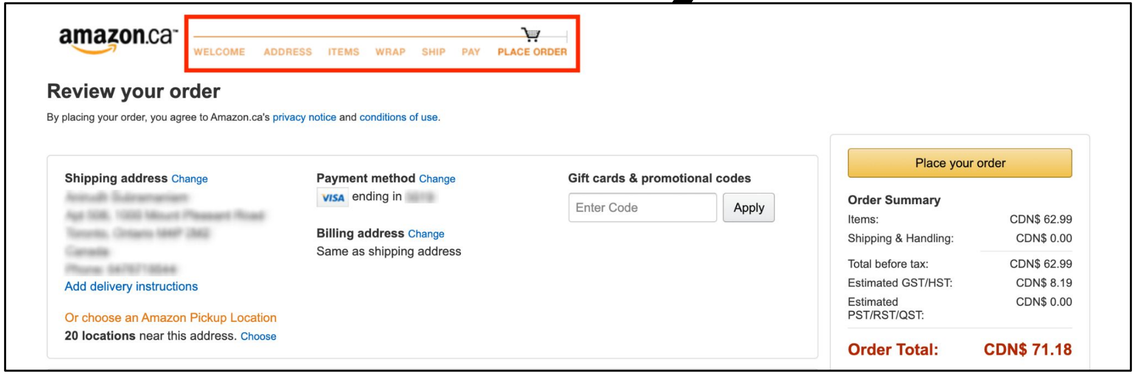

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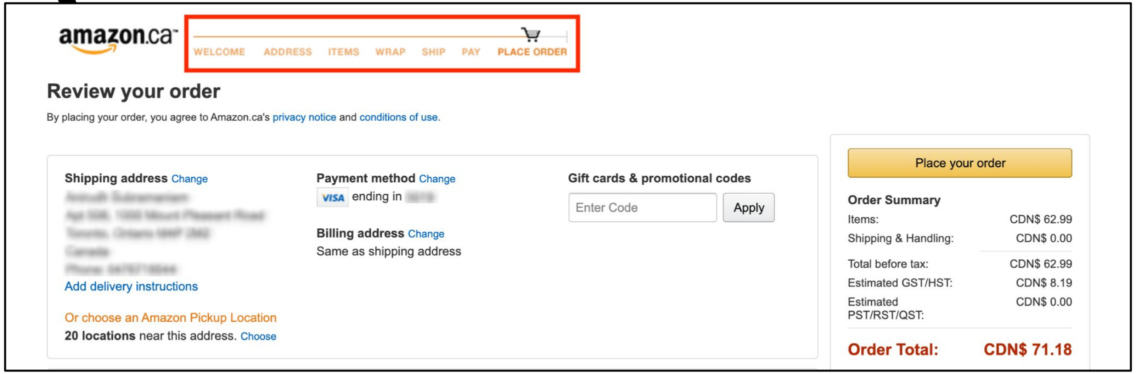

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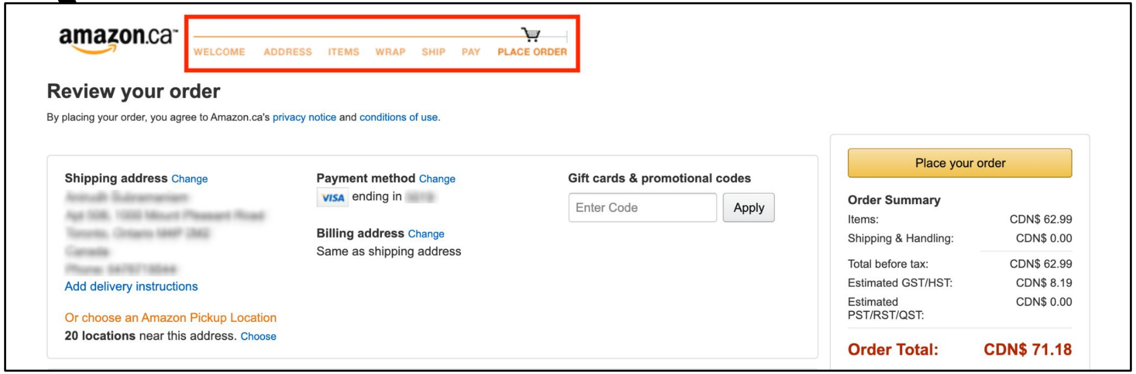

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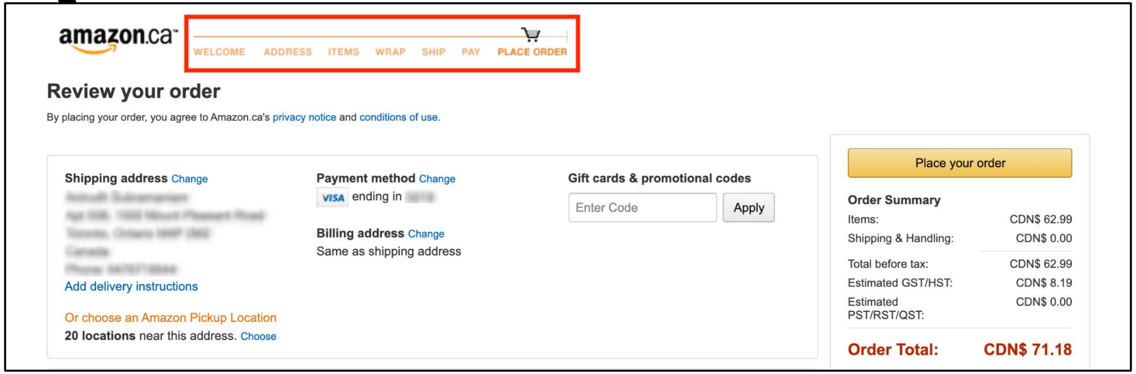
Treatment sequence



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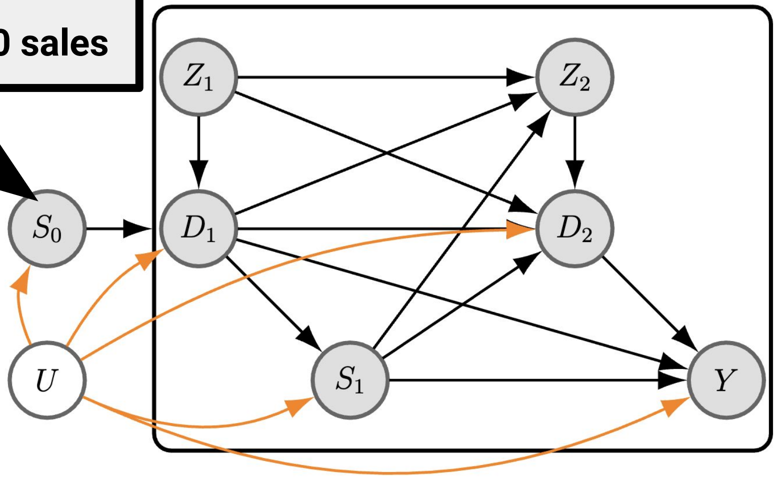
Did not visit checkout or buy, \$0 sales



Noncompliance in Dynamic Treatment Regimes

Encouragement: Z_1 = 0.
Not offered free trial at checkout

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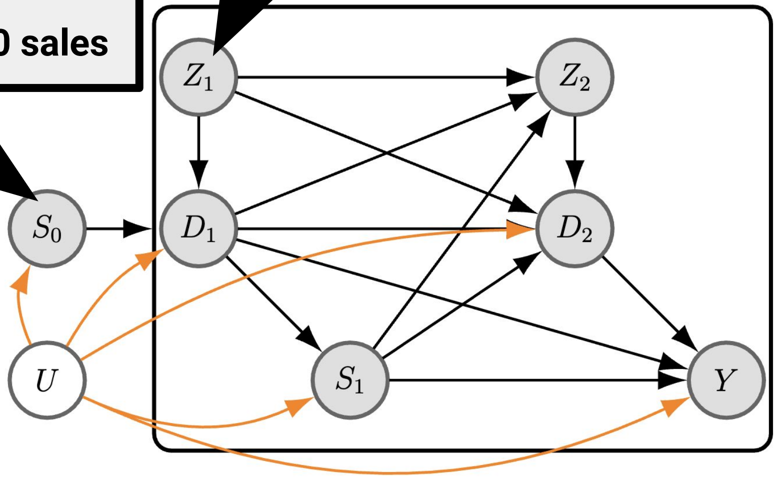

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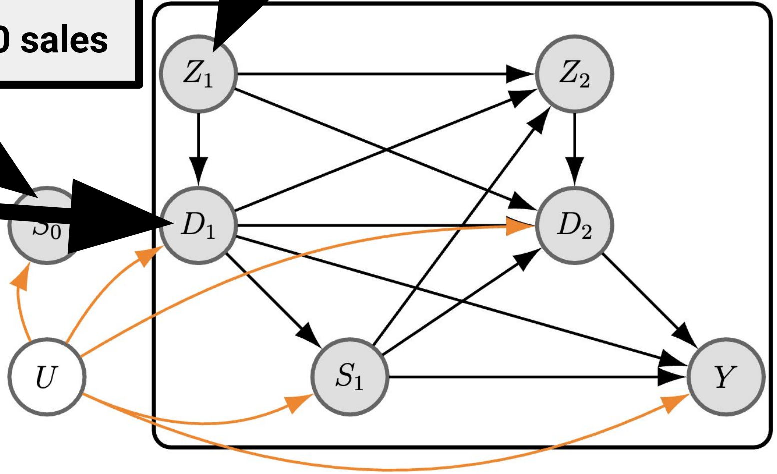

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State: S_1 = (1, 0, $0).
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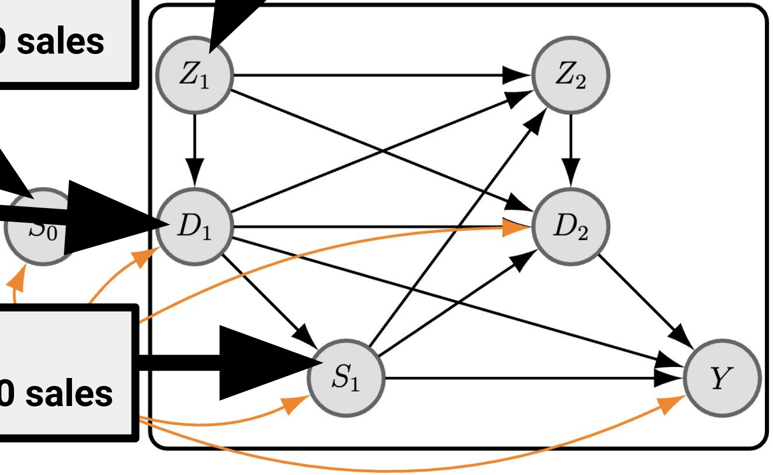

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Encouragement: Z_1 = 0.
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Encouragement: Z_2 = 1.
Offered free trial at checkout

State: S_0 = (0, 0, \$0).
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Treatment: D_1 = 0.
Did not redeem at checkout

State: S_1 = (1, 0, \$0).
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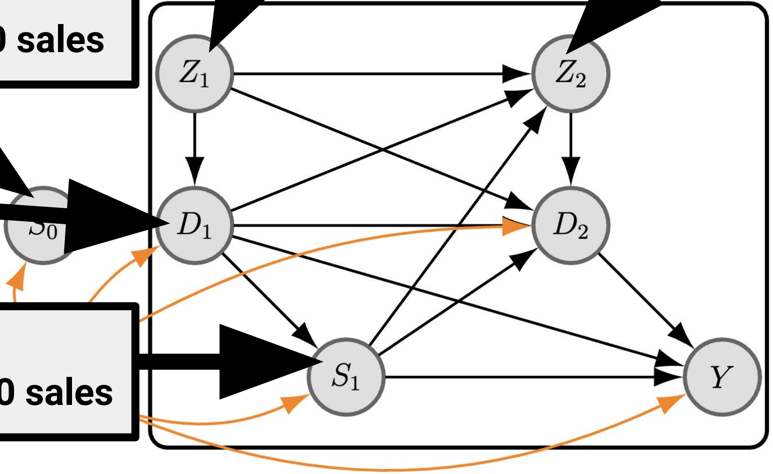

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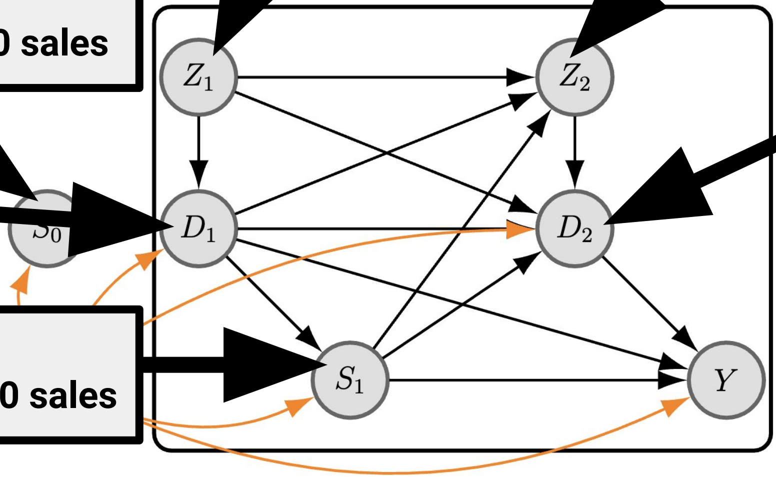

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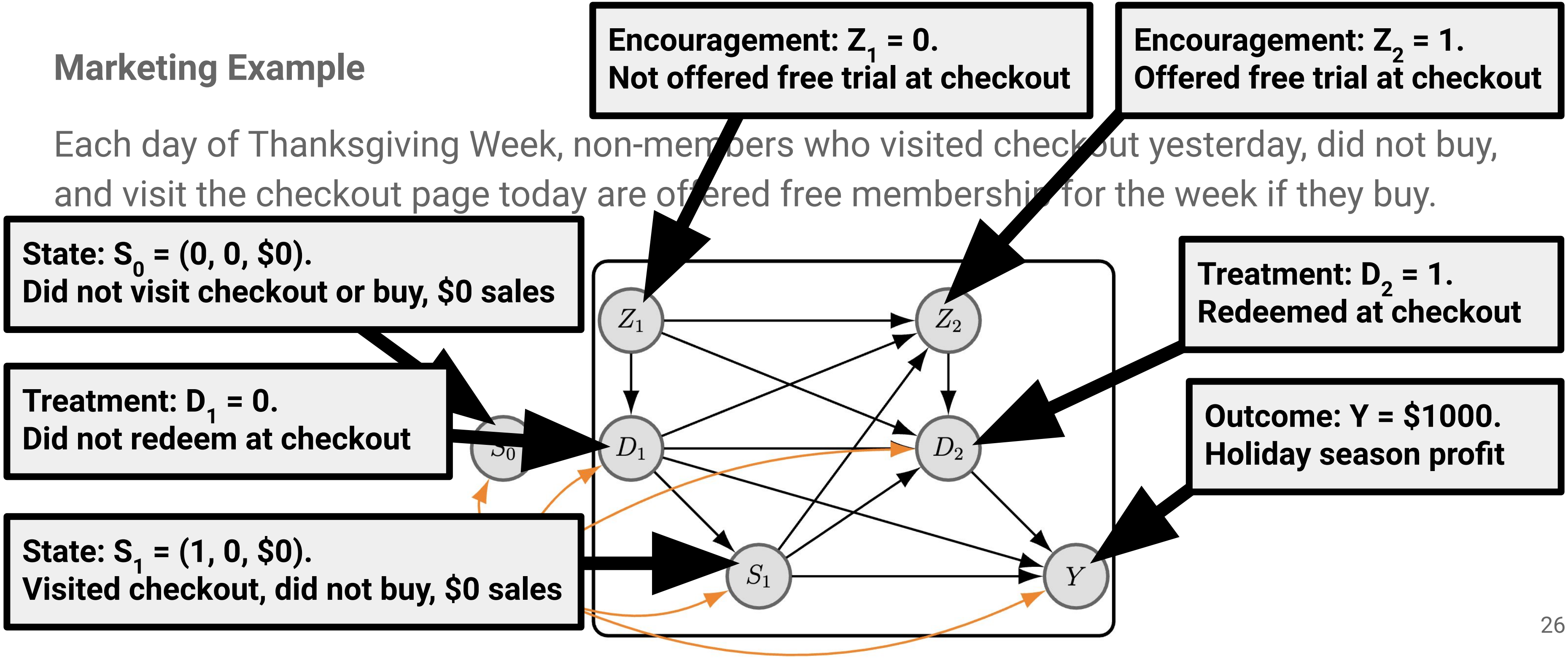


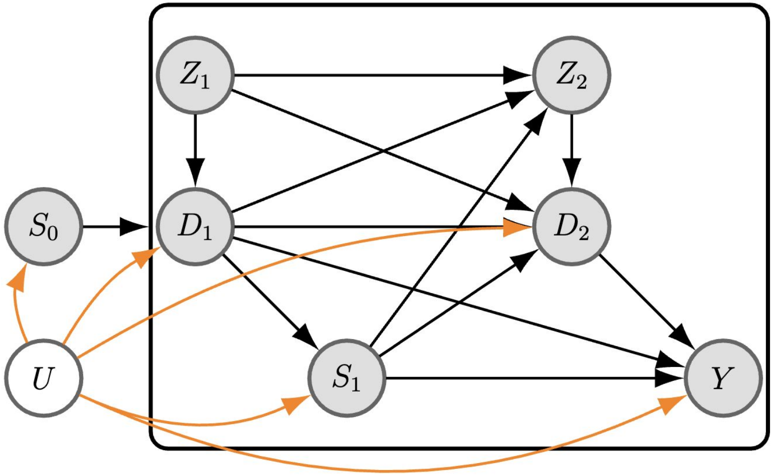
What should we offer non-members next year?

Marketing Example

Each day of Thanksgiving Week, non-members who visited checkout yesterday, did not buy, and visit the checkout page today are offered free membership for the week if they buy.

Question. Next year, how do we *improve the offer* for non-members to transact? On which days?

Perhaps name/describe the membership benefit/service that was most profitable because of the redemption?



Z_t: Offer

D_t: Redeem

S_t: Purchase Intent

Y: Gross profit

This is a DTR since offers depend on prior states.

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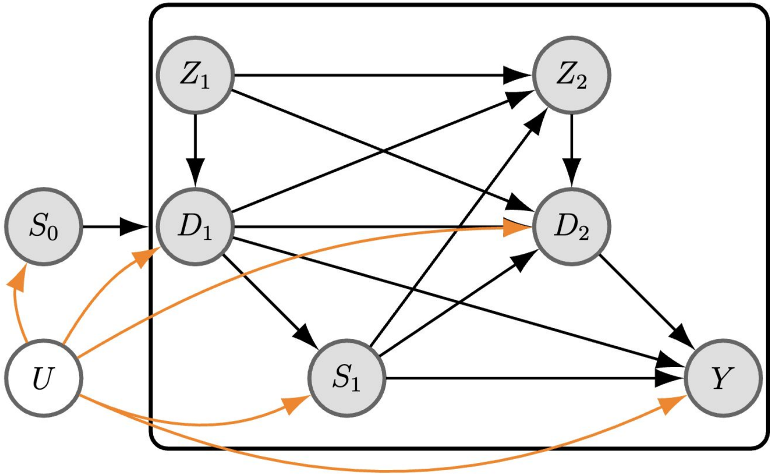

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There is noncompliance since a subset redeems.

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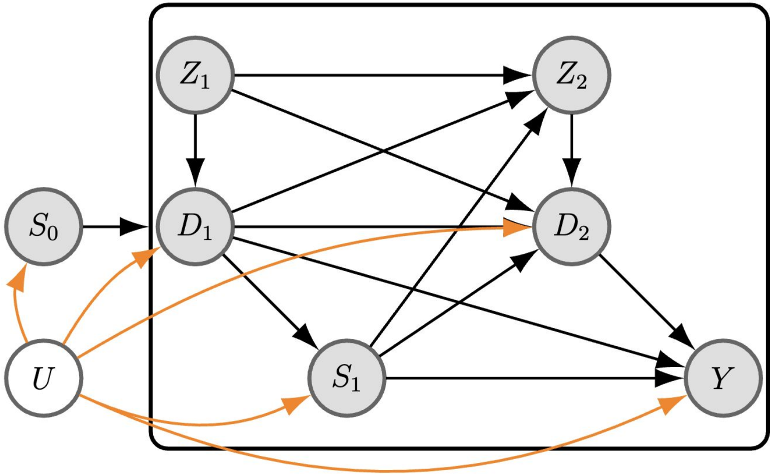

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Treatments, States, & Outcomes are confounded.

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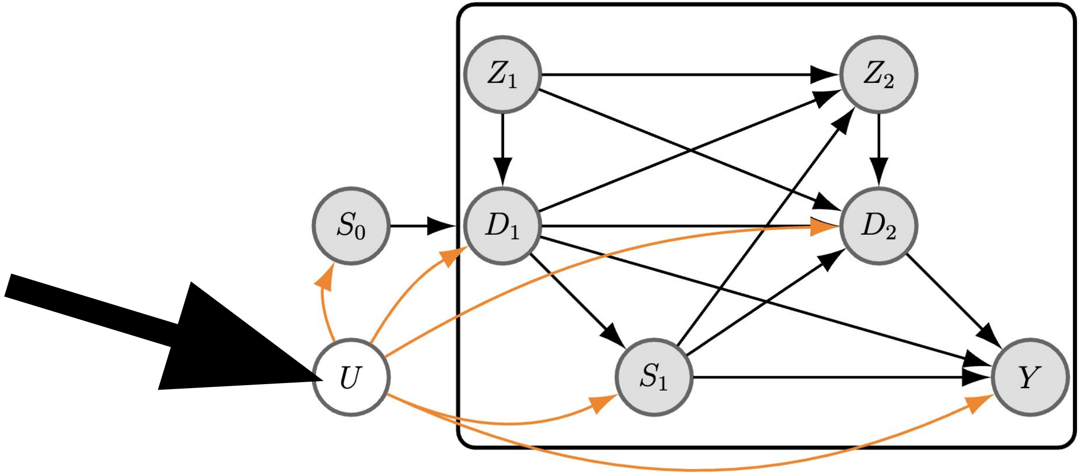

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Thm 1. One Sided Noncompliance enables $\tau_{1,0}, \tau_{0,1}$.

Theorem.

Under sequential extensions of identifying assumptions for LATE and replacing Monotonicity with One Sided Noncompliance, *for **d=z ∈ { (1,0), (0,1) }, we have***

$$\mathbb{E}[Y(d) - Y(0,0) \mid D(z) = d, D(0,0) = (0,0)] = \frac{\mathbb{E}[Y(D(z))] - \mathbb{E}[Y(D(0,0))]}{\Pr\{D(z) = d\}}$$

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Imbens and Angrist (1994)

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1. ***Sequential*** No-interference Consistency for Y, D, S, Z;
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$$\boxed{\begin{aligned} S_0 &\triangleq S_0(\cdot) \\ Z_1 &\triangleq Z_1(S_0) \\ D_1 &\triangleq D_1(Z_1, S_0) \\ S_1 &\triangleq S_1(D_1, S_0) \\ Z_2 &\triangleq Z_2(Z_1, D_1, S) \\ D_2 &\triangleq D_2(Z, D_1, S) \\ Y &\triangleq Y(D, S) \end{aligned}}$$

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$$\boxed{\begin{aligned} \{Y(D(z)), D(z)\} &\perp Z_1 \mid S_0 \\ \{Y(D_1, D_2(Z_1, z_2)), D_2(Z_1, z_2)\} &\perp Z_2 \mid S, D_1, Z_1 \end{aligned}}$$

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$$\boxed{\begin{gathered}\Pr\{Z_1 = z_1 \mid S_0\} > 0 \\ \Pr\{Z_2 = z_2 \mid Z_1 = z_1, D_1, S_0, S_1\} > 0, \forall z \in \mathcal{Z}\end{gathered}}$$

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$$\boxed{\begin{aligned} \mathbb{E}[D_1(1) - D_1(0) \mid S_0] > 0 \\ \mathbb{E}[D_2(Z_1, 1) - D_2(Z_1, 0) \mid Z_1, D_1, S] > 0 \end{aligned}}$$

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May (not) redeem if offered, but cannot redeem otherwise.

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Under sequential extensions of identifying assumptions for LATE and **replacing Monotonicity with One Sided Noncompliance**, *for **d=z ∈ { (1,0), (0,1) }, we have***

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$$\boxed{\Pr\{D_t(z_{<t}) \leq z_t\} = 1}$$

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One time period is treated

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Effect of D on Y for compliers

Effect of Z on D for everyone

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A causal effect is identified if it equals an estimable function of the observational data distribution.

Theorem.

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$$\mathbb{E}[Y(D(z))] = \mathbb{E}[\mathbb{E}[\mathbb{E}[Y \mid S, D_1, Z = z] \mid S_0, Z_1 = z_1]]$$
$$\Pr(D(z) = d) = \mathbb{E}[\mathbb{E}[\Pr(D = d \mid S, D_1, Z = z) \mid S_0, Z_1 = z_1]].$$

One Sided Noncompliance also gives Local-ITT.

Theorem.

Under sequential extensions of identifying assumptions for LATE[1] and replacing sequential Monotonicity with One Sided Noncompliance, *for **z≠(0,0), we have***

$$\mathbb{E}[Y(D(z)) - Y(0,0) \mid D(z) \neq (0,0), D(0,0) = (0,0)] = \frac{\mathbb{E}[Y(D(z))] - \mathbb{E}[Y(D(0,0))]}{1 - \Pr\{D(z) = (0,0)\}}$$

$$\mathbb{E}[Y(D(z))] = \mathbb{E}[\mathbb{E}[\mathbb{E}[Y \mid S, D_1, Z = z] \mid S_0, Z_1 = z_1]]$$
$$\Pr(D(z) = d) = \mathbb{E}[\mathbb{E}[\Pr(D = d \mid S, D_1, Z = z) \mid S_0, Z_1 = z_1]].$$

One Sided Noncompliance also gives Local-ITT.

People who comply at least once.

Theorem.

Under sequential extensions of identifying assumptions for LATE[1] and replacing sequential Monotonicity with One Sided Noncompliance, *for* ***z≠(0,0), we have***

$$\mathbb{E}[Y(D(z)) - Y(0,0) \mid D(z) \neq (0,0), D(0,0) = (0,0)] = \frac{\mathbb{E}[Y(D(z))] - \mathbb{E}[Y(D(0,0))]}{1 - \Pr\{D(z) = (0,0)\}}$$

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Effect of Z on Y for everyone

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Effect of D on Y for people comply at least once with an offer

Effect of Z on D for everyone

One Sided Noncompliance also gives Local-ITT.

A causal effect is identified if it equals an estimable function of the observational data distribution.

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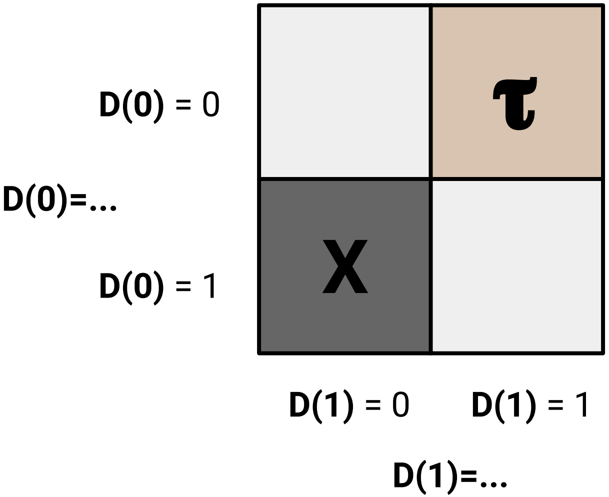
This is stronger than (Sequential) Montonicity.

$$\mathbb{E}[Y(1) - Y(0) \mid D(1) - D(0) = 1] = \frac{\mathbb{E}[Y \mid Z = 1] - \mathbb{E}[Y \mid Z = 0]}{\mathbb{E}[D \mid Z = 1] - \mathbb{E}[D \mid Z = 0]}$$

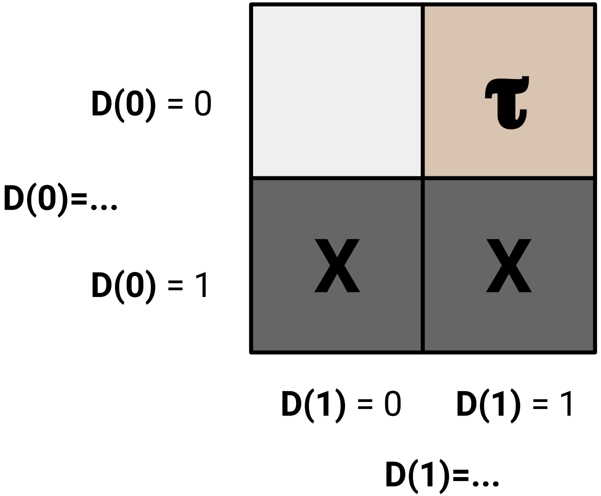
Monotonicity

$$D_1(1) \geq D_1(0)$$



One Sided Noncompliance

$$D_1(z_1) \leq z_1$$



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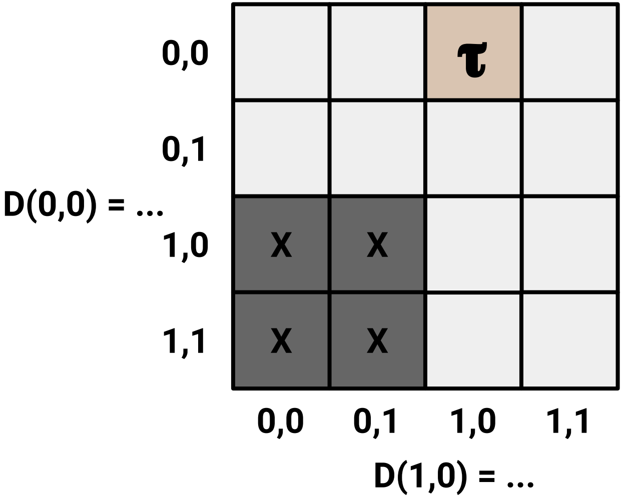
$$D_1(1) \geq D_1(0)$$
$$D_2(Z_1, 1) \geq D_2(Z_1, 0)$$

$$D_1(z_1) \leq z_1$$
$$D_2(z) \leq z_2$$

(Sequential) Monotonicity



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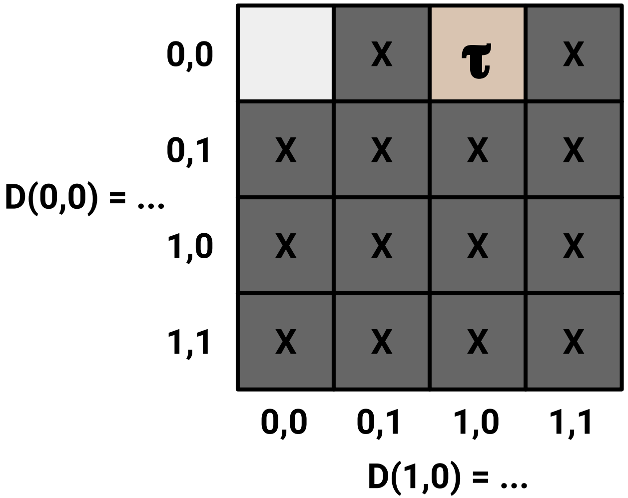

But, Sequential Monotonicity will not work: $\tau_{1,0}$.

We can construct the same observational data distribution under two different LATE values.

This means estimators cannot converge the true value in at least one of the scenarios.

This is why we used One Sided Noncompliance.

But, this is not sufficient for $\boldsymbol{\tau}_{1,1}$ identification.

One Sided Noncompliance is insufficient for $\tau_{1,1}$.

Suppose we want an identification result for E[Y(1,1) - Y(0,0) | D(1,1)=(1,1), D(0,0)=(0,0)].
The reduced form estimand has three subpopulations that do not cancel themselves out.

One Sided Noncompliance

D(0,0) = ... \ D(1,1) = ...	0,0	0,1	1,0	1,1
0,0		???	???	τ
0,1	X	X	X	X
1,0	X	X	X	X
1,1	X	X	X	X

Thm 2. $\mathrm{Cov}[\boldsymbol{\tau}_{i,1,0}, D_2(1,1) \mid D_1(1)=1, S_0] = 0$ gives $\boldsymbol{\tau}_{1,1}$.

Examples 1,2: Staggered Adoption & Compliance

Lemma.

Under the assumptions of the first theorem, if Staggered Adoption $\Pr[D_2=1 \mid S_0, Z_1=1, D_1=1] = 1$ or "Staggered Compliance" $\Pr[D_2(1,1)=1 \mid S_0, Z_1=1\ D_1=1] = 1$ holds, ***then***

$$\mathbb{E}[Y(1,1) - Y(0,0) \mid D(1,1) = (1,1), D(0,0) = (0,0)]$$

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Once treated, remain treated

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Ratio of effects (for everyone) of (Z adaptive wrt D) on Y, D

Thm 2. Cov[$\boldsymbol{\tau}_{i,1,0}$,$D_2(1,1)$ | $D_1(1)$=1,S_0] = 0 gives $\boldsymbol{\tau}_{1,1}$.

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Under the assumptions of the previous theorem, if the ***effect*** of not continuing to comply is independent (on average) ***of whether units*** remain compliers (Cov[$\boldsymbol{\tau}_{i,1,0}$,$D_2(1,1)$ | $D_1(1)$=1,S_0] = 0), ***then***

$$\mathbb{E}[Y(1,1) - Y(0,0) \mid D(1,1) = (1,1), D(0,0) = (0,0)]$$

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$$\Gamma \triangleq \mathbb{E}[\mathbb{E}[Y \mid H_2, Z_2 = D_1] \mid S_0, Z_1 = 1] - \mathbb{E}[\mathbb{E}[Y \mid H_2, Z_2 = 0] \mid S_0, Z_1 = 0]$$

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Effect of D on Y for compliers

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Effect of adaptive (wrt D) policy Z on Y for everyone

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Effect of Z on D for everyone

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Theorem.

Correction term

Under the assumptions of the previous theorem, if the ***effect*** of not continuing to comply is

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Effect of Z on D for everyone

Examples 3+: Put structure on confounders.

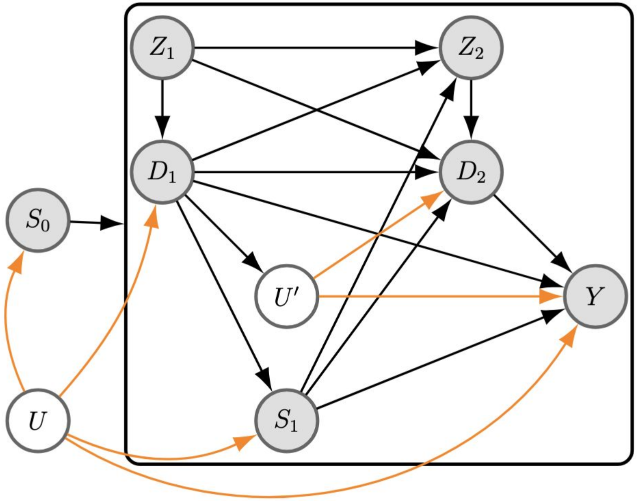

Figure 3: Example Directed Acyclic Graphs that satisfies assumption when combined with $Y(1,0) - Y(0,0) \perp\!\!\!\perp U'$. For example, $f_Y(d, s, u, u', \epsilon) = \tau_{10}(s, u, \epsilon)d_1 + \tau_{01}(s, u, u', \epsilon)d_2 + \tau_{11}(s, u, u', \epsilon)d_1 d_2 + Y_{0,0}(s, u, u', \epsilon)$.

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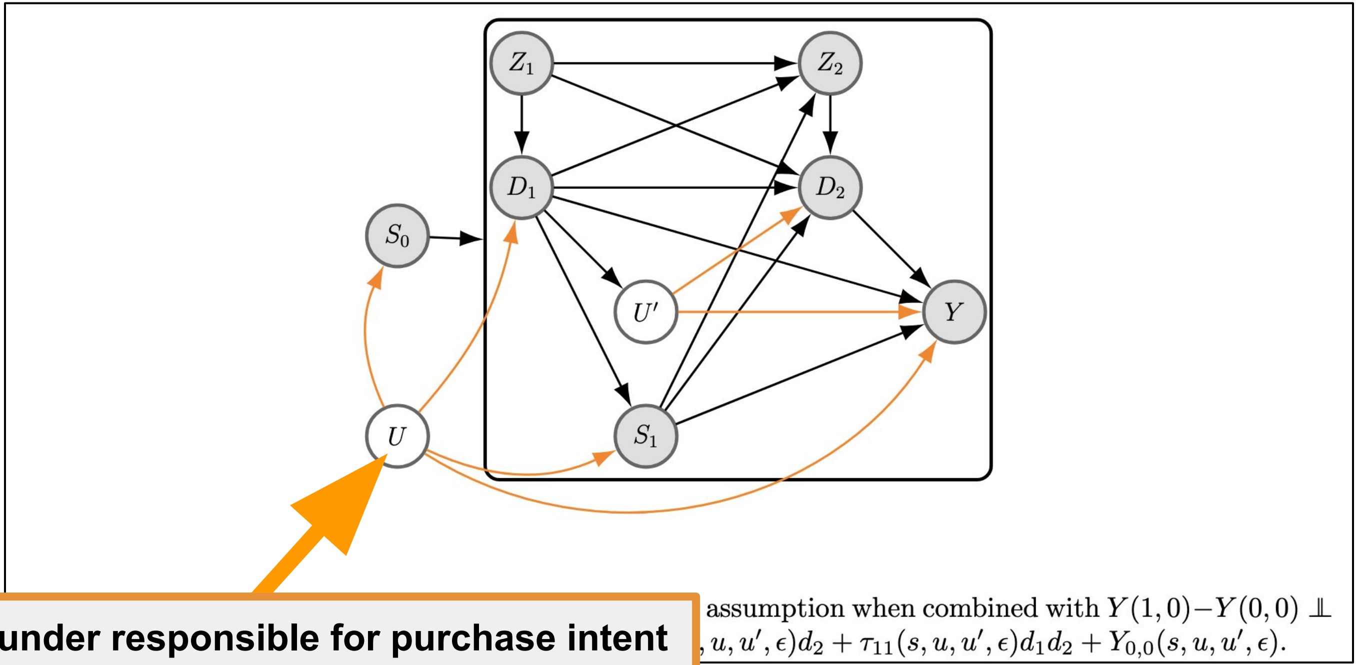
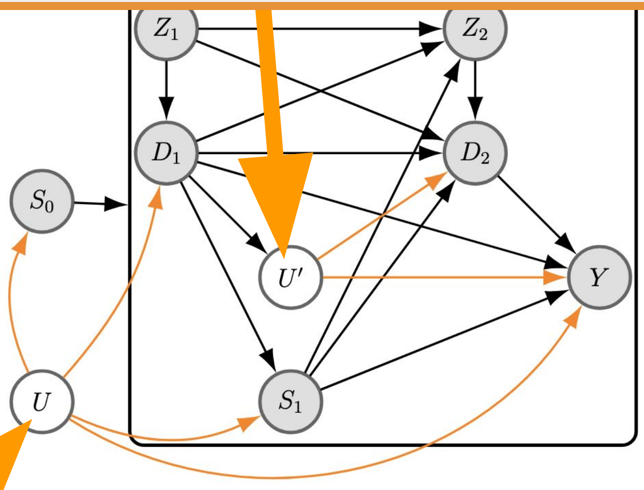


E.g. confounder responsible for purchase intent

Examples 3+: Put structure on confounders.

E.g. confounder responsible for subsequent platform side changes (e.g. free trial is extended)



E.g. confounder responsible for purchase intent

assumption when combined with $Y(1,0)-Y(0,0) \perp\!\!\!\perp$

$, u, u', \epsilon)d_2 + \tau_{11}(s, u, u', \epsilon)d_1 d_2 + Y_{0,0}(s, u, u', \epsilon).$

Our results hold *even if there are no dynamics*.

We also give alternatives to the second condition.

Under alternative conditions, you can further weaken overlap and ignorability (see next draft).

There is prior work on using IVs for ATE ID.

E.g. Han (2021), Heckman and Navarro (2007); James J Heckman, John Eric Humphries, and Gregory Veramen (2016); Cui, Michael, Tanser, and Tchetgen Tchetgen (2023); Michael, Cui, Lorch, and Tchetgen Tchetgen (2023).

Many of these impose restrictions on the confounders or unobservables. E.g. Michael et. al (2023) and Cui et al. (2023) use the following.

$$\mathbb{E}[D_t(1) - D_t(0) \mid U_t, H_t] = \mathbb{E}[D_t(1) - D_t(0) \mid H_t]$$

There is prior work on optimal DTR ID using IVs.

For example,

Han (2023) gives partial identification;

Chen and Zhang (2023) give identification of a regime better than baseline; and

Spicker et al. (2024) give identification assuming structure on confounding.

There are attempts at >1 time period LATE ID.

Ferman and Tecchio (2023). Encouragements Z are static and cannot depend on the history of states S or treatments D. Homogeneous effects for those who do not always comply.

Miquel (2002). Encouragements Z can be dynamic, but cannot depend on prior treatments. Treatment D_t in each time period can only depend on that time period's encouragement Z_t.

Recently since our paper, **Picchetti (2024)**. Treatment D_t in each time period can only depend on the IV Z_t in that time period.

Check out our paper for more results!

Formally, we nonparametrically identify the quantities below in a way that allows dynamics.

Local Intent to Treat: $E[Y(D(\mathbf{z})) - Y(\mathbf{0}) \mid D(\mathbf{z}) \neq \mathbf{0}]$ for every z not equal to **0**.

Dynamic LATE: $E[Y(\mathbf{d}) - Y(\mathbf{0}) \mid D(\mathbf{z})=\mathbf{d}]$.

- When-to-Treat: $\mathbf{d}=\mathbf{z}$ and $\mathbf{z},\mathbf{d}$ are standard basis vectors.
- Always-Treat: $\mathbf{d}=\mathbf{z}$ and $\mathbf{z},\mathbf{d}$ equal vectors of ones.
- When-to-Start: $\mathbf{d}=\mathbf{z}$, $\mathbf{z}$ non-decreasing in its coordinates, and one in **z**'s last (T'th) entry.
- When-to-Comply: $\mathbf{d}_{\geq t'}=\mathbf{z}_{\geq t'}$ for some t' and $d_t=0$ for $t<t'$.

We can also identify all of these conditional on baseline covariates (Heterogeneous LATEs).

We also provide estimation, inference, generalization to T time periods, and simulations.

Treatment dynamics & noncompliance are rife!

In panel data settings, assignment often depends on prior states, and there is noncompliance.

Example Setting	Education	Marketing	Medical Treatment
Outcome (Y)	Student SAT score	Customer spend	Patient tumor size
Treatment (D)	Enroll in advanced course	Redeem discount	Continue strong drug
Dynamic encouragement (Z)	Offer advanced course if grades were high last year	Offer discount if customer added-to-cart yesterday	Encourage status quo if toxicity is acceptable
Treatment Noncompliance (D≠Z)	Students may not enroll in advanced classes	Customers may not redeem offered discounts	Physician shifts to weaker drug
Unobserved D-Y confounder (U)	Student has limited time (e.g. provides childcare to siblings)	Customer already owns the specific product	Patient puts more weight on quality than quantity of life

Thank you!

Dynamic Local Average Treatment Effects

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