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1) Commutative Property:-

We know that $GCD(a, b) = GCD(b, a)$ and $LCM(a, b) = LCM(b, a)$.

i.e, $a \cdot b = b \cdot a$ and $a + b = b + a$.

Hence verified.

2) Associative Property:-

We know that
$$GCD(GCD(a, b), c) = GCD(a, b, c) = GCD(b, c, a)$$
$$= GCD(GCD(b, c), a)$$
$$= GCD(a, GCD(b, c)).$$

Similarly,
$$LCM(LCM(a, b), c) = LCM(a, b, c) = LCM(a, LCM(b, c))$$

i.e, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and $(a+b)+c = a+(b+c)$.

Hence verified.

3) Distributive Property:-

We need to show that, $a+(b \cdot c) = (a+b) \cdot (a+c)$

$a \cdot (b+c) = a \cdot b + a \cdot c$ and $\cancel{(a+b) \cdot c = a \cdot c + b \cdot c \; (a+c) \cdot (b+c)}$

Consider,
$$a \cdot (b+c) = GCD(a, LCM(b, c))$$
$$a \cdot b + a \cdot c = LCM(GCD(a, b), GCD(a, c))$$

Let $N = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_n^{\alpha_n}$ with $\alpha_1 > 0, \alpha_2 > 0, \ldots \alpha_n > 0$.

Since a, b, c are divisors of N, we have.

$a = P_1^{a_1} \cdot P_2^{a_2} \cdots P_n^{a_n}$

$b = P_1^{b_1} \cdot P_2^{b_2} \cdots P_n^{b_n}$ with $a_i, b_i, c_i \geq 0$.

$c = P_1^{c_1} \cdot P_2^{c_2} \cdots P_n^{c_n}$

Now, $GCD(a, LCM(b, c))$

$$= P_1^{\min(a_1, \max(b_1, c_1))} \cdot P_2^{\min(a_2, \max(b_2, c_2))} \cdots P_n^{\min(a_n, \max(b_n, c_n))}$$

$$(\because GCD(P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_n^{\alpha_n}, P_1^{\beta_1} P_2^{\beta_2} \cdots P_n^{\beta_n}) = P_1^{\min(\alpha_1, \beta_1)} \cdots P_n^{\min(\alpha_n, \beta_n)}).$$

$LCM(GCD(a, b), GCD(a, c))$

$$= P_1^{\max(\min(a_1, b_1), \min(a_1, c_1))} \cdot P_2^{\max(\min(a_2, b_2), \min(a_2, c_2))} \cdots P_n^{\max(\min(a_n, b_n), \min(a_n, c_n))}$$

Consider

$\min(a, \max(b, c))$ and $\max(\min(a, b), \min(a, c))$.

ORDER	$\min(a, \max(b, c))$	$\max(\min(a, b), \min(a, c))$
$a \leq b \leq c$	a	a
$b \leq c \leq a$	c	c
$c \leq a \leq b$	a	a
$b \leq a \leq c$	a	a
$c \leq b \leq a$	b	b
$a \leq c \leq b$	a	a

On observation we have

$$\min(a, \max(b,c)) = \max(\min(a,b), \min(a,c)).$$

i.e., we have

$$GCD(a, LCM(b,c)) = LCM(GCD(a,b), GCD(a,c)).$$

Consider,

$$a+(b\cdot c) = LCM(a, GCD(b,c))$$

$$(a+b)\cdot(a+c) = GCD(LCM(a,b), LCM(a,c)).$$

We have

$LCM(a, GCD(b,c)) =$

$$P_1^{\max(a_1, \min(b_1,c_1))} \cdot P_2^{\max(a_2, \min(b_2,c_2))} \cdots P_n^{\max(a_n, \min(b_n,c_n))}$$

$GCD(LCM(a,b), LCM(a,c))$

$$= P_1^{\min(\max(a_1,b_1), \max(a_1,c_1))} \cdot P_2^{\min(\max(a_2,b_2), \max(a_2,c_2))} \cdots$$

$$\cdots P_n^{\min(\max(a_n,b_n), \max(a_n,c_n))}.$$

Consider $\max(a, \min(b,c))$ and $\min(\max(a,b), \max(a,c))$.

~~ORDER~~

ORDER	$\max(a,\min(b,c))$	$\min(\max(a,b),\max(a,c))$
$a\le b\le c$	b	b
$b\le c\le a$	a	a
$c\le a\le b$	a	a
$a\le c\le b$	c	c
$c\le b\le a$	a	a
$b\le a\le c$	a	a

i.e, $\max(a,\min(b,c)) = \min(\max(a,b),\max(a,c))$

Hence,

$a+(b\cdot c) = (a+b)\cdot(a+c)$

i.e, distributive property holds.

4) Identity Property:-

We have, $I_{\cdot} = N$ and $I_{+} = 1$.

i.e, $a\cdot N = a$ and $a+1 = a$.

$a\cdot N = GCD(a, N) = a$ [$\because a | N$ as a is factor of N].

$a+1 = LCM(1, a) = a$.

i.e, Identity Property holds.

5) Complement Property:-

Given that $a' = N/a$

Now $a + a' = LCM(a, N/a)$

$a \cdot a' = GCD(a, N/a)$.

Suppose that $GCD(a, N/a) = g$.

Then $a = b_1 g$ where $b_1, b_2 \in \mathbb{N}$.

$\frac{N}{a} = b_2 g$

$N = b_1 b_2 g^2$. i.e, there exists a perfect square, g^2 that divides N.

If $g \neq 1$,

$a \cdot a' = g \neq 1$ i.e., $a \cdot a' \neq I+$

$a + a' = LCM(a, N/a) = \frac{N}{g} \neq I$.

If $g = 1$

$a \cdot a' = 1 = I+$, $a + a' = N = I$.

So, if there exists no perfect square g^2 greater than 1, such that g^2 divides N, the given-set forms a boolean algebra.

Since, my entry number is 2019CS10369.

$N = 69 = 3 \times 23$.

Divisors of $N = \{1, 3, 23, 69\}$.

Hence there is no perfect square greater than 1, that divides 69.

Hence the tuple $\langle B, +, \cdot, I_+, I_\cdot \rangle$ forms a boolean algebra for $N = 69$.