Problem 1

Bui Quang Tu

February 7, 2024

Problem

Let θ, β be 3×3 skew-symmetric matrices and σ be a 3×3 matrix. Find symmetric S, T such that:

$$(S-\theta)(T-\beta) = \sigma$$

Solution

Assuming nonsingularity whenever necessary

From $(S-\theta)(T-\beta) = \sigma$ we have $S = \theta + \sigma(T-\beta)^{-1}$

S is hermitian if and only if

$$S = \theta + \sigma(T-\beta)^{-1} = \theta^\dagger + ((T-\beta)^{-1})^\dagger \sigma^\dagger$$

or equivalently:

$$2\theta = (T+\beta)^{-1}\sigma^\dagger - \sigma(T-\beta)^{-1}$$

$$2(T+\beta)\theta(T-\beta) = \sigma^\dagger(T-\beta) - (T+\beta)\sigma$$

$$2T\theta T + (2\beta\theta - \sigma^\dagger)T + T(\sigma - 2\theta\beta) - 2\beta\theta\beta + \sigma^\dagger\beta + \beta\sigma = 0$$

We denote $R = 2\theta; A = (2\theta\beta - \sigma); Q = -2\beta\theta\beta + \sigma^\dagger\beta + \beta\sigma$

then $A^\dagger = (2\beta\theta - \sigma^\dagger)$

The equation becomes

$$TRT + A^\dagger T - TA + Q = 0$$

with given A and skew-hermitian R, Q.

To find 1 solution we assume that T is diagonal.

Denote R $= \begin{pmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{pmatrix}$, A $= \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}$, Q $= \begin{pmatrix} 0 & q_1 & q_2 \\ -q_1 & 0 & q_3 \\ -q_2 & -q_3 & 0 \end{pmatrix}$,

T $= \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$

Then we have $TRT = \begin{pmatrix} 0 & cxy & bxz \\ -cxy & 0 & ayz \\ -bxz & -ayz & 0 \end{pmatrix}$, $A^\dagger X = \begin{pmatrix} 0 & m_4 y & m_7 z \\ m_2 x & 0 & m_8 z \\ m_3 x & m_6 y & 0 \end{pmatrix}$

Then we have the explicit form of the equation:

$$cxy + m_4 y - m_2 x = q_1 (1)$$

$$bxz + m_7 z - m_3 x = q_2 (2)$$

$$ayz + m_8 z - m_6 y = q_3 (3)$$

This system of equations is solved by eliminate z (by (2) and (3)) then calculate y from x (by the identity of xy). Then we are left with a quadratic equation of x.

Have x we can solve y, z.

The explicit solution is obtainable but not worth calculated by hand.