The Method of Elimination

Here are two examples of system solving using the method of elimination.

Example 1. In this example, we'll solve the linear system

$$\left|\begin{array}{rcrcrcr} & & y & + & z & = & 1 \\ 2x & & & - & 10z & = & -14 \\ 3x & + & 2y & - & 13z & = & -19 \end{array}\right|$$

As we know, the overall goal is to manipulate this system to get it into the form

$$\left|\begin{array}{ccccc} x & & & = & ? \\ & y & & = & ? \\ & & z & = & ? \end{array}\right|$$

We start by trying to get the coefficient of x in the first equation to be 1. Unfortunately, the first equation doesn't include x at all! The simplest way to deal with this is to swap the order of the equations; in this case, the second and third equations both include x, so we can swap either of these with the first row. Let's swap the first and second rows:

$$\left|\begin{array}{rcrcrcr} & & y & + & z & = & 1 \\ 2x & & & - & 10z & = & -14 \\ 3x & + & 2y & - & 13z & = & -19 \end{array}\right| \begin{array}{l} \\ \text{swap with (I)} \\ \\ \end{array} \rightarrow \left|\begin{array}{rcrcrcr} 2x & & & - & 10z & = & -14 \\ & & y & + & z & = & 1 \\ 3x & + & 2y & - & 13z & = & -19 \end{array}\right|$$

Now, to make the coefficient of x in the first equation equal to 1, we can divide the first equation by 2:

$$\left|\begin{array}{rcrcrcr} 2x & & & - & 10z & = & -14 \\ & & y & + & z & = & 1 \\ 3x & + & 2y & - & 13z & = & -19 \end{array}\right| \begin{array}{l} \div 2 \\ \\ \\ \end{array} \rightarrow \left|\begin{array}{rcrcrcr} x & & & - & 5z & = & -7 \\ & & y & + & z & = & 1 \\ 3x & + & 2y & - & 13z & = & -19 \end{array}\right|$$

Next, we eliminate x from the other equations; we do this by subtracting multiples of the first equation:

$$\left|\begin{array}{rcrcrcr} x & & & - & 5z & = & -7 \\ & & y & + & z & = & 1 \\ 3x & + & 2y & - & 13z & = & -19 \end{array}\right| \begin{array}{l} \\ \\ -3(\text{I}) \end{array} \rightarrow \left|\begin{array}{rcrcrcr} x & & & - & 5z & = & -7 \\ & & y & + & z & = & 1 \\ & & 2y & + & 2z & = & 2 \end{array}\right|$$

Now, we repeat the process with the next variable (y); we first want to make the coefficient of y in the second equation equal to 1 (it already is); then, we'll eliminate y from the other equations.

$$\left|\begin{array}{rcrcrcr} x & & & - & 5z & = & -7 \\ & & y & + & z & = & 1 \\ & & 2y & + & 2z & = & 2 \end{array}\right| \begin{array}{l} \\ \\ -2(\text{II}) \end{array} \rightarrow \left|\begin{array}{rcrcrcr} x & & & - & 5z & = & -7 \\ & & y & + & z & = & 1 \\ & & & & 0 & = & 0 \end{array}\right|$$

At this stage, we haven't achieved our goal of making the three equations look like $x = ?$, $y = ?$, $z = ?$, but we also can't make any more progress; we've done as much as we can to simplify the equations.

Let's introduce some language we'll be using throughout the semester to describe such a situation. We have now reduced our original system to a system of three simplified equations; the last equation gives us no useful information, so we really have two useful equations. In each of these equations, the first variable is called a leading variable; so, the leading variables are the circled variables:

$$\left|\begin{array}{ccccccc} x & & & - & 5z & = & -7 \\ & & y & + & z & = & 1 \\ & & & & 0 & = & 0 \end{array}\right|$$

We can rearrange the two useful equations to solve for the leading variables: the first says $x = 5z - 7$ and the second says $y = -z + 1$. This shows us that x and y are both determined by z, while z can be any real number. We emphasize this by saying that z can equal t, where t is any real number. Then, $x = 5t - 7$ and $y = -t + 1$. Therefore, we can express our solutions in vector form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5t - 7 \\ -t + 1 \\ t \end{bmatrix}.$$

We will often try to emphasize which terms in our solution depend on t, like this:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5t \\ -t \\ t \end{bmatrix} + \begin{bmatrix} -7 \\ 1 \\ 0 \end{bmatrix} = \boxed{t \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -7 \\ 1 \\ 0 \end{bmatrix} \text{ where } t \text{ is any real number}}$$

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In general, the variables that are *not* leading variables are called free variables (in the previous example, z was the only free variable) because they are free to equal any real number.

Here's a summary of the method of elimination.

The method of elimination

The method of elimination amounts to strategically using three basic operations:

1. Swapping two equations.
2. Dividing an equation by a non-zero number.
3. Adding or subtracting a multiple of one equation from another equation.

The basic strategy is to:

1. First make the coefficient of the first variable in the first equation 1; then eliminate that variable from all of the other equations.
2. Make the coefficient of the next variable in the next equation 1; then eliminate that variable from all of the other equations.
3. Repeat Step 2 until you're out of equations and/or variables.

Example 2. As another example, we'll solve the following system for x_1, x_2, x_3, x_4, x_5:

$$\left|\begin{array}{ccccccccccc} 3x_1 & - & 6x_2 & + & 3x_3 & + & 9x_4 & - & 3x_5 & = & 3 \\ 3x_1 & - & 6x_2 & + & 3x_3 & + & 9x_4 & - & x_5 & = & -9 \\ 4x_1 & - & 8x_2 & + & 7x_3 & & & - & x_5 & = & -20 \end{array}\right|$$

First, we make the coefficient of x_1 in the first equation 1 and eliminate x_1 from the remaining equations:

$$\left|\begin{array}{rcrcrcrcrcr} 3x_1 & - & 6x_2 & + & 3x_3 & + & 9x_4 & - & 3x_5 & = & 3 \\ 3x_1 & - & 6x_2 & + & 3x_3 & + & 9x_4 & - & x_5 & = & -9 \\ 4x_1 & - & 8x_2 & + & 7x_3 & & & - & x_5 & = & -20 \end{array}\right| \begin{array}{l} \div 3 \\ \\ \\ \end{array}$$

$$\rightarrow \left|\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & 3x_4 & - & x_5 & = & 1 \\ 3x_1 & - & 6x_2 & + & 3x_3 & + & 9x_4 & - & x_5 & = & -9 \\ 4x_1 & - & 8x_2 & + & 7x_3 & & & - & x_5 & = & -20 \end{array}\right| \begin{array}{l} \\ -3(\mathrm{I}) \\ -4(\mathrm{I}) \end{array}$$

$$\rightarrow \left|\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & 3x_4 & - & x_5 & = & 1 \\ & & & & & & & & 2x_5 & = & -12 \\ & & & & 3x_3 & - & 12x_4 & + & 3x_5 & = & -24 \end{array}\right|$$

Now, we should make the coefficient of x_2 equal to 1 in the second equation, but there are no x_2 terms left to work with, so we'll move on to x_3. (The first equation has an x_2 term, but we've already "fixed" the first equation with a leading variable of x_1, so we don't want to mess with that variable any more.)

$$\left|\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & 3x_4 & - & x_5 & = & 1 \\ & & & & & & & & 2x_5 & = & -12 \\ & & & & 3x_3 & - & 12x_4 & + & 3x_5 & = & -24 \end{array}\right| \begin{array}{l} \\ \\ \text{swap with (II)} \end{array}$$

$$\rightarrow \left|\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & 3x_4 & - & x_5 & = & 1 \\ & & & & 3x_3 & - & 12x_4 & + & 3x_5 & = & -24 \\ & & & & & & & & 2x_5 & = & -12 \end{array}\right| \begin{array}{l} \\ \div 3 \\ \\ \end{array}$$

$$\rightarrow \left|\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & 3x_4 & - & x_5 & = & 1 \\ & & & & x_3 & - & 4x_4 & + & x_5 & = & -8 \\ & & & & & & & & 2x_5 & = & -12 \end{array}\right| \begin{array}{l} -(\mathrm{II}) \\ \\ \\ \end{array}$$

Now that we're done with x_3, we move on to the next available variable in the remaining equation, which is x_5:

$$\rightarrow \left|\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & & & + & 7x_4 & - & 2x_5 & = & 9 \\ & & & & x_3 & - & 4x_4 & + & x_5 & = & -8 \\ & & & & & & & & 2x_5 & = & -12 \end{array}\right| \begin{array}{l} \\ \\ \div 2 \end{array}$$

$$\rightarrow \left|\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & & & + & 7x_4 & - & 2x_5 & = & 9 \\ & & & & x_3 & - & 4x_4 & + & x_5 & = & -8 \\ & & & & & & & & x_5 & = & -6 \end{array}\right| \begin{array}{l} +2(\mathrm{III}) \\ -(\mathrm{III}) \\ \\ \end{array}$$

$$\rightarrow \left|\begin{array}{rcrcrcrcrcr} x_1 & - & 2x_2 & & & + & 7x_4 & & & = & -3 \\ & & & & x_3 & - & 4x_4 & & & = & -2 \\ & & & & & & & & x_5 & = & -6 \end{array}\right|$$

So, we have three leading variables (x_1, x_3, x_5) and two free variables (x_2, x_4). The free variables can take any values, so we'll say $x_2 = s$ and $x_4 = t$. Writing our leading variables in terms of s and t, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 7t - 3 \\ s \\ 4t - 2 \\ t \\ -6 \end{bmatrix} = \boxed{s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ -6 \end{bmatrix} \text{ where } s, t \text{ are any real numbers}}$$

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