Team ranking algorithms

Introduction

Suppose that an intramural volleyball league consists of four teams T_1, T_2, T_3 and T_4. Part way into the season, their results look like: where W is games won, L is games lost, PF is points for and PA is points

Team	W	L	PF	PA
T_1	4	0	60	28
T_2	2	2	50	50
T_3	2	2	50	50
T_4	0	4	28	60

Table 1: Overall wins, losses and points scored for and against each team.

against. At first, one can think that teams T_2 and T_3 are equal. But what if T_2 has played T_1 twice (losing) and T_3 twice (winning), and T_3 has played T_2 twice (losing) and T_4 twice winning? A way of represent such schedule is the following matrix.

	T_1	T_2	T_3	T_4
T_1	-	2	0	2
T_2	2	-	2	0
T_3	0	2	-	2
T_4	2	0	2	-

Table 2: Match-up table

Since T_1 seems to be the stronger team, it is plausible to argue that the two losses of T_2 against T_1 are less meaningful that the two losses of T_3 against T_2, and thus give the edge to team T_2 in the current rankings. A related question is to predict the outcome of future games. If team T_2 eventually plays against T_4, by how much do we expect team T_2 to win/lose? We'll try to answer these questions looking at two common team rating systems.

Massey's Method

Mathematics professor Kenneth Massey developed this method while an undergraduate student at Bluefield College in 1997. It is one of the methods currently used by the Bowl Championship Series, a rating system that determines which teams in NCAA college football play in each of the bowl games. Also referred to as the Point Spread Method, this method incorporates the number of games played by each team as well as the difference in points scored by teams that played against each other.

Massey's main idea is the following: when a game is played between two teams, the team with higher rating should win by a score proportional to the difference between ratings. Mathematically, letting r_i and r_j be the ratings of teams T_1 and T_2, then ideally it should be satisfied that

$$r_i - r_j = y,$$

where y is the point difference in the games played between teams i and j. However, writing this equation for each pair of teams i and j could lead to too many equations and a potentially inconsistent system.

Question 1 (5 points)**.** *The number of possible games in a group of n teams is much larger than n. What is that number?*

Example

[(Cont.)] We'll look back at Table 1 to illustrate the method. Since in our example there are four teams, we will use r_1, r_2, r_3 and r_4 to denote their ratings.

Team T_1 has played 2 games against T_2 and two against T_4. For the total of four games, team T_1 should have won by a margin of

$$(r_1 - r_2) + (r_1 - r_2) + (r_1 - r_4) + (r_1 - r_4) = 4r_1 - 2r_2 - 2r_4.$$

Equaling this to T_1 actual four-game point spread of $60 - 28$ we come up with the first equation:

$$4r_1 - 2r_2 - 2r_4 = 32.$$

Doing the same for teams T_2, T_3 and T_4, we have:

$$\begin{aligned} 4r_2 - 2r_1 - 2r_3 &= 0 \\ 4r_3 - 2r_2 - 2r_4 &= 0 \\ 4r_4 - 2r_1 - 2r_3 &= -32 \end{aligned}$$

Overall, we have come up with a system of four equations and four unknowns:

$$\begin{bmatrix} 4 & -2 & 0 & -2 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ -2 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 0 \\ 0 \\ 32 \end{bmatrix}$$

Perhaps unsurprisingly, the system is consistent, but has infinitely many solutions:

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Question 2 (10 points)**.** *Interpret the results. What does the answer above say about the rankings of the teams? Which team is ranked higher? Why are there infinitely many solutions?*

1 Colley's Method

Now we'll look at a different method, the Colley Matrix Method. This was one of the six computer methods incorporated into the national Bowl Championship Series rankings, which were used from 1998 - 2013 to decide which NCAA football teams would play in bowl games. In comparison to Massey's method of ranking, Colley's Matrix Method has two big differences in philosophy: first, point spreads are not used, only wins/losses; second, Colley's method takes into account the strength of opponents (that is, a win over a strong opponent should mean more than a win over a weak opponent).

In Colley's method, each team's rating is a number between 0 or 1, with higher numbers being stronger. Colley's method boils down to the equation below. Let r_i denote the ranking of a team T_i, then:

$$r_i = \frac{1 + \frac{\text{num. of wins} - \text{num. of losses}}{2} + \text{the sum of all its- opponents' ratings}}{2 + \text{num. of games played by team } i}.$$

Example

Again, we follow the example in Tables 1 and 2. Team T_1 has played a total of 4 games, 2 against T_2 and 2 against T_4, winning all four games. Hence its rating should satisfy

$$r_1 = \frac{1 + \frac{4-0}{2} + (2r_2 + 2r_4)}{2+4}.$$

Note that each opponent is counted once for each game played. The other three equations, are left for you to find!

Question 3 (10 points)**.** *We'd like to use Colley's method on the game results given in Table 1; write down a matrix equation $A\vec{r} = \vec{b}$ that you can solve to find each team's rating. You'll have to write the the three remaining equations, and re-write the system in matrix form.*

Question 4 (5 points)**.** *Give each team's rating (between 0 and 1), as well as its rank. For this very simple example, are there any differences between Colley's and Massey's methods?*