THE TITLE

by

Ravi Rai

Submitted in partial fulfillment of the requirements
for the degree of Master of Science

at

Dalhousie University
Halifax, Nova Scotia
August 2021

Optionally, the thesis can be dedicated to someone, and the student can enter the dedication content here.

Table of Contents

Abstract

This is a test document.

Acknowledgements

Thanks to all the little people who make me look tall.

Chapter 1

Introduction

Get it done! Use reference material by Lamport [**?**] or Gooses, Mittelback, and Samarin [**?**].

To test if the margins are satisfactory, let us generate a lot of garbage text: This sentence goes on, and and here we should be around top of the page 2, and we go on, and on, and on, and on, and on. This following line ______________ should be exactly 5cm long. It can be used to check the typesetting process. And now we go on, and on.

Chapter 2

Basic Techniques

2.1 Stabilizer Nullity

Definition 2.1.1 (Stabilizer)**.** Let $|\psi\rangle$ be a non-zero n-qubit state. The stabilizer of $|\psi\rangle$ is the sub-group of the Pauli group $\mathcal{P}_n$ on n qubits for which $|\psi\rangle$ is a +1 eigenstate, denoted by Stab$|\psi\rangle$. This means that Stab$|\psi\rangle = \{P \in \mathcal{P}_n \; P\,|\psi\rangle = |\psi\rangle\}$. The states for which the size of the stabalizer is $2^n$are called stabalizer states. States for which the stabilizer contains only the identity matrix are said to have a trivial stabilizer. If Pauli P is in Stab$|\psi\rangle$, we say that P stabilizes $|\psi\rangle$.

Note the following facts about the Stab$|\psi\rangle$:

- Stab$|\psi\rangle$ does not contain -I.
- All Pauli group elements contained in Stab$|\psi\rangle$ commute with each other and are Hermitian matrices.
- The size of the stabilizer is equal to some power of two, and given any Clifford Unitary C, the size of Stab$|\psi\rangle$ is always equal to the size of Stab($C\,|\psi\rangle$.
- Finally, the stabilizer is multiplicative for the tensor products of states, that is $|\text{Stab}(|\psi\rangle\,|\phi\rangle)| = |\text{Stab}\,|\psi\rangle\,| \cdot |\text{Stab}\,|\phi\rangle\,|$.

Definition 2.1.2. Let $|\psi\rangle$ be a non-zero n-qubit state. The Stabilizer nullity of $|\psi\rangle$ is $\nu(|\psi\rangle) = n - log_2|\text{Stab}\,|\psi\rangle\,|$.

Proposition 2.1.3. *Let $|\psi\rangle$ be a non-zero n-qubit state and let P be an n-qubit Pauli matrix and suppose that the probability of a +1 outcome when measuring P on $|\psi\rangle$ is non-zero. Then there are two alternatives for the state $|\phi\rangle$ after measurement: either $|Stab\,|\phi\rangle\,| = |Stab\,|\psi\rangle\,|$ or $|Stab\,|\phi\rangle\,| \geq 2|Stab\,|\psi\rangle\,|$, both of which satisfy $\nu(|\phi\rangle) \leq \nu(|\psi\rangle)$.*

Proof. First consider the simple case when P is in Stab$|\psi\rangle$. In this case, the "+1" measurement outcome occurs with probability 1 and $|\psi\rangle$ is unchanged. When P is not in Stab$|\psi\rangle$ we consider two alternatives. The first alternative is that P commutes with all elements of Stab$|\psi\rangle$. Recall we have the post-measurement state $|\phi\rangle = \frac{P|\psi\rangle}{\sqrt{\langle\psi|P^\dagger P|\psi\rangle}} = \frac{P|\psi\rangle}{\sqrt{\langle\psi||\psi\rangle}}$, and let $Q \in \text{Stab}\,|\psi\rangle$. Then $Q\,|\phi\rangle = \frac{QP|\psi\rangle}{\sqrt{\langle\psi||\psi\rangle}} = \frac{PQ|\psi\rangle}{\sqrt{\langle\psi||\psi\rangle}} = \frac{P|\psi\rangle}{\sqrt{\langle\psi||\psi\rangle}} = |\phi\rangle$. Note also that Stab$|\phi\rangle$ also contains PStab$|\psi\rangle$, and thus Stab$|\phi\rangle$ contains Stab$|\psi\rangle \cup P$Stab$|\psi\rangle$ and thus its size is at least $2|\text{Stab}\,|\psi\rangle\,|$.

The second alternative is when P anti-commutes with some element $Q \in \text{Stab}|\psi\rangle$. Note that $Q\,|\psi\rangle = |\psi\rangle$ and $QPQ = -P$, so the probability of the +1 outcome is $\langle\psi|\,(I+P)\,|\psi\rangle\,/2 = \langle\psi|\,Q(I+P)Q\,|\psi\rangle\,/2 = \langle\psi|\,(I-P)\,|\psi\rangle\,/2$, which is the probability of the -1 outcome. Thus the probability of the +1 outcome is $1/2$. Then $|\phi\rangle = (I+P)/\sqrt{2}\,|\psi\rangle$ where we fixed the normalization condition such that $\langle\phi|\,|\phi\rangle = \langle\psi|\,|\psi\rangle$. Also, observe that we can write $|\phi\rangle = (I+PQ)/\sqrt{2}\,|\psi\rangle$. Since $(I+PQ)/\sqrt{2}$ is a Clifford unitary equal to $\exp(i\pi P'/4)$ for $P' = iPQ$, we see that $|\phi\rangle$ and $|\psi\rangle$ differ by a Clifford and therefore $|\text{Stab}|\psi\rangle\,| = |\text{Stab}|\phi\rangle\,|$. $\square$

2.2 Next Step

Do it!

Of course, you have to have pictures to show how you did it to make people understand things better.

Chapter 3

Conversion of Resource States

Theorem 3.0.1. *Let $|U\rangle$ be an n-qubit magic state for a diagonal unitary U from the 3^{rd} level of the Clifford hierarchy, and let $\tau(U)$ be the minimum number of T gates needed to implement U using the gate set $\{CNOT, S, T\}$. The following resource conversion is possible*

$$|U\rangle \xRightarrow{|T\rangle^{\otimes \tau(U)-\nu(|U\rangle}} |T\rangle^{\otimes 2\nu(|U\rangle)-\tau(U)}$$

Proof. Recall the following phase polynomial formalism. For any diagonal unitary in the 3^{rd} level Clifford hierarchy we have $U_f = \sum_x exp(if(x)\pi/4)\,|x\rangle\langle x|$, where $f : \mathbb{Z}_2^n \to \mathbb{Z}_8$ is of cubic form and so can be decomposed as the phase polynomial $f(x) = \sum_{a_k \neq 0} a_k \lambda_k(x)$ (mod 8) where $a_k \in \mathbb{Z}_8$ and each λ_k is a $\mathbb{Z}_2$ linear function. That is, each λ_k has the form $\lambda_k(x) = (P_{1,k}x_1) \oplus (P_{2,k}, x_2) \ldots (P_{n,k}, x_n)$ (mod 2) where $P_{j,k}$ are binary. Thus we can describe the function by a binary matrix P and vector a, with columns corresponding to nonzero a_k (so the number of columns is the number of terms in f). For a function with a single term $f(x) = a_k \lambda_k(x)$, an easily verified circuit decomposition is $U_{\lambda_k} = \sum_x exp(i\lambda_k(x)\pi/4)\,|x\rangle\langle x| = V^\dagger_{CNOT(\lambda_k)} T_1^{a_k} V_{CNOT(\lambda_k)}$ where T_1 is a T gate acting on qubit 1 and $V_{CNOT(\lambda_k)}$ is a cascade of CNOT gates such that

$$V_{CNOT(\lambda_k)}\,|x\rangle = V_{CNOT(\lambda_k)}\,|x_1, x_2, \ldots x_n\rangle = |\lambda_k(x_1), x_2, \ldots x_n\rangle.$$

Now note that if a_k is even then $T_a^{a_k} = S_1^{a_k/2}$ is a Clifford and the whole circuit is Clifford. But if a_k is odd then $T_1^{a_k} = T_1 S_1^{(a_k-1)/2}$ and only a single T gate is used. Now, generalizing to a phase polynomial f with many terms we have $U_f = \prod_k U_{\lambda_k}$ and so the T-count for the associated circuit is equal to the number of odd valued a_k (so if all values are even then the unitary is Clifford).

This allows us to split the unitary U_f into a Clifford and non-Clifford part. For each a_k coefficient, we define $b_k \in \mathbb{Z}_4$ and $c_k \in \mathbb{Z}_2$ such that $a_k = 2b_k + c_k$ (so $c_k = 1$ if and only if a_k is odd). Now for functions $g(x) = \sum_{c_k \neq 0} c_k \lambda_k(x)$ and $h(x) = \sum_{b_k \neq 0} b_k \lambda_k(x)$ we have that $f = g + h$ and $U_f = U_{g+2h} = U_g U_{2h}$ where U_{2h} is a Clifford Unitary. The non-Clifford part is U_g and all the terms have odd valued co-coefficients, so the number of terms in g gives an upper bound on $\tau(U_g)$ as discussed earlier. It follows that if the function g has m (odd-valued) terms then the state can be prepared using m many T gates/states. Note that for any given unitary U_g there is an equivalence class of different functions g that all result in the same unitary but with different numbers of terms. From now on we will assume that g is the optimal representative with the fewest number of terms, denoted by $\tau(U_g)$. Furthermore, there is a binary matrix P description of g with a number of columns also equal to $\tau(U_g)$. A trivial but relevant example is $U = T^{\otimes n}$ for which $P = \mathbb{1}_n$ and $\tau(T^{\otimes n}) = n$.

The next important step is that given a unitary U_g we may also be able to remove terms from g by applying inverse T gates. More generally, given two such unitaries U_g and $U_{g'}$ with phase polynomials g and g', we have that $U_{g'} = U_g U_\Delta$ where $\Delta = g - g'$. Therefore,

$$|U_{g'}\rangle = U_\Delta |U_g\rangle \tag{3.1}$$

and

$$|T\rangle^{\otimes\tau(U_\Delta)} |U_{g'}\rangle \rightarrow |U_g\rangle \tag{3.2}$$

The number of T states needed is $\tau(U_\Delta)$, which just the number of terms where g and g' differ.

Using arguments from [**?**], given any P we can always bring it into row-reduced echelon form using a CNOT circuit. Then

$$P = \begin{pmatrix} \mathbb{1}_r & A \\ 0 & 0 \end{pmatrix} \tag{3.3}$$

where $\mathbb{1}_r$ is an identity matrix of size $r := \mathrm{rank}(P)$. If P is full rank the additional 0 padding is not present. Note that if P has any 0 rows then the unitary acts trivially on the corresponding qubits leaving them in the $|+\rangle$ state, meaning that $|U\rangle = U|+\rangle = |\psi\rangle |+\rangle^{\otimes(n-r)}$ for some state $|\psi\rangle$. Also, for an n qubit stabilizer state $|\phi\rangle$,

$$\nu(|\phi\rangle) = 0 \Rightarrow \log_2 |Stab\,|\phi\rangle\,| = n \tag{3.4}$$

Next, observe that

$$\begin{aligned}
\log_2 |Stab\,|U\rangle\,| =& \log_2 |Stab(|\psi\rangle\,|+\rangle^{\otimes(n-r)})| \\
=& \log_2 (|Stab\,|\psi\rangle\,| \cdot |Stab\,|+\rangle^{\otimes(n-r)}\,|) \\
=& \log_2 |Stab\,|\psi\rangle\,| + \log_2 |Stab\,|+\rangle^{\otimes(n-r)}\,| = \quad \log_2 |Stab\,|\psi\rangle\,| + (n-r) = \alpha + n - r
\end{aligned}$$

for some positive integer $\alpha = \log_2|Stab\,|\psi\rangle\,|$. Hence $\log_2|Stab\,|U\rangle\,| \geq n - r$, so rearranging we have that $n - \log_2|Stab\,|U\rangle\,| = \nu(|U\rangle) \leq r$.

Using our earlier argument, we can always remove from P the columns corresponding to the matrix A using a number of T states equal to the number of columns in A. Since A has $\tau(U_g) - r$ columns, this requires the same quantity of T states. The resulting $U_{g'}$ has $P' = \mathbb{1}_r$ (with possibly some 0 row padding) which corresponds to r copies of T states. Therefore, we can perform

$$|U_g\rangle\,|T\rangle^{\otimes(\tau(U_g)-r)} \to |T\rangle^{\otimes r} \tag{3.5}$$

If $r = \nu(U_g)$ then we have the result of the theorem. If $r > \nu(U_g)$ then the result is even stronger than the theorem, and so the theorem holds in either case. $\square$

Chapter 4

Conclusion

Did it!