LOWER BOUNDS NON-CLIFFORD RESOURCES

by

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2.1 Stabilizer Nullity

Definition 2.1.1. The *Pauli matrices* X, Y, and Z are defined as:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Definition 2.1.2. The *Pauli group on n qubits* $\mathcal{P}(n)$ is a group with elements $\{\pm I, \pm X, \pm Y, \pm Z, \pm iI, \pm iX, \pm iY, \pm iZ\}^{\otimes n}$, where $\otimes n$ represents tensors of these 2×2 matrices.

The Pauli group defines the $1st - level$ of the Clifford Hierarchy, which is $\mathcal{C}^{(1)} = e^{i\theta} \cdot \{I, X, Y, Z\}^{\otimes n}$, so it is as above but with all phases. The group of automorphisms of the Pauli group is the $2nd-level$ of the Clifford Hierarchy, $\mathcal{C}^{(2)}$, and is also known as the Clifford Group. It is important to note that circuits made from only gates in $\mathcal{C}^{(2)}$ are not universal for quantum computing. To get around this issue, we can use gates from the $3rd - level$ of the Clifford Hierarchy: $\mathcal{C}^{(3)} = \{U | UPU^\dagger \in \mathcal{C}^{(2)}, \forall P \in \mathcal{P}(n)$, and so we will use unitaries from $\mathcal{C}^{(3)}$ frequently. Finally, we now define the $kth-level$ of the Clifford Hierarchy.

Definition 2.1.3. The k-th level of the Clifford Hierarchy is:

$$\mathcal{C}^{(k)} = \{U | UPU^\dagger \in \mathcal{C}^{(k-1)}, \forall P \in \mathcal{P}(n)\} \tag{2.1}$$

Proposition 2.1.4. *We have* $|\mathcal{P}(n)| = 4^{n+1}$.

Proof. By induction, first let $n = 1$. Then from definition 2.1.2 we easily see that there are $16 = 4^2$ elements in $\mathcal{P}(1)$, so the base case is true. Now assume true for m qubits, i.e. $|\mathcal{P}(m)| = 4^{m+1}$. Then to $\mathcal{P}(m+1)$ has elements that are tensor products of elements in $\mathcal{P}(m)$ and I, X, Y, and Z. Thus there are $4 \cdot 4^{m+1} = 4^{(m+1)+1}$ elements in $\mathcal{P}(m+1)$. Note we do not take tensor products with $-I, \pm iI, -X, \ldots$, as we would then quadruple count the elements in $\mathcal{P}(m+1)$. □

Definition 2.1.5 (Stabilizer)**.** Let $|\psi\rangle$ be a non-zero n-qubit state. The stabilizer of $|\psi\rangle$ is the sub-group of the Pauli group $\mathcal{P}_n$ on n qubits for which $|\psi\rangle$ is a $+1$ eigenstate, denoted by Stab$|\psi\rangle$. This means that Stab$|\psi\rangle = \{P \in \mathcal{P}_n \, P\,|\psi\rangle = |\psi\rangle\}$. The states for which the size of the stabilizer is $2^n$are called stabilizer states. States for which the stabilizer contains only the identity matrix are said to have a trivial stabilizer. If Pauli P is in Stab$|\psi\rangle$, we say that P stabilizes $|\psi\rangle$.

Proposition 2.1.6. *Let $|\psi\rangle$ be a non-zero n qubit state. Then we have the following facts about Stab$|\psi\rangle$:*

1. *Stab$|\psi\rangle$ does not contain -I.*

2. *All Pauli group elements contained in Stab$|\psi\rangle$ commute with each other and are Hermitian matrices.*

3. *The size of the stabilizer is equal to some power of two.*

4. *Given any Clifford Unitary C, the size of Stab$|\psi\rangle$ is always equal to the size of Stab$(C\,|\psi\rangle)$.*

5. *Finally, the size of the stabilizer is multiplicative for the tensor products of states, that is $|Stab(|\psi\rangle\,|\phi\rangle)| = |Stab\,|\psi\rangle\,| \cdot |Stab\,|\phi\rangle\,|$.*

Proof.

1. If $I \in \text{Stab}\,|\psi\rangle$, then $-\,|\psi\rangle = -I\,|\psi\rangle = |\psi\rangle$, which of course is not true for non-zero states.

2. First note that for any two Pauli's P, Q, they either commute or anti-commute. Now suppose $P, Q \in \text{Stab}\,|\psi\rangle$ anti-commute. Then $|\psi\rangle = PQ\,|\psi\rangle = -QP\,|\psi\rangle = -\,|\psi\rangle$. This implies that $-I \in \text{Stab}\,|\psi\rangle$, which from above can't be true, so P and Q must commute.

3. It is known that the Pauli group's cardinality is a power of two, and since $\text{Stab}\,|\psi\rangle$ is a subgroup of the Pauli group, $|\text{Stab}\,|\psi\rangle\,|$ must divide a power of two, thus it must also be a power of two.

4. First note that Clifford unitaries normalize pauli matrices, i.e. for some Clifford unitary C, and some pauli P, $CPC^\dagger = P'$, where P' is also a pauli. Now let $P \in \text{Stab}\,|\psi\rangle$ and let C be some Clifford unitary. Then $P'C\,|\psi\rangle = CPC^\dagger C\,|\psi\rangle = CP\,|\psi\rangle = C\,|\psi\rangle$, so $P' \in \text{Stab}(C\,|\psi\rangle)$. Now consider the map $\theta_C : \text{Stab}\,|\psi\rangle \to \text{Stab}(C\,|\psi\rangle)$ which takes elements $P \mapsto CPC^\dagger = P'$. This map has an inverse, $\theta_{C^\dagger} : \text{Stab}(C\,|\psi\rangle) \longrightarrow \text{Stab}(C^\dagger C\,|\psi\rangle)$, which takes elements $P' \mapsto C^\dagger P' C$ (where we note that $\text{Stab}(C^\dagger C\,|\psi\rangle) = \text{Stab}\,|\psi\rangle$). Thus θ_C is a bijection, and so we have that $|\text{Stab}\,|\psi\rangle\,| = |\text{Stab}(C\,|\psi\rangle)|$.

5. Let $|\phi\rangle$ be another non-zero state on n qubits, and let $P \in \text{Stab}\,|\psi\rangle$ and $Q \in \text{Stab}\,|\phi\rangle$. Then $P \otimes Q\,|\psi\rangle\,|\phi\rangle = P\,|\psi\rangle \otimes Q\,|\phi\rangle = |\psi\rangle\,|\phi\rangle$. So $P \otimes Q \in \text{Stab}\,|\psi\rangle\,|\phi\rangle$. Now let $R \in \text{Stab}\,|\psi\rangle\,|\phi\rangle$. Then since R is a Pauli, we can write $R = R_1 \otimes R_2$, and $R\,|\psi\rangle\,|\phi\rangle = |\psi\rangle\,|\phi\rangle = R_1 \otimes R_2\,|\psi\rangle\,|\phi\rangle = R_1\,|\psi\rangle \otimes R_2\,|\phi\rangle$. Now let $|\psi'\rangle = R_1\,|\psi\rangle$ and $|\phi'\rangle = R_2\,|\phi\rangle$, and since R_1 and R_2 are both Clifford unitaries, fact 4 gives us that $|\text{Stab}\,|\psi'\rangle\,| = |\text{Stab}(R_1\,|\psi\rangle)|$ and $|\text{Stab}\,|\phi'\rangle\,| = |\text{Stab}(R_2\,|\phi\rangle)|$. We also have that $|\text{Stab}(R\,|\psi\rangle\,|\phi\rangle)| = |\text{Stab}\,|\psi'\rangle\,|\phi'\rangle\,|$.

   So every element in $\text{Stab}\,|\psi\rangle\,|\phi\rangle$ is of the form $R_1 \otimes R_2$ as above, thus $\text{Stab}\,|\psi\rangle\,|\phi\rangle = \text{Stab}\,|\psi\rangle \otimes \text{Stab}\,|\phi\rangle$. Then we have a bijection (from the direct product) $\theta : \text{Stab}\,|\psi\rangle \times \text{Stab}\,|\phi\rangle \longrightarrow \text{Stab}\,|\psi\rangle \otimes \text{Stab}\,|\phi\rangle$, which gives us $|\text{Stab}\,|\psi\rangle\,|\phi\rangle\,| = |\text{Stab}\,|\psi\rangle \otimes \text{Stab}\,|\phi\rangle\,| = |\text{Stab}\,|\psi\rangle\,| \cdot |\text{Stab}\,|\phi\rangle\,|$.

□

An example of a stabilizer state is the $|0\rangle$ state, since there are 2^1 pauli's that stabilize it, namely I and Z. Note that, from fact 4 in Proposition 2.1.6, for any Clifford C, $|\text{Stab}\,|0\rangle\,| = |Stab(C\,|0\rangle)| = 2^1 \implies C\,|0\rangle$ is a stabilizer state. And an example of a non-stabilizer state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + \frac{i}{2}\,|11\rangle)$, which through computation one can find that it has the following stabilizers: $I \otimes I$ and $Z \otimes Z$. Since there are only 2 stabilizers, and $2 \neq 2^2$, $|\psi\rangle$ cannot be a stabilizer state.

Corollary 2.1.7. *The computational basis state $|00\ldots0\rangle$ on n-qubits is a stabilizer state. If $|\psi\rangle$ is a basis state, then it is a stabilizer state.*

Proof. First we prove that $|00\ldots0\rangle$ is a stabilizer state by induction, where the base case is the $|0\rangle$ state which we know is a stabilizer state from above (and $|\text{Stab}\,|0\rangle\,| =$

2^1). Now assume that $|00\ldots0\rangle$ is a stabilizer state on n-qubits with $|\text{Stab}\,|00\ldots0\rangle\,| = 2^n$. Let $P \in \text{Stab}\,|0\rangle$ and $Q \in \text{Stab}\,|00\ldots0\rangle$, then $(P \otimes Q)\,|0\rangle\,|00\ldots0\rangle = P\,|0\rangle \otimes Q\,|00\ldots0\rangle = |0\rangle\,|00\ldots0\rangle = |00\ldots00\rangle$ (n+1 qubits). So $P \otimes Q \in \text{Stab}\,|00\ldots00\rangle$, and from above arguments we know that every element in $\text{Stab}\,|00\ldots00\rangle$ is of this form, thus there are 2^{n+1} elements in $\text{Stab}\,|00\ldots00\rangle$, making it a stabilizer state.

Next, if $|\psi\rangle$ is an n-qubit basis state, then it differs from the n-qubit computational basis state $|00\ldots0\rangle$ by only a Clifford unitary, i.e. $C\,|00\ldots0\rangle = |\psi\rangle$ for some Clifford circuit C. Then as remarked above, it follows that $|\psi\rangle$ is a stabilizer state. □

Definition 2.1.8. Let $|\psi\rangle$ be a non-zero n-qubit state. The Stabilizer nullity of $|\psi\rangle$ is $\nu(|\psi\rangle) = n - log_2|\text{Stab}\,|\psi\rangle\,|$.

Proposition 2.1.9. *Let $|\psi\rangle$ be a non-zero n-qubit state and let P be an n-qubit Pauli matrix and suppose that the probability of a $+1$ outcome when measuring P on $|\psi\rangle$ is non-zero. Then there are two alternatives for the state $|\phi\rangle$ after measurement: either $|Stab\,|\phi\rangle\,| = |Stab\,|\psi\rangle\,|$ or $|Stab\,|\phi\rangle\,| \geq 2|Stab\,|\psi\rangle\,|$, both of which satisfy $\nu(|\phi\rangle) \leq \nu(|\psi\rangle)$.*

Proof. First consider the simple case when P is in $\text{Stab}|\psi\rangle$. In this case, the "+1" measurement outcome occurs with probability 1 and $|\psi\rangle$ is unchanged. When P is not in $\text{Stab}|\psi\rangle$ we consider two alternatives. The first alternative is that P commutes with all elements of $\text{Stab}|\psi\rangle$. Recall we have the post-measurement state $|\phi\rangle = \frac{P|\psi\rangle}{\sqrt{\langle\psi|P^\dagger P|\psi\rangle}} = \frac{P|\psi\rangle}{\sqrt{\langle\psi||\psi\rangle}}$, and let $Q \in \text{Stab}\,|\psi\rangle$. Then $Q\,|\phi\rangle = \frac{QP|\psi\rangle}{\sqrt{\langle\psi||\psi\rangle}} = \frac{PQ|\psi\rangle}{\sqrt{\langle\psi||\psi\rangle}} = \frac{P|\psi\rangle}{\sqrt{\langle\psi||\psi\rangle}} = |\phi\rangle$. Note also that $\text{Stab}|\phi\rangle$ also contains $P\text{Stab}|\psi\rangle$, and thus $\text{Stab}|\phi\rangle$ contains $\text{Stab}|\psi\rangle \cup P\text{Stab}|\psi\rangle$ and thus its size is at least $2|\text{Stab}\,|\psi\rangle\,|$.

The second alternative is when P anti-commutes with some element $Q \in \text{Stab}|\psi\rangle$. Note that $Q\,|\psi\rangle = |\psi\rangle$ and $QPQ = -P$, so the probability of the $+1$ outcome is $\langle\psi|\,(I+P)\,|\psi\rangle\,/2 = \langle\psi|\,Q(I+P)Q\,|\psi\rangle\,/2 = \langle\psi|\,(I-P)\,|\psi\rangle\,/2$, which is the probability of the -1 outcome. Thus the probability of the $+1$ outcome is $1/2$. Then $|\phi\rangle = (I+P)/\sqrt{2}\,|\psi\rangle$ where we fixed the normalization condition such that $\langle\phi\rangle = \langle\psi\rangle$. Also, observe that we can write $|\phi\rangle = (I+PQ)/\sqrt{2}\,|\psi\rangle$. Since $(I+PQ)/\sqrt{2}$ is a Clifford unitary equal to $\exp(i\pi P'/4)$ for $P' = iPQ$, we see that $|\phi\rangle$ and $|\psi\rangle$ differ by a Clifford and therefore $|\text{Stab}|\psi\rangle\,| = |\text{Stab}|\phi\rangle\,|$. □

Definition 2.1.10 (Pauli Spectrum)**.** Let $|\psi\rangle$ be a non-zero n-qubit state. The Pauli spectrum Spec$|\psi\rangle$ of ψ is:

$$\text{Spec}\,|\psi\rangle = \left\{ \frac{|\,\langle\psi|\,P\,|\psi\rangle\,|}{\langle\psi|\psi\rangle}, \forall P \in \{I,X,Y,Z\}^{\otimes n} \right\} \tag{2.2}$$

The Pauli spectrum is a list of 4^n real numbers each between 0 and 1 which is invariant under Clifford gates. Consider the following example.

Proposition 2.1.11. *The Pauli spectrum of the state* $|\theta\rangle = (|0\rangle + e^{i\theta}\,|1\rangle)/\sqrt{2}$ *is* $\{1, cos\theta, sin\theta, 0\}$*. The state* $|\theta\rangle$ *is therefore a stabilizer state only for* $\theta = m\pi/2$ *for some integer* m*.*

Proof. First note that $|\theta\rangle$ is normalized so $\langle\theta|\,|\theta\rangle = 1$. Now by direct computation, we have:

- $\langle\theta|\,I\,|\theta\rangle = \langle\theta|\theta\rangle = 1$
- $\langle\theta|\,X\,|\theta\rangle = (\langle 1|\,e^{-i\theta} + \langle 0|)(|1\rangle + e^{i\theta}\,|0\rangle)/2 = (e^{-i\theta} + e^{i\theta})/2 = cos\theta$
- $\langle\theta|\,Y\,|\theta\rangle = (\langle 1|\,e^{-i\theta} + \langle 0|)(i\,|1\rangle - ie^{i\theta}\,|0\rangle)/2 = i(e^{-i\theta} - e^{i\theta})/2 = i(-2sin\theta)/2 = sin\theta$
- $\langle\theta|\,Z\,|\theta\rangle = (\langle 1|\,e^{-i\theta} + \langle 0|)(|0\rangle - e^{i\theta}\,|1\rangle)/2 = 1 - 1 = 0$

Moreover, if $\theta = 2k\pi/2$ for some integer k, then $X \in \text{Stab}\,|\theta\rangle$, and if $\theta = (2k+1)\pi/2$, then $Y \in \text{Stab}\,|\theta\rangle$. Observe that $\forall\theta$, $I \in \text{Stab}\,|\theta\rangle$ and $Z \notin \text{Stab}\,|\theta\rangle$, thus $|\text{Stab}\,|\theta\rangle\,| = 2$ if and only if either X or $Y \in \text{Stab}\,|\theta\rangle$, or more generally if $\theta = m\pi/2$, for some integer m. □

Note that the number of 1s in the Pauli spectrum of $|\psi\rangle$ is $|\text{Stab}\,|\psi\rangle\,|$.

2.2 Catalysis

Theorem 2.2.1. *Let* F *be a number field which contains* $\mathbb{Q}(i)$ *and which is closed under complex conjugation. Any stabilizer circuit applied to a density matrix with all entries in* F *produces a density matrix with all entries in* F*, with both density matrices written in the computational basis.*

For example, no stabilizer circuit on any number of $|CS\rangle$ or $|CCZ\rangle$ states (which have density matrices with all entries in $\mathbb{Q}(i)$) can be used to produce a $|T\rangle$ state (which has a density matrix with all entries in $\mathbb{Q}(\zeta_8)$). Similarly, no stabilizer circuit on any number of $|T\rangle$ states can be used to produce a $|\sqrt{T}\rangle$ state (with entries in $\mathbb{Q}(\zeta_{16})$).

Proof. Suppose our stabilizer circuit acts upon N qubits initially in the $|0\rangle$ state. Clearly the density matrix $\rho_{initial} = (|0\rangle \langle 0|)^{\otimes n}$ has entries over $\mathbb{Q}$. We point out that all Clifford unitaries can be written as matrices with entries over $\mathbb{Q}(i)$, and therefore as matrices with entries over F. Explicitly, the Clifford group is generated by H, CZ, and S which are defined as:

$$H = \frac{1}{1+i}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S : |0\rangle \mapsto |0\rangle, |1\rangle \mapsto |1\rangle$$

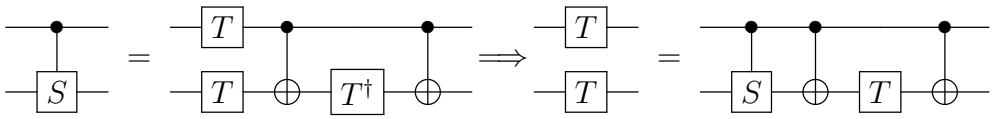
$$CZ : |ab\rangle \mapsto (-1)^{a \wedge b} |ab\rangle$$

Given any gate U in the circuit is a tensor product of a unitary with entries over F and I and ρ has entries over F the product $U\rho U^\dagger$ is a density matrix with entries over F. Therefore applying the gates in the circuit preserves the required property. Note that measurement with or without post-selection can be described as:

$$\rho \mapsto \frac{P\rho P}{Tr\rho P}$$

$$\rho \mapsto \sum_{P \in \mathcal{P}} P\rho P$$

The projectors P above correspond to measurement in the computational basis and therefore can be written as matrices with entries over $\mathbb{Q}(i)$ and therefore over F. The product of matrices over F is a matrix over F. The trace of a matrix over F is also in F by the definition of a field. The quotient of a matrix over F and an element of F is again a matrix over F because any field is closed under the division operation. This completes the proof. □



Definition 2.2.2 (Conversion Notation)**.** The equation $|A\rangle \to |B\rangle$ indicates that resource state $|A\rangle$ can be converted into resource state $|B\rangle$ with stabilizer operations in the absences of a catalyst. On the other hand, $|A\rangle \overset{|C\rangle}{\Longrightarrow} |B\rangle$, which is equivalent to $|A\rangle |C\rangle \to |B\rangle |C\rangle$, indicates the conversion can proceed with the use of a catalyst $|C\rangle$ (which may sometimes be omitted above the arrow). When a process is impossible, we strike through the arrow, for example $|A\rangle \not\Rightarrow |B\rangle$ signifies that $|A\rangle$ cannot be converted to $|B\rangle$ by stabilizer operations even in the presence of an arbitrary catalyst. In cases involving multiple copies of a given state such as $|A\rangle^{\otimes 2} \overset{|C\rangle}{\Longrightarrow} |B\rangle$, we sometimes write $2\,|A\rangle \overset{|C\rangle}{\Longrightarrow} |B\rangle$ to avoid clutter.

Chapter 3

Conversion of Resource States

Theorem 3.0.1. *Let $|U\rangle$ be an n-qubit magic state for a diagonal unitary U from the 3^{rd} level of the Clifford hierarchy, and let $\tau(U)$ be the minimum number of T gates needed to implement U using the gate set $\{CNOT, S, T\}$. The following resource conversion is possible*

$$|U\rangle \xrightarrow{|T\rangle^{\otimes \tau(U)-\nu(|U\rangle)}} |T\rangle^{\otimes 2\nu(|U\rangle)-\tau(U)}$$

Proof. Recall the following phase polynomial formalism. For any diagonal unitary in the 3rd level Clifford hierarchy we have $U_f = \sum_x exp(if(x)\pi/4)\,|x\rangle\langle x|$, where $f : \mathbb{Z}_2^n \to \mathbb{Z}_8$ is of cubic form and so can be decomposed as the phase polynomial $f(x) = \sum_{a_k \neq 0} a_k \lambda_k(x) \pmod 8$ where $a_k \in \mathbb{Z}_8$ and each λ_k is a $\mathbb{Z}_2$ linear function. That is, each λ_k has the form $\lambda_k(x) = (P_{1,k}x_1) \oplus (P_{2,k}, x_2) \ldots (P_{n,k}, x_n) \pmod 2$ where $P_{j,k}$ are binary. Thus we can describe the function by a binary matrix P and vector a, with columns corresponding to nonzero a_k (so the number of columns is the number of terms in f). For a function with a single term $f(x) = a_k\lambda_k(x)$, an easily verified circuit decomposition is $U_{\lambda_k} = \sum_x exp(i\lambda_k(x)\pi/4)\,|x\rangle\langle x| = V^\dagger_{CNOT(\lambda_k)} T_1^{a_k} V_{CNOT(\lambda_k)}$ where T_1 is a T gate acting on qubit 1 and $V_{CNOT(\lambda_k)}$ is a cascade of CNOT gates such that

$V_{CNOT(\lambda_k)}\,|x\rangle = V_{CNOT(\lambda_k)}\,|x_1, x_2, \ldots x_n\rangle = |\lambda_k(x_1), x_2, \ldots x_n\rangle$.

Now note that if a_k is even then $T_a^{a_k} = S_1^{a_k/2}$ is a Clifford and the whole circuit is Clifford. But if a_k is odd then $T_1^{a_k} = T_1 S_1^{(a_k-1)/2}$ and only a single T gate is used. Now, generalizing to a phase polynomial f with many terms we have $U_f = \prod_k U_{\lambda_k}$ and so the T-count for the associated circuit is equal to the number of odd valued a_k (so if all values are even then the unitary is Clifford).

This allows us to split the unitary U_f into a Clifford and non-Clifford part. For each a_k coefficient, we define $b_k \in \mathbb{Z}_4$ and $c_k \in \mathbb{Z}_2$ such that $a_k = 2b_k + c_k$ (so $c_k = 1$ if

and only if a_k is odd). Now for functions $g(x) = \sum_{c_k \neq 0} c_k \lambda_k(x)$ and $h(x) = \sum_{b_k \neq 0} b_k \lambda_k(x)$ we have that $f = g + h$ and $U_f = U_{g+2h} = U_g U_{2h}$ where U_{2h} is a Clifford Unitary. The non-Clifford part is U_g and all the terms have odd valued co-coefficients, so the number of terms in g gives an upper bound on $\tau(U_g)$ as discussed earlier. It follows that if the function g has m (odd-valued) terms then the state can be prepared using m many T gates/states. Note that for any given unitary U_g there is an equivalence class of different functions g that all result in the same unitary but with different numbers of terms. From now on we will assume that g is the optimal representative with the fewest number of terms, denoted by $\tau(U_g)$. Furthermore, there is a binary matrix P description of g with a number of columns also equal to $\tau(U_g)$. A trivial but relevant example is $U = T^{\otimes n}$ for which $P = \mathbb{1}_n$ and $\tau(T^{\otimes n}) = n$.

The next important step is that given a unitary U_g we may also be able to remove terms from g by applying inverse T gates. More generally, given two such unitaries U_g and $U_{g'}$ with phase polynomials g and g', we have that $U_{g'} = U_g U_\Delta$ where $\Delta = g - g'$. Therefore,

$$|U_{g'}\rangle = U_\Delta |U_g\rangle \tag{3.1}$$

and

$$|T\rangle^{\otimes \tau(U_\Delta)} |U_{g'}\rangle \to |U_g\rangle \tag{3.2}$$

The number of T states needed is $\tau(U_\Delta)$, which just the number of terms where g and g' differ.

Using arguments from [**?**], given any P we can always bring it into row-reduced echelon form using a CNOT circuit. Then

$$P = \begin{pmatrix} \mathbb{1}_r & A \\ 0 & 0 \end{pmatrix} \tag{3.3}$$

where $\mathbb{1}_r$ is an identity matrix of size $r := \text{rank}(P)$. If P is full rank the additional 0 padding is not present. Note that if P has any 0 rows then the unitary acts trivially on the corresponding qubits leaving them in the $|+\rangle$ state, meaning that $|U\rangle = U|+\rangle = |\psi\rangle |+\rangle^{\otimes(n-r)}$ for some state $|\psi\rangle$. Also, for an n qubit stabilizer state $|\phi\rangle$,

$$\nu(|\phi\rangle) = 0 \Rightarrow \log_2 |Stab\,|\phi\rangle\,| = n \tag{3.4}$$

Next, observe that

$$\begin{aligned}\log_2|Stab\,|U\rangle\,| =& \log_2|Stab(|\psi\rangle\,|+\rangle^{\otimes(n-r)})| \\ =& \log_2(|Stab\,|\psi\rangle\,|\cdot|Stab\,|+\rangle^{\otimes(n-r)}\,|) \\ =& \log_2|Stab\,|\psi\rangle\,| + \log_2|Stab\,|+\rangle^{\otimes(n-r)}\,| \\ =& \log_2|Stab\,|\psi\rangle\,| + (n-r) \\ =& \alpha + n - r\end{aligned}$$

for some positive integer $\alpha = \log_2|Stab\,|\psi\rangle\,|$. Hence $\log_2|Stab\,|U\rangle\,| \geq n-r$, so rearranging we have that $n - \log_2|Stab\,|U\rangle\,| = \nu(|U\rangle) \leq r$.

Using our earlier argument, we can always remove from P the columns corresponding to the matrix A using a number of T states equal to the number of columns in A. Since A has $\tau(U_g) - r$ columns, this requires the same quantity of T states. The resulting $U_{g'}$ has $P' = \mathbb{1}_r$ (with possibly some 0 row padding) which corresponds to r copies of T states. Therefore, we can perform

$$|U_g\rangle\,|T\rangle^{\otimes(\tau(U_g)-r)} \to |T\rangle^{\otimes r} \tag{3.5}$$

If $r = \nu(U_g)$ then we have the result of the theorem. If $r > \nu(U_g)$ then the result is even stronger than the theorem, and so the theorem holds in either case. □

Claim 1: Let U be a diagonal unitary from the third level of the Clifford hierarchy with phase polynomial matrix P. If all rows of P have even Hamming weight then $U\,|+\rangle^{\otimes n} \not\to |T\rangle$.

To see this, note that every diagonal unitary from the third level of the Clifford hierarchy is (up to Cliffords) a product of T, CS, CCZ gates []. In the special case that U has phase polynomial matrix with even rows, then the unitary is a product of CS and CCZ gates. Such a unitary has elements in the ring $\mathbb{Q}(i)$ and so $U\,|+\rangle^{\otimes n} \not\to |T\rangle$ follows. Though this transform is impossible without a catalyst, Theorem 3.0.1 allows us to easily construct concrete examples.

For any $n \geq 2$, we define W_n as the unitary with phase polynomial matrix

$$P_n = (\mathbb{1}_n, 1) = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & 0 & 1 \\ 0 & 0 & \dots & 1 & 1 \end{pmatrix}$$

which is the identity matrix padded with an all-one column. More explicitly, we have

$$W_n = \sum_x exp(i\pi g(x)/4) \left|x\right\rangle \left\langle x\right| \tag{3.6}$$

with $g(x) = (\oplus_{i=1}^{n} x_i) + \sum_{i=1}^{n} x_i$, where $\oplus$ is addition modulo 2. Now we can introduce and prove the following lemma.

Lemma 3.0.2. $\tau(W_n) = n + 1$

Proof. Since P has a width of $n + 1$ columns, we have $\tau(W_n) \leq n + 1$. The only full rank phase polynomial matrices that give a unitary that is Clifford equivalent to $T^{\otimes n}$ are square. Since this is not the case we conclude $\tau(W_n) = n + 1$. □

Chapter 4

Conclusion

Did it!