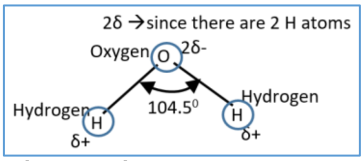
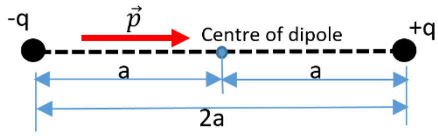
Dipole

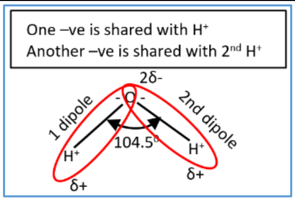
- Before we study what is a dipole, we need see a few definitions from Chemistry knowledge.
- **Electronegativity** : In a covalent bond, there is sharing of electrons (we know also this from Semiconductor chapter). The tendency of a "bonded" atom to attract electrons is called "electronegativity"
- **Non-polar covalent bon**d: (eg: CO_2, CH_4, Cl-Cl) When a covalent bond is formed between two atoms of the same "element" (as in cl-cl), the shared pair of electrons between the chlorine atoms will be attracted equally by the two atoms. Such a bond is called "non-polar covalent bond".
- **Polar covalent bond** : In the case of a covalent bond formed between atoms of different "elements", the covalent bond has a partial ionic character. For example, in the formation of 2H-O (water) bond, the oxygen atom which has a higher electronegativity (3.5) as compared to the hydrogen atom (2.1) attracts the shared pair of electrons toward itself. This causes electrical imbalance.
  - As a result oxygen carries a partial negative charge (δ-) and the hydrogen atoms carry a partial positive charge (δ+) as shown below



  - The two dipoles [O-H & O-H] are inclined to each other at an angle 104.5^0
  - H_2O is a highly polar molecule due to high electronegativity of oxygen atom
  - H_2O → its "electric dipole moment" $|\vec{p}| = 1.84D$; $D \rightarrow Debye$ is the unit of "electric dipole moment"
  - Water (H_2O) has a "permanent electric dipole moment" (even in the absence of external electric field). Such molecules are called "polar molecules".
- Permanent dipole : We mean that "electric dipole moment" $\vec{p}$ exists in the absence of external electric field. $\vec{p}$ is not induced by external electric field. eg: H_2O, HCl
- Several molecules (or atoms) are not electric dipoles by nature (hence non-polar). However, when placed in an electric field the positive centres and negative centres in the atom are displaced relative to each other and the atom becomes an electric dipole.

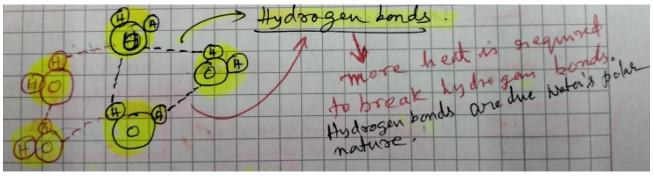


- **Electric dipole** :
- An electric dipole is a pair of equal and opposite point-charges at a short distance apart.
- The product of one charge and the distance between the charges is called the magnitude of the "electric dipole moment" $\vec{p}$
- Suppose the charges of dipole are –q and +q and the small distance between them is 2a, then the magnitude of the "electric dipole moment" is $\vec{p} = q\ x\ 2a = 2q\vec{a}$
- The midpoint of the dipole is called "centre of the dipole"
- The magnitude of the "electric dipole moment" is $|\vec{p}| = 2qa$
- Direction: Since the electric dipole moment is a vector $\vec{p}$, the convention used is the direction pointing from –ve charge to the +ve charge. This direction (-q to +q) is known as "direction of dipole axis"
- Unit of $\vec{p}\ is\ Cm\ \ (Coulomb - meter)$
- Also $\vec{p}$ is expressed in Debye (D)
- 1 D = 3.33564 x 10^{-30} Cm

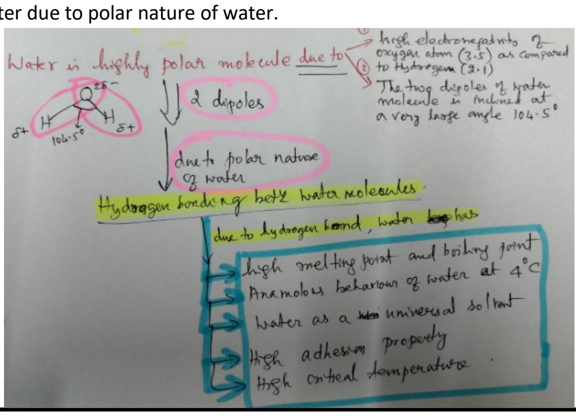


- **Example : water molecule**
- Water has a permanent electric dipole moment [even in the absence of external electric field]. Such molecules (H_2O, HCl …) are called "polar molecules"
- Since in water, the two dipoles are inclined to each other at an angle 104.5^0, therefore water is a highly polar molecule.
- The "electric dipole moment" of water = 1.84 D
- 1 D = 3.33564 x 10^{-30} Cm
- Water = 1.84D = (1.84) x (3.33564 x 10^{-30}) Cm = 6.14 Cm

Application of water being an electric dipole

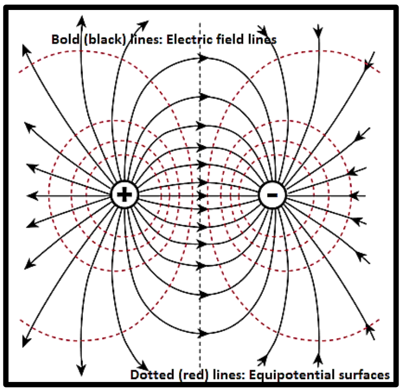


- Water would be in liquid state at room temperature. Due to polar nature, the water molecules attract each other (positive areas being attracted to negative areas) and to other polar molecules. This attraction is known as hydrogen bonding.
- Due to polar in nature, there is hydrogen bonding and hence water has relatively high melting and boiling point temperatures. So, more heat is required to break hydrogen bonds between water molecules.
- Anomalous behaviour of water at 4^0C whereby density is maximum is due to hydrogen bonding. This will help aquatic life.
- High adhesion property due to polar in nature of water.
- Water as an universal solvent where ionic and polar compounds can easily dissolve in water. [water has high dielectric constant 79.39 and high dipole moment 1.84D.
- High critical temperature of water due to polar nature of water.
- Consolidating we have



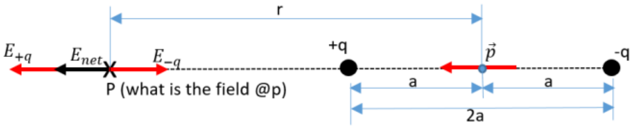
Dipole (contd..)

- Several molecules such as HCl, H2O etc..are electric dipoles. We know that each molecule has two or more atoms. Each atom has +vely charged nucleus and –vely charged electrons. In some molecules, the arrangement of the nuclei and the electrons of the atoms is such that one end of the molecule is +vely charged and the other end is equally –vely charged (however, the net charge on the molecule remains zero). Such molecules are "electric dipoles".
- In an atom, the centre of the +ve charges (nucleus) and the centre of the –ve charges (electrons) coincide each other. Hence the atom is not a dipole. But if the atom is placed in an electric field, then the +ve and the –ve centres are displaced relative to each other and the atom becomes a dipole.
- **Field lines of a dipole**: The field lines originate at the positive pole and terminate at the negative pole. The concentration of lines is greatest in the region between the charges, where the electric field is the strongest.



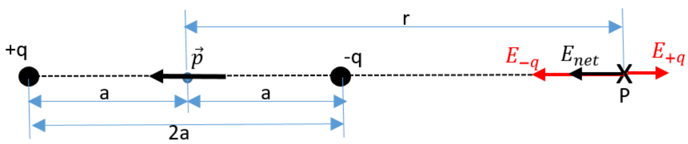
- We consider 2 aspects of electric dipole, namely
  - Nature of the electric field created by the dipole (two special cases). E can be found out using coulomb's law and superposition principle.
    - Intensity of electric field at a point on the axis of a dipole
    - Intensity of electric field at a point on the equatorial line of a dipole.
    - *General : In any other case, electric field E at any general point p is obtained by adding the electric fields $\vec{E}_{-q}$ due to the charge –q and $\vec{E}_{+q}$ due to charge +q using parallelogram law of vectors.*
  - The force (and torque) acting on a dipole when placed in an external electric field.

Intensity of electric field at a point on the axis of a dipole



- Field at p due to +q = $E_{+q} = \left[\frac{q}{4\pi\epsilon_0}\right]\frac{1}{(r-a)^2}$ *along the dipole axis*
- Field at p due to -q = $E_{-q} = \left[\frac{q}{4\pi\epsilon_0}\right]\frac{1}{(r+a)^2}$ *opposite to the dipole axis*
- Net field at p is (since $E_{+q} > E_{-q}$)
- $E = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{(r^2+a^2+2ar)-(r^2+a^2-2ar)}{(r-a)^2(r+a)^2}\right]$
- $E = \frac{q}{4\pi\epsilon_0}\left[\frac{4ar}{(r^2+a^2-2ar)(r^2+a^2+2ar)}\right]$
- $E = \frac{q}{4\pi\epsilon_0}\left[\frac{4ar}{r^4+r^2a^2+2ar^3+a^2r^2+a^4+2a^3r-2ar^3-2a^3r-4a^2r^2}\right]$
- $E = \frac{q}{4\pi\epsilon_0}\left[\frac{4ar}{r^4-2r^2a^2+a^4}\right]$
- $E = \frac{q}{4\pi\epsilon_0}\left[\frac{4ar}{(r^2-a^2)^2}\right]\hat{p}$; direction of E is along the dipole axis (from –ve towards +ve charge) ----------------- (1)
- $For\ r \gg a,\ E = \frac{q}{4\pi\epsilon_0}\left[\frac{4ar}{(r)^4}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{4a}{(r)^3}\right]$
- *However dipole moment vector is given by* $\vec{p} = 2aq\,\hat{p}$
- $\therefore E = \frac{2\vec{p}}{4\pi\epsilon_0 r^3}$ *for* $r \gg a$ *and* $\vec{p} = 2q\vec{a}$ -- (2)
- Note that at large distances, field of an electric dipole falls off not as $\frac{1}{r^2}$ *but as* $\frac{1}{r^3}$.
- Furthermore, the magnitude and the direction of the dipole field depends not only on distance r but also on the angle between the position vector $\vec{r}$ and the dipole moment $\vec{p}$.

For the point on the right side of the dipole, E_{net} is same in magnitude and direction as described above.
Since $|E_{-q}| > |E_{+q}|$, the magnitude is same as when point p was on the left side of the axial line
Since $E_{-q} > E_{+q}$, the net direction is dictated by –q (field lines into the charge) ; hence E_{net} direction towards left side (in line with dipole axis which is same as when point p is on the left side)



In page 25, problem 2, field at point B, we can use the above equation (1) $E = \frac{q}{4\pi\epsilon_0}\left[\frac{4ar}{(r^2-a^2)^2}\right]$; here r = 0.1 m, a = 0.05m, q = 10^{-8} C, $\frac{1}{4\pi\epsilon_0} = 9x10^9$
Substituting these values in the equation and simplifying we get E = 3.2 x 10^4 NC^{-1} directed towards left.

Physical significance of electric dipoles:

- In most molecules, the centres of +ve charges and of –ve charges lie at the same place. Therefore, their dipole moment is zero. CO_2 and CH_4 are of this type of molecules. However, they develop a dipole moment when an electric field is applied.
- But in some molecules, the centres of –ve charges and of +ve charges do not coincide. Therefore, they have a permanent electric dipole moment, even in the absence of an electric field. Such molecules are called polar molecules. Water molecule H_2O is an example of this type.
- Various materials give rise to interesting properties and important applications in the presence or absence of an external electric field.

Dipole (contd..)

Intensity of electric field at a point on the equatorial line of a dipole

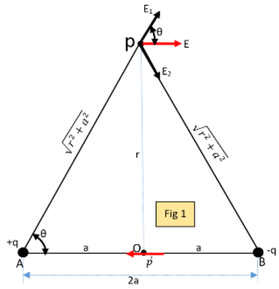


Fig 1

- Suppose point p is situated on the right bisector of the dipole AB at a distance r meter from its mid-point O (See fig 1)
- Let E_1 be the magnitude of the intensity electric field at p due to charge +q
- Let E_2 be the magnitude of the intensity electric field at p due to charge –q
- The directions $\vec{E}_1$ and $\vec{E}_2$ are shown in the figure 1
- $E_1 = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{q}{r^2+a^2}$ *away from + q* ----------------------- (1)
- $E_2 = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{q}{r^2+a^2}$ *towards – q* ------------------------ (1)
- Magnitudes of E_1 and E_2 are equal but directions are different (see fig1)
- On resolving vectors $\vec{E}_1$ and $\vec{E}_2$ into its horizontal and vertical components, as we see from fig (2), the vertical components (E_1 sinθ and E_2 sinθ) cancel each other (because they are equal in magnitude and opposite in direction).
- The two horizontal components (fig2) being in the same direction, add up. Hence the resultant intensity electric field at point p is

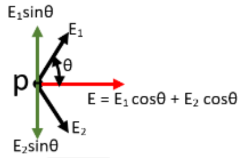


Fig 2

- E = E_1 cosθ + E_2 cosθ where $cos\theta = \frac{a}{\sqrt{r^2+a^2}}$ --------------- (2)
- Since E_1 = E_2 in magnitude, therefore $E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{q}{r^2+a^2} x\ 2\ cos\theta$; plug cosθ from (2), we get
- $E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{2qa}{[r^2+a^2]^{\frac{3}{2}}}$; but $2qa = \vec{p}$ (moment of electric dipole)
- $E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{p}{[r^2+a^2]^{\frac{3}{2}}}$; From fig (2), we can conclude the direction of E is antiparallel to the dipole axis --------------- (3)
- If r >> 2a, the a^2 may be neglected in comparison to r^2, then electric intensity at the point p due to dipole is
- $E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{p}{r^3}\ NC^{-1}$; in vector notation $\vec{E} = -\left[\frac{1}{4\pi\epsilon_0}\right]\frac{\vec{p}}{r^3}$; minus sign indicates $\vec{E}$ is antiparallel to $\vec{p}$ --------------- (4)
- Comparing E on axial point and equatorial point due to a dipole, E is "double" in magnitude on an axial point as compared to equatorial point, but opposite in direction.
- Further, in comparing the fields of a point-charge and of a dipole, the $\frac{1}{r^2}$ dependence is replaced by $\frac{1}{r^3}$ dependence.

In page 25, problem 2, field at point C, we can use the above equation $E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{2qa}{[r^2+a^2]^{\frac{3}{2}}}$

here $\sqrt{r^2+a^2}$ = 0.1 m, a = 0.05m, q = 10^{-8} C, $\frac{1}{4\pi\epsilon_0} = 9x10^9$; $[r^2+a^2]^{\frac{3}{2}} = 10^{-3}$

Substituting these values in the equation and simplifying we get the net magnitude of E at C = E_C = $9\ x\ 10^3 NC^{-1}$.

As shown in figure in page 25, the direction of E at C is towards right.

Problem : The following data was obtained for the dependence of the magnitude of electric field with distance from a reference point O within the charge distribution in the shaded area.

Field point	A	B	C	A`	B`	C`
Magnitude of electric field	E	$\frac{E}{8}$	$\frac{E}{27}$	$\frac{E}{2}$	$\frac{E}{16}$	$\frac{E}{54}$

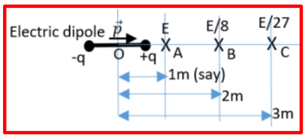
(1) Identify the charge distribution and justify your answer

(2) If the potential due to this charge distribution by a charge at O has a value "V" at point "A", what is its value at the point "A`"

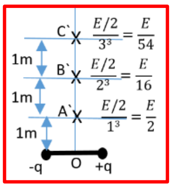
Solution: (1)

- If we see the magnitude of electric field intensity in the given table, there is a pattern as given below due to some charge distribution at O.

Field point	A	B	C	A`	B`	C`
Magnitude of electric field	$\frac{E}{1^3}$	$\frac{E}{2^3}$	$\frac{E}{3^3}$	$\frac{E/2}{1^3}$	$\frac{E/2}{2^3}$	$\frac{E/2}{3^3}$



- From the above table, we observe $|\vec{E}|$ at points A, B and C varies as the inverse cube of the distance of the field point along axial line.
  - We know that $|\vec{E}|$ at a point on the axis of a dipole is given by (approx. formula) $E = \left[\frac{2p}{4\pi\epsilon_0}\right]\left[\frac{1}{r^3}\right]$
  - This formula agrees with the given table values.
  - Hence at O there must be a electric dipole whose centre is at O



- Similarly, A`, B` and C` points are on equatorial line of the dipole. We know that $|\vec{E}|$ at a point on the equatorial line of a dipole is given by
  - $E = \left[\frac{p}{4\pi\epsilon_0}\right]\left[\frac{1}{r^3}\right]$. Intensity of electric field at A`, B`, C` = half of intensity of electric field at A,B,C for the same distance.
  - Therefore, at O, there must be an electric dipole whose centre is at O.
- Therefore, the charge distribution given in the problem is due to an **electric dipole whose centre is at O**

(2) Potential due to an electric dipole is given by $V = \frac{\vec{p}.\vec{r}}{4\pi\epsilon_0 r^3}$, where $\vec{p}.\vec{r}$ is the dot product of $\vec{p}$ and $\vec{r}$ = pr cosθ ; θ is the angle between $\vec{p}$ and $\vec{r}$

$V = \frac{prcos\theta}{4\pi\epsilon_0 r^3} = \frac{pcos\theta}{4\pi\epsilon_0 r^2}$; Given in the problem, at A, the potential is V (axial line)

Points A (B & C also) lie on the axial line of the electric dipole, hence θ = 0^0 and cos0^0=1

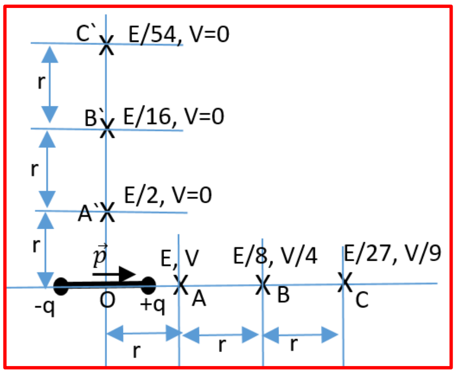
∴ At "A", $V = \frac{pcos\theta}{4\pi\epsilon_0 r^2}$; $V\ \alpha\ \frac{1}{r^2}$;

At "B", V= V/4 and at "C", V=V/9

At "A`" on the equatorial line of the electric dipole, θ = 90^0 and cos90^0 = 0

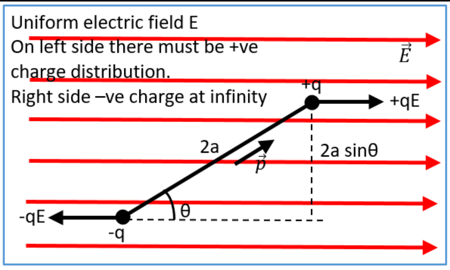
Therefore, at A`, V= 0

Electric potential due to a dipole is "zero" at all points on the equatorial line of the electric dipole.



Dipole (contd..)

Force (and Torque) acting on an "electric dipole" placed in a "uniform external electric field"



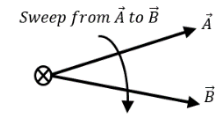
- Let us consider an electric dipole placed in a uniform $\vec{E}$ as shown in the figure (red parallel arrows from left to right).
- Let the electric dipole moment $\vec{p}$ be making an angle θ with the field $\vec{E}$
- The –q and +q charges are forming the dipole and 2a is the distance between them.
- Due to external electric field E, the charge +q experiences a force qE (in the direction of the field E, since it is getting attractive force to –ve charge at infinity on the right side) and the charge –q experiences an equal and opposite force (-qE) opposite to external field E. Due to uniform electric field, the two forces are equal in magnitude and opposite in direction. Therefore, the net force on the dipole is zero.
- However, the two forces act at different points and hence they form a couple which tends to set the dipole parallel to $\vec{E}$. This is called restoring couple. The moment of this restoring couple is known as 'torque' τ on the dipole. The magnitude of the torque = force * perpendicular distance.
- $|\vec{\tau}| = (qE)(2a\ sin\theta) = (2qa)E\ sin\theta$. (2qa) is the magnitude of the dipole moment. Hence
- $|\vec{\tau}| = [pE\ sin\theta]Nm.$ *In vector form* $\vec{\tau} = \vec{p} \times \vec{E}$, *the cross product of* $\vec{p}$ *and* $\vec{E}$
- Thus, in a uniform electric field a dipole experiences a torque, but no net force. In the above figure, the torque is perpendicular to the page pointing downwards (right hand screw rule). *Note :* $\vec{\tau} = \vec{p} \times \vec{E}$ *into the page and* $\vec{\tau} = \vec{E} \times \vec{\tau}$ *out of the page*
- If dipole is placed perpendicular to $\vec{E}$, θ = 90⁰ and sin90⁰ = 1, then the torque acting on the dipole is maximum. τ_{max} = pE. ∴ p = τ_{max}/E
- If dipole is placed parallel to E, θ = 0⁰ and sin0⁰ = 0, then there is no torque acting on the dipole and also no net force. (note that if the dipole is placed in a non-uniform electric field, it experiences a net force in addition to torque).

Note :

$\vec{\tau} = \vec{p} \times \vec{E}$ (cross product of $\vec{p}$ *and* $\vec{E}$). In a cross product, the torque τ is perpendicular to the plane containing $\vec{p}$ *and* $\vec{E}$. If $\vec{p}$ *and* $\vec{E}$ are in the plane of paper, $\vec{\tau}$ is perpendicular to paper. Whether into the page or out of page is depending on whether it is $\vec{p} \times \vec{E}$ *or* $\vec{E} \times \vec{p}$. In the present case of $\vec{p} \times \vec{E}$, $\vec{\tau}$ is into the page.

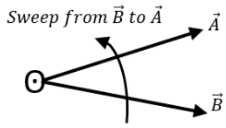
Cross product: (Informative)

$\vec{C} = \vec{A} \times \vec{B}$; Assume $\vec{A}$ *and* $\vec{B}$ are on the plane of the paper. $\vec{C}$ will be perpendicular to the plane of the paper and pointing into the paper. It is depicted as ⊗



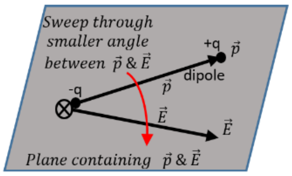
$\vec{C} = \vec{B} \times \vec{A}$; Assume $\vec{A}$ *and* $\vec{B}$ are on the plane of the paper. $\vec{C}$ will be perpendicular to the plane of the paper and pointing out of the paper.

It is depicted as ʘ



$(A \times B) = -(B \times A)$. Cumulative law fails for cross product.

Cross product $\vec{\tau} = \vec{p} \times \vec{E}$, where $\vec{p}$ is the "electric dipole moment" vector ; $\vec{E}$ is the "electric field strength" vector ; $\vec{\tau}$ is the torque vector.



- Imagine a plane containing $\vec{p}$ *and* $\vec{E}$ vectors
- In cross product $\vec{p} \times \vec{E}$, the net vector $\vec{\tau}$ is perpendicular to the plane containing $\vec{p}$ *and* $\vec{E}$
- Question is whether $\vec{\tau}$ is into the plane or out of the plane.
- Use right hand; Sweep 4 fingers only (not thumb) from $\vec{p}$ to $\vec{E}$ through the smaller angle between $\vec{p}$ and $\vec{E}$; the thumb gives the direction of $\vec{\tau}$. [Note: We can also use right hand screw rule]
- If the plane containing $\vec{p}$ *and* $\vec{E}$ vectors is the plane of paper, then $(\vec{p} \times \vec{E})$ is into the paper and $(\vec{E} \times \vec{p})$ is out of the paper

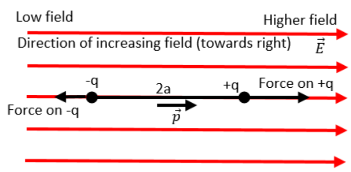
If the external electric field $\vec{E}$ is not uniform, then the net force is non-zero. In addition, there will be a torque as before.

Let us consider two simple cases. They are …

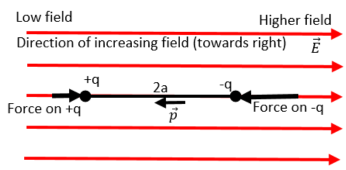
- *When* $\vec{p}$ *is parallel to* $\vec{E}$. [θ = 0⁰ and sin0⁰ = 0 ; τ = pE sinθ = 0]
- *When* $\vec{p}$ *is antiparallel to* $\vec{E}$ [θ = 180⁰ and sin180⁰ = 0 ; τ = pE sinθ = 0]

In either case, the net torque is zero, but there is a net force on the dipole.

When $\vec{p}$ *is parallel to* $\vec{E}$, the dipole net force is in the direction of increasing field (towards right in the figure). Dipole has translatory motion towards right. Since +q is situated at higher $\vec{E}$ as compared to –q, therefore force on +q > force on –q. Hence the net force is towards right.
Note the length of force vectors shown in the figure.



When $\vec{p}$ *is antiparallel to* $\vec{E}$, the dipole net force is in the direction of decreasing field (towards left in the figure). Dipole has translatory motion towards left. Since -q is situated at higher $\vec{E}$ as compared to +q, therefore force on -q > force on +q. Hence the net force is towards left.
Note the length of force vectors shown in the figure.



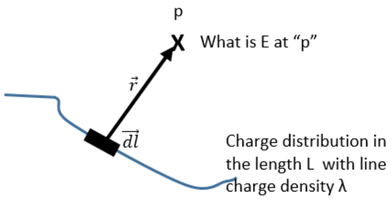
In general, the force on the dipole depends on the orientation of $\vec{p}$ *wrt* $\vec{E}$.
If the dipole is placed at an angle θ ≠ 0 to E in a non uniform electric field, there will be both net force and torque.

Continuous charge distribution

- So far, we are dealing with discrete charges q_1, q_2,….since it was mathematically simpler and does not involve calculus.
- However, it is impractical to work in terms of discrete charges and we need to work with continuous charge distributions. For example, on the surface of a charged conductor, it is impractical to work on microscopic levels (discrete charges), but it is more feasible to consider an area element ΔS on the surface of the conductor. ΔS is still very small on macroscopic level, but still contains large number of electrons.
- Surface charge density $\sigma = \frac{\Delta Q}{\Delta S}\ Cm^{-2}$, where C is Coulomb
- Line charge density $\lambda = \frac{\Delta Q}{\Delta l}\ Cm^{-1}$
- Volume charge density $\rho = \frac{\Delta Q}{\Delta V}\ Cm^{-3}$
- Field due to continuous charge distribution is worked out the same way as we did for discrete charges.

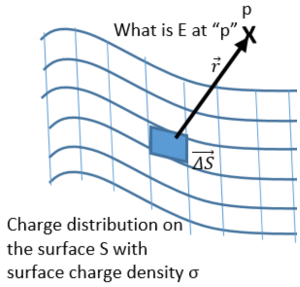
Line Charge Density (λ)

- Consider a length L of a conductor with continuous charge distribution with line charge density λ
- Consider a small line element Δl. The charge in this line element Δl is (λ Δl). That is $\Delta Q = \lambda\,\Delta l$
- Consider a point "p" with displacement vector $\vec{r}$. We need to find the electric field strength at p due to complete length of the conductor.
- Electric field due to Δl is $\Delta\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{(\lambda\,\Delta l)}{r^2}\ \hat{r}$, where $\hat{r}$ is the unit vector in the direction from charge element Δl *to the point p*
- From superposition principle, the total intensity of electric field due to complete length is
- $\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{all\ \Delta l}\frac{(\lambda\,\Delta l)}{r^2}\ \hat{r}$
- As $\Delta l \rightarrow 0$, summation is replaced by integration.



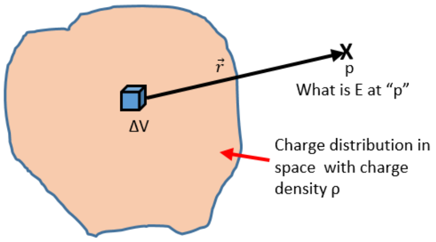
Surface charge density (σ)

- Consider an area S in space with continuous charge distribution with surface charge density σ
- Consider a small area element ΔS. The charge in this area element ΔS is (σ ΔS). That is $\Delta Q = \sigma\,\Delta S$
- Consider a point "p" with displacement vector $\vec{r}$. We need to find the electric field strength at p due to complete area S.
- Electric field due to ΔS is $\Delta\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{(\sigma\,\Delta S)}{r^2}\ \hat{r}$, where $\hat{r}$ is the unit vector in the direction from area element ΔS *to the point p*
- From superposition principle, the total intensity of electric field due to complete length is
- $\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{all\ \Delta S}\frac{(\sigma\,\Delta S)}{r^2}\ \hat{r}$
- As $\Delta S \rightarrow 0$, summation is replaced by integration.



Volume Charge Density (ρ)

- Consider a volume in space with continuous charge distribution with charge density ρ
- Consider a small volume element ΔV. The charge in this volume element ΔV is (ρ ΔV). That is ΔQ = ρ ΔV
- Consider a point "p" with displacement vector $\vec{r}$. We need to find the electric field strength at p due to complete volume.
- Electric field due to ΔV is $\Delta\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{(\rho\,\Delta V)}{r^2}\ \hat{r}$, where $\hat{r}$ is the unit vector in the direction from charge element ΔV to the point p
- From superposition principle, the total intensity of electric field due to complete volume is
- $\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{all\ \Delta V}\frac{(\rho\,\Delta V)}{r^2}\ \hat{r}$
- As ΔV → 0, summation is replaced by integration.



Problem: If a net charge of 1/3 C charge is uniformly distributed in a sphere of radius 1 m, what is the volume charge density ?

Solution :

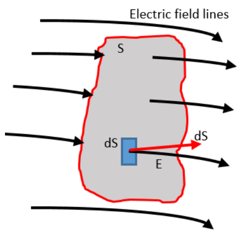
Given total charge inside the sphere of radius $1\ m = \frac{1}{3}\ C$

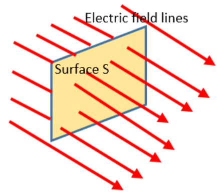
$$\rho = \frac{Q}{V} = \frac{\frac{1}{3}}{\frac{4}{3}\pi r^3} = \frac{\frac{1}{3}}{\frac{4}{3}\pi(1)^3} = \frac{1}{4\pi}\ Cm^{-3}$$

<u>**Note**</u> : We normally define **density = mass/volume**. A body has typical characteristics like mass, charge etc…Therefore, when we talk about charge density, we need to replace "m" by "Q"

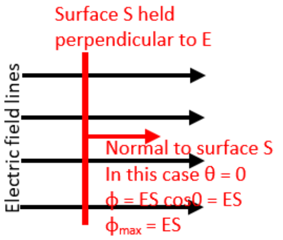
Electric Flux

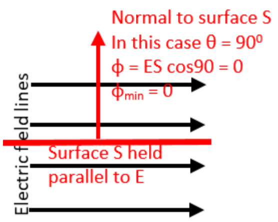
The electric flux is a property of electric field. We know that an electric field can be visualized by "field lines" ; E is stronger where the field lines are closer and vice-versa. The electric flux is a measure of the number of field lines passing through some surface held in the electric field. It is denoted by ϕ_E

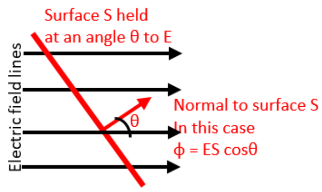




- As shown in the figure, consider a small area element $\overrightarrow{dS}$ (magnitude $|\overrightarrow{dS}|$) within an area S held in the electric field E, then the scalar (dot) product $\vec{E}.\overrightarrow{dS}$ is defined as "electric flux" through that area element $\overrightarrow{dS}$.
  - $\therefore \Delta\varphi_E = \vec{E}.\overrightarrow{dS}$, where $\overrightarrow{dS}$ is the area vector of the small element dS on the surface S.
- The electric flux through the entire surface is $\varphi_E = \int_S E.dS$, where $\int_s$ is the surface integral over the entire surface.
- ϕ_E is positive for the filed lines "leaving the surface"
- ϕ_E is negative for the filed lines "entering the surface"
- In $\Delta\varphi_E = \vec{E}.\overrightarrow{dS}$, **where $\overrightarrow{dS}$ is normal to the surface element dS**
- The angle between $\vec{E}$ and $\overrightarrow{dS}$ is θ
- $\Delta\varphi_E = \vec{E}.\overrightarrow{dS}$ = E dS cosθ (dot product of $\vec{E}$ and $\overrightarrow{dS}$)
- Total flux through the surface S is $\varphi_E = \int_S E\ dS\ \cos\theta = E\cos\theta \int_S dS$; but $\int_S dS = S$
- $\therefore$ ϕ_E = ES cosθ, where θ is the angle between E and perpendicular to S
- For the field lines of E entering the plane surface normally (θ = 180^0) , ϕ = ES cos 180 = -ES
- For the field lines of E leaving the plane surface normally (θ = 0^0) , ϕ = ES cos0 = +ES
- Flux is maximum when the surface S is held perpendicular to E (see figure below)
- Flux is zero when the surface S is held parallel to E (see figure below)







SI unit of electric flux:

ϕ_E = ES = (F/q)S $\Longrightarrow$ Nm^{2}C^{-1}

Also $E = \frac{V}{l}$ (where V is the voltage and l is the length → See next chapter)

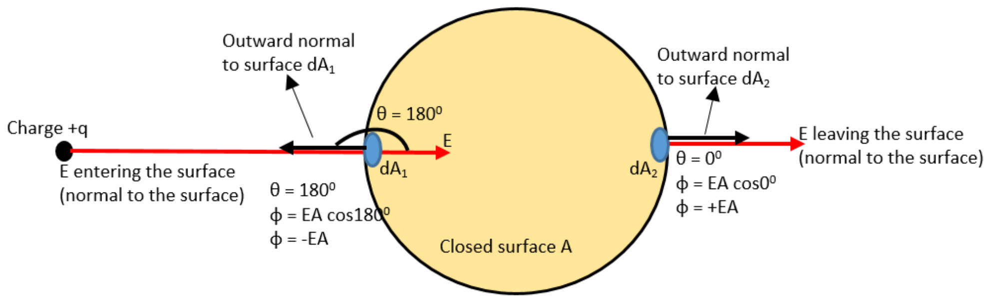
ϕ_E = ES = $\frac{V}{l}\ S = \frac{V}{m}\ m^2 = Vm$ [$(volt)(meter)$

***SI unit of electric flux*: Nm^2C^{-1} or Vm**

Dimension of ϕ_E

- SI unit of electric flux = Nm^{2}C^{-1} = (kg ms^{-2}) (m^2) (A^{-1}S^{-1}) [since we know that I = q/t and hence q = I t]
- = M L^3 T^{-3} A^{-1}

For the field lines of E entering the plane surface normally (θ = 180^0) , ϕ = EA cos 180 = -EA
For the field lines of E leaving the plane surface normally (θ = 0^0) , ϕ = EA cos0 = +EA
How ?



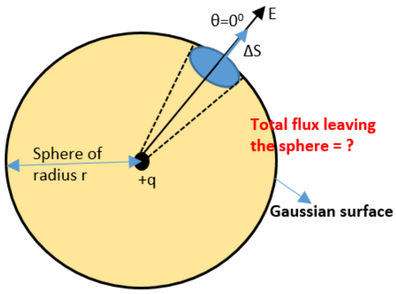
Consider a charge +q outside the "closed surface A", the total flux through the surface is zero because

- Flux entering through dA_1 is EA cos180^0 = -E dA_1
- Flux leaving through dA_2 is EA cos0^0 = +E dA_2
- Hence, for the whole closed surface where no charge is enclosed by the closed surface, the total flux = 0

Electric flux density : In an electric field, the ratio of electric flux ϕ_E through the surface to the area S is called "electric flux density" at the location of the surface. $\Longrightarrow$ electric flux density = $\frac{\varphi_E}{S} = \frac{ES}{S} = E$

- For a plane surface "normal" to the electric field , ϕ_E = ES. $\therefore$ electric flux density = $\frac{\varphi_E}{S} = \frac{ES}{S} = E$
- The unit of "electric flux density" is same as that of electric field. Since E is a vector, electric flux density is also a vector (whereas the electric flux is a scalar quantity.

Gauss's law in electrostatics



As a simple application of the notion of electric flux, let us consider the total flux through a sphere of radius r, which encloses a point charge +q at its centre.

- Divide the sphere into small area elements
- Flux through an area element ΔS is Δϕ = E.ΔS (dot product) = $\left[\frac{q}{4\pi\epsilon_0 r^2}\right][\hat{r}.\vec{\Delta S}] - - - - - (1)$
- The unit vector $\hat{r}$ is along the radius vector from the centre to the area element ΔS
- Since the normal to a sphere at every point is along radius vector at that point, ΔS and r have same direction. Therefore, $\theta = 0, cos0 = 1 \,\&\, \hat{r} = 1$
- $\hat{r}.\vec{\Delta S} = \hat{r}\ \Delta S\ cos0 = \ \Delta S$. Therefore equation (1) becomes
- Δϕ = $\left[\frac{q}{4\pi\epsilon_0 r^2}\right][\Delta S]$
- Total flux of all area elements on the sphere is
- $\varphi = \sum_{all\ \Delta S}\left[\frac{q}{4\pi\epsilon_0 r^2}\right][\Delta S]$; since $\sum \Delta S = 4\pi r^2$; $surface\ area\ of\ sphere$
- $\varphi = \left[\frac{q}{4\pi\epsilon_0 r^2}\right]4\pi r^2 = \frac{q}{\epsilon_0}$; **$\therefore\ \varphi = \frac{q}{\epsilon_0}$, where q is the net charge ENCLOSED by the surface**
- Gauss's law states that the electric flux through a closed surface S = $\frac{q}{\epsilon_0}$ where q is the total net charge enclosed by S
  - If no charge is enclosed by the surface, q = 0 and hence ϕ = 0
  - If there are several charges enclosed by the surface like $+q_1$, $+q_2$, $+q_3$, $-q_4$, $-q_5$, the total flux through the surface due to all charges = $\frac{1}{\epsilon_0}[q_1 + q_2 + q_3 - q_4 - q_5] = \frac{1}{\epsilon_0}\sum q$, where $\sum q$ is the algebraic sum of all charges enclosed by the closed surface.
  - If net charge enclosed by the surface = 0, then ϕ = 0. If the enclosed charges are +q, +q, +q, -q, -q, -q, then the net charge = 0, hence ϕ=0

ϕ = ES cos0 = ES = E $(4\pi r^2)$ (see the above figure)

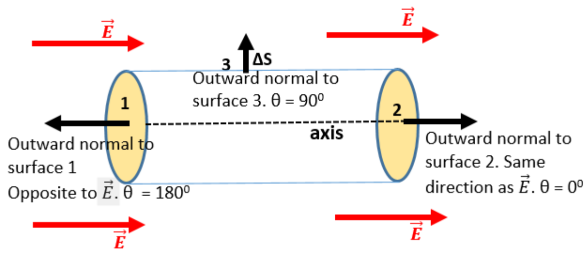
But ϕ = q/ϵ_0

∴ E $(4\pi r^2)$ = q/ϵ_0

$E = \frac{q}{4\pi\epsilon_0 r^2}$ … we got coulomb's law.

Coulomb's law and Gauss's law are mutually equivalent. They are not 2 physical independent laws, rather the same law is expressed in different ways.

To prove that net flux = 0 when there is no charge is enclosed in the closed surface.



- Consider a cylindrical surface with its axis parallel to E.
- Flux through the surface ϕ = $\phi_1 + \phi_2 + \phi_3$, where ϕ_1 and ϕ_2 are flux through 2 cross sectional circular area and ϕ_3 is the flux through curved cylindrical part of the closed surface.
- Normal to surface 3 at every point is perpendicular to E, so by definition $\phi_3 = 0$ [since θ = 90⁰, cos90⁰ = 0 and hence ϕ_3 = ES cosθ = 0]
- Outward normal to surface 2 is along E (θ = 0⁰) and the outward normal to surface 1 is opposite to E (θ = 180⁰)
- So, $\phi_1 = -ES_1$ and $\phi_2 = +ES_2$; since $S_1 = S_2 = S$ (area of circular cross section), **∴ $\phi_1 = -\phi_2$**
- Since ϕ = $\phi_1 + \phi_2 + \phi_3$, ϕ = 0
- ∴ whenever "net flux" through a closed surface = 0, we can conclude that the "total net charge" contained in the closed surface is zero.

- The great significance of Gauss's law $\boldsymbol{\varphi = \frac{q}{\epsilon_0}}$ is that it is true, in general, for any closed surface , no matter what its shape or size.
- The term q in ϕ = q/ϵ_0 includes "sum" of all charges "enclosed by the surface". The charges may be located anywhere inside the surface.
- In a situation when the surface S is so chosen that there are some charges inside and some outside, the electric field **[whose flux appears on the LHS of $\boldsymbol{\varphi = \frac{q}{\epsilon_0}}$]** is due to all the charges both inside and outside S.
  - However, the term q on the RHS of $\boldsymbol{\varphi = \frac{q}{\epsilon_0}}$ represents only the total charge inside S.
- The surface that we choose for the application of Gauss's law is called the "Gaussian surface". You may choose any Gaussian surface and apply Gauss's law. Take care not to let Gaussian surface pass through any discrete charge (not well defined). However, Gaussian surface can pass through a continuous charge distribution.
- Gauss's law is very useful to calculate electric field when some system has symmetry (choose suitable Gaussian surface).
- Gauss's law is based on inverse square dependence on distance contained in the coulomb's law. Any violation of Gauss's law will indicate departure from the inverse square law.

Applications of Gauss's law – Comprehensive list

- Electric field due to a uniformly charged infinite long straight wire
- $E = \left[\frac{\lambda}{2\pi\epsilon_0 r}\right]\hat{r}$
- Problems on this application [see further pages]

- Electric field due to a uniformly charged thin infinite plane sheet
- $E = \left[\frac{\sigma}{2\epsilon_0}\right]\hat{r}$
- Problems on this application [see further pages]

- Electric field 'just' outside a charged conductor
- $E = \left[\frac{\sigma}{\epsilon_0}\right]\hat{r}$

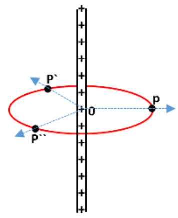
- Electric field due to TWO infinite parallel sheets of charge (both positively charged)
  - Points between sheets E = 0
  - Points outside the sheets $E = \left[\frac{\sigma}{\epsilon_0}\right]\hat{r}$

- Electric field due to TWO infinite parallel sheets of charge (one positively charged and the other negatively charged ; like a capacitor)
  - Points between sheets
    - E is uniform
    - $E = \left[\frac{\sigma}{\epsilon_0}\right]\hat{r}$
  - Points outside the sheets
    - $E = 0$

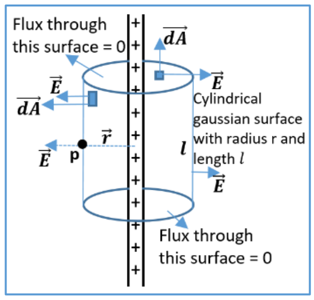
- Electric field due to a uniformly charged thin spherical shell of radius R
  - Points outside the shell
    - r > R
    - $E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{q}{r^2} = \left[\frac{\sigma}{\epsilon_0}\right]\frac{R^2}{r^2}$
  - Points on the shell
    - r = R
    - $E = \left[\frac{\sigma}{\epsilon_0}\right]$
  - Points inside the shell
    - r < R
    - $E = 0$

- Electric field due to a uniformly charged sphere of radius R
  - Points outside the sphere
    - r > R
    - $E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{q}{r^2} = \left[\frac{\rho}{3\epsilon_0}\right]\frac{R^3}{r^2}$
  - Points on the sphere
    - r = R
    - $E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{q}{R^2} = \left[\frac{\rho R}{3\epsilon_0}\right]$
  - Points inside the sphere
    - r < R
    - $E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{qr}{R^3} = \left[\frac{\rho r}{3\epsilon_0}\right]$

Applications of Gauss's law – Electric field due to a uniformly charged infinite long straight wire



- Consider an infinitely long thin straight wire with uniform linear charge density λ. Electric field has same magnitude at points p, p\`, p\`\` etc.
- The direction of E at every point must be radial ; outward if λ > 0, inward if λ < 0 [in other words whether the charge is +ve or –ve]
- The electric field is everywhere radial in the plane cutting the wire normally and its magnitude depends only on the radial distance r.
- To calculate the field, imagine cylindrical Gaussian surface. Since E is radial, flux through the 2 ends of cylindrical Gaussian surface is zero. [since the outward normal from 2 ends are perpendicular to E ; $\cos 90^0 = 0$ and hence flux = 0].
- At the cylindrical part, E is normal to the surface at every point and its magnitude is constant, since it depends only on r. The surface area of the curved part = $2\pi rl$, where l is the length of the cylinder.
- Flux through the Gaussian surface = flux through the curved cylindrical part of the surface = EA cosθ = EA cos0 = EA = $E(2\pi rl)$
- The Gaussian surface includes charge equal to "λl". Gauss's law then gives
- $E(2\pi rl) = \frac{\lambda l}{\epsilon_0} \quad \Longrightarrow \quad \boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_0 r}$ (is the magnitude)
- In vector form, $\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\boldsymbol{r}} - - - (\mathbf{1})$ where r is the (radial) unit vector in the plane normal to the wire passing through the point.
- E is directed outward if λ is positive (positive charge) and inward if λ is negative (negative charge).
- Note that though only charge enclosed by the Gaussian surface (λl) is included above in RHS, the electric field E is due to the charge on the entire wire. Equation (1) is a very good approximation even for a long wire (**not infinite**) for the electric field around the central portions of a long wire, where the end effects may be ignored.



Problem : What is the electric field 50 cm away from a conductor that has a linear charge density (λ) of 25 μC per meter ?

Solution: Formula : $\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_0 r}$

Given r = 50 cm = 0.5 m and λ = 25 μC m^{-1} = 25 x 10^{-6} C m^{-1}

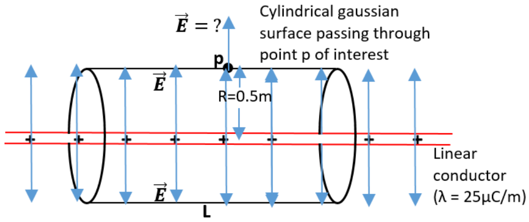
We know that $\frac{1}{4\pi\epsilon_0} = 9x10^9\ Nm^2C^{-2}$

We can write $E = \frac{\lambda}{2\pi\epsilon_0 r}\ as\ E = \left[\frac{1}{4\pi\epsilon_0}\right]\frac{2\lambda}{r}$; Substituting the above values, we get

$$E = \frac{(2\ x\ 25\ x\ 10^{-6}\ Cm^{-1})\ (9x10^9\ Nm^2C^{-2})}{0.5\ m} = 9\ x\ 10^5\ NC^{-1}$$

Solve the above problem from first principles

By using Gauss's law, one can avoid rigorous mathematics like calculus and integration. **Let us solve this problem using Gauss's law concepts**



- Given a very long linear conductor with λ = 25μC/m = 25 x 10^{-6} C/m. We need to find the electric field (magnitude and direction) at a point p which is 0.5m away from the linear straight conductor.
- This linear conductor contains a certain amount of +ve charge (say) and assuming that there no current in the conductor which implies charges do not move (electrostatics). Given λ = 25μC/m = 25 x 10^{-6} C/m
- Consider a Gaussian surface, a cylindrical surface passing our point p of interest. You can see the electric field line E emanating out from the central axis of the Gaussian cylinder (hypothetical cylinder)
- $\oint \vec{E}.\overrightarrow{dA} = \frac{q_{enclosed}}{\epsilon_0}$; since E is same anywhere on the surface of the cylinder and hence it is a constant for this assumed Gaussian cylindrical surface and hence taken out of integration (in LHS). So, $\vec{E}\oint\overrightarrow{dA} = \frac{q_{enclosed}}{\epsilon_0}$
- $\oint\overrightarrow{dA}$ for cylindrical surface (except 2 circular cross-sectional area at its ends → which we ignore assuming that given linear conductor is long enough so that edge effects can ignored.
- $\therefore \oint\overrightarrow{dA} = 2\pi RL$;
- $\therefore E\ x2\pi RL = \frac{q_{enclosed}}{\epsilon_0}$; we need to find out $q_{enclosed}$; how much charge is enclosed in the assumed cylinder. If L is the length of cylinder and λ is the charge density $\Longrightarrow \lambda\ = \frac{\Delta Q}{\Delta L}$; $therefore\ \Delta Q\ =\ \lambda\,\Delta L$
- $\therefore\ q_{enclosed}\ =\ \lambda L$; substituting, we get
- $E\ x2\pi RL = \frac{\lambda L}{\epsilon_0}$; $\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_0 R}$; substituting the values, we get
- $\boldsymbol{E} = \frac{(2\,x\,25\,x\,10^{-6}\ Cm^{-1})\,(9x10^9\ Nm^2C^{-2})}{0.5\ m} = \boldsymbol{9\ x\ 10^5\ NC^{-1}}$ **(magnitude)**
- $\boldsymbol{Direction\ of\ E = radially\ away\ from\ the\ axis\ of\ cylinder.}$

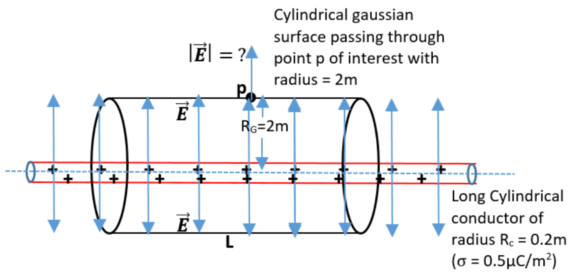
A point charge Q is kept in free space at the centre of the a cubic Gaussian surface of side "l". What is the electric flux through one of its faces?

Cube has 6 faces; Total flux through the complete cube = $\varphi = \frac{Q}{\epsilon_0}$; where $\epsilon_0 = 8.85\ x\ 10^{-12}N^{-1}m^{-2}C^2$

- Flux through 1 face of cube = $\varphi = \left(\frac{1}{6}\right)\left(\frac{Q}{\epsilon_0}\right)\ Nm^2C^{-1}$
- Flux through 2 faces of cube = $\varphi = \left(\frac{2}{6}\right)\left(\frac{Q}{\epsilon_0}\right) = \left(\frac{1}{3}\right)\left(\frac{Q}{\epsilon_0}\right)Nm^2C^{-1}$
- Flux through 3 faces of cube = $\varphi = \left(\frac{3}{6}\right)\left(\frac{Q}{\epsilon_0}\right) = \left(\frac{1}{2}\right)\left(\frac{Q}{\epsilon_0}\right)Nm^2C^{-1}$
- Flux through 4 faces of cube = $\varphi = \left(\frac{4}{6}\right)\left(\frac{Q}{\epsilon_0}\right) = \left(\frac{2}{3}\right)\left(\frac{Q}{\epsilon_0}\right)Nm^2C^{-1}$
- Flux through 5 faces of cube = $\varphi = \left(\frac{5}{6}\right)\left(\frac{Q}{\epsilon_0}\right)Nm^2C^{-1}$
- Flux through 6 faces of cube = $\varphi = \left(\frac{6}{6}\right)\left(\frac{Q}{\epsilon_0}\right) = \left(\frac{Q}{\epsilon_0}\right)Nm^2C^{-1}$

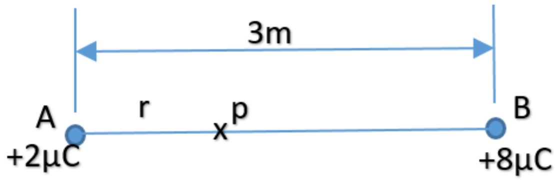
Problem: in the previous problem, we considered thin long linear wire (conductor). Let us consider cylindrical conductor of radius 20 cm and a charge density σ = 0.5 μC/m². What is the electric field 2 m away from this cylindrical conductor ?

Solution :



- Assume cylindrical conductor is long enough to ignore side edge effects. So, only we consider cylindrical surface.
- Imagine a hypothetical Gaussian cylinder passing through p of interest so that R_G = 2 m
- Direction of E is everywhere radially outwards (since cylindrical conductor has +ve charges enclosed)
- $\therefore \oint \vec{E}.\overrightarrow{dA} = \frac{Q_{enclosed}}{\epsilon_0} \implies \vec{E} \oint \overrightarrow{dA} = \frac{Q_{enclosed}}{\epsilon_0}$
- $\oint \overrightarrow{dA} = 2\pi R_G L$ and $Q_{enclosed} = (2\pi R_C L)\sigma$
- $E(2\pi R_G L) = \frac{(2\pi R_C L)\sigma}{\epsilon_0}$
- $E = \frac{R_C\,\sigma}{R_G\,\epsilon_0} \quad \left[\frac{1}{4\pi\epsilon_0} = k = 9\,x\,10^9 \implies \epsilon_0 = \frac{1}{4\pi k}\right]$ $using\ this, we\ get$
- $E = \frac{R_C}{R_G}\sigma\,(4\pi k); substituing\ the\ values\ in\ this, we\ get$
- $E = \frac{(0.2)(0.5)(10^{-6})\,4\pi\,(9\,x\,10^9)}{2}$; simplifying this , we get
- $E = 5650\,NC^{-1}\,\hat{r}$

Problem : Two point charges +2μC and +8μC are placed at x = 0m and x = 3m on x-axis respectively. Find the position between the two charges where the net electric field is zero.



Solution: Let p be the point where net E = 0. Let r be the distance from x = 0

E_1 is the electric field at point p due to A = $E_1 = \frac{k\,(2\,\mu C)}{r^2}$

E_1 is the electric field at point p due to A = $E_1 = \frac{k\,(8\,\mu C)}{(3-r)^2}$

At point p, $E_1 = E_2$ and since they are in opposite direction, net field = 0

$\therefore \frac{k\,(2\,\mu C)}{r^2} = \frac{k\,(8\,\mu C)}{(3-r)^2}$

$4\,r^2 = (3-r)^2$; $\implies (2r)^2 - (3-r)^2 = 0$

[2r + (3-r)][2r –(3-r)] = 0

[r+3][3r-3] = 0

∴ r = -3 or r = 1

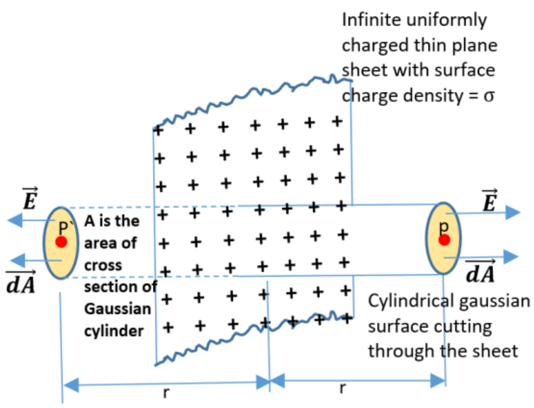
r = -3 is not possible

Hence the answer is r = 1m

At a distance 1 m from x = 0, the net electric field is zero.

Electric field due to a "uniformly" charged "Infinite" "Thin" plane sheet

- Let us consider a uniformly charged infinite thin non conducting plane sheet having surface charge density σ (σ = charge per unit area)
- As shown in the figure, choose points p and p` equidistant from the sheet. We need to find the electric field intensity at p (or/and p`)
- Draw a Gaussian cylinder (with cross sectional area A) cutting through the sheet and passing through the points p and p`.
- Let A be the area of each plane end of Gaussian cylinder.
- Let the sheet be positively charged.
- Since the sheet is infinite in extension, the electric field intensity $\vec{E}$ at all points on either side near the sheet will be perpendicular to the sheet and directed outwards away from the sheet (sheet is +vely charged). If the sheet is –vely charged, $\vec{E}$ is directed inwards towards the sheet.
- Thus as shown in the figure, the $\vec{E}$ is perpendicular to the two plane ends of the Gaussian cylinder. The magnitude of $\vec{E}$ at p and p` will be same as p and p` are equidistant from the sheet.



  - Therefore, the flux through the two plane ends of the cylinder is
  - $\varphi_E = \int_A \vec{E}.\vec{dA} + \int_A \vec{E}.\vec{dA}$, where $\vec{dA}$ is the area vector at the plane ends of the cylinder.
  - Also, the sheet is uniformly charged, $\vec{E}$ is constant at p and p`
  - Since $\vec{E}$ and $\vec{dA}$ are parallel, θ = 0 ∴ cos0 = 1. Therefore $\varphi_E = E\int_A \vec{dA} + E\int_A \vec{dA} = EA + EA = 2EA$
    - Since the flux (field lines) through both plane ends of the cylinder are going out of cylinder, flux at both ends are added. Therefore, $\varphi_E = 2EA$
- The flux through the curved surface of the Gaussian surface is ZERO because $\vec{E}$ and $\vec{dA}$ are perpendicular to each other everywhere on the curved surface (cos 90 =0).
- Hence the total flux is the contribution from only 2 plane ends of the cylinder. $\boldsymbol{\varphi_E = 2EA ------(1)}$
- As per Gauss's law, $\varphi_E = \frac{q}{\epsilon_0}$, where q is the net charge enclosed by the Gaussian cylinder. Here, q = σA and so $\varphi_E = \frac{\sigma A}{\epsilon_0} ------ (2)$
- From (1) and (2), $\boldsymbol{2EA} = \frac{\sigma A}{\epsilon_0}$
- $\therefore \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{r} ----(3)$; ***where $\hat{r}$ is the unit vector normal to the plane sheet***
- Direction : $\vec{E}$ at any point (as p and p`) is directed away from the sheet of positive charge. If the sheet is –vely charged, E is directed inwards towards the sheet.
- Equation (3) is independent of r. It means that the magnitude of the electric field intensity is same for all points "close" to the plane sheet of charge. This result is not surprising. As we move away and away from the sheet more and more charge comes in our "field of view" and this offset the decrease in the field due to increasing distance.
- For a finite sheet, equation (3) is approximately true in the middle of the plane sheet, away from the ends.

Problem : Plane sheet is positively charged with uniform σ = 2 x 10^{-6} Cm^{-2}. What is the electric field near both ends of plane sheet.

Solution : Formula $\boldsymbol{E} = \frac{\sigma}{2\epsilon_0}\hat{r}$ ***where $\hat{r}$ is the unit vector normal to the plane sheet and away from the sheet.***

Magnitude $|\vec{E}| = \frac{\sigma}{2\epsilon_0}$; multiply and divide RHS by 2π, we get

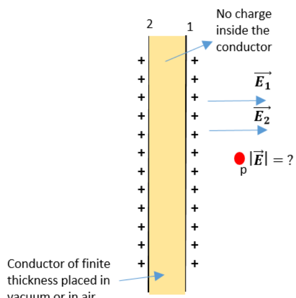
$|\vec{E}| = \left[\frac{1}{4\pi\epsilon_0}\right]2\pi\sigma$; $\left[\frac{1}{4\pi\epsilon_0}\right] = 9\ x\ 10^9\ Nm^2C^{-2}$ and given $\sigma = 2\ x\ 10^{-6}\ Cm^{-2}$

$|\vec{E}| = [9\ x\ 10^9\ Nm^2C^{-2}]2\pi\ x\ 2\ x\ 10^{-6}\ Cm^{-2} = 36\pi\ x\ 10^3 NC^{-1} = 1.13\ x\ 10^5 NC^{-1}$; this is the magnitude of Electric field.

Direction of $\vec{E}$ is perpendicular to the plane sheet surface on both sides and away from the sheet (since sheet is +vely charged)

Electric field 'just' outside a charged "conductor"

- We have seen above a thin infinite sheet and arrived at the intensity of electric field as $\boldsymbol{E} = \frac{\sigma}{2\epsilon_0}\hat{r}$
- Now, consider a large positively charged plane conductor of finite thickness placed in vacuum or air. Since the conductor is plane and thick, it total charge distributes itself "uniformly" on the "external surface of the conductor". There is no charge inside the conductor.
- Let σ be the surface charge density on each plate.
- Consider a point 'p' close to the conductor at which the intensity of the electric field is to be determined.
- Since there is no charge inside the conductor, this conductor can be assumed as equivalent to two thin plane sheets of charge 1 and 2.

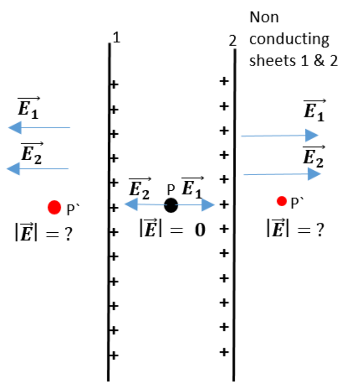


- The magnitude of field intensity $\vec{E_1}$ at 'p' due to sheet 1 is $\vec{E_1} = \frac{\sigma}{2\epsilon_0}$ $(\boldsymbol{away\ from\ sheet\ 1})$
- The magnitude of field intensity $\vec{E_2}$ at 'p' due to sheet 2 is $\vec{E_2} = \frac{\sigma}{2\epsilon_0}$ $(\boldsymbol{away\ from\ sheet\ 2})$
- Since $\vec{E_1}$ *and* $\vec{E_2}$ are in the same direction, the resultant intensity $\vec{E}$ at 'p' due to both the sheets is given by $\vec{E} = \vec{E_1} + \vec{E_2}$
- $\therefore E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$; $\boldsymbol{E = \frac{\sigma}{\epsilon_0}}$ away from the conductor (since +vely charged). If the conductor is –vely charged, E is directed towards the conductor .
- The formula $\boldsymbol{E = \frac{\sigma}{\epsilon_0}}$ is for plane charged thick conductor. In fact it holds for the electric field intensity 'just' outside a charged conductor of any shape
- As per the formula $\boldsymbol{E = \frac{\sigma}{\epsilon_0}}$, the intensity of electric field E close to a charged conductor is directly proportional to the surface charge density σ

Extra : if the surface of the conductor is pointed somewhere, then there σ is very high & so E is very high in the neighbouring air.

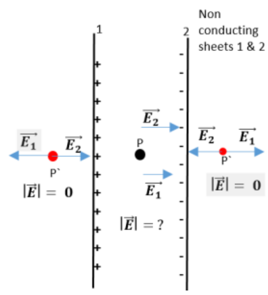
- In air some +ve and –ve ions are always present due to cosmic rays. If the conductor is +vely charged, the –ve ions in the air is accelerated towards the conductor. On their way, they collide with air molecules and produce more ions.
- Thus, the air in the neighbourhood of the pointed part of the conductor becomes conducting due to which the conductor is discharged very rapidly.

Electric field due to a two infinite parallel sheets of charge (same charge)



- Suppose 2 infinite, plane, non-conducting sheets of positive charge, 1 and 2, are placed parallel to each other in vacuum or in air
- Assume same surface charge density σ for both the sheets 1 and 2.
- We already know the magnitude of E on either side of sheet (close to sheet) of charge density σ is $\boldsymbol{E = \frac{\sigma}{2\epsilon_0}}$
- Let $\vec{E_1}$ be the intensity of $\vec{E}$ due to sheet 1 and $\vec{E_2}$ be the intensity of $\vec{E}$ due to the sheet 2
- For points p` (outside the sheets – see figure), we have
- $\vec{E_1} = \frac{\sigma}{2\epsilon_0}$ $(\boldsymbol{away\ from\ sheet\ 1})$ and $\vec{E_2} = \frac{\sigma}{2\epsilon_0}$ $(\boldsymbol{away\ from\ sheet\ 2})$
- Since $\vec{E_1}$ *and* $\vec{E_2}$ are in the same direction, the resultant intensity $\vec{E}$ at 'p' due to both the sheets is given by $\vec{E} = \vec{E_1} + \vec{E_2}$
- $\therefore E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$; $\boldsymbol{E = \frac{\sigma}{\epsilon_0}}$ away from both the sheet 1 and 2 (since +vely charged). If the sheets are –vely charged, E is directed towards the sheets.
- At a point p between the plates, $\vec{E_1}$ *and* $\vec{E_2}$ have the same magnitude but oppositely directed. ∴ $\vec{E} = \vec{E_1} - \vec{E_2} = \frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = 0$
- Therefore, the intensity of electric field between plates = 0
- If surface charge densities are different on the two plates (plate 1 has σ_1 and plate 2 has σ_2)
  - At points p` (outside plates) → $\boldsymbol{E = \frac{1}{2\epsilon_0}(\sigma_1 + \sigma_2)}$
  - At any point between the plates → $\boldsymbol{E = \frac{1}{2\epsilon_0}(\sigma_1 - \sigma_2)}$
  - If $\boldsymbol{\sigma_1 = \sigma_2}$, we get at p` → $\boldsymbol{E = \frac{\sigma}{\epsilon_0}}$; at 'p' inside the plates, $\boldsymbol{E = 0}$

Electric field due to a two infinite parallel sheets of charge (oppositely charged)



- Suppose 2 infinite, plane, non-conducting sheets of opposite charge, 1 and 2, are placed parallel to each other in vacuum or in air
- Assume same surface charge density σ for both the sheets 1 and 2.
- We already know the magnitude of E on either side of sheet (close to sheet) of charge density σ is $\boldsymbol{E = \frac{\sigma}{2\epsilon_0}}$
- Let $\vec{E_1}$ be the intensity of $\vec{E}$ due to sheet 1 and $\vec{E_2}$ be the intensity of $\vec{E}$ due to the sheet 2

For points p` (outside the sheets – see figure), we have

- $\vec{E_1} = \frac{\sigma}{2\epsilon_0}$ $(\boldsymbol{away\ from\ sheet\ 1})$ and $\vec{E_2} = \frac{\sigma}{2\epsilon_0}$ $(\boldsymbol{towards\ sheet\ 2})$
- Since $\vec{E_1}$ *and* $\vec{E_2}$ at p` are oppositely charged, the resultant intensity $\vec{E}$ at p` due to both the sheets is $\vec{E} = \vec{E_1} - \vec{E_2} = \frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = 0$
- Therefore, the intensity of electric field outside the plates = 0 everywhere.

For point p in between the sheets, we have $\vec{E_1} = \frac{\sigma}{2\epsilon_0}$ $(\boldsymbol{away\ from\ sheet\ 1})$ and $\vec{E_2} = \frac{\sigma}{2\epsilon_0}$ $(\boldsymbol{towards\ sheet\ 2})$

- Since $\vec{E_1}$ *and* $\vec{E_2}$ are in the same direction, the resultant intensity $\vec{E}$ at 'p' due to both the sheets is given by $\vec{E} = \vec{E_1} + \vec{E_2}$
- $\therefore E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$; $\boldsymbol{E = \frac{\sigma}{\epsilon_0}}$; Direction → points from +ve sheet to –ve sheet.
- In the above equation, the magnitude of E is independent on the position p between the sheets. Hence the electric field intensity between the plates (except near the edges) is uniform everywhere, independent of the separation between the sheets. E points from +ve towards –ve plate
- Thus, in this case, a uniform electric field exists only in the region between the plates and is zero elsewhere. In fact, this is the way to produce uniform E in a limited region of space. The only requirement is that the plates should be much larger in linear dimension than the separation between them.
- Capacitor is a typical example of this in steady state condition of the circuit (not during charging and discharging. Electric field inside the capacitor plates is $\boldsymbol{E = \frac{\sigma}{\epsilon_0}}$
- If A is the area of each plate of capacitor and the total charge on each plate = q, then q = σA ; σ = q/A ; ∴ $\boldsymbol{E = \frac{q}{\epsilon_0 A}}$ ------- **(1)**
- We will know from next chapter that for uniform E between the plates of a capacitor, the potential difference V is given by $\boldsymbol{E = \frac{V}{d}}$, where d is the distance between the two plates of the capacitor ; E is uniform within the distance d; **V = Ed ---- (2)**
- We know that $C = \frac{q}{v} = \frac{\epsilon_0 AE}{Ed} = \frac{\epsilon_0 A}{d}$; $\therefore C = \frac{\epsilon_0 A}{d}$; So capacitance of a capacitor depends only the the geometry of its production.

****

- Example#1 : if A = 1m² and d= 1mm, then C = 8.85 nF (where $\boldsymbol{\epsilon_0 = 8.85\ x\ 10^{-12} N^{-1} m^{-2} C^2}$)
- Example#2 : If C = I F , d = 1cm, what is A ?
  - $A = \frac{Cd}{\epsilon_0} = \frac{1\ F\ x\ 10^{-2} m}{8.85\ x\ 10^{-12} N^{-1} m^{-2} C^2}$
  - $1\ F = 1\ CV^{-1} = 1\ C\ (NC^{-1}m)^{-1} = 1\ N^{-1}m^{-1}C^2$
  - $A = \frac{1\ N^{-1}m^{-1}C^2\ x\ 10^{-2}m}{8.85\ x\ 10^{-12} N^{-1} m^{-2} C^2} = \frac{10^{10}}{8.85} \approx 10^9 m^2 = 10^9\ x\ (10^{-3} km)^2 \approx 1000\ km \approx 30\ x\ 30\ km\ in$ length & breadth .

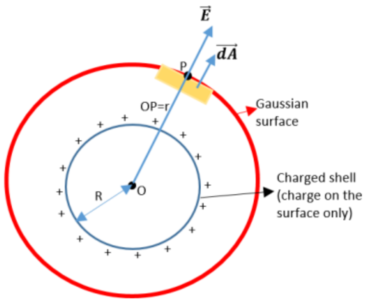
  This shows 1F is too big a unit practically.

- **Formulae :**
  - **For parallel plate capacitor $E = \frac{q}{\epsilon_0 A}$; where A is the area of each capacitor plate.**
  - **E = V/d ; V = Ed =** $\frac{qd}{\epsilon_0 A}$ and C = q/V = $\frac{\epsilon_0 A}{d}$

Electric field due to a uniformly charged Thin Spherical Shell

- Let O be the centre and R the radius of a thin, isolated spherical shell carrying a charge +q (say) which is uniformly distributed on its surface. **Note that the charges are on the surface of the shell only ; there is no charge inside the shell.**
- We have to determine "intensity of electric field" due to this shell at
  - Points outside the shell
  - On the surface of the shell
  - Inside the shell.

Field at a point p outside the shell (r > R)

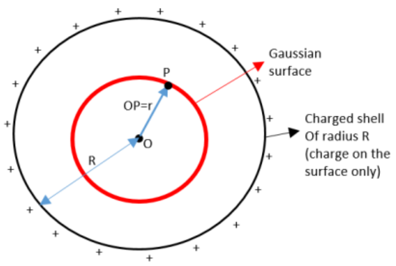


- Let p be the point, distant r (>R) from the centre O. we need to determine intensity of $\vec{E}$ at p.
- Let us draw a concentric spherical Gaussian surface of radius r through the point p. all points on this Gaussian surface are equidistant from the surface of the charged shell.
- Because of spherical symmetry, the magnitude E of the electric field intensity is same at all points on the Gaussian surface and directly radially <u>outwards</u>. (since we have taken charge as +q)
- Let us consider a small area element dA around point p. Both the electric field intensity vector E and the area vector dA at point p are directly radially outwards. Therefore the angle between them is zero. Therefore, the electric flux through the area element dA is
- $d\varphi_E = \vec{E}.\vec{dA} = E\ dA\ cos0 = E\ dA$
- Therefore, the flux through the entire Gaussian surface is
- $\varphi_E = \oint E\ dA = E \oint dA = E(4\pi r^2)$
- From Gauss's law, $\varphi_E = \frac{q}{\epsilon_0}$, where q is the total charge enclosed by the Gaussian surface
- $\therefore\ E(4\pi r^2) = \frac{q}{\epsilon_0}$
- $\therefore E = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q}{r^2}$ for r > R ---------------------------- (1)
- This is also the formula for the electric intensity at a point which is at distance r from a point charge q at the centre. It therefore follows that for points outside the charged spherical shell, the shell behaves as if all the charge on its surface were concentrated at its centre.
- If σ is the uniform charge density at the surface of the shell, then the total charge on the shell = q = σ (surface area of the sphere) = $\sigma\ (4\pi R^2)$
- Equation (1) becomes $E = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\sigma\ (4\pi R^2)}{r^2} = \left(\frac{\sigma}{\epsilon_0}\right)\left[\frac{R^2}{r^2}\right]$; $\therefore E = \left(\frac{\sigma}{\epsilon_0}\right)\left[\frac{R^2}{r^2}\right]$ ------------------------------ (2)
- Equations (1) and (2) are the required relations for r > R

Field at the surface of the shell (r = R)

- If point p lies on the surface of the shell, then r = R
- Eq (1) becomes $E = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q}{R^2}$
- Eq (2) becomes $E = \left(\frac{\sigma}{\epsilon_0}\right)\left[\frac{R^2}{R^2}\right] = \left(\frac{\sigma}{\epsilon_0}\right)$; $\therefore E = \frac{\sigma}{\epsilon_0}$

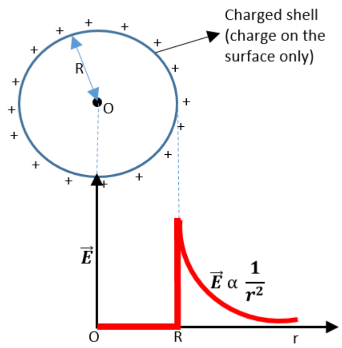
Field inside the shell (r < R)



- What is the intensity E at p inside the shell. p is at a distance r from centre O. Clearly r < R (see figure)
- Draw a concentric spherical Gaussian surface of radius r passing though the point p.
- This Gaussian surface does not enclose any charge since all charges on the surface of the shell.
- As per Gauss's theorem, we have
- $\varphi_E = E(4\pi r^2) = \frac{q}{\epsilon_0}$; since no charge is enclosed by Gaussian surface, q = 0 and hence $\varphi_E = 0\ \ and\ E = 0$
- The electric field intensity is zero everywhere inside the "uniformly charged thin shell"

Graph of E Vs r

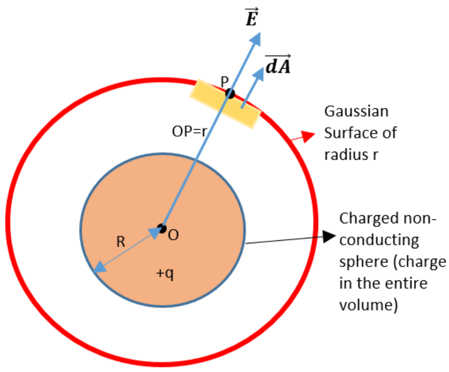
- The variation of electric filed E of a uniformly charged spherical shell with distance r from the centre of the shell is shown in the figure.
- As we can see from the graph, the electric field E is zero everywhere inside the shell (from r = 0 to r < R)
- E is maximum at the surface of the shell (r = R).
- E decreases rapidly outside the shell ($E\ \alpha\ \frac{1}{r^2}$)



Electric field due to a uniformly charged non-conducting Sphere

- Let O be the centre and R the radius of an isolated non-conducting charged sphere carrying a charge +q (say) which is uniformly distributed in the entire volume of the sphere. **[Note that if the sphere is conducting, the charge will get collected on the surface only].**
- We have to determine "intensity of electric field" due to this sphere at
    - Points outside the sphere
    - On the surface of the sphere
    - Inside the sphere.

Field at a point p outside the sphere (r > R)

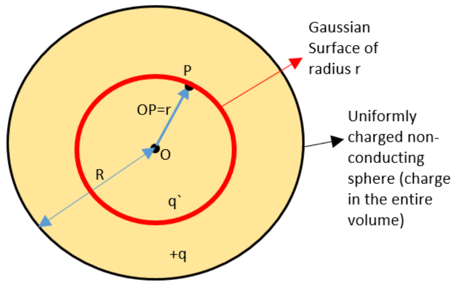


- Let p be the point, distant r (>R) from the centre O. we need to determine intensity of $\vec{E}$ at p.
- Let us draw a concentric spherical Gaussian surface of radius r through the point p. All points on this Gaussian surface are equidistant from the surface of the charged sphere.
- Because of spherical symmetry, the magnitude E of the electric field intensity is same at all points on the Gaussian surface and directly radially outwards. (since we have taken charge as +q)
- Let us consider a small area element dA around point p on the Gaussian surface. Both the electric field intensity vector E and the area vector dA at point p are radially outwards. Therefore the angle between them is zero. Therefore, the electric flux through the area element dA is
- $d\varphi_E = \vec{E}.\overrightarrow{dA} = E\ dA\ cos0 = E\ dA$
- Therefore, the flux through the entire Gaussian surface is
- $\varphi_E = \oint E\ dA = E \oint dA = E(4\pi r^2)$ [E is constant at all points on the Gaussian surface, hence taken out of integral]
- From Gauss's law, $\varphi_E = \frac{q}{\epsilon_0}$, where q is the total charge enclosed by the Gaussian surface
- $\therefore\ E(4\pi r^2) = \frac{q}{\epsilon_0}$
- $\therefore E = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q}{r^2}$ for r > R ---------------------------- (1)
- This is also the formula for the electric intensity at a point which is at distance r from a point charge q at the centre. It therefore follows that for points outside the charged spherical sphere, the sphere behaves as if all the charge on its surface were concentrated at its centre.
- Since the sphere is non-conducting , the charge is distributed throughout the volume of the sphere.
- If ρ is the volume charge density (unit Cm^{-3}), then q = ρ x (volume of the sphere) = ρ x $4/3\pi r^3$; substituting this in equation (1), we get
- $E = \left(\frac{1}{4\pi\epsilon_0 r^2}\right)\frac{\rho\,(4\pi R^3)}{3} = \left(\frac{\rho}{3\epsilon_0}\right)\frac{R^3}{r^2}$ ----------------- (2)
- Equations (1) and (2) are the required relations for r > R for non-conducting charged sphere.
- In case of a metallic (conducting) sphere, the entire charge will lie on the surface of the sphere and hence equation 1 will only hold. In this case equation (2) is not applicable.

Field at the surface of the sphere (r = R)

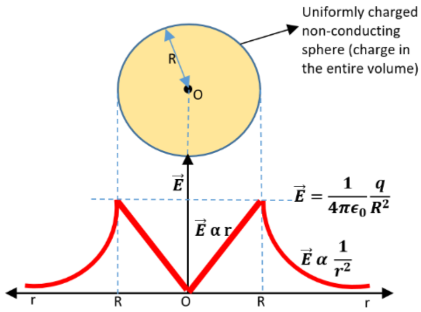
- If point p lies on the surface of the shell, then r = R
- Eq (1) becomes $E = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q}{R^2}$
- Eq (2) becomes $E = \left(\frac{\rho}{3\epsilon_0}\right)\left[\frac{R^3}{R^2}\right] = \left(\frac{\sigma R}{3\epsilon_0}\right)$; $\therefore E = \frac{\sigma R}{3\epsilon_0}$ -----------------------(3)

Field inside the sphere (r < R)



- What is the intensity E at p inside the sphere. p is at a distance r from centre O. Clearly r < R (see figure)
- Draw a concentric spherical Gaussian surface of radius r passing though the point p.
- Electric flux through this Gaussian surface (r < R)
- $\varphi_E = E(4\pi r^2);$ *where r is the radius of the* Gaussian surface
- As per Gauss's theorem, we have $\varphi_E = \frac{q`}{\epsilon_0}$, where q\` is that part of q which is contained within the Gaussian surface of radius r
- $\therefore E(4\pi r^2) = \frac{q`}{\epsilon_0}$ *and hence* $E = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q`}{r^2}$ ------------------------- (4)
- Since the sphere is non-conducting and uniformly charged, the volume charge density ρ is constant throughout the charged sphere
- $\Rightarrow \rho = \frac{q}{\frac{4}{3}\pi R^3} = \frac{q`}{\frac{4}{3}\pi r^3} \Rightarrow\ q` = q\left[\frac{r}{R}\right]^3 ----(5)$; plugging 5 in 4, we get
- $E = \left(\frac{1}{4\pi\epsilon_0 r^2}\right) q \left[\frac{r}{R}\right]^3 = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{qr}{R^3}$ -- (6)
- Since $q = \left[\frac{4}{3}\pi R^3\right]\rho$, then $E = \left(\frac{r}{4\pi\epsilon_0 R^3}\right)\left[\frac{4}{3}\pi R^3\right]\rho\ \ \Rightarrow E = \frac{\rho r}{3\epsilon_0}$ -------- (7)
- Equations 6 and 7 give electric field intensity at points inside the non-conducting charged sphere at an internal point and is proportional to the distance r of the point from the centre of the sphere.
    - *In case of metallic conducting sphere, the entire charge will reside on the outer surface of the sphere. Thus there will be no charge within the Gaussian surface (r < R). So by Gauss's theorem, we have $\varphi_E = 0$ and $E = 0$. Thus, the electric field throughout the interior of a charged "conducting" sphere is zero.*

Graph E Vs r



- E increases linearly from 0 inside the sphere (from r = 0 to r = R) becomes maximum at the surface of the sphere and decreases rapidly with distance ($\alpha\ \frac{1}{r^2}$) outside the sphere.
- For conducting sphere (whose charge lies on the outer surface), the variation is same as for a uniformly charged thin spherical SHELL.
- If we compare spherical shell and solid non conducting sphere, we see that in both cases the electric field at external point is the same (it is as if the entire charge is concentrated at the centre). But for the internal points, the two cases differ. For the shell the field is zero. For solid sphere, the field linearly decreases from its maximum value at the surface to ZERO at the centre.