Problem: Let G, be a group such that for all $a, b \in G$, we have:

$$(ab)^m = a^m b^m \quad \text{—— (*)}$$

and $$(ab)^n = a^n b^n \quad \text{—— (ii) (**)}$$

where $m (> n)$ and n are two positive integers.

How should m and n be chosen such that G be-comes Abelian?

Solution: Since $m > n$ we can write:

$$(ab)^m = (ab)^{m-n} (ab)^n$$

$$= a^{m-n} b^{m-n} a^n b^n = a^m b^m \quad \text{—— (*)(*)}$$

$$\Rightarrow \quad a^n b^{m-n} = b^{m-n} a^n. \quad \text{—— (1)}$$

Similarly,

$$(ab)^m = (ab)^n (ab)^{m-n} = a^m b^m$$

$$\Rightarrow \quad a^{m-n} b^n = b^n a^{m-n}. \quad \text{—— (2)}$$

2 implies that:

$$b^n a^n = a^{m-n} b^n a^{2n-m}$$

or $$b^n a^n = a^{m-n} (b^n a^n) a^{-(m-n)} \quad \text{—— (3)}$$

1 implies that:

$$a^n b^n = b^{m-n} a^n . b^{2n-m}$$

or $$a^n b^n = b^{m-n} (a^n b^n) b^{-(m-n)} \quad \text{—— (4)}$$

4 implies that: (by replacing a with a^{-1} and b with b^{-1}):

$$a^{-n} b^{-n} = b^{-(m-n)} (a^{-n} b^{-n}) b^{m-n} \quad \text{—— (5)}$$

Multiplying LHS's and RHS's of 3 and 5 we get:

$$1 = a^{m-n} (b^n a^n) a^{-(m-n)}$$
$$* \; b^{-(m-n)} (a^{-n} b^{-n}) b^{m-n}.$$

$= a^{m-n} (ba)^n (ba)^{-(m-n)} (ab)^{-n} b^{n-m}$

$= a^{m-n} (ba)^{2n-m} (ab)^{-m} b^{n-m}$

$\Rightarrow a^{-(m-n)} b^{-(m-n)} = (ba)^{2n-m} (ab)^{-n}$

$\Rightarrow (ba)^{-(m-n)} = (ba)^{2n-m} (ab)^{-n}$

$\Rightarrow (ab)^n = (ba)^{2n-m+m-n}$

$\Rightarrow (ab)^n = (ba)^n$

$\Rightarrow a^n b^n = b^n a^n$ —— 6

We again multiply the LHS's and the RHS's of 3 & 5 to get.

$1 = a^{m-n} (b^n a^n) a^{-(m-n)}$

$\times b^{-(m-n)} (a^{-n} b^{-n}) b^{m-n}$

$= a^{m-n} (b^n a^n) (a^{-(m-n)} b^{-(m-n)})$

$(a^{-n} b^{-n}) b^{m-n}$

$$= a^{m-n}(ba)^{n}(ba)^{-m}\, b^{m-n}$$

$$= a^{m-n}(ba)^{-(m-n)}\, b^{m-n}.$$

$$\Rightarrow (ba)^{-(m-n)} = a^{-(m-n)}\, b^{-(m-n)}$$

$$\Rightarrow (ba)^{m-n} = b^{m-n} a^{m-n}$$

Interchanging a & b we get that:

$$(ab)^{m-n} = a^{m-n} b^{m-n}$$

$$\forall\ a, b \in G \quad \text{—— } 7.$$

Now letting $m' = m$

and $n' = m-n$.

and repeating the initial 4 steps we obtain an equation similar to equation ~~4~~ 6 with n' in place of n:

$$a^{n'} b^{n'} = b^{n'} a^{n'}.$$

Thus we have manged to find two numbers n and n' such that

$$\left\{ \begin{array}{l} a^n b^n = b^n a^n \\ \text{and} \quad a^{n'} b^{n'} = b^{n'} a^{n'}. \end{array} \right\} \quad \text{— } (8)$$

Next we refer to the article (paper) published in arxiv. org.

(8) is exactly the property P stated on page (2) of the paper.

Next we refer to the Theorem 3.1 in this paper. The theorem states that:

Theorem 3.1: Let G be a finite group satisfying property P. Then G is Abelian.

So in order to make our given group Abelian, we only have to ensure that m & n are chosen in such a way that n & n' become co-prime.