Supplementary Guide for Lab 1: Gaussian Distributions & Covariance

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1 Univariate Gaussian Density

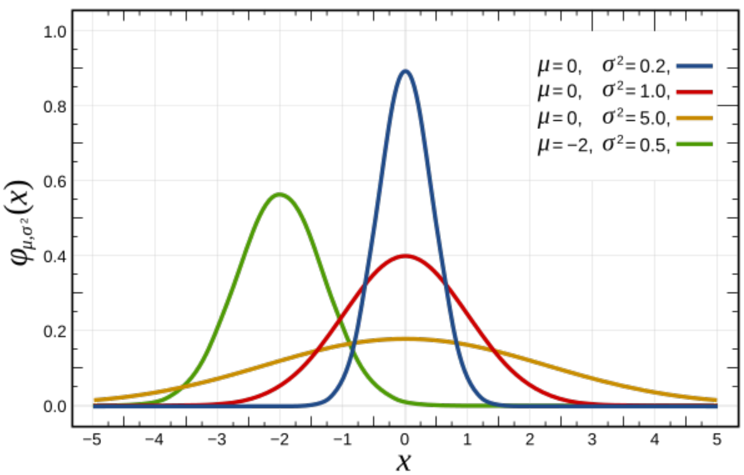

Figure 1: Univariate Gaussian Density

A *univariate Gaussian density* (also called the *normal distribution*) is a continuous PDF on the real line, parameterized by mean μ and variance $\sigma^2 > 0$.

Definition:

$$f(x) \;=\; \frac{1}{\sqrt{2\pi\,\sigma^2}}\exp\!\Bigl(-\tfrac{(x-\mu)^2}{2\,\sigma^2}\Bigr), \quad x \in \mathbb{R}.$$

- μ is the location (center of the bell).
- σ^2 is the spread (width of the bell).
- The normalizing factor ensures $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Properties:

- **Shape & Symmetry:** Bell curve centered at $x = \mu$, symmetric: $f(\mu+t) = f(\mu-t)$.
- **Moments:**
  - Mean: $E[X] = \mu$.
  - Variance: $\mathrm{Var}(X) = \sigma^2$.

- **Standard Gaussian:** When $\mu = 0$, $\sigma^2 = 1$,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\Bigl(-\tfrac{x^2}{2}\Bigr),$$

  and any $X \sim \mathcal{N}(\mu, \sigma^2)$ can be standardized via $Z = (X - \mu)/\sigma \sim \mathcal{N}(0, 1)$.

- **Visual Intuition:**
  - High $\sigma \to$ wide, flat bell.
  - Low $\sigma \to$ tall, narrow bell.

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2 Multivariate Gaussian Distribution

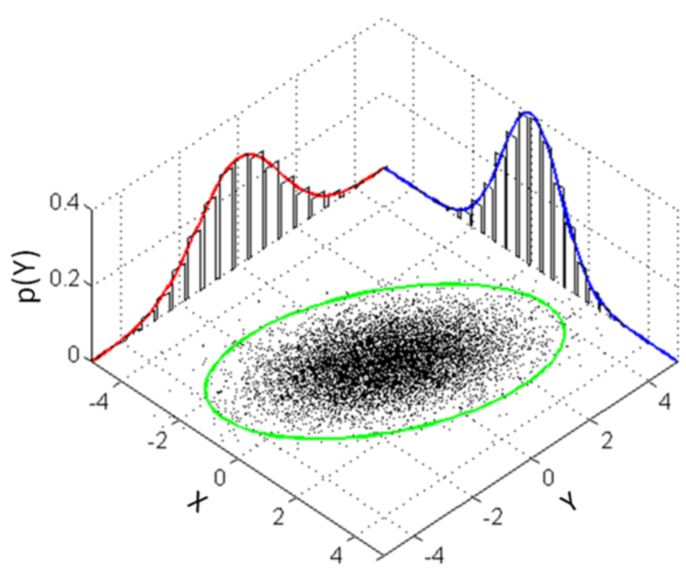

Figure 2: Multivariate Gaussian Distribution

A *multivariate Gaussian distribution* generalizes the normal distribution to d-dimensional vectors. It is defined by a mean vector $\mu \in \mathbb{R}^d$ and a covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$ (symmetric, positive semidefinite).

For $d = 2$, let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma), \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\,\sigma_1\sigma_2 \\ \rho\,\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

- Each marginal $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$.

- The correlation is $\rho = \dfrac{\mathrm{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$.

- Contours of the joint density are ellipses oriented and scaled by the eigenvectors and eigenvalues of Σ.

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3 Central Limit Theorem (CLT)

The CLT explains why sums (or averages) of many small, independent random effects tend toward a Gaussian distribution.

1. **Informal Statement:**
   Taking a large number of i.i.d. random variables (finite mean and variance) and summing them (with rescaling) yields an approximately Gaussian distribution, regardless of the originals' shape.

2. **Formal Version:**
   Let $X_1, \ldots, X_n$ be i.i.d. with $E[X_i] = \mu$, $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Define

   $$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu).$$

   Then as $n \to \infty$,

   $$S_n \xrightarrow{d} \mathcal{N}(0, \sigma^2) \quad \text{or} \quad \frac{S_n}{\sigma} \xrightarrow{d} \mathcal{N}(0, 1).$$

3. **Key Points:**
   - *Distribution-free:* Original X_i can be uniform, exponential, etc.
   - *Universality:* Explains prevalence of bell-shaped noise/averages.
   - *Inference Foundation:* Justifies normal-based confidence intervals/tests for large n.

4. **Illustration:**
   - Sum of 1 $\mathrm{Uniform}(0, 1)$: flat histogram.
   - Sum of 2 uniforms: triangular.
   - Sum of 4, 12 uniforms: increasingly bell-shaped.

5. **Lab Connection:**
   In Section 1 of the lab, summing $\mathrm{Uniform}(0, 1)$ samples and plotting histograms for increasing n demonstrates the CLT in action.

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4 Covariance

Covariance generalizes variance to pairs of variables, measuring how they vary together.

1. **Variance (univariate):**

$$\mathrm{Var}(X) = E\big[(X-\mu)^2\big],$$

   always nonnegative.

2. **Covariance (two variables):**

$$\mathrm{Cov}(X,Y) = E\big[(X-\mu_X)(Y-\mu_Y)\big].$$

   - Positive: X, Y tend to move together.
   - Negative: they move oppositely.
   - Zero: uncorrelated (not necessarily independent).

3. **Variance as Special Case:**

$$\mathrm{Cov}(X,X) = \mathrm{Var}(X).$$

4. **Covariance Matrix (multivariate):**
   For $\mathbf{X} = (X_1, \ldots, X_d) \sim \mathcal{N}(\mu, \Sigma)$,

$$\Sigma_{ij} = \mathrm{Cov}(X_i, X_j).$$

   - Diagonals: variances.
   - Off-diagonals: covariances.
   - Σ must be positive semidefinite.

Let

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim \mathcal{N}\Big(\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \Sigma = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 2 \end{bmatrix}\Big).$$

1. **Variances & Covariance:**

$$\mathrm{Var}(X_1) = 1, \quad \mathrm{Var}(X_2) = 2, \quad \mathrm{Cov}(X_1, X_2) = 0.8 > 0.$$

2. **Joint Density:**

$$f(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det\Sigma}} \exp\Big(-\tfrac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\Big),$$

   with $\det\Sigma = 1.36$ and

$$\Sigma^{-1} = \frac{1}{1.36} \begin{bmatrix} 2 & -0.8 \\ -0.8 & 1 \end{bmatrix}.$$

3. **Geometric Interpretation:**
   - Contours are ellipses.
   - Axis lengths = $\sqrt{\lambda_1}, \sqrt{\lambda_2}$ (eigenvalues).
   - Tilt toward the line $x_1 = x_2$ due to positive covariance.
4. **Relation to Univariate:**
   If $\Sigma_{12} = 0$, the bivariate factors into two independent univariate Gaussians.
5. **Marginals & Conditionals:**

$$X_1 \sim \mathcal{N}(0,1), \quad X_2 \sim \mathcal{N}(0,2), \quad X_1 \mid X_2 = x_2 \sim \mathcal{N}(0.4\,x_2,\ 1 - 0.32).$$

Use this guide alongside your lab exercises for quick reference on definitions, properties, and geometric insights.