Chapter 2 HW 7

Statistical Mechanics: EP 400

Section: 02DB

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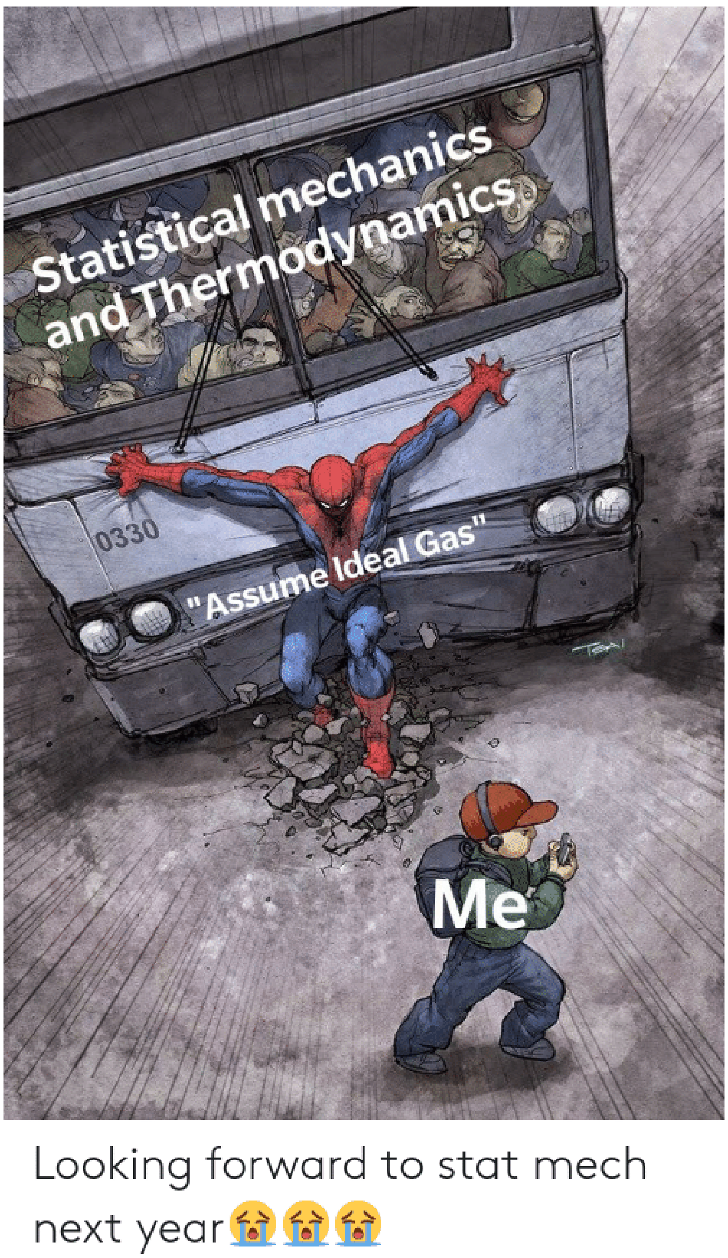

1 2.26:

Instead of deriving the multiplicity in 2 dimensions, we can compare the 3 dimensional equation 2.40 where d=3N:

$$\Omega_N = \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{(3N/2)!}(3mU)^{3N/2}$$

We can purely just make d=2N and instead volume we now have area, making the equation:

$$\Omega_N = \frac{1}{N!}\frac{A^N}{h^{2N}}\frac{\pi^{2N/2}}{(2N/2)!}(2mU)^{2N/2} = \frac{1}{N!}\frac{A^N}{h^{2N}}\frac{\pi^N}{(N)!}(2mU)^N$$

Though this is not a sufficient proof, so let's start by allowing the momentum space to be 2 dimensional, $p_x + p_y = \sqrt{2mU}$, assuming we'd have the factor of $A \cdot A_p$ since it is not volumetric anymore and having the multiplicity being proportional to $1/h^2$, instead of $1/h^3$. With this we get $\Omega_1 = \frac{A \cdot A_p}{h^2}$ similar to equation 2.34, $\Omega_1 = \frac{V \cdot V_p}{h^3}$ and having to account for the multiple counting of states with the factor of $1/N!$ and the momentum space factor with area of 2N and a radius of $\sqrt{2mU}$, getting an area from the general hyper-sphere equation, $A = \frac{2\pi^{d/2}}{(d/2-1)!}r^{d-1}$, where d=2N, $= \frac{2\pi^N}{(N-1)!}r^{2N-1}$ and we assume some factors to be negligible due to N being a very large number. From equation 2.38 and finding similarities from equation 2.40, we can put this together:

$$\Omega_N = \frac{1}{N!}\frac{A^N}{h^{2N}} \cdot area = \frac{1}{N!}\frac{A^N}{h^{2N}}\frac{\pi^N}{N!}(2mU)^N$$

2 2.28:

We have 52 cards that are face-up only, where N=52 and the number of possibilities, or multiplicity of all the macrostates, is $\Omega = N! = 52! = 8.066 \times 10^{67}$ Using equation 2.45 for the definition of entropy $S \triangleq kln(\Omega) = kln(N!) =$ thus we get the entropy in fundamental units:

$$\frac{S}{k} = ln(8.066 \times 10^{67}) = 156.4$$

Such in S.I. units entropy is:

$$S = 156.4k = 2.16 \times 10^{-21} J/K$$

This is the entropy created when shuffling/dealing cards, which is a very small number, especially on terms of thermal motions with gases. Since entropy is proportional to the number of particles, making it a relatively large value.

3 2.30:

Since the Einstein solids are identical, they have equal oscillators, $N = N_A = N_B$, because of this when they go into thermal equilibrium, their energies are equal $q_A = q_B$.

3.1

For $N = 10^{23}$, we can find the entropy from equation 2.45, $S/k = ln(\Omega_T)$, because we are finding the entropy if all the macrostates are allowed. Using the solution to problem 2.22b, $\Omega_T = \frac{2^{4N}}{\sqrt{8\pi N}}$:

$$\frac{S}{k} = ln(\frac{2^{4N}}{\sqrt{8\pi N}}) = ln(2^{4N}) - ln(\sqrt{8\pi N}) = 4Nln(2) - ln(\sqrt{8\pi N}) = 4(10^{23})ln(2) - ln(\sqrt{8\pi 10^{23}}) = 2.773 \times 10^{23} - 28.1$$

Since we are dealing with large numbers, the final solution is 2.773×10^{23}

3.2

When the system is in its most likely state, its probability is highest and that is when it is in thermal equilibrium. Using the solution from problem 2.22c and the same entropy equation from a: $\Omega = \frac{2^{4N}}{\sqrt{4\pi N}}$:

$$\frac{S}{k} = ln(\frac{2^{4N}}{\sqrt{4\pi N}}) = ln(2^{4N}) - ln(\sqrt{4\pi N}) = 4Nln(2) - ln(\sqrt{4\pi N}) = 4(10^{23})ln(2) - ln(\sqrt{4\pi 10^{23}}) = 2.773 \times 10^{23} - 55.44$$

Again since we are dealing with large numbers, the solution comes to be 2.773×10^{23}

3.3

In parts 3.1 and 3.2, the entropy changed by a factor of $55.44 - 28.1 = 27.4$, which is a fairly small number considering the very large value of 3×10^{23} that is in the entropy solution in both parts, thus the effects of time scales of the system are negligible.

3.4

Same as the previous part, the difference is only 27.4 units, saying that when you go from only particles from one side to particles evenly distributes, its a difference of 27.4. Thus this violation is insignificant and I will sleep like a baby tonight, after I finish studying of course.

4 2.31:

Deriving the Sackur-Tetrode (S-T) equation, equation 2.49 $S = Nk[ln(V/N(\frac{4\pi mU}{3Nh^2})^{3/2}) + 5/2]$ starting from the definition of entropy from equation 2.40, $S \triangleq kln(\Omega)$, where $\Omega = \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{(3N/2)!}(3mU)^{3N/2}$ arranging this equation differently:

$$\frac{S}{k} = ln(\frac{V^N}{(N)!(3N/2)!}\frac{\pi^{3N/2}(2mU)^{3N/2}}{h^{3N}}) = ln(\frac{V^N}{(N)!(3N/2)!}(\frac{2\pi mU}{h^2})^{3N/2}) = ln(V^N) - ln(N)! - ln(\frac{3N}{2})! + ln(\frac{2\pi mU}{h^2})^{3N/2}$$

using Law of Logs and Sterling's approximation from equation 2.16 $ln(N!) \approx Nln(N) - N$:

$$\frac{S}{k} \approx Nln(V) - Nln(N) + N - \frac{3N}{2}ln(\frac{3N}{2}) + \frac{3N}{2} + \frac{3N}{2}ln(\frac{2\pi mU}{h^2}) = N[ln(V/N) + 1 + 3/2 + \frac{3}{2}ln(\frac{2\pi mU}{h^2} \cdot \frac{2}{3N})]$$

$$= N[ln(\frac{V}{N}(\frac{2\pi mU}{h^2} \cdot \frac{2}{3N})^{3/2}) + \frac{3+2}{2}] \therefore S = Nk[ln(\frac{V}{N}(\frac{4\pi mU}{3Nh^2})^{3/2}) + \frac{5}{2}]$$

We can say this is equal and not approximate due to N being a very large number.

5 2.32:

Using problem 2.26 to derive a 2 dimensional formula for entropy in terms on U, A, N. The multiplicity solved for in problem 2.26 is given:

$$\Omega = \frac{1}{N!}\frac{A^N}{h^{2N}}\frac{\pi^N}{N!}(2mU)^N$$

Using the definition of entropy from equation 2.45, $S \triangleq kln(\Omega)$, thus the entropy of a 2 dimensional ideal gas:

$$\frac{S}{k} = ln(\frac{1}{N!}\frac{A^N}{h^{2N}}\frac{\pi^N}{N!}(2mU)^N) = ln(\frac{1}{N^2!}\frac{(A\pi)^N}{h^{2N}}(2mU)^N) = ln(N)^2! + ln(A\frac{2\pi mU}{h^2})^N$$

Using Sterling's approximation from equation 2.16 and the Law of Logs:

$$\frac{S}{k} \approx Nln(A\frac{2\pi mU}{h^2}) - 2(Nln(N) - N) = N[ln(\frac{2\pi AmU}{N^2h^2}) + 2] \therefore \frac{S}{k} = N[ln(\frac{A}{N}\frac{2\pi mU}{Nh^2}) + 2]$$

Again, we can say this is equal due to N being a very large number.

6 2.33:

The entropy of argon gas (Ar) can be found by using the S-T equation, equipartition theorem, and the ideal gas law. Given, 1 mol of gas, an average room temperature of 300k, and mass of an Ar molecule of 40u, or Z=40, at 1 atm. Where Ar is a monatomic gas, and assuming an ideal gas with indistinguishable particles, and the equipartition theorem states: $U = 3/2NkT$ and ideal gas law states: $PV = NkT = nRT$, where the S-T equation is $S = Nk[ln(V/N(\frac{4\pi mU}{3Nh^2})^{3/2}) + 5/2]$, plugging in and simplifying:

$$S = nR[ln(\frac{kT}{P}(\frac{4\pi m}{3Nh^2})^{3/2}(\frac{3NkT}{2})^{3/2} + 5/2] = nR[ln(\frac{kT}{P}(\frac{2\pi mkT}{h^2})^{3/2} + 5/2]$$

Where $m = Zm_p = 40(1.67 \cdot 10^{-27}kg)$,$P = 1atm = 10^5Nm^{-2}$,$n = 1$mol,$T = 300K$, and of course our constants:

$$S = 8.315[ln(\frac{1.38 \times 10^{-23} \cdot 300}{10^5}(\frac{2\pi \cdot 40 \cdot 1.67 \times 10^{-27} \cdot 1.38 \times 10^{-23} \cdot 300}{(6.626 \times 10^{-34})^2})^{3/2}) + 5/2] = 8.315[18.64] = 155J/K$$

Everything else in the solution is either the same or a constant, where the only thing changing is the masses. Thus the difference between Argon and Helium, where Argon has mass of 40u and Helium of 4u, is $(\frac{40}{4})^{3/2} = 31.6$

7 2.35:

From equation 2.49 we know the S-T equation to be

$$S/k = N[ln(V/N(\frac{4\pi mU}{3Nh^2})^{3/2}) + 5/2]$$

Assuming the gas is an ideal gas and using a monatomic gas Helium to solve for the minimum temperature where the S-T breaks down, causing entropy to become negative. The gas is at atmospheric pressure with constant density, starting at room temperature of around 300K, solving for how low it can go until negative. Because this is a logarithmic expression, we know that if $x < 1$, where we have $ln(x)$, the outcome is a negative number. Doing all this we re-write the S-T equation by writing 5/2 as $ln(e^{5/2})$, thus the S-T equation becomes:

$$S = Nkln[V/Ne^{5/2}(\frac{4\pi mU}{3Nh^2})^{3/2}]$$

Setting the logarithm equal to 1, so we can find the minimum of ln(x), using the equipartition theorem, $U = 3/2NkT$ and setting equal to T:

$$1 = V/Ne^{5/2}(\frac{4\pi mU}{3Nh^2})^{3/2} = V/Ne^{5/2}(\frac{2\pi mkT}{h^2})^{3/2} \therefore T = (\frac{N}{V})^{2/3}\frac{h^2}{2\pi e^{5/2}mk}$$

Assuming the term N/V is constant, so we can write it as P_0/kT_0, where $P_0 = 10^5Nm^{-2}$ and $T_0 = 300K$, where the mass of a Helium atom is 4u, which is 6.64×10^{-27}kg, thus the temperature where the S-T equation breaks down:

$$T = (\frac{P_0}{kT_0})^{2/3}\frac{h^2}{2\pi e^{5/2}mk} = (2.415)^{2/3}\frac{(6.626 \times 10^{-34})^2}{2\pi e^{5/2} \cdot 6.64 \times 10^{-27} \cdot 1.38 \times 10^{-23}} = 0.012K$$

This is nearly absolute zero, thus the S-T equation won't work in the vacuum of space in a dust cloud of some sorts.

8 2.38:

From the section on dipole moments of para-magnets, we know $\Omega(N, N_\uparrow) = \frac{N!}{N_\uparrow!N_\downarrow!} = \binom{N}{N_\uparrow}$, now putting this in terms of 2 interacting gases A and B we get $\Omega(N, N_A) = \frac{N!}{N_A!N_B!}$ From the definition of entropy, equation 2.45, we know $S = kln(\Omega(N))$, thus when we have 2 bodies you can say:

$$\Delta S = kln(\Omega(N, N_A)) = kln\binom{N}{N_A} = kln(\frac{N!}{N_A!N_B!})$$

Assuming both N values are large, we can use Sterling's approximation from equation 2.16 $ln(N)! \approx Nln(N) - N$:

$$\Delta S = k[ln(N)!-ln(N_A)!-ln(N_B)!]k[Nln(N)-N-N_Aln(N_A)+N_A-N_Bln(N_B)+N_B] = k[Nln(N)-N_Aln(N_A)-N_Bln(N_B)]$$

Substituting $N_A = N(1-x)$ and $N_B = xN$ into the equation:

$$\Delta S \approx k[Nln(N) - N(1-x)ln(N(1-x)) - xNln(xN)] = k[Nln(N) - N(1-x)(ln(1-x) + ln(N) - xNln(x) - xNln(N)]$$

$$= kN[ln(N) - (1-x)ln(N) - (1-x)ln(1-x) - xlnl(x) - xln(N)] = kN[ln(N) - ln(N) + xln(N) - (1-x)ln(1-x) - xln(x) - xln(N)]$$

$$\therefore \Delta S = -Nk[xln(x) + (1-x)ln(1-x)]$$

Assuming again that N is very large number, setting them equal and not approximately equal.

9 2.42:

The book tells us that adding mass increases the entropy of a black hole (BH) and that the entropy of a BH must be greater than any type of matter that could have been used to create it.

9.1

Using dimensional analysis, we can show that the mass M should have a radius of order GM/c^2, where the radius of a BH is called the Schwarzschild radius and every object has one:

$$RBH = \frac{GM}{c^2} = \frac{m^3kg^{-1}s^{-2}kg}{m^2s^{-2}} = m$$

Thus the equation shows the radius should have S.I. units in meters. Now calculating the radius if the mass is equal to that of the sun, where $M_\odot = 1.989 \times 10^{30}kg$ and speed of light $c = 2.998 \times 10^8 m/s$, thus we get:

$$R_{BH} = \frac{6.67 \times 10^{-11} \cdot 1.989 \times 10^{30}}{(2.998 \times 10^8)^2} = 1477m = 1.477km$$

9.2

The entropy of a BH is on the order of the maximum number of particles that could have made it because if we take N number of particles and collapse them to an infinitely dense point, their Schwarzschild radius, we'd still have the same amount of particles. Thus we'd have same amount of mass and energy, not destroying or creating anything, energy is conserved here. For this to happen, the final entropy must be on the order of the maximum number of particles N, such that these are the largest number of particles that it could have been formed from.

9.3

Aligning this with the previous question, to maximize the number of particles that make a BH, or N, we want to have a long wavelength, such that they are massless particles. But this wavelength cannot be larger than that of the radius of the BH, thus it is equal to its radius. Setting the energy of the particles (E_p) equal to mc^2, where $m = M/N$, we can estimate the maximum number of photons needed to make a BH of mass M:

$$R_{BH} = \lambda = \frac{GM}{c^2}$$

We know that the total energy of the BH is: $E = hf = \frac{hc}{\lambda}$, where λ is the de-broglie wavelength, equal to that of the BH's radius, thus we can say:

$$\lambda = \frac{GM}{c^2} = \frac{hc}{E} \therefore \frac{GM}{c^2} = \frac{hc}{mc^2} = \frac{hN}{Mc}$$

Solving for N:

$$N = \frac{GM}{c^2}\frac{Mc}{h} = \frac{GM^2}{hc}$$

This is in terms of the fundamental units of entropy such that $N = S/k$, putting it in S.I.:

$$S = k\frac{GM^2}{hc}$$

Where Hawking received the factor of $8\pi^2$, thus:

$$S = k\frac{8\pi^2 GM^2}{hc}$$

9.4

Using the formula derived in the previous part, we can solve for the entropy of a one solar mass BH, where $M_\odot = 1.989 \times 10^30 kg$:

$$S = k\frac{8\pi^2 GM^2}{hc} = 1.38 \times 10^{-23} \cdot \frac{8\pi^2 \cdot 6.67 \times 10^{-11} \cdot (1.989 \times 10^{30})^2}{6.626 \times 10^{-34} \cdot 2.998 \times 10^8} = 1.447 \times 10^54$$

This is a massive number! The sun is not even a relatively large star compared to super giants, which are sometimes 100's of times larger. I do not know the difference in entropy from a super massive BH made from a super giant star, but I can imagine it is around a billion or trillion times the entropy of the sun, or one solar mass star. Which is absolutely stunning, but insane to think about!