HW 2: Appendix B10-B16

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1 B17:

Derive the General Integration formulas from B.36

From B.32 for n=1 we know: $\int_0^\infty \frac{x}{e^x \pm 1} dx \approx \int_0^\infty [xe^{-x} \mp xe^{-2x} + xe^{-3x}]dx$
Making a general integral for the solution for n $\geq$ 0 and k > 0:

$$\int_0^\infty x^n e^{-kx} dx // u = kx; du = kdx \rightarrow \frac{1}{k}\int_0^\infty (\frac{u}{k})^n e^{-u} du = \frac{1}{k \cdot k^n}\int_0^\infty u^n e^{-u} du$$

From B.12 we know: $\int_0^\infty x^n e^{-x} dx = \Gamma(n+1)$

$$\therefore \frac{1}{k^{n+1}}\int_0^\infty u^n e^{-u} du = \frac{1}{k^{n+1}}\Gamma(n+1)$$

Expanding the k term for $k > 0$:

$$\frac{1}{k^{n+1}}\Gamma(n+1) \approx [1 \mp \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} \mp \frac{1}{4^{n+1}}]\Gamma(n+1)$$

Showing that from B.32 for n=1: $\int_0^\infty \frac{x}{e^x \pm 1} dx \approx 1 \mp \frac{1}{2^2} + \frac{1}{3^2} \mp \frac{1}{4^2}$ we can generalize it into:
$\int_0^\infty \frac{x^n}{e^x \pm 1} dx \approx 1 \mp \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} \mp \frac{1}{4^{n+1}}$

From all this, we can say:

$$\int_0^\infty \frac{x^n}{e^x \pm 1} dx \approx [1 \mp \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} \mp \frac{1}{4^{n+1}}]\Gamma(n+1)$$

From these derivations, we can solve for both signs, starting with (-):

$$\int_0^\infty \frac{x^n}{e^x - 1} dx \approx [1 + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{4^{n+1}}]\Gamma(n+1) = \underline{\zeta(n+1)\Gamma(n+1)}$$

Where $\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + ...$ from B.33

Now solving for (+) using the approximation from B.35:

$$\int_0^\infty \frac{x^n}{e^x + 1} dx \approx [1 - \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} - \frac{1}{4^{n+1}}]\Gamma(n+1)$$

$$\approx [(1 + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{4^{n+1}}) - 2(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} + \frac{1}{6^{n+1}} + \frac{1}{8^{n+1}})]\Gamma(n+1)$$

$$\approx [\zeta(n+1) - \frac{2}{2^{n+1}}(1 + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{4^{n+1}})]\Gamma(n+1)$$

$$\approx [\zeta(n+1) - \frac{1}{2^n}\zeta(n+1)]\Gamma(n+1) = \underline{(1 - \frac{1}{2^n})\zeta(n+1)\Gamma(n+1)}$$

The reason we had to do extra algebra for the (+) sequence was b/c we had alternating signs, which is not included in the definition of $\zeta(n)$

2 B19:

Integrate eq B.42 twice more and solve for $x^{'} = \pi/2$ to obtain a formula for $\sum_{odd} 1/k^4$:

From B.42 we know: $\frac{\pi x^{'}}{4} = \sum_{odd} \frac{1}{k}\int_0^{x^{'}} sin(kx)dx = \sum_{odd} \frac{1-cos(kx^{'})}{k^2}$, integrating the right side:

$$\sum_{odd}\int_0^x \frac{1-cos(kx^{'})}{k^2}dx^{'} = \sum_{odd}\frac{1}{k^2}[\int_0^x dx^{'} - \int_0^x cos(kx)dx^{'}] = \sum_{odd}\frac{1}{k^2}[x - \frac{1}{k}sin(kx)]$$

$$\rightarrow \sum_{odd}\frac{1}{k^2}[\int_0^{x^{'}} xdx - \frac{1}{k}\int_0^{x^{'}} sin(kx)dx] = \sum_{odd}\frac{1}{k^2}[\frac{x^{'2}}{2} - \frac{1}{k}(\frac{-1}{k}cos(kx^{'}) + \frac{1}{k})]$$

$$= \sum_{odd}\frac{1}{k^2}[\frac{x^{'2}}{2} + \frac{1}{k^2}(cos(kx^{'}) - 1)]$$

Plugging $x^{'} = \pi/2$, where $cos(\frac{k\pi}{2}) = 0$ we get:

$$\sum_{odd}\frac{1}{k^2}[\frac{(\pi/2)^2}{2} + \frac{1}{k^2}(cos(\frac{k\pi}{2}) - 1)] = \sum_{odd}\frac{1}{k^2}[\frac{\pi^2}{2\cdot 4} - \frac{1}{k^2}] = \frac{\pi^2}{8}[\sum_{odd}\frac{1}{k^2}] - \sum_{odd}\frac{1}{k^4}$$

From B.43 we know $\sum_{odd}\frac{1}{k^2} = \frac{\pi^2}{8}$, thus we get:

$$\frac{\pi^2}{8}\cdot\frac{\pi^2}{8} - \sum_{odd}\frac{1}{k^4} = \frac{\pi^4}{64} - \sum_{odd}\frac{1}{k^4}$$

Notice we have all fourth power exponents, now integrating the left side of B.42 twice:

$$\int_0^x \frac{\pi x^{'}}{4}dx^{'} = \frac{\pi x^2}{2\cdot 4} \rightarrow \int_0^{x^{'}}\frac{\pi x^2}{8}dx = \frac{\pi x^{'3}}{8\cdot 3} = \frac{\pi x^{'3}}{24}$$

Plugging in $x^{'} = \pi/2$ like we did to the right side we get:

$$\frac{\pi\cdot\pi^3}{24\cdot 2^3} = \frac{\pi^4}{192}$$

Another fourth power; putting it all together:

$$\frac{\pi^4}{192} = \frac{\pi^4}{64} - \sum_{odd}\frac{1}{k^4} \rightarrow \sum_{odd}\frac{1}{k^4} = \frac{\pi^4}{64} - \frac{\pi^4}{192} = \frac{3\pi^4 - \pi^4}{192} = \frac{2\pi^4}{192} = \underline{\frac{\pi^4}{96}}$$

From B.44 we know $\zeta(n) = \sum_{odd}\frac{1}{k^n} + \sum_{even}\frac{1}{k^n}$; finding the solution to $\zeta(4)$:

$$\zeta(4) = \sum_{odd}\frac{1}{k^4} + \sum_{even}\frac{1}{k^4} \approx \frac{\pi^4}{96} + [\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4}] \approx \frac{\pi^4}{96} + \frac{1}{16}[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4}]$$

And from B.33 we know $\zeta(n) \approx 1 + \frac{1}{2^n} + \frac{1}{3^n}$, thus we can say:

$$\zeta(4) \approx \frac{\pi^4}{96} + \frac{\zeta(4)}{16} \rightarrow \frac{\pi^4}{96} = \zeta(4) - \frac{\zeta(4)}{16} = \zeta(4)[1 - 1/16] = \frac{15\zeta(4)}{16}$$

$$\therefore \zeta(4) = \frac{16\pi^4}{15 \cdot 96} = \underline{\frac{\pi^4}{90}}$$

Now evaluating integrals from B.36, for n=3, and using the gamma definition $\Gamma(n+1) = n!$ we get:

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \zeta(3+1)\Gamma(3+1) = \zeta(4)3! = \frac{3 \cdot 2 \cdot 1 \cdot \pi^4}{90} = \frac{6\pi^4}{90} = \underline{\frac{\pi^4}{15}}$$

$$\int_0^\infty \frac{x^3}{e^x + 1} dx = [1 - \frac{1}{2^3}]\zeta(4)\Gamma(3+1) = [1 - \frac{1}{8}]\frac{\pi^4}{90}3! = \frac{7 \cdot 6}{8}\frac{\pi^4}{90} = \underline{\frac{7\pi^4}{120}}$$

The reasoning of why we can't use this to evaluate $\zeta(3)$ is because plugging $x = \pi/2$ into the sum, which we saw from earlier by taking the first integral, which is the following:

$$\frac{\pi(\pi/2)^3}{4} = \frac{\pi^3}{32} = \sum_{odd} \frac{1}{k^2}[\frac{\pi}{2} - \frac{1}{k} sin(\frac{k\pi}{2})]$$

And we know $\sum_{odd} \frac{1}{k^2} = \frac{\pi^2}{8}$:

$$\therefore \frac{\pi^3}{16} - \frac{\pi^3}{32} = \sum_{odd} \frac{sin(\frac{k\pi}{2})}{k^3} = \frac{2\pi^3 - \pi^3}{32} = \frac{\pi^3}{32}$$

b/c this is a sin function, when you have k=1,5,9,etc it is positive, but if k=3,7,11,etc it is negative, thus alternating terms, which a series for does not exist for $\zeta(n)$

3 B20:

Evaluate eq. B.41 for $x = \pi/2$ and obtain π up to 3 sig figs:

As I started doing this problem by hand I quickly noticed I would need an enumerable amount of terms in this series, which is $\frac{\pi}{4} = \sum_{odd} \frac{kx}{k}$, to obtain a number relatively equal to that of π, therefore I wrote code to do this for me:

```
import math

def odd_series(x, k):
    result = 0
    for i in range(1, k+1, 2):
        result += math.sin(i * x)/i
    return result
k = 500
x = math.pi/2
ans = odd_series(x,k)
print('THE APP. ANSWER IS: ', 4*ans)
print('WHERE THE EXACT VALUE FOR PI: ', math.pi)
print('THE ERROR:', abs((4*ans - math.pi)/math.pi)*100,'%')
```

```
THE APP. ANSWER IS:  3.137592669589472
WHERE THE EXACT VALUE FOR PI:  3.141592653589793
THE ERROR: 0.12732344518791894 %
```

As you can see if k=500 I get 3.13759, which is 3 sig figs off of 3.14159, the exact value of π. Repeating this for k=100 and 250 I received values of 3.12159 and 3.14959, respectfully. So you can see as k increases the approximate value of π waivers within the 3 sig fig range from the exact value.

4 B21:

Show that the integrand given is an even function and evaluate with bounds $x = 0, \infty$:

By looking at the function, I see the squared power, thus it'll come out positive if x is a negative integer, and there are exponential functions. But proving mathematically by algebraically simplifying the integrand:

$$\frac{x^2e^e}{(e^x+1)^2} = \frac{x^2e^x}{(e^x+1)(e^x+1)} = \frac{x^2}{(e^x+1)(e^x+1)e^{-x}} = \frac{x^2}{(e^x+1)(e^x \cdot e^{-x} + e^{-x})} = \frac{x^2}{(e^x+1)(e^{-x}+1)}$$

To test, just have x equal any positive or negative integer. b/c of the positive and negative exponentials, the solution will always be positive, thus an even function. Now we are able to continue with the evaluation:

$$2\int_0^\infty \frac{x^2e^e}{(e^x+1)^2}dx = 2\int_0^\infty x^2 \cdot e^x(e^x+1)^{-2}//u = x^2; dv = e^x(e^x+1)^{-2}$$

Solving $\int_0^\infty e^x(e^x+1)^{-2}dx$ we get $-(e^x+1)^{-1}\Big|_0^\infty$ (which goes to 0), by applying the chain rule, continuing with IBP:

$$2[0 - (\int_0^\infty -2x(e^x+1)^{-1}dx)] = 2[2\int_0^\infty \frac{x}{(e^x+1)}dx]$$

From B.35 we know $\int_0^\infty \frac{x}{(e^x+1)}dx = \frac{\zeta(2)}{2}$ and from B.45 we know $\zeta(2) = \frac{\pi^2}{6}$, plugging in we get:

$$2[\frac{2\zeta(2)}{2}] = 2\zeta(2) = \frac{2\pi^2}{6} = \underline{\frac{\pi^2}{3}}$$

5 1.1:

5.1 a.

$$\frac{T_f - 32}{212-32} = \frac{T_c}{100} = \frac{T_k - 273}{100}$$

$$\frac{T_f - 32}{212-32} = \frac{T_c}{100} \to T_f = \frac{180T_c}{100} + 32 = \frac{9T_c}{5} + 32$$

$$\to T_c = \frac{5T_f}{9} - 32$$

5.2 b.

in Celsius, Absolute Zero is -273C, converting into Fahrenheit:

$$T_f = \frac{9 \cdot (-273C)}{5} + 32 = -459.4F$$

6 1.2:

In the previous problem we solved for Absolute Zero in degrees F, which is equivalent to 0R: $\therefore T_r = T_f + 459.4$ and converting Rankine to Kelvin is thus: $T_r = \frac{9T_k}{5}$

7 1.3:

7.1 a.

$T_f \approx 97.8F$ for the human body, which is what google gave $\therefore T_c = \frac{5 \cdot 97.8}{9} - 32 = 22.3C$ giving $T_k = 273 + 22.3 = 295.3K$

7.2 b.

Boiling point of what is 100 degrees C and freezing point is 100 degrees C, meaning $T_k = 273 + 100 = 373K$

7.3 c.

I went camping in below freezing weather over winter break, after the new year, where the lowest it got was -3 degrees C: $T_k = 273 - 3 = 270K$

7.4 d.

$$T_k = 273 - 196 = 77K$$

7.5 e.

$$T_k = 327 + 273 = 600K$$

8 1.4:

The Celsius scale is derived from the boiling (100 Celsius) and freezing (0 Celsius) points of water, chosen arbitrarily, thus saying something is twice as hot as the freezing point of water does make much sense. Same thing applies to Fahrenheit because it is derived from the human body temperature, making it arbitrary. Where Kelvin is defined from Absolute Zero, where nothing can get colder than. Making it the fundamental unit for nature because it is the limit. Thus it makes sense to say something is twice as hot if it is being described in Kelvin.

9 1.5:

My mom always told me to leave the thermometer in my mouth for a few minutes. But the temperature only takes a few seconds to reach the sensor, or the bulb. Meaning the numerical value of the "relaxation time" could be anywhere from 5 or 30 seconds to over a minute. It is dependent on how we define it.

10 1.6:

When I run into the street, which is made from cement and black paint, on a very hot day in Daytona Beach my feet start to burn immediately, so I jump in the grass b/c it is cooler, even though they are at the same temperature. This is a good example of different materials having different heat capacities.

11 1.7:

Since I am not 100 years of age, I do not have a mercury thermometer lying around, thus I looked up some dimensions on google. The thermometer I am using from google has a bulb radius of 1.5mm and length of 6mm, $\therefore V_0 = \pi r^2 \cdot l = 42.4mm^3$

11.1 a.

$$\Delta V = \beta V_0 \Delta T = (1.81 \cdot 10^{-4} K^{-1})(42.4mm^3)(1K) = 0.00767mm^3$$

Where the actual radii of the liquid Mercury is the following (Assuming l=10mm after expanding):

$$\Delta V = \pi r^2 l \rightarrow r = \sqrt{\frac{\Delta V}{\pi l}} = \sqrt{\frac{0.00767mm^3}{\pi 10mm}} = 0.156mm \therefore d = 2r = 0.0313mm$$

11.2 b.

Lakes freeze from the top down, usually freezing the top layer then floating on the water below b/c the density of ice is less than that of water. But theoretically if β were always positive, the process would flip and the water would freeze from the bottom up. Then chunks of ice would probably float to the surface.

12 1.8:

12.1 a.

Saying on a hot day $T_h = 30C$ and a cold day $T_c = 0C$, for simplicity:

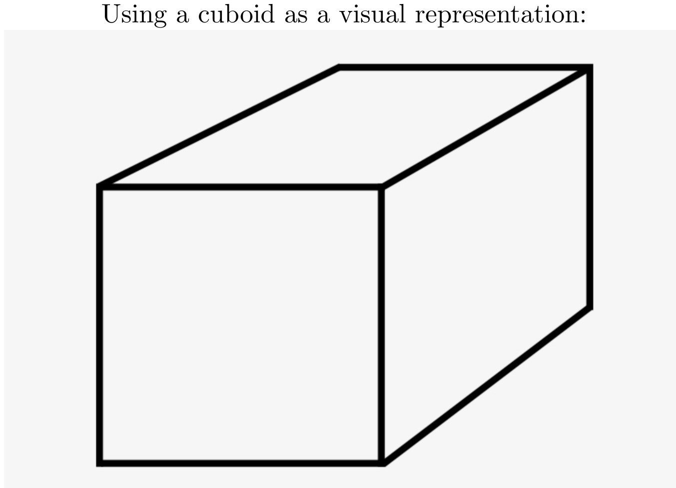
$$\Delta L = \alpha L_0 \Delta T = (1.1 \cdot 10^{-5} K^{-1})(1000m)(30K) = 0.33m$$

12.2 b.

The 2 metals have different thermal expansion coefficients. Thus one of the metals expands faster as the temperature increases, rotating the coil and moving the dial.

12.3 c.

Using a cuboid as a visual representation:



Where we have x, y, and z components for length:

$$\Delta x = \alpha_x x_0 \Delta T / \Delta y = \alpha_y y_0 \Delta T / \Delta z = \alpha_z z_0 \Delta T$$

$$\Delta V = \Delta x y_0 z_0 + \Delta y x_0 z_0 + \Delta z x_0 y_0 = (\alpha_x x_0 \Delta T) y_0 z_0 + (\alpha_y y_0 \Delta T) x_0 z_0 + (\alpha_z z_0 \Delta T) x_0 y_0$$

Where $V_0 = x_0 y_0 z_0$:

$$\therefore \Delta V = (\alpha_x + \alpha_y + \alpha_z) V_0 \Delta T$$

Where $\beta = (\alpha_x + \alpha_y + \alpha_z)$