Chapter 3 HW 8

Statistical Mechanics: EP 400

Section: 02DB

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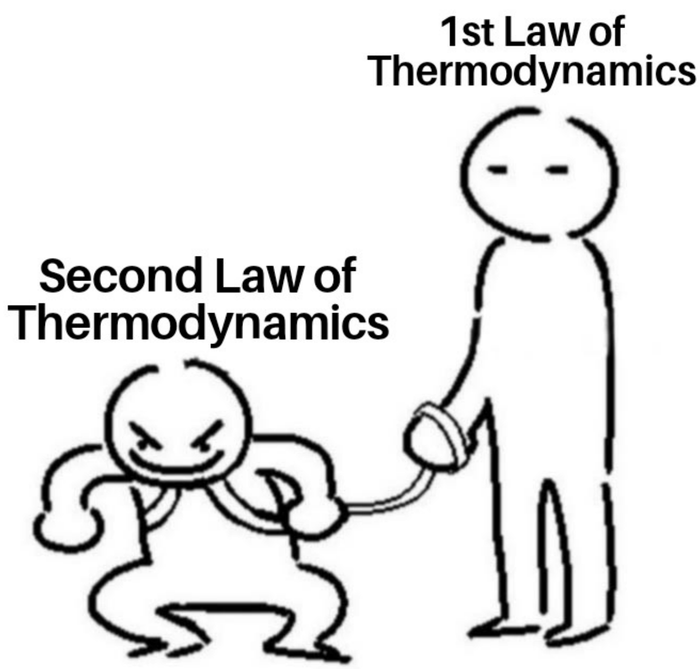


1 3.2:

The zeroth Law of Thermodynamics states that when in equilibrium, the temperatures of bodies A and B are equal and same for bodies B and C, thus temperatures of body A and C are equal, $T_A = T_B = T_C$. Proving this by using the definition of temperature, defined in equation 3.5,

$$\frac{1}{T} \triangleq (\frac{\partial S}{\partial U})_{N,V} \Rightarrow \frac{1}{T_A} = \frac{\partial S_A}{\partial U_A} = \frac{1}{T_B},$$

Where,

$$\frac{1}{T_B} = \frac{\partial S_B}{\partial U_B} = \frac{1}{T_C},$$

$$\therefore \frac{\partial S_A}{\partial U_A} = \frac{\partial S_C}{\partial U_C}.$$

2 3.3:

Body A has a steeper slope than that of body B, meaning the energy (U) wants to spontaneously flow into A from B. Effectively increasing the entropy (S) in body A. The flow of energy into A stops when the slopes of the 2 S vs. U plots are equal, thus in thermal equilibrium. An image of this equalizing is shown below,

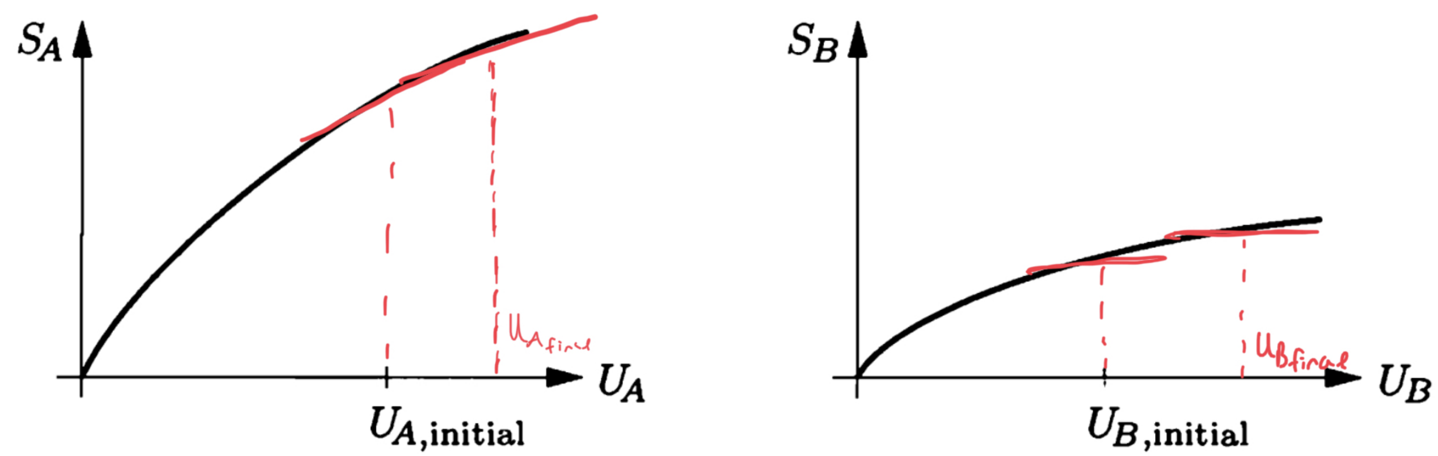


Figure 3.3. Graphs of entropy vs. energy for two objects.

Figure 1: Copied from the book on page 90 of chapter 3

3 3.5:

The result from problem 2.17 in chapter 2 is multiplicity when $q << N$, given by,

$$\Omega = (\frac{eN}{q})^q,$$

We also know from the section 'Real World Examples' in chapter 3 that $U = q\epsilon$. Given the definitions of entropy and temperature from eq 2.45 and 3.5, respectively, are,

$$S \triangleq kln(\Omega), \qquad \frac{1}{T} \triangleq (\frac{\partial S}{\partial U})_{N,V},$$

We can solve for Energy as a function of Temperature,

$$S = kln(\frac{eN}{q})^q = kqln(\frac{eN}{q}) = kq(ln(e) + ln(N) - ln(q)),$$

Plugging in $U = q\epsilon$,

$$S = \frac{kU}{\epsilon}(ln(N) - ln(U) + ln(\epsilon) + 1),$$

Taking the partial w.r.t. the energy (U) and solving for U by using Law of Logs and the Power Rule,

$$\frac{1}{T} = \frac{\partial}{\partial U}[\frac{kU}{\epsilon}(ln(N) - ln(U) + ln(\epsilon) + 1)] = \frac{\partial}{\partial U}(\frac{kU}{\epsilon}[ln(N\epsilon) + 1]) - \frac{\partial}{\partial U}(\frac{kUln(U)}{\epsilon}) = \frac{k}{\epsilon}[ln(N\epsilon) + 1] - (\frac{kln(U)}{\epsilon} + \frac{k}{\epsilon}) =$$

$$\frac{k}{\epsilon}[ln(\frac{N\epsilon}{U}) + 1 - 1] = \frac{k}{\epsilon}ln(\frac{N\epsilon}{U}),$$

Moving all the terms outside of the log to the other side and solving for U,

$$\frac{\epsilon}{kT} = ln(\frac{N\epsilon}{U}) \Rightarrow e^{\epsilon/kT} = e^{ln(N\epsilon/U)} = \frac{N\epsilon}{U},$$

$$\therefore U = N\epsilon e^{-(\epsilon/kT)}.$$

4 3.6:

We know that the multiplicity is proportional to the energy to the power of half of the total degrees of freedom, with some constant of proportionality l,

$$\Omega = l \cdot U^{Nf/2},$$

Using the definition of entropy and temperature from eq. 2.45 and 3.5, respectively, we can find the Energy in terms of its Temperature,

$$S \triangleq kln(\Omega) = k[ln(l) + ln(U)^{(Nf/2)}] = k[ln(l) + \frac{Nf}{2}ln(U)],$$

Plugging into the definition of Temperature,

$$\frac{1}{T} \triangleq (\frac{\partial S}{\partial U})_{N,V} = \frac{\partial S}{\partial U}[kln(l)] + \frac{\partial S}{\partial U}[k\frac{Nf}{2}ln(U)] = 0 + [k\frac{Nf}{2U}],$$

$$\therefore U = \frac{NfkT}{2}.$$

This is known as the Equipartition Theorem, for every DoF (Nf) there is an average Kinetic Energy of $\frac{1}{2}kT$.

5 3.7:

From problem 2.42 in chapter 2 we solved for the entropy of a Black Hole (B.H.), which is found to be,

$$S = \frac{8\pi^2 G U^2 k}{hc^5},$$

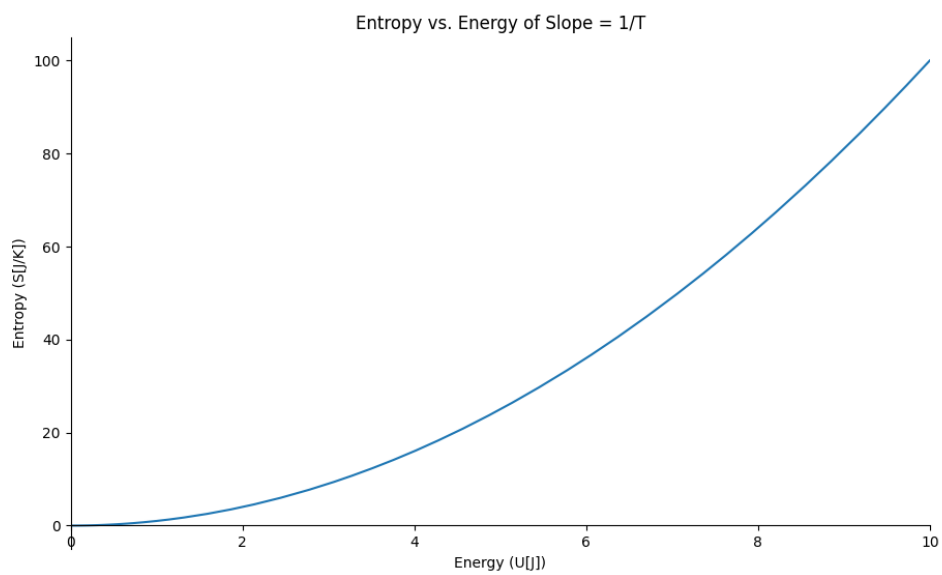
The question gives tells us that $U = Mc^2$, where for this question, the mass of the B.H. is 1 solar mass, $M_\odot = 1.989 \times 10^{30} kg$. Calculating the Temperature of the B.H. from the definition of Temperature, eq. 3.5,

$$\frac{1}{T} \triangleq (\frac{\partial S}{\partial U})_{N,V} = \frac{8\pi^2 Gk}{hc^5}\frac{\partial}{\partial U}(U^2) = \frac{8\pi^2 Gk}{hc^5} \cdot 2M_\odot c^2 = \frac{16\pi^2 G M_\odot k}{hc^3}.$$

Solving for the reciprocal and plugging in the knowns,

$$T = \frac{hc^3}{16\pi^2 G M_\odot k} = \frac{(6.626 \times 10^{-34} Js)(2.998 \times 10^8 m/s)^3}{16\pi^2 (6.67 \times 10^{-11} m^3/kg \cdot s^2)(1.989 \times 10^{30} kg)(1.38 \times 10^{-23} J/K)} = 6.176 \times 10^{-8} K = 0.6176 nK.$$

Because the entropy equation is proportional to the squared value of U, the graph is parabolic,



The slope of the graph is steep which means it has a low temperature, as we saw from our solution of the temperature of a B.H., and that the energy wants to flow into it. The parabola is concave up, meaning the Entropy increases as Energy increases, increasing the temperature. This is a greedy little guy, since he wants more energy without giving any; this is called a miserly system, described in Chapter 3 from the analogy.