Chapter 2 HW 6

Statistical Mechanics: EP 400

Section: 02DB

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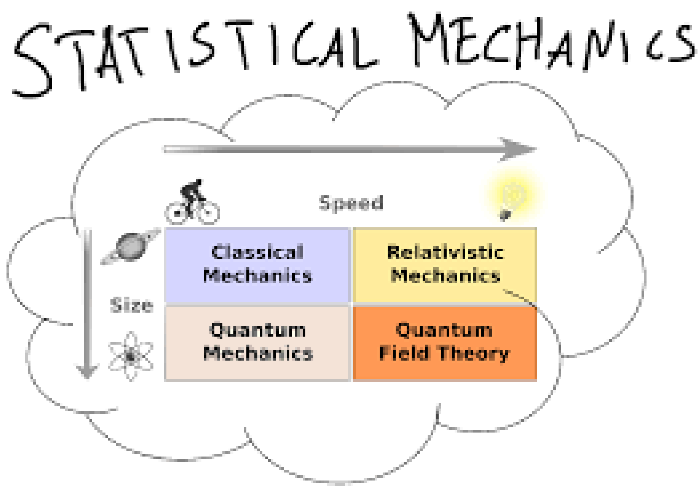


1 2.8:

If the system has 2 Einstein solids, A and B, each with 10 oscillators, we know $N_A = N_B = 10$, *sharing* a total of 20 units of energy, meaning $q_T = q_A + q_B = 20$. When we assume a weakly coupled system, we assume the bodies exchange heat very slowly. We are also assuming the total energy is constant, these are both common assumptions.

1.1

A macrostate is defined as the possible values for the total energy of the system, in this case the macrostate is equal to 21, because the macrostate $= 0 + 1 + 2 + 3 + 4 + ...$

1.2

A microstate is defined as the number of ways the macrostates can be described, in other words, the microstate is equal to the multiplicity of all the macrostates: $\Omega(N, q_T) = \frac{(q_T+N-1)!}{q_T!(N-1)!}$ (From eq. 2.9 of the text), thus

$$\Omega(20, 20) = \Omega(all) = \frac{(20 + 20 - 1)!}{20!19!} = \underline{6.892 \cdot 10^{10}}$$

1.3

Your chances of finding all of the energy in body A are highly unlikely, due to energy wanting to travel to the colder body and become evenly distributed (or be in thermal equilibrium), called the fundamental assumption of statistical mechanics. For this situation the probability is: $P(N_A, q_A) = \frac{\Omega(N_A,q_A)}{\Omega(all)}$ (From eq. 2.1 of the text), where $\Omega(N_A, q_A) = \frac{(q_A+N_A-1)!}{q_A!(N-1)!} = \frac{(20+10-1)!}{20!9!} = 1 \cdot 10^7$ and from 1.2 we know $\Omega(all) = 6.892 \cdot 10^{10}$, thus

$$P(N_A, q_A) = P(10, 20) = \frac{1 \cdot 10^7}{6.892 \cdot 10^{10}} = 1.45 \cdot 10^{-4} = \underline{0.0145\%}$$

1.4

If half of the energy is in body A, $q_A = 10$, thus $q_B = 10$ and the system is in thermal equilibrium, which is the state of highest probability, which is: $P(N_A, q_A) = P(N_B, q_B) = P(N, q) = \frac{\Omega_T}{\Omega(all)}$, we know from fig. 2.4 of the text
$\Omega_T = \Omega_A \Omega_B = \frac{(q_A+N_A-1)!}{q_A!(N_A-1)!} \cdot \frac{(q_B+N_B-1)!}{q_B!(N_B-1)!} = [\frac{(10+10-1)!}{10!9!}]^2 = 8.534 \cdot 10^9$ and from 1.2 we know $\Omega(all) = 6.892 \cdot 10^{10}$, thus

$$P(N, q) = \frac{8.534 \cdot 10^9}{6.892 \cdot 10^{10}} = 0.1238 = \underline{12.38\%}$$

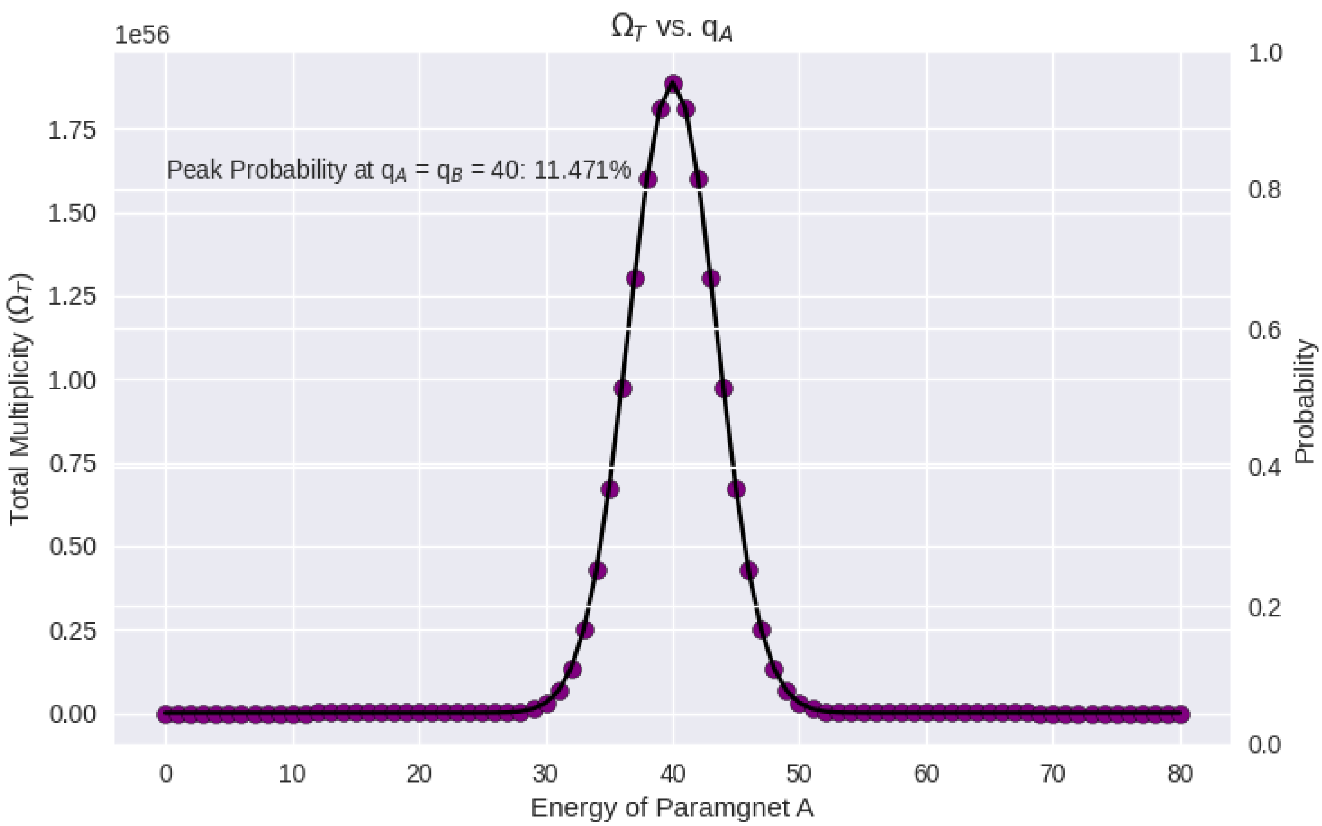
Which is a lot larger than if all the energy was distributed to body A or body B, or any other configuration.

1.5

If the system starts with all the energy in body B, it will eventually go into thermal equilibrium and have an evenly distributed energy across the system. If the system starts in equilibrium the probability for the energy to get unevenly distributed among the bodies is very small. This means it is not inevitable for the energy to be distributed completely to body B, but highly in-probable. Thus it is sort-of irreversible if this happens and the energy goes from equilibrium completely to body B, which is highly unlikely.

2 2.11:

For a 2 state para-magnet with N dipoles and q units of energy: $\Omega(N,q) = \frac{N!}{q!(N-q)!}$, where $N_A = N_B = 100$ and $q_T = q_A + q_B = 80$, thus there are 81 macrostates and because $N_A = N_B$, the maximum probability will be at $q_A = q_B = 40$. Showing the Gaussian plot of the probability and total multiplicity as a function of the macrostate:



Looking at the probability curve, the peak is at 11.471% when $q_A = 40$, with total multiplicity of $1.89 \cdot 10^{56}$, which is a *very large* number. That is the most probable macrostate, where the least probable macrostate is when $q_A = 0$ or $q_B = 0$ with a total multiplicity of $5.36 \cdot 10^{20}$, which has a probability of $3.254 \cdot 10^{-37}$, which is essentially *impossible* to happen.

The chart showing the multiplicities, macrostates, and probabilities of the macrostates is shown in the Appendix at the end of the document

3 2.13:

Fun with LOGS!

3.1

Simplifying the expression $e^{aln(b)}$ by using the Law of Logs:

$$e^{aln(b)} = ln(e^{aln(b)}) = ln(e^{ln(b)^a}) = b^a$$

3.2

Assuming $b << a$, we can prove $ln(a+b) \approx ln(a) + b/a$ by simplifying the left side to $ln(a(1+b/a)) = ln(a) + ln(1+b/a)$, using the T.S expansion: $ln(1+x) = x + \frac{1}{2}x^2 + ...$, we get $ln(1+b/a) = \frac{b}{a} + \frac{b^2}{2a^2} + ...$, thus we can say: $ln(a+b) \approx ln(a) + [\frac{b}{a} + \frac{b^2}{2a^2}]$, because $b << a$, we can say the term $\frac{b^2}{2a^2}$ is negligible:

$$ln(a+b) \approx ln(a) + \frac{b}{a}$$

4 2.14:

The problem basically asks to prove $e^{10^{23}} = 10^x$, which can be done using the Law of Logs:

$$ln(e^{10^{23}}) = ln(10^x) = xln(10) \therefore x = \frac{ln(e^{10^{23}})}{ln(10)} = \frac{10^{23}}{ln(10)} = 4.343 \cdot 10^{22}$$

Thus we can say:

$$e^{10^{23}} = 10^{4.343 \cdot 10^{22}}$$

5 2.16:

When we have 1000 coins we can say N=1000, since our only option is heads or tails this is a 2 state system, thus $\Omega(all) = 2^{1000}$ and the macrostate with the highest probability will be when there are 500 heads and 500 tails.

5.1

Number of heads is $n_H = 500$ and tails $n_T = 500$, where $N = n_H + n_T$, thus $\Omega(N, n_H) = \frac{N!}{n_H!(N-n_H)!} = \frac{N!}{n_H! n_T!} = \Omega(N, n_T) = \Omega(N, n)$, using the strong form of Stirling's approximation of $N! \approx N^N e^{-N}\sqrt{2\pi N}$ from eq 2.14:

$$\Omega(N,n) = \Omega(1000, 500) = \frac{1000!}{500!500!} = \frac{1000!}{(500!)^2} = \frac{1000^{1000}e^{-1000}\sqrt{2\pi 1000}}{(500^{500}e^{-500}\sqrt{2\pi 500})^2} = \frac{1000^{1000}e^{-1000}}{500^{1000}e^{-1000}\sqrt{500\pi}} = \frac{2^{1000}}{\sqrt{500\pi}} = \Omega(all)(500\pi)^{-1/2}$$

We know that the probability is $P(N,n) = \frac{\Omega(N,n)}{\Omega(all)}$ from an extension of eq 2.1, thus we say the max probability:

$$P(1000,500) = \frac{\Omega(all)}{\Omega(all)}(500\pi)^{-1/2} = (500\pi)^{-1/2} = 0.0252 = \underline{2.52\%}$$

5.2

Doing the same procedure, but if $n_H = 600$ and $n_T = 400$ by using the same equations from 5.1:

$$\Omega(N,n_H) = \Omega(1000,600) = \frac{1000!}{600!400!} = \frac{1000^{1000}e^{-1000}\sqrt{2\pi 1000}}{(600^{600}e^{-600}\sqrt{2\pi 600})(400^{400}e^{-400}\sqrt{2\pi 400})} = \frac{1000^{1000}e^{-1000}\sqrt{2000\pi}}{(600^{600}400^{400}e^{-1000}\sqrt{800\pi}\sqrt{1200\pi}} =$$

$$\frac{1000^{1000}}{600^{600}400^{400}\sqrt{480\pi}} \therefore P(1000,600) = P(1000,400) = \frac{1000^{1000}}{(2^{1000})(600^{600}400^{400}\sqrt{480\pi})} = \frac{500^{1000}}{(600^{600}400^{400}\sqrt{480\pi})}$$

$$= \frac{500^{600}500^{400}}{(600^{600}400^{400}\sqrt{480\pi})} = (\frac{500}{600})^{600}(\frac{500}{400})^{400}(480\pi)^{-1/2} = \underline{4.64 \cdot 10^{-11}}$$

Which is much less than the maximum probability, meaning it is not inevitable for this event to occur, but highly unlikely.

6 2.17:

Using methods similar from the section *Multiplicity of a Large Einstein Solid* to derive eq. 2.21: $\Omega(N,q) \approx (\frac{eq}{N})^N$ if $q << N$, rather than $q >> N$:

We know $\Omega(N,q) = \frac{(q+N-1)!}{q!(N-1)!} \approx \frac{(q+N)!}{q!N!}$ from eq 2.17 and saying $\Omega(N,q) = \Omega$ for simplicity and taking the natural log: $ln(\Omega) = ln(\frac{(q+N)!}{q!N!}) = ln(q+N)! - ln(q!N!) = ln(q+N)! - ln(q)! - ln(N)!$, using the T.S. expansion: $ln(x)! \approx xln(x) - x$ for all terms, we get $ln(\Omega) \approx (q+N)ln(q+N) - q - N - qln(q) + q - Nln(N) + N = (q+N)ln(N(q/N+1)) - qln(q) - Nln(N)$ $= (q+N)[ln(N) + ln(q/N+1)] - qln(q) - Nln(N)$

Using the T.S. expansion: $ln(1+x) \approx x$ for $ln(q/N+1) \approx q/N$ neglecting the rest of the terms because $q << N$, we now see that $ln(\Omega) \approx (q+N)ln(N) - (q+N)[q/N] - qln(q) - Nln(N) = qln(N) - qln(q) + N + N^2/q = Nln(N/q) + N + N^2/q$, dropping the last term because $q << N$ thus it is negligible and using the Law of Logs:

$$e^{ln(\Omega)} \approx e^{ln(N/q)^N + N} \therefore \Omega \approx (\frac{N}{q})^N e^N = (\frac{eN}{q})^N$$

Because $q << N$, in the therm $ln(q+N)$, we took out the N, since it is much larger than q, where in eq. 2.21 they took out the q, because it is much larger than the N.

7 2.18

From eq 2.17 we know: $\Omega(N,q) = \frac{(q+N-1)!}{q!(N-1)!}$, multiplying by $\frac{q+N}{N}$ and assuming $n(n-1)! = n!$, we get: $\Omega(N,q) = \frac{(q+N)!}{q!N!}$. We can continue by saying: $\Omega = \Omega(N,q) \cdot \frac{N}{q+N}$, which we can see is true by plugging in the original: $\Omega = \frac{(q+N)!}{q!N!} \cdot \frac{N}{q+N} = \frac{(q+N-1)!}{q!(N-1)!} \cdot \frac{q+N}{N} \cdot \frac{N}{q+N} = \frac{(q+N-1)!}{q!(N-1)!} = \Omega(N,q)$, using this solution, we can use the strong

form of Stirling's approximation: $N! \approx N^N e^{-N}\sqrt{2\pi N}$ **from eq 2.14, to solve for** $\Omega(N,q)$**:**

$$\Omega(N,q) \approx \frac{(q+N)^{(q+N)}e^{-(q+N)}\sqrt{2\pi(q+N)}}{(q^q e^{-q}\sqrt{2\pi q})(N^N e^{-N}\sqrt{2\pi N})} \cdot \frac{N}{q+N} = \frac{(q+N)^q (q+N)^N e^{-q} e^{-N}\sqrt{2\pi(q+N)}}{(q^q e^{-q}\sqrt{2\pi q})(N^N e^{-N}\sqrt{2\pi N})} \cdot \frac{N}{q+N} =$$

$$(\frac{q+N}{q})^q (\frac{q+N}{N})^N \frac{\sqrt{2\pi(q+N)}}{\sqrt{2\pi q}\sqrt{2\pi N}} \cdot \frac{N}{q+N}$$

Solving for $\frac{\sqrt{2\pi(q+N)}}{\sqrt{2\pi q}\sqrt{2\pi N}} \cdot \frac{N}{q+N}$ to conserve space and plugging back in later:

$$\frac{\sqrt{2\pi(q+N)}}{\sqrt{2\pi q}\sqrt{2\pi N}} \cdot \frac{N}{q+N} = \frac{\sqrt{2\pi}\sqrt{q+N}}{\sqrt{(2\pi)^2}\sqrt{qN}} \cdot \frac{N}{q+N} = (2\pi)^{-1/2}\sqrt{\frac{q+N}{qN}} \cdot \frac{N}{q+N} = (2\pi)^{-1/2}\sqrt{\frac{(q+N)N^2}{qN(q+N)^2}} = \sqrt{\frac{N}{2\pi q(q+N)}}$$

Putting all the terms back together, we get:

$$\Omega(N,q) \approx (\frac{q+N}{q})^q (\frac{q+N}{N})^N \sqrt{\frac{N}{2\pi q(q+N)}}$$

8 2.19:

Using Stirling's weak approximation from eq 2.15: $N! \approx N^N e^{-N}$ **we can find an approximate formula for the multiplicity of a 2-state para-magnet, where we assume** $N_\downarrow << N$ **to get** $\Omega \approx (\frac{Ne}{N})^{N_\downarrow}$**. We know from the section *The Two-State Paramagnet* that** $N = N_\uparrow + N_\downarrow$ **and using eq 2.7 of that section we see that** $\Omega(N_\downarrow) = \frac{N!}{N_\downarrow! N_\uparrow!} = \frac{N!}{N_\downarrow!(N-N_\downarrow)!}$ **using all of this, we have some fun:**

$$\Omega(N_\downarrow) \approx \frac{N^N e^{-N}}{(N_\downarrow^{N_\downarrow} e^{-N_\downarrow})((N-N_\downarrow)^{(N-N_\downarrow)} e^{-(N-N_\downarrow)})} = \frac{N^N e^{N_\downarrow} e^N}{(N_\downarrow^{N_\downarrow})((N-N_\downarrow)^{(N-N_\downarrow)} e^{N_\downarrow}) e^N} = \frac{N^N}{(N_\downarrow^{N_\downarrow})(N-N_\downarrow)^{(N-N_\downarrow)}}$$

Taking the natural log of both sides:

$$ln(\Omega(N_\downarrow)) \approx ln(N^N N_\downarrow^{-N_\downarrow}(N-N_\downarrow)^{-(N-N_\downarrow)}) = Nln(N) - N_\downarrow ln(N_\downarrow) - (N-N_\downarrow)ln(N-N_\downarrow)$$

Where $(N-N_\downarrow)ln(N-N_\downarrow) = (N-N_\downarrow)ln(N(1-N_\downarrow/N)) = (N-N_\downarrow)(ln(N)+ln(1-N_\downarrow/N))$, using the T.S. expansion $ln(1-x) \approx -x$ neglecting the rest of the terms because $N_\downarrow << N$ for $ln(1-N_\downarrow/N) \approx -(N_\downarrow/N)$, thus $(N-N_\downarrow)ln(N-N_\downarrow) \approx (N-N_\downarrow)(ln(N)-(N_\downarrow/N)) = Nln(N) - N_\downarrow - N_\downarrow ln(N) + N_\downarrow^2/N$, neglecting the last term because $N_\downarrow << N$ and plugging it back:

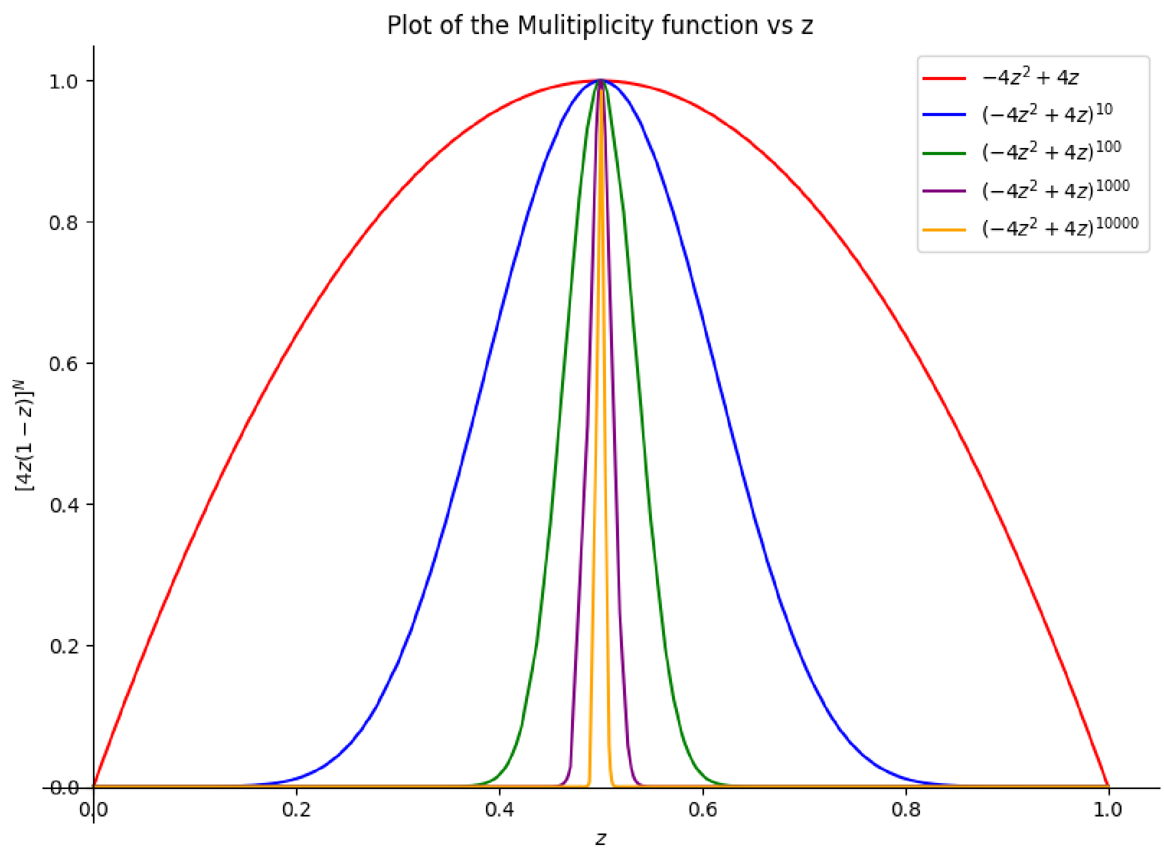
$$ln(\Omega(N_\downarrow)) \approx Nln(N) - N_\downarrow ln(N_\downarrow) - (Nln(N) - N_\downarrow - N_\downarrow ln(N)) = N_\downarrow ln(N)) - N_\downarrow ln(N_\downarrow) + N_\downarrow = N_\downarrow ln(N/N_\downarrow) + N_\downarrow$$

Using the Law of Logs we can simplify:

$$e^{ln(\Omega(N_\downarrow))} \approx e^{N_\downarrow ln(N/N_\downarrow)+N_\downarrow} = e^{ln(N/N_\downarrow)^{N_\downarrow}+N_\downarrow} = (\frac{N}{N_\downarrow})^{N_\downarrow} e^{N_\downarrow} \therefore \Omega(N_\downarrow) \approx (\frac{Ne}{N_\downarrow})^{N_\downarrow}$$

9 2.21:

We know from eq 2.22 that the multiplicity as a function of the macrostate: $\Omega = (\frac{e}{N})^{2N}(q_A q_B)^N$, where given in the problem, $q_A = qz$ and $q_B = q(1-z)$ where z is some variable ranging from 0 to 1. The equation then becomes: $\Omega = (\frac{e}{N})^{2N}[(qz)(q(1-z))]^N = (\frac{e}{N})^{2N}(q^2)^N z(1-z)^N = (\frac{qe}{N})^{2N} z(1-z)^N$, the problem then states we need a factor of 4 to ensure that the height of the peak is equal to 1 for any N, giving: $\Omega = (\frac{qe}{2N})^{2N} 4z(1-z)^N$, where $(\frac{qe}{2N})^{2N}$ is constant, thus does not need to be included in the plot, only plotting $4z(1-z)^N$, which is given as a function of z with N=1, 10, 100, 1000, and 10000:



From this plot, we can see that as N increases, the width of the peak, or the full width half max (FWHM), decreases in size. Meaning that as you have more oscillators, the probability of the macrostate becomes more defined due to it being so small for most of the macrostates and drastically larger for only a few macrostates. The peak eventually looks as if it is only at the maximum probability, starting and falling off at nearly a linear pace.

10 Appendix