HW 2: Appendix B10-B16

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EP 400: 02

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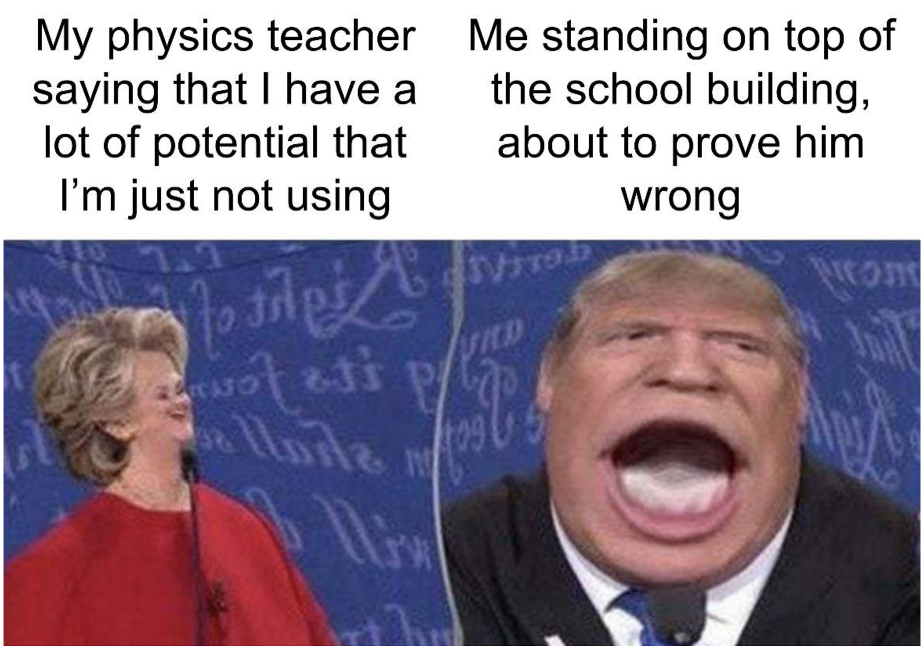


1 B10:

Choose the limits to the integral B.18 to derive a more precise app. of n!

I am using the limits (1,n+1/2) b/c at 0 the function approaches $-\infty$ and in Fig B.4 the histogram increases by 1/2

$$ln(n!) = \int_1^{n+1/2} ln(x)dx = (xln(x) - x)\Big|_1^{n+1/2} = (n+1/2)ln(n+1/2) - n - 1/2 - (0-1)$$

$$= (n+1/2)\underline{ln(n+1/2)} - n + 1/2$$

T.S. for $ln(n+1/2) = ln(n(1+1/2n))$ is the following:

$$ln(n) + ln(1+1/2n) = ln(n) + 1/2n \therefore (n+1/2)(ln(n) + 1/2n) - n + 1/2$$

$$= nln(n) + ln(n)/2 + 1/2 + 1/4n + 1/2 - n = n^n + n^{1/2} + 1 - n + 1/4n$$

where $1/4n$ approaches 0

$$n^n n^{1/2} e^1 e^{-n} = \underline{(n/e)^n \sqrt{e^2 n}}$$

2 B11:

Prove $x^n e^{-x}$ reaches maximum value at x=n

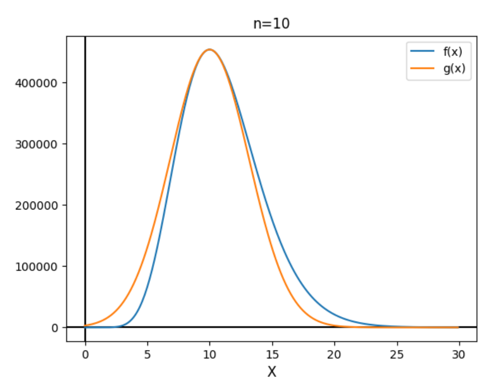
$$\frac{d}{dx} x^n e^{-x} = 0 \to nx^{n-1}e^{-x} = x^n e^{-x} \to nx^{n-1} = x^n \therefore \underline{n = x^{n+1-n} = x}$$

3 B12:

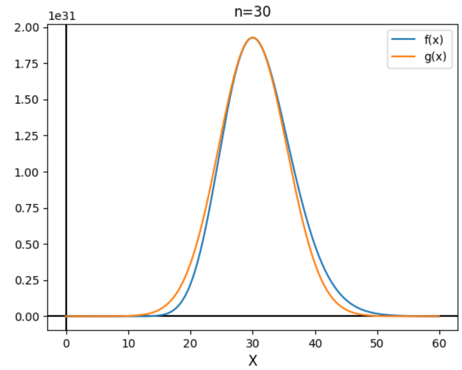
Plot the function $x^n e^{-x}$ and the Gaussian approximation to the function

Here is my following code with the respective graph:

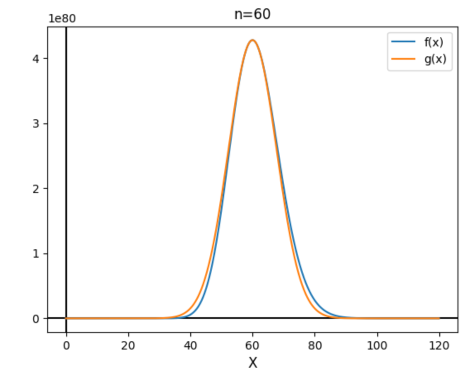
```
import numpy as np
import matplotlib.pyplot as plt
import math as m
from scipy.optimize import fsolve


def plot1(n=10):
    x=np.arange(0,30,0.1)
    plt.axhline(color='k')
    plt.axvline(color='k')
    line1,=plt.plot(x,x**n*np.exp(-x))
    line2,=plt.plot(x,n**n*np.exp(-n)*np.exp(-(x-n)**2/(2*n)))
    plt.grid
    plt.xlabel('X',fontsize=12)
    plt.title('n=10')
    plt.legend((line1,line2), ('f(x)','g(x)'))
    return 0
plt.show
plt.savefig('n=10')
plot1()
```



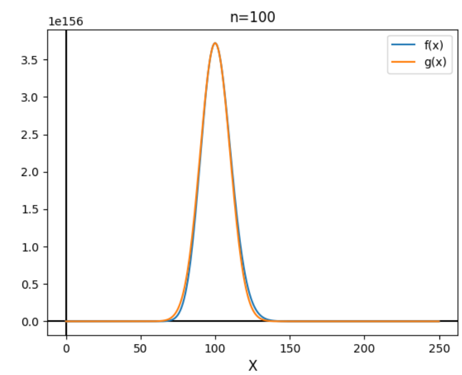
```python
def plot2(n=30):
    x=np.arange(0,60,0.1)
    plt.axhline(color='k')
    plt.axvline(color='k')
    line1,=plt.plot(x,x**n*np.exp(-x))
    line2,=plt.plot(x,n**n*np.exp(-n)*np.exp(-(x-n)**2/(2*n)))
    plt.grid
    plt.xlabel('X',fontsize=12)
    plt.title('n=30')
    plt.legend((line1,line2), ('f(x)','g(x)'))
    return 0
plt.show
plt.savefig('n=30')
plot2()
```




```python
def plot3(n=60):
    x=np.arange(0,120,0.1)
    plt.axhline(color='k')
    plt.axvline(color='k')
    line1,=plt.plot(x,x**n*np.exp(-x))
    line2,=plt.plot(x,n**n*np.exp(-n)*np.exp(-(x-n)**2/(2*n)))
    plt.grid
    plt.xlabel('X',fontsize=12)
    plt.title('n=60')
    plt.legend((line1,line2), ('f(x)','g(x)'))
    return 0
plt.show
plt.savefig('n=60')
plot3()
```




```python
def plot4(n=100):
    x=np.arange(0,250,0.1)
    plt.axhline(color='k')
    plt.axvline(color='k')
    line1,=plt.plot(x,x**n*np.exp(-x))
    line2,=plt.plot(x,n**n*np.exp(-n)*np.exp(-(x-n)**2/(2*n)))
    plt.grid
    plt.xlabel('X',fontsize=12)
    plt.title('n=100')
    plt.legend((line1,line2), ('f(x)','g(x)'))
    return 0
plt.show
plt.savefig('n=100')
plot4()
```



4 B13:

Improve Stirling's Approximation by including more terms of the T.S.

Proving $n! \approx n^n e^{-n}\sqrt{2n\pi}(1 + 1/12n)$

T.S. expansion for $ln(1 + y/n)$ is the following:

$$ln(1+y/n) = \frac{y}{n} - \frac{y^2}{2n^2} + \frac{y^3}{3n^3} - \frac{y^4}{4n^4}$$

Plugging into Eq. B.21 we get:

$$nln(n) + n(\frac{y}{n} - \frac{y^2}{2n^2} + \frac{y^3}{3n^3} - \frac{y^4}{4n^4}) - n - y = nln(n) - n - \frac{y^2}{2n} + \frac{y^3}{3n^2} - \frac{y^4}{4n^3}$$

$$\therefore n^n e^{-n} e^{\frac{-y^2}{2n}} \underline{e^{(\frac{y^3}{3n^2} - \frac{y^4}{4n^3})}}$$

Where the T.S of $e^{(\frac{y^3}{3n^2} - \frac{y^4}{4n^3})}$ is:

$$1 + (\frac{y^3}{3n^2} - \frac{y^4}{4n^3}) + (\frac{y^3}{3n^2} - \frac{y^4}{4n^3})^2$$

b/c we are assuming $y = \sqrt{n}$, we cancel any terms that are >y $\therefore \frac{y^4}{4n^3}$ in the squared term $\rightarrow 0$ and you have $1 + \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^6}{18n^4}$ left, finally:

$$n^n e^{-n} \underline{\int_{-\infty}^{\infty} e^{-y^2/2n}} [1 + \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^6}{18n^4}] \rightarrow \int_{-\infty}^{\infty} e^{-y^2/2n} = \sqrt{2n\pi}$$

Now integrating the rest of the sequence; from the Gaussian Substitution rules: $\frac{1}{3n^2}\int_0^{\infty} x^3 e^{-ax^2}dx = 0$, b/c the exponential is odd, from question B.3a

From problem B.2: $\frac{1}{4n^3}\int_0^{\infty} x^4 e^{-ax^2}dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$

Using the last solution and differentiating it:
$\frac{1}{18n^4}\int_0^{\infty} \frac{\partial}{\partial a} x^4 e^{-ax^2}dx = \frac{1}{18n^4}\int_0^{\infty} x^6 e^{-ax^2}dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$

$$\therefore n^n e^{-n}[\sqrt{2n\pi} - \frac{3\sqrt{\pi}}{16n^3 a^{5/2}} + \frac{15\sqrt{\pi}}{144n^4 a^{7/2}}], a = 1/2n \rightarrow n^n e^{-n}\sqrt{2n\pi}[1 - \frac{3(2n)^{4/2}}{16n^3} + \frac{15(2n)^{6/2}}{144n^4}]$$

$$= n^n e^{-n}\sqrt{2n\pi}[1 - \frac{12}{16n} + \frac{120}{144n} = n^n e^{-n}\sqrt{2n\pi}[1 - \frac{3}{4n} + \frac{5}{6n} = \underline{n^n e^{-n}\sqrt{2n\pi}[1 + \frac{1}{12n}]}$$

For n=1: $1 \cdot e^{-1} \cdot \sqrt{2\pi} \cdot 13/12 = 0.9989 \approx \underline{1!}$
For n=10: $10^{10} \cdot e^{-10} \cdot \sqrt{20\pi} \cdot 121/120 = 3,628,684.7 \approx \underline{10!}$

5 B14:

5.1 Check the formula B.28 for n=0,1:

For n=0: $\frac{\sqrt{\pi}\Gamma(1/2)}{\Gamma(1)} = \underline{\pi}$ AND n=1: $\frac{\sqrt{\pi}\Gamma(1)}{\Gamma(3/2)} = \frac{2\sqrt{\pi}}{\Gamma(1/2)} = \underline{2}$

5.2 Show that $\int_0^\pi sin(\theta)^n d\theta = \frac{n-1}{n}\int_0^\pi sin(\theta)^{n-2}d\theta$:

$$sin(\theta)^n = (sin(\theta))^{n-2}(1-cos(\theta)^2) \rightarrow \int_0^\pi sin(\theta)^{n-2}d\theta - \int_0^\pi \underline{cos(\theta)(cos(\theta)sin(\theta)^{n-2})}d\theta$$

$$\rightarrow IBP : u = cos(\theta) \text{ and } dv = cos(\theta)sin(\theta)^{n-2}$$

$$\therefore cos(\theta)\frac{sin(\theta)^{n-1}}{n-1}\Big|_0^\pi + \int_0^\pi sin(\theta)\frac{sin(\theta)^{n-1}}{n-1}d\theta = \frac{1}{n-1}\int_0^\pi sin(\theta)^n d\theta$$

$$\rightarrow \int_0^\pi sin(\theta)^{n-2}d\theta - \frac{1}{n-1}\int_0^\pi sin(\theta)^n d\theta = \int_0^\pi sin(\theta)^n d\theta$$

$$\rightarrow [1+\frac{1}{n-1}]\int_0^\pi sin(\theta)^n d\theta = \int_0^\pi sin(\theta)^{n-2}d\theta$$

$$\therefore \underline{\int_0^\pi sin(\theta)^n d\theta = \frac{n}{n-1}\int_0^\pi sin(\theta)^{n-2}d\theta}$$

5.3 Using .1 and .2 prove formula B.28 by induction:

From .2: $\int_0^\pi sin(\theta)^{n-2}d\theta = \frac{\sqrt{\pi}\Gamma(\frac{n-2}{2}+\frac{1}{2})}{\Gamma(\frac{n-2}{2}-1)}\frac{n-1}{n} = \int_0^\pi sin(\theta)^n d\theta$

$$\rightarrow \frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})(n-1)}{\Gamma(\frac{n}{2})n} = \frac{\sqrt{\pi}\Gamma(n/2-1/2)(n/2-1/2)}{\Gamma(n/2)(n/2)} = \frac{\sqrt{\pi}\Gamma(n/2-1/2+1)}{\Gamma(n/2+1)} = \underline{\frac{\sqrt{\pi}\Gamma(n/2+1/2)}{\Gamma(n/2+1)}}$$

6 B15:

A cleaner and trickier derivation to EQ. B.25

6.1 Evaluate in Cartesian, proving the integrand $e^{-r^2} = \pi^{d/2}$ for ALL dimensions:

$$b/c \int_{-\infty}^{\infty} e^{-x^2}dx = \pi^{1/2}, d=1 \rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(-x_1^2-x_2^2)}dx_1dx_2 = \pi^{2/2} = \pi, d=2$$

$$\rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(-x_1^2-x_2^2-x_3^2)}dx_1dx_2dx_3 = \pi^{3/2}, d=3$$

$$\therefore \int e^{-r^2}dx^d = [\int e^{-x^2}dx]^d = \underline{\pi^{d/2}}$$

6.2 Integrate in dimensional spherical:

$$(\int e^{-x^2}dx)^d = \int e^{-r^2}dr$$

b/c we are using spherical coordinates, we need the area and using dimensional analysis we know area is proportional to the radius:

$$\int_0^\infty e^{-r^2}A_d(r)dr, A_d(r) \propto r^{d-1} \rightarrow \int_0^\infty r^{d-1}e^{-r^2}A_d(r)dr$$

From the question, we see $A_d(1)$, meaning r=1 for the given factor:

$$\therefore \underline{A_d(1)\int_0^\infty r^{d-1}e^{-r^2}dr}$$

6.3 Evaluate the previous integral w.r.t r in terms of $\Gamma(x)$

constraining a few variables, $c = r^2, dr = dc/2r \therefore r^{d-1} \cdot r^{-1} = r^{d-2}, u = d-2$

$$\rightarrow \frac{A_d(1)}{2}\int_0^\infty c^{u/2}e^{-c}dc, n = u/2$$

From EQ. B.12 $\int_0^\infty c^n e^{-c}dc \equiv \Gamma(n+1)$

$$\rightarrow \frac{A_d(1)}{2}\Gamma(\frac{d-2}{2}+1) = \frac{A_d(1)}{2}\Gamma(\frac{d}{2}) = \pi^{d/2}$$

$$A_d(1) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \therefore \underline{A_d(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)}r^{d-1}}$$

7 B16:

Derive a formula for the volume of a d-dimensional hyper-sphere

Area = A$_d$(r) AND Width = r $\therefore dV = A_d(r)dr$

$$\int dV = \int A_d(r)dr \rightarrow V = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int r^{d-1}dr = \underline{\frac{2\pi^{d/2}}{\Gamma(d/2)} \cdot \frac{r^d}{d}}$$