HW 1: Appendix B1-B9

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EP 400: 02

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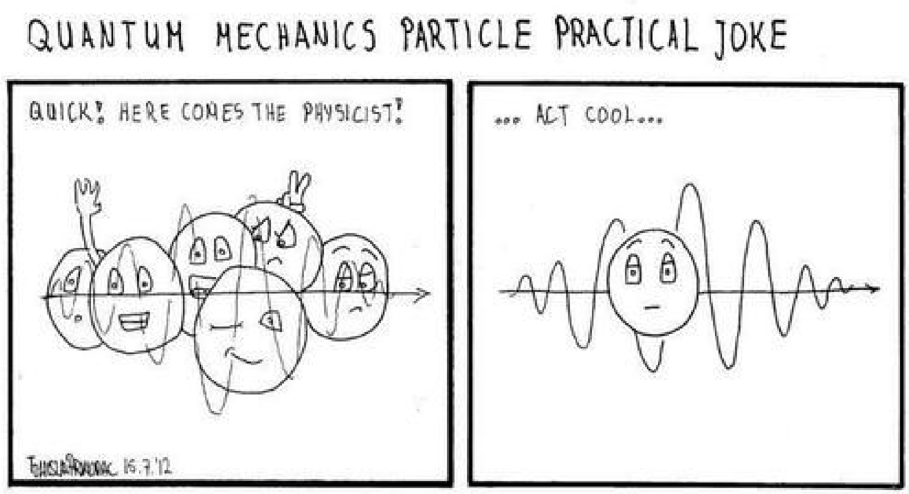
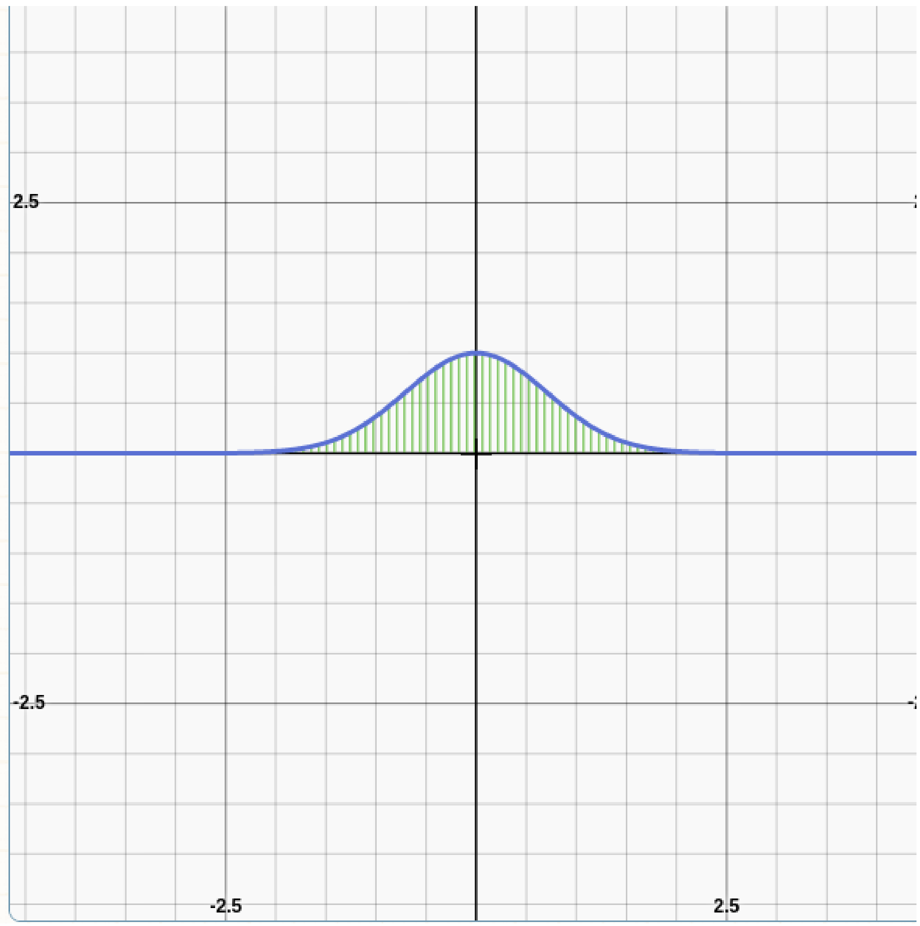


1 B1:

Sketch the anti-derivative of e^{-x^2}

Assuming this is the same function being described by equation **B.1**, the sketch of the Integral, with bounds $(-\infty, \infty)$, will be the following:



2 B2:

Take another derivative of eq. B.8 to evaluate $\int_0^\infty x^4 e^{-ax^2} dx$

Solution:

$$\int_0^\infty x^2 \frac{\partial}{\partial a} e^{-ax^2} dx = \frac{\sqrt{\pi}}{4} \frac{d}{da} a^{\frac{-3}{2}} = \frac{\sqrt{\pi}}{4} a^{\frac{-5}{2}} = \underline{\frac{1}{4}\sqrt{\frac{\pi}{a^5}}}$$

3 B3:

The Integral of $x^n e^{-ax^2} dx$ is easier to evaluate when n is odd
Solution:
a. Evaluate $\int_{-\infty}^{\infty} xe^{-ax^2} dx$ without computation:

Due to the odd exponent n, the solution is 0. If it were an even exponential n, the solution would be some value > 0

b. Evaluate the indefinite Integral of xe^{-ax^2}

$$\int_{-\infty}^{\infty} xe^{-ax^2} dx \to u = -ax^2, du = -2axdx \to \frac{-1}{2a} \int_{-\infty}^{\infty} e^u du = \frac{-1}{2a} e^{-ax^2} \Big|_{-\infty}^{\infty}$$

, because $e^{-\infty} = 0$ (converges to 0)
c. Evaluate $\int_0^{-\infty} xe^{-ax^2} dx$

$$\int_0^{\infty} xe^{-ax^2} dx = \frac{-1}{2a} e^{-ax^2} \Big|_0^{\infty} = \underline{\frac{1}{2a}}, a > 0$$

d. Differentiate the previous result to evaluate $\int_0^{\infty} x^3 e^{-ax^2} dx$

$$\int_0^{\infty} x \frac{\partial}{\partial a} e^{-ax^2} dx = \frac{1}{2} \frac{d}{da} a^{-1} \to \int_0^{\infty} x^3 e^{-ax^2} dx = \underline{\frac{1}{2a^2}}$$

4 B4:

Integrate the tail of the Gaussian function $\int_x^{\infty} e^{-t^2} dt, x >> 1$

$$s = t^2, ds = 2tdt \therefore dt = \frac{ds}{2\sqrt{s}} : \int_x^{\infty} e^{-t^2} dt = \frac{1}{2} \int_{x^2}^{\infty} s^{-1/2} e^{-s} ds$$

Using the Taylor Series expansion:

$$s^{-1/2} = \frac{1}{x} - \frac{s - x^2}{2x^3} + \frac{3(s - x^2)^2}{8x^5} = \frac{15}{8x} - \frac{5s}{4x^3} + \frac{3s^2}{8x^5}$$

$$\to \frac{15}{16x} \int_{x^2}^{\infty} e^{-s} ds - \frac{5}{8x^3} \int_{x^2}^{\infty} se^{-s} ds + \frac{3}{16x^5} \int_{x^2}^{\infty} s^2 e^{-s} ds = \underline{e^{-x^2} [\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5}]}$$

5 B5:

Find asymptotic expansion for the integral of $t^2 e^{-t^2}$

$$s = t^2, ds = 2tdt \therefore dt = \frac{ds}{2\sqrt{s}} : \int_x^{\infty} t^2 e^{-t^2} dt = \frac{1}{2} \int_{x^2}^{\infty} s^{1/2} e^{-s} ds$$

Using Integration By Parts:

$$u = s^{1/2}, dv = e^{-s} \to -s^{1/2}e^{-s} - \frac{1}{2}s^{-1/2}e^{-s} - \frac{1}{4}s^{-3/2}e^{-s}$$

$$\to \underline{-e^{-x^2}[\frac{x}{2} + \frac{1}{4x} + \frac{1}{8x^3}]}$$

6 B6:

By definition, the error function is the integral of e^{-x^2}, set equal to x=0 and multiplied by $\frac{2}{\sqrt{\pi}}$: $erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

a. Show that $erf(\pm\infty) = \pm 1$

$$erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$$

$$erf(-\infty) = \frac{2}{\sqrt{\pi}} \int_0^{-\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \cdot \frac{-\sqrt{\pi}}{2} = -1$$

$$\therefore erf(\pm\infty) = \pm 1$$

b. Evaluate $\int_0^x t^2 e^{-t^2} dt$ in terms of erf(x)

$$\int_0^x t^2 e^{-t^2} dt = \frac{-te^{-t^2}}{2} - \int_0^x \frac{-e^{-t^2}}{2} dt = \frac{-te^{-t^2}}{2} + \frac{\sqrt{\pi}}{4} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} dt = \frac{-te^{-t^2}}{2} + \frac{\sqrt{\pi}}{4} erf(x)$$

$$\underline{\frac{\sqrt{\pi} erf(x) - 2te^{-t^2}}{4}}$$

c. Use B.4 to find an expression for erf(x) when $x >> 1$

$$if \int_x^{\infty} e^{-t^2} dt = e^{-x^2}[\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5}] \to erf(x) = \frac{2e^{-x^2}}{\sqrt{\pi}}[\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5}], x >> 1$$

7 B7:

Prove the Recursion formula

$$\Gamma(n+1) \equiv \int_0^{\infty} x^n e^{-x} dx = -x^n e^{-x}\Big|_0^{\infty} + \int_0^{\infty} n x^{n-1} e^{-x} dx = n \int_0^{\infty} x^{n-1} e^{-x} dx = \underline{n\Gamma(n), n > 0}$$

8 B8:

Evaluate $\Gamma(\frac{1}{2})$ then, using recursion, evaluate $\frac{3}{2}$ and $\frac{-1}{2}$

Solving for $\frac{1}{2}$:

$$x = t^2 \therefore dx = 2tdt \rightarrow \int_0^\infty x^{1/2-1}e^{-x}dx = 2\int_0^\infty \frac{t}{\sqrt{t^2}}e^{-t^2}dt = \sqrt{\pi}erf(\infty) = \underline{\sqrt{\pi}}$$

Solving for the others:

$$\Gamma(\frac{3}{2}) = \Gamma(\frac{3}{2}-1)\cdot(\frac{3}{2}-1) = \Gamma(\frac{1}{2})\cdot\frac{1}{2} = \underline{\frac{\sqrt{\pi}}{2}}$$

$$\Gamma(\frac{-1}{2}) = -2\Gamma(\frac{-1}{2}+1) = \underline{-2\sqrt{\pi}}$$

9 B9:

Solve integral B.12 to evaluate $\Gamma(1/3)$ and $\Gamma(2/3)$

HW1.py > ...

```python
# FOR PROBLEM B9

import math

var1 = 1/3
var2 = 2/3

print('The gamme func value of 1/3: ' + str(math.gamma(var1)))
print('The gamme func value of 2/3: ' + str(math.gamma(var2)))
```

After the program ran, it gave the following solution:

The Gamma func. value of 1/3: 2.678938534707748

The Gamma func. value of 2/3: 1.3541179394264005