MATH 4630 / 6632 3.0 - Fall 2022
Solution for Assignment 2

Question 1:

a. Model $\underset{\sim}{X}_{ijk} = \underset{\sim}{\mu} + \underset{\sim}{\tau}_i + \underset{\sim}{\beta}_j + \underset{\sim}{\gamma}_{ij} + \underset{\sim}{\epsilon}_{ijk}$ for $i = 1, 2, 3, 4, \quad j = 1, 2, \quad k = 1, 2, 3, 4$ and $g = 3, b = 2, n = 4, p = 2$, where

$$\begin{array}{ll} \underset{\sim}{X}_{ijk} = & k^{th} \text{ observation of } (L_i, V_j) \\ \underset{\sim}{\mu} = & \text{overall mean} \\ \underset{\sim}{\tau}_i = & \text{effect of } L_i \\ \underset{\sim}{\beta}_j = & \text{effect of } V_j \\ \underset{\sim}{\gamma}_{ij} = & \text{interaction effect of } (L_i, V_j) \\ \underset{\sim}{\epsilon}_{ijk} = & k^{th} \text{ random error of } (L_i, V_j) \end{array}$$

Assumptions:

- $\underset{\sim}{\epsilon}_{ijk}$ iid $N_2(\underset{\sim}{0}, \Sigma)$
- $\sum_{i=1}^{3} \underset{\sim}{\tau}_i = \sum_{j=1}^{2} \underset{\sim}{\beta}_j = \sum_{i=1}^{3} \underset{\sim}{\gamma}_{ij} = \sum_{j=1}^{2} \underset{\sim}{\gamma}_{ij} = 0$

b.

$$SSE = \begin{pmatrix} 3.1532 & -14.9535 \\ -14.9535 & 536.9725 \end{pmatrix}, \ SSTotal = \begin{pmatrix} 5.49389 & -9.8710 \\ -9.8710 & 952.9895 \end{pmatrix}, \ SSTr = \begin{pmatrix} 2.3457 & 5.0825 \\ 5.0825 & 416.01705 \end{pmatrix}$$

$$SSL = \begin{pmatrix} 1.6927 & -9.6989 \\ -9.6989 & 56.3233 \end{pmatrix}, \ SSV = \begin{pmatrix} 0.6240 & 14.8834 \\ 14.8834 & 354.9704 \end{pmatrix}, \ SSLV = \begin{pmatrix} 0.0290 & -0.1020 \\ -0.1020 & 4.7233 \end{pmatrix}$$

c. H_0 : no treatment effect vs H_a : treatment effect exists
Observed Wilks Lambda is $\frac{|SSE|}{|SSE+SSTr|} = 0.2857$
Observed test statistic (adjusted observed Wilks Lambda) is 23.8005
p-value $\approx P(\chi^2_5 > 23.8005) = 0.0002$
A small p-value indicates strong evidence of existence of treatment effect.

d. Answers are from SAS. H_0 : no interaction effect versus H_a : interaction effect exists
p-value is 0.9881. A large p-value indicates no evidence of existence of interaction effect.
H_0 : no L effect versus H_a : L effect exists
p-value is 0.1082. A large p-value indicates no evidence of existence of lubricants effect.
H_0 : no V effect versus H_a : V effect exists
p-value is 0.0009 A small p-value indicates no evidence of existence of velocity effect.

Question 2:

a. Calculations done in SAS.
H_0 : no L effect versus H_a : L effect exists
p-value is less than 0.0779 Small p-value indicates mild evidence of lubricant effect.

b. We have $a = 3, n_1 = n_2 = n_3 = n = 8, p = 2$,

$$E = \begin{pmatrix} 3.8062 & -0.1721 \\ -0.1721 & 895.6663 \end{pmatrix}, \quad \bar{\underset{\sim}{L}}_1 = \begin{pmatrix} 7.4613 \\ 85.2375 \end{pmatrix}, \quad \bar{\underset{\sim}{L}}_2 = \begin{pmatrix} 7.4425 \\ 84.9125 \end{pmatrix},$$

$dfE = 24 - 3 = 21, S_{pooled} = \frac{E}{dfE}$ and $\hat{\underset{\sim}{\delta}} = \bar{\underset{\sim}{L}}_1 - \bar{\underset{\sim}{L}}_2$. Thus, the 95% exact confidence $\underset{\sim}{\delta}$ is

$$\left\{ \underset{\sim}{\delta} : \quad \frac{n_1 + n3 - p - 1}{(n_1 + n_3 - 2)p} (\hat{\underset{\sim}{\delta}} - \underset{\sim}{\delta})' \left[\left(\frac{1}{n_1} + \frac{1}{n_3} \right) S_{pooled}^{-1} \right]^{-1} (\hat{\underset{\sim}{\delta}} - \underset{\sim}{\delta}) \leq F_{p,n_1+n_3-p-1,0.05} \right\}$$

c. $H_0 : \Sigma_1 = \Sigma_2 = \Sigma_3 = \Sigma$ versus H_a : not all equal
Using the likelihood ratio test, the test statistic is

$$\Lambda = \prod_{i=1}^{2} \left[\frac{|S_i|}{|S_{pooled}|} \right]^{(n_i - 1)/2}$$

In this case, $n_1 = n_2 = n_3 = 8$,

$$S_1 = \begin{pmatrix} 0.1194 & 0.1018 \\ 0.1018 & 29.7713 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0.2999 & -0.6726 \\ -0.6726 & 67.1913 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0.1245 & 0.5462 \\ 0.5462 & 30.9762 \end{pmatrix}$$

Also $S_{pooled} = E/21$ and E is given in part (b).
Therefore, the observed test statistic is $\Lambda_{obs} = 0.1141$.
Without making any adjustment, $M_{obs} = -2 \log \Lambda_{obs} = 4.3414$ and the adjusted observed test statistic is $(1 - 0.1376) * M = 3.7442$.
The p-value is 0.7112
large p-value, this implies no evidence of heterogeneity in variance.

Question 3: Let $\underset{\sim}{D} = \underset{\sim}{C}_1 - \underset{\sim}{C}_2$. Assume $\underset{\sim}{d}_1, \ldots, \underset{\sim}{d}_{15}$ is a sample from a bivariate normal with mean $\underset{\sim}{\mu}_d$ and variance Σ_d
We want to test $H_0 : \underset{\sim}{\mu}_d = 0 \quad \text{vs} \quad H_a : \underset{\sim}{\mu}_d \neq 0$.
Based on the R output, the observed test statistic is 5.0231 and the p-value is 0.0242.
Small p-value indicates that there is mild evidence of against H_0.

Question 4: Answer is in the solution of the test.