MATH 4630 / 6632 3.0 - Fall 2022
Solution for Assignment 1

Question 1:

a1. To obtain the eigenvalues, we need to solve

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 4-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0.$$

Therefore we have

$$\begin{aligned} & (2-\lambda)(4-\lambda)((2-\lambda)-(2-\lambda)-(2-\lambda)=0 \\ \Rightarrow\ & (2-\lambda)(\lambda^2-6\lambda-6)=0 \\ \Rightarrow\ & \lambda=2 \quad \text{or} \quad \lambda=\frac{6\pm\sqrt{6^2+4(6)}}{2}=4.7321 \quad \text{or} \quad 1.2679 \end{aligned}$$

For $\lambda = 4.7321$, the corresponding eigenvector $\underset{\sim}{e} = (e_1, e_2, e_3)'$ must satisfies

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} \underset{\sim}{e} = 4.7321\underset{\sim}{e}.$$

We have $e_2 = 2.7321e_1 = 2.7321e_3$. By setting $e_2 = 1$, we have $e_1 = e_3 = \frac{1}{2.7321}$.
Thus, for eigenvalue is 4.7321, the normalized eigenvector is $(0.3251, 0.8881, 0.3251)'$.
Similarly, for eigenvalue is 2, the normalized eigenvector is $(0.7071, 0, -0.7071)'$, and for eigenvalue is 1.2679, the normalized eigenvector is $(0.6280, -0.4597, 0.6280)'$.

a2. Let

$$L = \begin{pmatrix} l_1 & 0 & 0 \\ l_2 & l_3 & 0 \\ l_4 & l_5 & l_6 \end{pmatrix}$$

We have

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} = LL' = \begin{pmatrix} l_1^2 & l_1l_2 & l_1l_4 \\ l_1l_2 & l_2^2+l_3^2 & l_2l_4+l_3l_5 \\ l_1l_4 & l_2l_4+l_3l_5 & l_4^2+l_5^2+l_6^2 \end{pmatrix}$$

From the first row, we have

$$\begin{aligned} l_1^2 = 2 &\ \Rightarrow\ l_1 = 1.4142 \\ l_1l_2 = 1 &\ \Rightarrow\ l_2 = 0.7071 \\ l_1l_4 = 0 &\ \Rightarrow\ l_4 = 0 \end{aligned}$$

From the second row, we have

$$\begin{aligned} l_2^2 + l_3^2 = 4 &\ \Rightarrow\ l_3 = 1.8708 \\ l_2l_4 + l_3l_5 = 1 &\ \Rightarrow\ l_5 = 0.5345 \end{aligned}$$

And from the last row, we have

$$l_4^2 + l_5^2 + l_6^2 = 2 \quad \Rightarrow \quad l_6 = 1.3093$$

Thus, by Cholesky decomposition,

$$A^{1/2} = L = \begin{pmatrix} 1.4142 & 0 & 0 \\ 0.7071 & 1.8708 & 0 \\ 0 & 0.5345 & 1.3093 \end{pmatrix}.$$

a3. Let

$$P = \begin{pmatrix} 0.3251 & 0.7071 & 0.6280 \\ 0.8881 & 0 & -0.4597 \\ 0.3251 & -0.7071 & 0.6280 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 4.7321 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1.2679 \end{pmatrix}.$$

Then

$$D^{1/2} = \begin{pmatrix} 2.1753 & 0 & 0 \\ 0 & 1.4142 & 0 \\ 0 & 0 & 1.1260 \end{pmatrix}.$$

Therefore,

$$A^{1/2} = PD^{1/2}P' = \begin{pmatrix} 1.3810 & 0.3029 & -0.0332 \\ 0.3029 & 1.9536 & 0.3029 \\ -0.0332 & 0.3029 & 1.3810 \end{pmatrix}.$$

b. Since all eigenvalues are position, the matrix A is a positive definite matrix.

c. See the R script which verifies all the results in part (a).

Question 2:

a. Let I_k be a $(k \times k)$ identity matrix. Since

$$I_p = AB = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}.$$

we have

$$\begin{aligned} A_{11}B_{11} + A_{12}B_{21} &= I_{p_1} && (1) \\ A_{11}B_{12} + A_{12}B_{22} &= 0 && (2) \\ A_{21}B_{11} + A_{22}B_{21} &= 0 && (3) \\ A_{21}B_{12} + A_{22}B_{22} &= I_{p_2} && (4) \end{aligned}$$

where 0 is the zero matrix with the corresponding dimension.

From (2), we have $B_{12} = -A_{11}^{-1}A_{12}B_{22}$

Substitute this into (4), we have

$$B_{22} = (A_{22} - A_{21}A_{22}^{-1}A_{12})^{-1}, \quad \text{and} \quad B_{12} = -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{22}^{-1}A_{12})^{-1}$$

Similarly, from (3), we have $B_{21} = -A_{22}^{-1}A_{21}B_{11}$

Substitute this into (1), we have

$$B_{11} = (A_{11} - A_{12}A_{11}^{-1}A_{21})^{-1}, \quad \text{and} \quad B_{21} = -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{11}^{-1}A_{21})^{-1}.$$

b. Using the same logic as in part (a), we have

$$\begin{aligned} B_{11} = (A_{11} - A_{12}A_{11}^{-1}A_{21})^{-1} \quad &\text{and} \quad B_{12} = -(A_{11} - A_{12}A_{11}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ B_{22} = (A_{22} - A_{21}A_{22}^{-1}A_{12})^{-1}, \quad &\text{and} \quad B_{21} = -(A_{22} - A_{21}A_{22}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} \end{aligned}$$

Notes:

1. **There are other equivalent representation of the B matrix (which is the A^{-1} matrix) as well.**
2. **See part (d) for another representation.**

c . Recall from matrix algebra, we know

1.

$$\begin{vmatrix} E & 0 \\ 0 & G \end{vmatrix} = \begin{vmatrix} E & 0 \\ F & G \end{vmatrix} = \begin{vmatrix} E & H \\ 0 & G \end{vmatrix} = |E||G|$$

2.

$$\begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}^{-1} = \begin{pmatrix} E^{-1} & 0 \\ 0 & G^{-1} \end{pmatrix}$$

Let

$$D = \begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}.$$

Then $|D| = |E| = 1$. Moreover, let

$$H = DAE = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}.$$

Then

$$|DAE| = |H| \quad \Rightarrow \quad |A| = |A_{11}||A_{22} - A_{21}A_{11}^{-1}A_{12}|.$$

Similarly, let

$$K = EAD = \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{pmatrix}.$$

Then

$$|A| = |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{21}|.$$

d. Using the idea in part (c), we have

$$A^{-1} = EH^{-1}D = DK^{-1}E$$

and performing the matrices multiplications, you will have both results.

Question 3: The joint density of $(\underset{\sim}{X}_1, \underset{\sim}{X}_2)$ is

$$f(\underset{\sim}{x}_1, \underset{\sim}{x}_2) = (2\pi)^{-p/2}|\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}\begin{pmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{pmatrix}\right\}$$

and the marginal density of $\underset{\sim}{X}_2$ is

$$f(\underset{\sim}{x}_2) = (2\pi)^{-p_2/2}|\Sigma_{22}|^{-1/2} \exp\left\{-\frac{1}{2}(\underset{\sim}{x}_2 - \underset{\sim}{\mu}_2)'\Sigma_{22}^{-1}(\underset{\sim}{x}_2 - \underset{\sim}{\mu}_2)\right\}$$

Therefore, the conditional density for $\underset{\sim}{X}_1$ given $\underset{\sim}{X}_2 = \underset{\sim}{x}_2$ is

$$f(\underset{\sim}{x}_1|\underset{\sim}{x}_2) = \frac{f(\underset{\sim}{x}_1, \underset{\sim}{x}_2)}{f(\underset{\sim}{x}_2)}$$

which takes the form

$$(2\pi)^{-p_1/2}\frac{|\Sigma|^{-1/2}}{|\Sigma_{22}|^{-1/2}} \exp\left\{-\frac{1}{2}\left[\begin{pmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{pmatrix} - (\underset{\sim}{x}_2 - \underset{\sim}{\mu}_2)'\Sigma_{22}^{-1}(\underset{\sim}{x}_2 - \underset{\sim}{\mu}_2)\right]\right\}$$

For simplicity, let

$$\begin{aligned}
\underset{\sim}{y}_i &= \underset{\sim}{x}_i - \underset{\sim}{\mu}_i \quad i = 1, 2 \\
\Sigma &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \\
\Sigma^{-1} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\
C &= \Sigma_{12}\Sigma_{22}^{-1}.
\end{aligned}$$

We know Σ is a symmetric matrix, and, hence, $\Sigma_{12}' = \Sigma_{21}$. Thus, $C' = \Sigma_{22}^{-1}\Sigma_{21}$.
From Question (2), we have

$$\begin{aligned}
|\Sigma| &= |\Sigma_{22}|\ |\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}| \\
A_{11} &= (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} \\
A_{12} &= -(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1} = -A_{11}C \\
A_{21} &= -\Sigma_{22}^{-1}\Sigma_{21}(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} = -C'A_{11} \\
A_{22} &= \Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{21}(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1} = \Sigma_{22}^{-1} + C'A_{11}C.
\end{aligned}$$

Now, we will consider the conditional density term-by-term.

- $$\begin{aligned}\frac{|\Sigma|^{-1/2}}{|\Sigma_{22}|^{-1/2}} &= \left\{\frac{|\Sigma|}{|\Sigma_{22}|}\right\}^{-1/2} = \left\{\frac{|\Sigma_{22}|\ |\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|}{|\Sigma_{22}|}\right\}^{-1/2} \\ &= |\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|^{-1/2} = |A_{11}^{-1}|^{-1/2}\end{aligned}$$

- $$\begin{aligned}& \begin{pmatrix} \underset{\sim}{x}_1-\underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2-\underset{\sim}{\mu}_2 \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} \underset{\sim}{x}_1-\underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2-\underset{\sim}{\mu}_2 \end{pmatrix} \\ =\ & \begin{pmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \end{pmatrix}' \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \end{pmatrix} \\ =\ & \underset{\sim}{y}_1' A_{11}\underset{\sim}{y}_1 + \underset{\sim}{y}_2' A_{21}\underset{\sim}{y}_1 + \underset{\sim}{y}_1' A_{12}\underset{\sim}{y}_2 + \underset{\sim}{y}_2' A_{22}\underset{\sim}{y}_2 \\ =\ & \underset{\sim}{y}_1' A_{11}\underset{\sim}{y}_1 - \underset{\sim}{y}_2' C' A_{11}\underset{\sim}{y}_1 - \underset{\sim}{y}_1' A_{11} C\underset{\sim}{y}_2 + \underset{\sim}{y}_2'(\Sigma_{22}^{-1} + C'A_{11}C)\underset{\sim}{y}_2 \\ =\ & \underset{\sim}{y}_1' A_{11}\underset{\sim}{y}_1 - \underset{\sim}{y}_2' C' A_{11}\underset{\sim}{y}_1 - \underset{\sim}{y}_1' A_{11} C\underset{\sim}{y}_2 + \underset{\sim}{y}_2' C'A_{11}C\underset{\sim}{y}_2 + \underset{\sim}{y}_2'\Sigma_{22}^{-1}\underset{\sim}{y}_2 \\ =\ & (\underset{\sim}{y}_1 - C\underset{\sim}{y}_2)' A_{11} (\underset{\sim}{y}_1 - C\underset{\sim}{y}_2) + \underset{\sim}{y}_2'\Sigma_{22}^{-1}\underset{\sim}{y}_2 \\ =\ & \left(\underset{\sim}{x}_1 - [\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{x}_2-\underset{\sim}{\mu}_2)]\right)' \left(A_{11}^{-1}\right)^{-1} \left(\underset{\sim}{x}_1 - [\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{x}_2-\underset{\sim}{\mu}_2)]\right) + \\ & (\underset{\sim}{x}_2-\underset{\sim}{\mu}_2)'\Sigma_{22}^{-1}(\underset{\sim}{x}_2-\underset{\sim}{\mu}_2)\end{aligned}$$

- $$\begin{aligned}& \begin{pmatrix} \underset{\sim}{x}_1-\underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2-\underset{\sim}{\mu}_2 \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} \underset{\sim}{x}_1-\underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2-\underset{\sim}{\mu}_2 \end{pmatrix} - (\underset{\sim}{x}_2-\underset{\sim}{\mu}_2)'\Sigma_{22}^{-1}(\underset{\sim}{x}_2-\underset{\sim}{\mu}_2) \\ =\ & \left(\underset{\sim}{x}_1 - [\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{x}_2-\underset{\sim}{\mu}_2)]\right)' \left(A_{11}^{-1}\right)^{-1} \left(\underset{\sim}{x}_1 - [\underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{x}_2-\underset{\sim}{\mu}_2)]\right)\end{aligned}$$

Let $\underset{\sim}{\delta} = \underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{x}_2-\underset{\sim}{\mu}_2)$. The conditional density for $\underset{\sim}{X}_1$ given $\underset{\sim}{X}_2 = \underset{\sim}{x}_2$ is

$$f(\underset{\sim}{x}_1|\underset{\sim}{x}_2) = (2\pi)^{-p_1/2}\left|A_{11}^{-1}\right|^{-1/2}\exp\left\{-\frac{1}{2}(\underset{\sim}{x}_1-\underset{\sim}{\delta})'\left(A_{11}^{-1}\right)^{-1}(\underset{\sim}{x}_1-\underset{\sim}{\delta})\right\}$$

which is the density of a p_1-dimensional normal distribution with mean

$$\underset{\sim}{\delta} = \underset{\sim}{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{x}_2-\underset{\sim}{\mu}_2)$$

and variance

$$A_{11}^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

Question 4: It is given $n = 6$ and $p = 2$.

a. We obtained

$$\underset{\sim}{\bar{x}} = \begin{pmatrix} 4.8333 \\ 12.8117 \end{pmatrix}.$$

b.

$$S = \begin{pmatrix} 3.7667 & -7.3517 \\ -7.3517 & 15.7175 \end{pmatrix}.$$

c. Since $d_i = (\underset{\sim}{x}_i - \underset{\sim}{\bar{x}})' S^{-1} (\underset{\sim}{x}_i - \underset{\sim}{\bar{x}})$, we have

$$\underset{\sim}{d} = (2.6705, 1.4201, 0.3246, 0.1644, 2.2191, 3.2013)'.$$

d1. We know $p = 2$ and $\chi^2_{p,0.5} = 1.3863$.
We also know $(\underset{\sim}{\bar{X}} - \underset{\sim}{\mu})' \left(\frac{1}{n}\Sigma\right)^{-1} (\underset{\sim}{\bar{X}} - \underset{\sim}{\mu}) \sim \chi^2_p$. Therefore, the required contour is by finding all $\underset{\sim}{\mu}$ satisfying

$$(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu})' \left(\frac{1}{n}S\right)^{-1} (\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}) = \chi^2_{p,0.5}$$

where $\underset{\sim}{\bar{x}}$ and S are obtained in parts (a) and (b).

d2. $H_0 : \underset{\sim}{\mu} = (3, 10)'$ versus $H_a : \underset{\sim}{\mu} \neq (3, 10)'$.
We can obtain $T^2 = 131.3229$ and therefore $F_{obs} = 52.5291$ and the corresponding p-value $= P(F_{2,4} > 52.5291) = 0.0013$.
Since p-value is less than the given $\alpha = 0.05$, we reject H_0.

Note: The T^2 reported in R is the F_{obs} and not the Hotelling T^2.

Question 5: Refer to the output generated from the *R* scripts.

a. Normal Q-Q plot of North and South seem to have a possible outlier and, thus, it does not resemble a straight line and hence the normality assumption is very doubtful.
Normal Q-Q plot of East does not resemble a straight line and hence the normality assumption is very doubtful.
Normal Q-Q plot of West, mildly, resembles a straight line and hence the normality assumption is probably fine.

b.
1. Assume North is from $N(\mu_N, \sigma_N^2)$.
A 95% confidence interval for the mean of North is $(1321.866, 1606.034)$.
2. Assume South is from $N(\mu_S, \sigma_S^2)$.
A 95% confidence interval for the mean of South is $(1726.799, 2050.401)$.
3. Assume East is from $N(\mu_E, \sigma_E^2)$.
A 95% confidence interval for the mean of East is $(1574.316, 1894.484)$.
4. Assume West is from $N(\mu_W, \sigma_W^2)$.
A 95% confidence interval for the mean of West is $(1542.455, 1861.445)$.

c. The sample mean, and sample variance matrix are

$$\underset{\sim}{\bar{x}} = \begin{pmatrix} 1463.95 \\ 1888.60 \\ 1734.40 \\ 1701.95 \end{pmatrix} .S = \begin{pmatrix} 92165.84 & 91525.08 & 76724.18 & 93988.10 \\ 91525.08 & 119521.09 & 108840.91 & 103275.98 \\ 76724.18 & 108840.91 & 115998.04 & 85358.18 \\ 93988.10 & 103275.98 & 85358.18 & 116138.68 \end{pmatrix}$$

The sample correlation matrix is

$$R = \begin{pmatrix} 1.0000 & 0.8720 & 0.7389 & 0.9084 \\ 0.8720 & 1.0000 & 0.9204 & 0.8766 \\ 0.7389 & 0.9204 & 1.0000 & 0.7323 \\ 0.9084 & 0.8766 & 0.7323 & 1.0000 \end{pmatrix}$$

d. Based on the multivariate normal Q-Q plot is the plot of the theoretical quantiles vs the sample quantiles, it looks like a straight line, and, hence, the multivariate normal assumption seems feasible.

e. With $n = 20, p = 4,$, we want to find $\underset{\sim}{\mu}$ satisfying

$$\frac{n-p}{(n-1)p}(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu})' \left(\frac{1}{n}S\right)^{-1} (\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}) \le F_{p,n-p,1-\alpha} = 3.0069$$

f. $H_0 : \underset{\sim}{\mu} = (1400, 1900, 1700, 1700)'$ versus $H_a : \underset{\sim}{\mu} \neq (1400, 1900, 1700, 1700)'$.
From output, we have $F_{obs} = 0.8352$ and the corresponding p-value is 0.5225, which is greater than $\alpha = 0.05$.
Thus, we fail to reject H_0.

g. Since we fail to reject H_0 at $\alpha = 0.05$, $\underset{\sim}{\mu}_0 = (1400, 1900, 1700, 1700)'$ must falls within the 95% confidence region of $\underset{\sim}{\mu}$.