Probability

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1 Central Limit Theorem

1.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (1.1.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
https://github.com/RavishaJain/
    Assignment_2/blob/main/Codes
    /exrand.c
https://github.com/RavishaJain/
    Assignment_2/blob/main/Codes
    /coeffs.h
```

1.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution:

```
https://github.com/RavishaJain/
    Assignment_2/blob/main/Codes
    /cdf_plot.py
```

The cumulative distribution function $F_X(x)$ of a random variable has the following important properties:

- Every CDF is non decreasing and right continuous $\lim_{x\to-\infty} F(x) = 0$ $\lim_{x\to+\infty} f(x) = 1$
- For all real numbers a and b with continuous random variable X, then the function f_x is equal to the derivative of F_X,such that

$$F_x(b) - F_x(a) = P(a \le X \le b) = \int_a^b f_X(x)dx$$

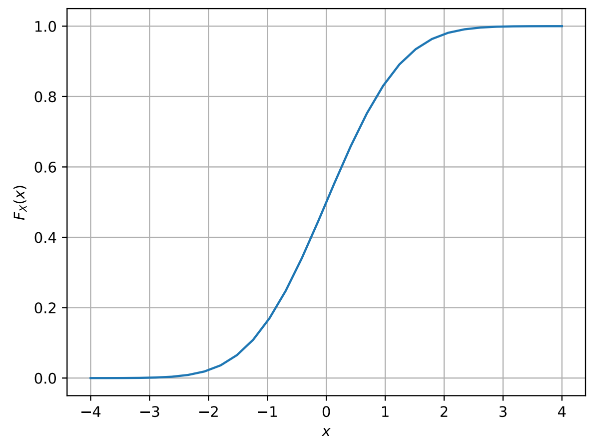

Fig. 1.2. The CDF of X

- If X is a completely discrete random variable, then it takes the values x1, x2, x3,... with probability p_i = p(xi), and the CDF of X will be discontinuous at the points x_i:

$$F_X(x) = P(X \le x)$$
$$\sum_{x_i \le x} P(X = x_i) = \sum_{x_i \le x} p(x_i)$$

1.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (1.3.1)$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 1.3 using the code below

```
https://github.com/RavishaJain/
    Assignment_2/blob/main/Codes
    /pdf_plot.py
```

The Probability density function formula is given as,

$$P(a < X < b) = \int_a^b f(x)dx \qquad (1.3.2)$$

or

$$P(a \le X \le b) = \int_a^b f(x)dx \qquad (1.3.3)$$

Let x be the continuous random variable with density function f(x), and the probability density function should satisfy the following conditions:

- For a continuous random variable that takes some value between certain limits, say a and b, the PDF is calculated by finding the area under its curve and the X-axis within the lower limit (a) and upper limit (b). Thus, the PDF is given by

$$P(x) = \int_a^b f(x)dx$$

- The probability density function is non-negative for all the possible values, i.e. f(x)≥ 0 for all x.
- The area between the density curve and horizontal X-axis is equal to 1, i.e

$$\int_{-\infty}^{\infty} dx = 1$$

- Due to the property of continuous random variables, the density function curve is continued for all over the given range. Also, this defines itself over a range of continuous values or the domain of the variable.

1.4 Find the mean and variance of X by writing a C program. **Solution:**

```
https://github.com/RavishaJain/
    Assignment_2/blob/main/Codes
    /Variance.c
https://github.com/RavishaJain/
    Assignment_2/blob/main/Codes
    /Mean.c
```

1.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \qquad (1.5.1)$$

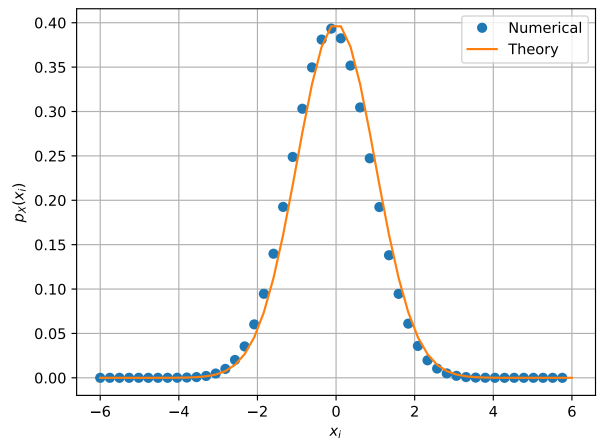

Fig. 1.3. The PDF of X

repeat the above exercise theoretically.
Solution: The expected value of a standard normal random variable X is:
E $[X]$=0 It can be derived as follows:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(\frac{-x^2}{2}\right)dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} x \exp\left(-\frac{x^2}{2}\right)dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x \exp\left(\frac{-x^2}{2}\right)dx$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{-x^2}{2}\right)_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left(\frac{-x^2}{2}\right)_{0}^{\infty}$$
$$= \frac{1}{\sqrt{2\pi}}[-1+0] + \frac{1}{\sqrt{2\pi}}[0+1]$$
$$= \frac{1}{\sqrt{2\pi}} - \frac{1}{\sqrt{2\pi}}$$
$$= 0$$

The variance can be calculated by $Var[X] = E[X^2] - E[X]^2$
So we have,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(\frac{-x^2}{2}\right)dx$$
$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} x^2 \exp\left(\frac{-x^2}{2}\right)dx\right] + \left]\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 \exp\left(\frac{-x^2}{2}\right)\right.$$

Upon integrating by parts the first part and second part in the above equation, we get

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[-x\exp\left(\frac{-x^2}{2}\right)\right]_{-\infty}^{0} + \int_{-\infty}^{0}\exp\left(\frac{-x^2}{2}\right)dx\right\} + \frac{1}{\sqrt{2\pi}}\left\{\left[-x\exp\left(\frac{-x^2}{2}\right)\right]_{0}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\exp\left(\frac{-x^2}{2}\right)dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{(0-0) + \int_{-\infty}^{0}\exp\left(\frac{-x^2}{2}\right)dx + (0-0) + \int_{0}^{\infty}\exp\left(\frac{-x^2}{2}\right)dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{2}\right)dx$$

The probability density fucntion over the entire region is 1

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{2}\right)dx$$

$$= 1$$

From above, we know that
$E[X]^2 = 0^2 = 0$
$\mathrm{Var}[X] = E[X^2] - E[X]^2 = 1 - 0 = 1$
Hence Proved.