Matrices

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Contents

1 Least Squares

Consider the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{1.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{1.0.2}$$

1.1. If the two lines intersect, show that

$$\mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \tag{1.1.1}$$

where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}. \tag{1.1.2}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix} \tag{1.1.3}$$

1.2. Find the rank of the augmented matrix in (1.1.1). Show that the lines in (1.0.2),(1.0.2) do not intersect.

1.3. Let

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \tag{1.3.1}$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \tag{1.3.2}$$

be the closest points on L_1 and L_2 respectively. Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1,L_2 and passing through A and B. Show that

$$\mathbf{M}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{1.3.3}$$

From (1.3.2) and (1.1.3)

$$\mathbf{A} - \mathbf{B} = \mathbf{x_1} - \mathbf{x_2} + \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{1.3.4}$$

1.4. Show that

$$\mathbf{M}^T \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{M}^T (\mathbf{x}_2 - \mathbf{x}_1) \tag{1.4.1}$$

1.5. Obtain λ_1 and λ_2.

1.6. Show that the distance between the lines is

$$\frac{3}{\sqrt{2}} \tag{1.6.1}$$

2 Singular Value Decomposition

2.1. Find $\mathbf{M}^T\mathbf{M}$ and $\mathbf{M}\mathbf{M}^T$.

2.2. Obtain the eigen decomposition

$$\mathbf{M}^T \mathbf{M} = \mathbf{P}_1 \mathbf{D}_1 \mathbf{P}_1^T \tag{2.2.1}$$

and

$$\mathbf{M}\mathbf{M}^T = \mathbf{P}_2 \mathbf{D}_2 \mathbf{P}_2^T \tag{2.2.2}$$

2.3. Perform the QR decompositions

$$\mathbf{P}_1 = \mathbf{U}\mathbf{R}_1 \mathbf{P}_2 = \mathbf{V}\mathbf{R}_2 \tag{2.3.1}$$

2.4. The singular value decomposition is the given by

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T, \tag{2.4.1}$$

where Σ has the same shape as $\mathbf{M}$ and

$$\Sigma = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{2.4.2}$$

2.5. Let

$$\mathbf{b} = \mathbf{x}_2 - \mathbf{x}_1 \tag{2.5.1}$$

$$\mathbf{y} = \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{2.5.2}$$

(1.1.1) can then be expressed as

$$\mathbf{U}\Sigma\mathbf{V}^T \mathbf{y} = \mathbf{b} \tag{2.5.3}$$

$$\implies \mathbf{y} = \mathbf{V}\Sigma^{-1}\mathbf{U}^T \mathbf{b} \tag{2.5.4}$$

where $\mathbf{\Sigma}^{-1}$ is obtained by inverting only the non-zero elements of $\mathbf{\Sigma}$.

3 Vector Calculus

3.1. *Definition:* Let $\mathbf{x} \in \mathbb{R}^2$, $f(\mathbf{x}) \in \mathbb{R}$. Then,

$$\frac{df(\mathbf{x})}{d\mathbf{x}} = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \end{pmatrix} \tag{3.1.1}$$

3.2. Show that

$$\begin{aligned} \frac{d\left(\mathbf{a}^T \mathbf{y}\right)}{d\mathbf{y}} &= \mathbf{a} \\ \frac{d\left(\mathbf{y}^T \mathbf{A} \mathbf{y}\right)}{d\mathbf{x}} &= 2\mathbf{A}\mathbf{y} \end{aligned} \tag{3.2.1}$$

3.3. Show that

$$f(\mathbf{y}) = \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{b}\|^2 + \|\mathbf{M}\mathbf{y}\|^2 \tag{3.3.1}$$

$$- 2\mathbf{b}^\top \mathbf{M}\mathbf{y} \tag{3.3.2}$$

3.4. Show that

$$\frac{d(f(\mathbf{y}))}{d\mathbf{y}} = \mathbf{0} \tag{3.4.1}$$

yields (1.4.1).

3.5. Verify if $\mathbf{M}^\top \mathbf{M} \geq 0$.