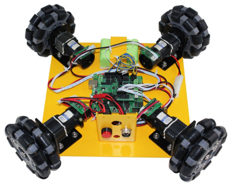
TSFS 12
Modelling of Ground Vehicles

Jan Åslund
Department of Electrical Engineering
Division of Vehicular Systems

Today I will consider different types of ground vehicles



The main question: In which directions can the different vehicles move?

Systems of ordinary differential equations

I will derive models in the form $\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \mathbf{u})$ for different kinds ground vehicles, where $\mathbf{q}$ is the state vector and $\mathbf{u}$ is the input vector.

First I will describe how a system of differential equations $\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \mathbf{u})$ can be interpreted geometrically. I will use the model

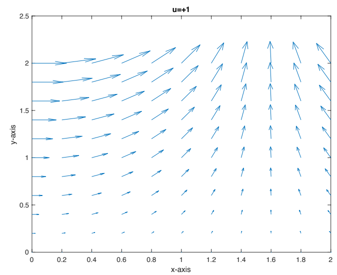
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \cos x \\ uy \sin x \end{pmatrix}$$

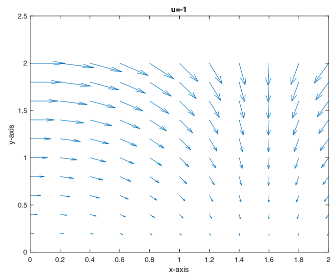
for illustration.

For every point (x, y) and control signal u, the right-hand side is a vector

$$\mathbf{f}(\mathbf{q}, \mathbf{u}) = \begin{pmatrix} y \cos x \\ uy \sin x \end{pmatrix}$$

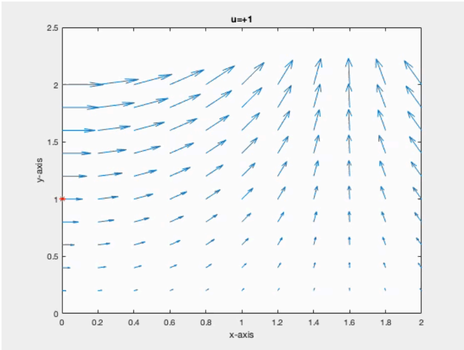
The vector fields for two values $u = \pm 1$:





The left-hand side of $\dot{\mathbf{q}} = f(\mathbf{q}, \mathbf{u})$ can be interpreted as the velocity vector of a particle and the solution is the trajectory of the particle moving in the vector field.

A particle moving first in the vector field corresponding to $u = +1$ and then the control signal switches to $u = -1$:



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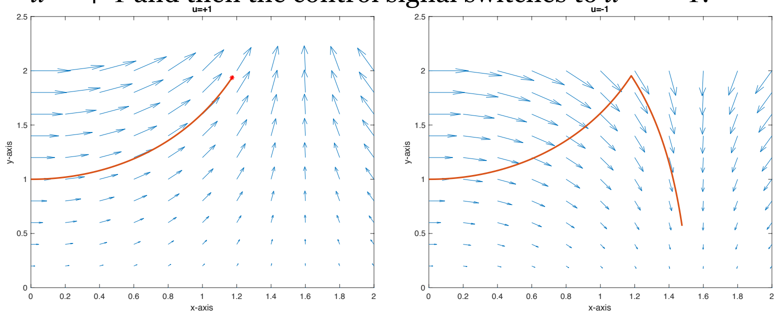
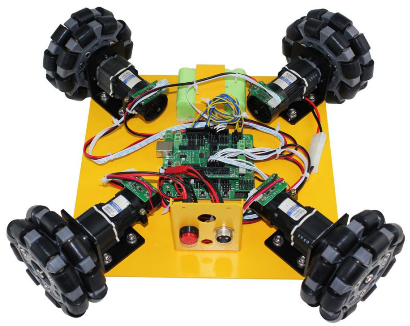
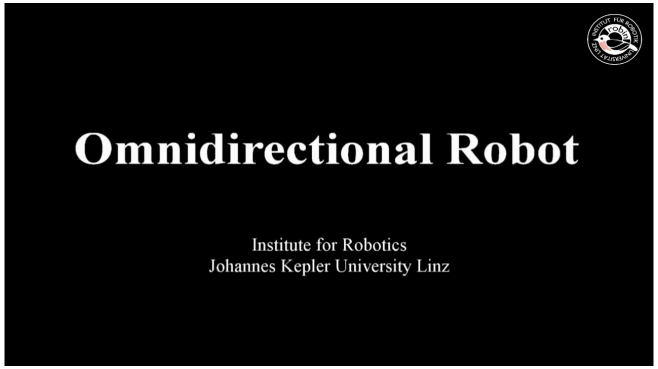


The Omnidirectional robot

The Omnidirectional Robot



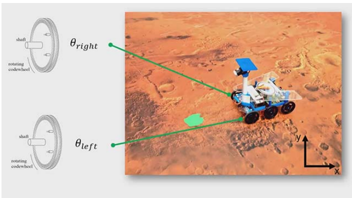
First example: The omnidirectional robot



The Differentially-Driven Mobile Robot

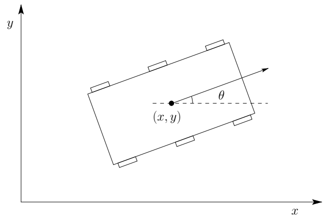
Second example: The Differentially-Driven Mobile Robot

Has two separately driven wheels placed on either side of the body of the robot.



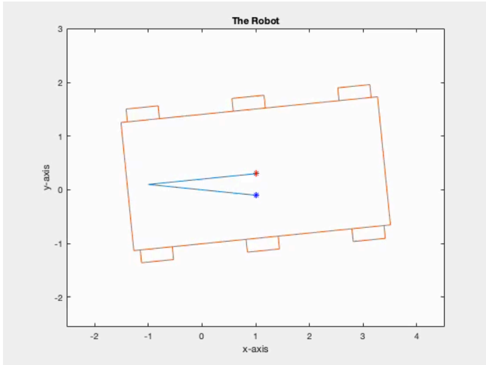
If the wheels rotate in the same direction, then the robot moves forward, and if they rotate in opposite direction, then the robot rotates.

For this robot, we shall a kinematic model with three states (position (x, y) and orientation θ with respect to x -axis), and two actuators. The system is called **underactuated** since the number of actuators is less the the dimension of the state space



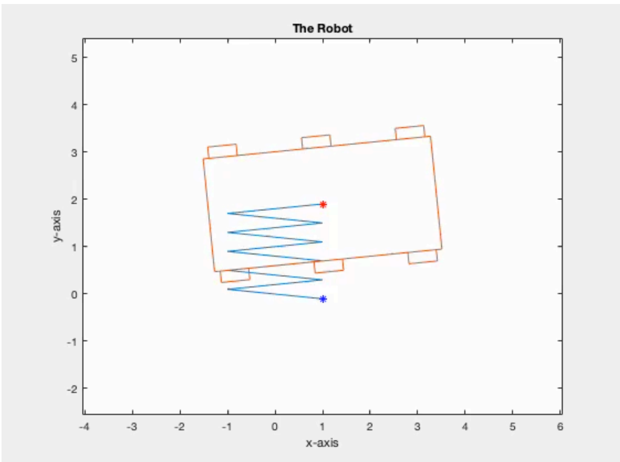
How can we accomplish a movement in the direction perpendicular the orientation of the vehicle?

Solution inspired by parallel parking:

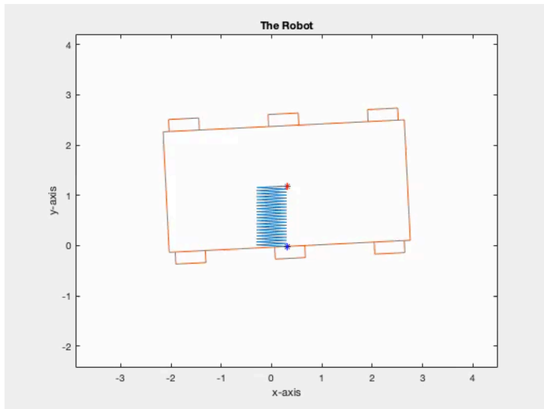


Conclusion: The robot moves one step upwards

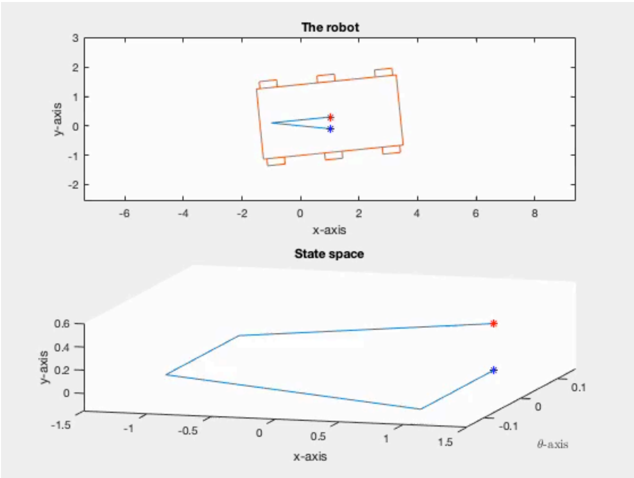
Repeating the procedure:



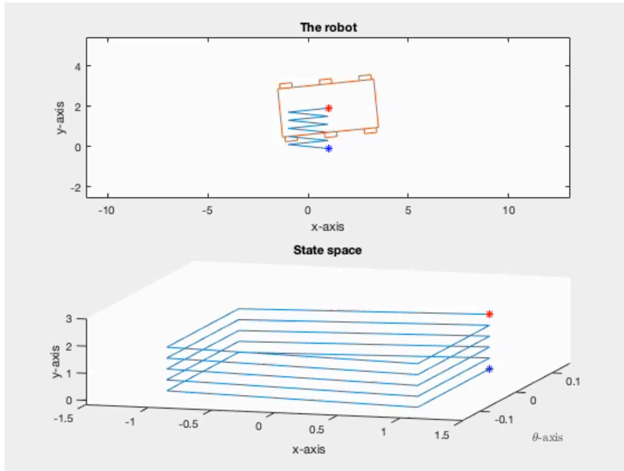
By taking smaller and smaller steps, the procedure can approximate a motion in the y-direction:



Add the trajectory in the state space:



Repeat the procedure:



Model of the robot

With the state vector $\mathbf{q} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$ and control vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, a model of the differential driven mobile robot can be written in the form

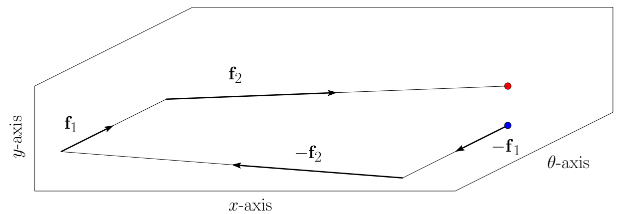
$$\dot{\mathbf{q}} = \mathbf{f}(\dot{\mathbf{q}}, \mathbf{u}) = u_1 \mathbf{f}_1(\mathbf{q}) + u_2 \mathbf{f}_2(\mathbf{q})$$

where $\mathbf{f}_1(\mathbf{q}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{f}_2(\mathbf{q}) = \begin{pmatrix} \cos\theta \\ sin\theta \\ 0 \end{pmatrix}$

The speed of the driven wheels are $v_r = u_2 + bu_1$ and $v_l = u_2 - bu_1$ where b is half of the distance between the right and left wheels.

If there is a limitation $|v_r|, |v_l| \leq v_{max}$ on the speeds of the driven wheels, then the corresponding limitation on the control signal is $|u_2| + b|u_1| \leq v_{max}$.

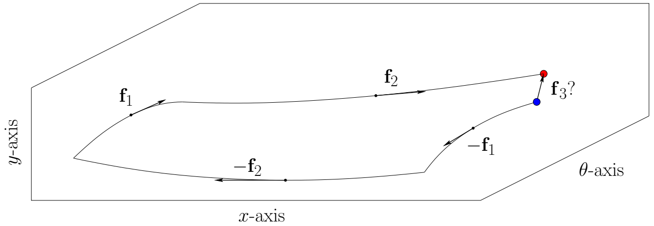
In the maneuvers described above, the robot moved in the directions $\mathbf{f}_1(\mathbf{q}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{f}_2(\mathbf{q}) = \begin{pmatrix} \cos\theta \\ sin\theta \\ 0 \end{pmatrix}$



$$\mathbf{q} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$$

This is the solution of $\dot{\mathbf{q}} = u_1 \mathbf{f}_1(\mathbf{q}) + u_2 \mathbf{f}_2(\mathbf{q})$ with the control sequence $(u_1, u_2) = (-1,0), (0,-1), (1,0), (0,1)$

Now we shall investigate what happens if we repeat the procedure for a system $\dot{\mathbf{q}} = u_1 f_1(\mathbf{q}) + u_2 f_2(\mathbf{q})$ with more general vector fields $f_1(\mathbf{q})$ and $f_2(\mathbf{q})$ and arbitrary small time steps $\Delta t = \varepsilon$.



Question: How can we compute an approximation of $\mathbf{f}_3$?

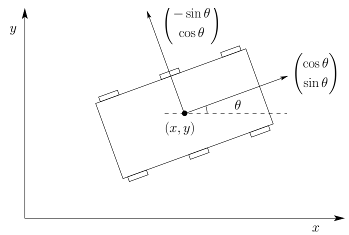
Answer: $\mathbf{f}_3 = \varepsilon^2((\mathbf{f}_2)_\mathbf{q}\mathbf{f}_1 - (\mathbf{f}_1)_\mathbf{q}\mathbf{f}_2) + \mathcal{O}(\varepsilon^3)$
where $(\mathbf{f}_i)_\mathbf{q} = \begin{pmatrix} \frac{\partial \mathbf{f}_i}{\partial x} & \frac{\partial \mathbf{f}_i}{\partial y} & \frac{\partial \mathbf{f}_i}{\partial \theta} \end{pmatrix}$ is the Jacobian matrix.

The expression $(\mathbf{f}_2)_\mathbf{q}\mathbf{f}_1 - (\mathbf{f}_1)_\mathbf{q}\mathbf{f}_2$ is called the Lie bracket and is denoted $[\mathbf{f}_1, \mathbf{f}_2]$

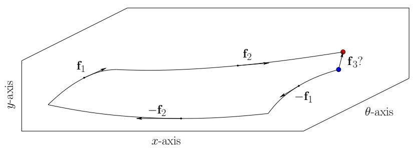
For example, with $\mathbf{f}_1(\mathbf{q}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{f}_2(\mathbf{q}) = \begin{pmatrix} \cos\theta \\ sin\theta \\ 0 \end{pmatrix}$ the result is

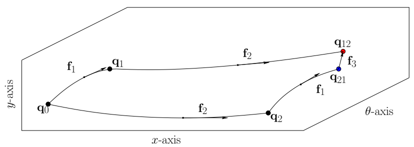
$$[\mathbf{f}_1, \mathbf{f}_2] = (\mathbf{f}_2)_\mathbf{q}\mathbf{f}_1 - (\mathbf{f}_1)_\mathbf{q}\mathbf{f}_2 = \begin{pmatrix} \frac{\partial \mathbf{f}_2}{\partial x} & \frac{\partial \mathbf{f}_2}{\partial y} & \frac{\partial \mathbf{f}_2}{\partial \theta} \end{pmatrix} \mathbf{f}_1 - \begin{pmatrix} \frac{\partial \mathbf{f}_1}{\partial x} & \frac{\partial \mathbf{f}_1}{\partial y} & \frac{\partial \mathbf{f}_1}{\partial \theta} \end{pmatrix} \mathbf{f}_2$$

$$= \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ cos\theta \\ 0 \end{pmatrix}$$

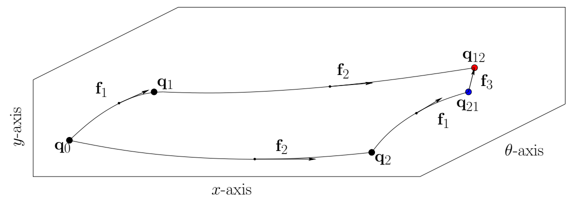


Sketch of proof





Instead of moving as in the figure to the left, we start at position $\mathbf{q}_0$ and first move as in the figure to the right to the two positions $\mathbf{q}_{12}$ and $\mathbf{q}_{21}$ and then compute $\mathbf{f}_3 = \mathbf{q}_{12} - \mathbf{q}_{21}$



$$\mathbf{q} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$$

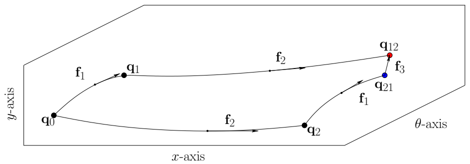
First move from $\mathbf{q}_0$ to $\mathbf{q}_1$ along $\mathbf{f}_1$ ($\Delta t = \varepsilon$):

$$\mathbf{q}_1 = \mathbf{q}_0(0) + \varepsilon\dot{\mathbf{q}}(0) + \frac{\varepsilon^2}{2}\ddot{\mathbf{q}}(0) + \mathcal{O}(\varepsilon^3)$$

$$= \mathbf{q}_0 + \varepsilon\mathbf{f}_1(\mathbf{q}_0) + \frac{\varepsilon^2}{2}(\mathbf{f}_1)_q(\mathbf{q}_0)\mathbf{f}_1(\mathbf{q}_0) + \mathcal{O}(\varepsilon^3)$$

where $(\mathbf{f}_1)_q = \left[\frac{\partial \mathbf{f}_1}{\partial x} \; \frac{\partial \mathbf{f}_1}{\partial y} \; \frac{\partial \mathbf{f}_1}{\partial \theta} \right]$ is the Jacobian matrix. We used that $\dot{\mathbf{q}} = \mathbf{f}_1(\mathbf{q})$ and

$$\ddot{\mathbf{q}} = \frac{d}{dt}\dot{\mathbf{x}} = \frac{d}{dt}\mathbf{f}_1(\mathbf{q}) = \frac{\partial \mathbf{f}_1}{\partial x}(\mathbf{q})\dot{x} + \frac{\partial \mathbf{f}_1}{\partial y}(\mathbf{q})\dot{y} + \frac{\partial \mathbf{f}_1}{\partial \theta}(\mathbf{q})\dot{\theta} = (\mathbf{f}_1)_q(\mathbf{q})\mathbf{f}_1(\mathbf{q})$$



By repeating the process we get

$$\mathbf{q}_1 = \mathbf{q}_0 + \varepsilon \mathbf{f}_1(\mathbf{q}_0) + \frac{\varepsilon^2}{2}(\mathbf{f}_1)_q(\mathbf{q}_0)\mathbf{f}_1(\mathbf{q}_0) + \mathcal{O}(\varepsilon^3)$$

$$\mathbf{q}_{12} = \mathbf{q}_1 + \varepsilon \mathbf{f}_2(\mathbf{q}_1) + \frac{\varepsilon^2}{2}(\mathbf{f}_2)_q(\mathbf{q}_1)\mathbf{f}_2(\mathbf{q}_1) + \mathcal{O}(\varepsilon^3)$$

Substitute $\mathbf{q}_1$ into $\mathbf{q}_{12}$:

$$\begin{aligned}\mathbf{q}_{12} = \mathbf{q}_0 + \varepsilon \mathbf{f}_1(\mathbf{q}_0) &+ \frac{\varepsilon^2}{2}(\mathbf{f}_1)_q(\mathbf{q}_0)\mathbf{f}_1(\mathbf{q}_0)\\ &+\varepsilon \mathbf{f}_2(\mathbf{q}_0 + \varepsilon \mathbf{f}_1(\mathbf{q}_0))\\ &+\frac{\varepsilon^2}{2}(\mathbf{f}_2)_q(\mathbf{q}_0)\mathbf{f}_2(\mathbf{q}_0) + \mathcal{O}(\varepsilon^3)\end{aligned}$$

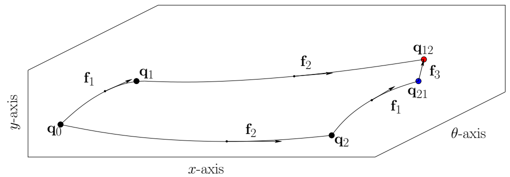
The Taylor expansion $\mathbf{f}_2(\mathbf{q}_0 + \varepsilon \mathbf{f}_1(\mathbf{q}_0)) = \mathbf{f}_2(\mathbf{q}_0) + \varepsilon(\mathbf{f}_2)_\mathbf{q}((\mathbf{q}_0)\mathbf{f}_1(\mathbf{q}_0) + \mathcal{O}(\varepsilon^2)$ gives

$$\begin{aligned}\mathbf{q}_{12} = \mathbf{q}_0 &+ \varepsilon(\mathbf{f}_1(\mathbf{q}_0) + \mathbf{f}_2(\mathbf{q}_0))\\ &+\frac{\varepsilon^2}{2}((\mathbf{f}_1)_q(\mathbf{q}_0)\mathbf{f}_1(\mathbf{q}_0) + (\mathbf{f}_2)_q(\mathbf{q}_0)\mathbf{f}_2(\mathbf{q}_0))\\ &+\varepsilon^2(\mathbf{f}_2)_\mathbf{q}(\mathbf{q}_0)\mathbf{f}_1(\mathbf{q}_0) + \mathcal{O}(\varepsilon^3)\end{aligned}$$

The corresponding expression for $\mathbf{q}_{21}$ is obtained by letting $\mathbf{f}_1$ and $\mathbf{f}_2$ switch places:

$$\begin{aligned}\mathbf{q}_{12} = \mathbf{q}_0 &+ \varepsilon(\mathbf{f}_1(\mathbf{q}_0) + \mathbf{f}_2(\mathbf{q}_0))\\ &+\frac{\varepsilon^2}{2}((\mathbf{f}_1)_q(\mathbf{q}_0)\mathbf{f}_1(\mathbf{q}_0) + (\mathbf{f}_2)_q(\mathbf{q}_0)\mathbf{f}_2(\mathbf{q}_0))\\ &+\varepsilon^2(\mathbf{f}_2)_\mathbf{q}(\mathbf{q}_0)\mathbf{f}_1(\mathbf{q}_0) + \mathcal{O}(\varepsilon^3)\end{aligned}$$

$$\begin{aligned}\mathbf{q}_{21} = \mathbf{q}_0 &+ \varepsilon(\mathbf{f}_2(\mathbf{q}_0) + \mathbf{f}_1(\mathbf{q}_0))\\ &+\frac{\varepsilon^2}{2}((\mathbf{f}_2)_q(\mathbf{q}_0)\mathbf{f}_2(\mathbf{q}_0) + (\mathbf{f}_1)_q(\mathbf{q}_0)\mathbf{f}_1(\mathbf{q}_0))\\ &+\varepsilon^2(\mathbf{f}_1)_\mathbf{q}(\mathbf{q}_0)\mathbf{f}_2(\mathbf{q}_0) + \mathcal{O}(\varepsilon^3)\end{aligned}$$

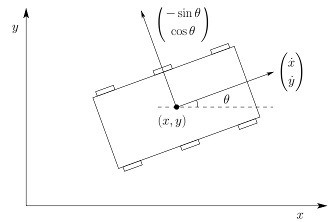


Finally: $\mathbf{f}_3 = \mathbf{q}_{12} - \mathbf{q}_{21} = \varepsilon^2((\mathbf{f}_2)_\mathbf{q}\mathbf{f}_1 - (\mathbf{f}_1)_\mathbf{q}\mathbf{f}_2) + \mathcal{O}(\varepsilon^3)$

Holonomic and Non-holonomic Systems

A non-holonomic example

The differentially driven robot is an example a non-holonomic systems



The directions which the robot can move are given by the condition

$$-\dot{x}\sin\theta + \dot{y}\cos\theta + \dot{\theta}0 = 0$$

A nonholonomic example

Nonholonomic means that the condition

$$-\dot{x}\sin\theta + \dot{y}\cos\theta + \dot{\theta}0 = 0$$

can not be written in the form

$$\frac{dG(x, y, \theta)}{dt} = 0$$

for any function $G(x, y, \theta)$.

A holonomic example

Consider a robot moving in three dimension and the kinematic constraint is

$$\dot{x}x + \dot{y}y + \dot{z}z = 0$$

This constraint is holonomic since it can be rewritten as

$$\frac{1}{2}\frac{d}{dt}(x^2 + y^2 + z^2) = 0$$

which is equivalent to

$$x^2 + y^2 + z^2 = r^2,$$

for some constant r, i.e., the equation of a sphere where the radius is given by the the initial state of the system.

A holonomic example

Two vector field that span the possible direction to move with the restriction $\dot{x}x + \dot{y}y + \dot{z}z = 0$ are

$$\mathbf{f}_1 = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} \text{ and } \mathbf{f}_2 = \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}$$

Can we use a the trick describe above to move in a third direction? The answer is no.

The Lie bracket $[\mathbf{f}_1, \mathbf{f}_2] = \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix}$ also fulfils the restriction and does not add an extra direction.

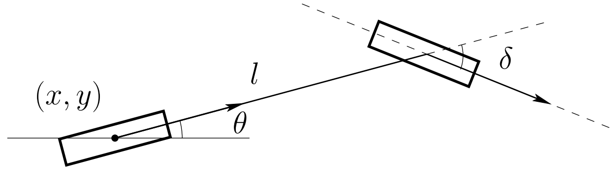
Conclusion: We can move in any direction as long as we do not leave the sphere.

Car with Front Wheel Steering

Third example: A car with front wheel steering

The Kinematic Single Track Model

A simple kinematic two wheel model of a car with front wheel steering. The model will have at least three states, position (x, y) of the center of the rear wheel, and orientation θ with respect to x-axis.

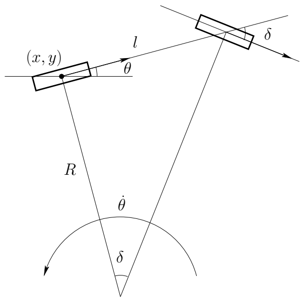


In the the figure, δ is the steering angle and l is the wheel base.

It is assumed that wheels are rolling without slipping, i.e., The velocity vector is parallel to the direction of the wheels.

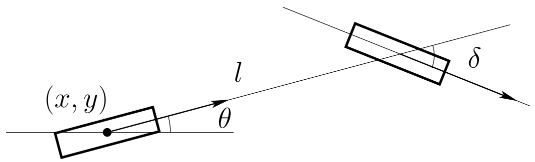
Assume that the car is moving and the speed at the rear wheels is v

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = v\frac{\tan\delta}{l}$$



In this case we have $\dot{x} = v\cos\theta$ and $\dot{y} = v\sin\theta$. Furthermore $\dot{\theta} = v\tan\delta/l$ follows from by eliminating the turning radius R from the equations $\tan\delta = l/R$ (the triangle) and $R\dot{\theta} = v$ (circular motion).

In addition to the three dynamic equations, we need to specify some control inputs.



$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = v\frac{\tan\delta}{l}$$

One example is to choose the acceleration and tangent of the steering angle

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = vu_1/l$$
$$\dot{v} = u_2$$
$$u_1 \in [-\tan\delta_{max}, \tan\delta_{max}]$$

Note that we have added restrictions on the steering angle.

One drawback with the model is that allows discontinuities in the steering angle. One way to get a smoother solution is to use the differentiated steering angle as input $\dot{\delta}$ and use the steering angle δ as an additional state.

$$
\begin{aligned}
\dot{x} &= v\cos\theta \\
\dot{y} &= v\sin\theta \\
\dot{\theta} &= v\tan\delta/l \\
\dot{\delta} &= u_1 \\
\dot{v} &= u_2 \\
\delta &\in [-\delta_{max}, \delta_{max}] \\
u_1 &\in [-\dot{\delta}_{max}, \dot{\delta}_{max}]
\end{aligned}
$$

Note that we have added restrictions on the steering angle

The two models can be written in the form $\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \mathbf{u})$ with

$$
\mathbf{q} = \begin{pmatrix} x \\ y \\ \theta \\ v \end{pmatrix}, \quad \mathbf{f}(\mathbf{q}, \mathbf{u}) = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ vu_1/l \\ u_2 \end{pmatrix}, \quad u_1 \in [-\tan\delta_{max}, \tan\delta_{max}]
$$

and

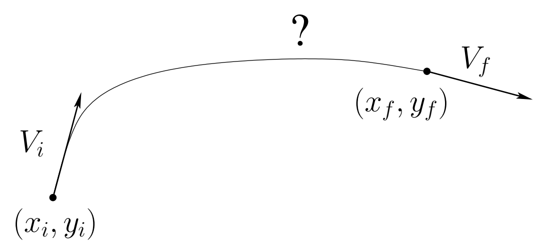
$$
\mathbf{q} = \begin{pmatrix} x \\ y \\ \theta \\ \delta \\ v \end{pmatrix}, \quad \mathbf{f}(\mathbf{q}, \mathbf{u}) = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ v\tan\delta/l \\ u_1 \\ u_2 \end{pmatrix}, \quad \begin{array}{c} \delta \in [-\delta_{max}, \delta_{max}] \\ u_1 \in [-\dot{\delta}_{max}, \dot{\delta}_{max}] \end{array}
$$

respectively.

Motion Planning: Two Classical Examples

Dubins path

One classic motion planning problem: Determine the shortest path between two points with the orientation specified at the initial and final point, and the turning radius limited from below



This problem can be formulated as an optimal control problem with the model

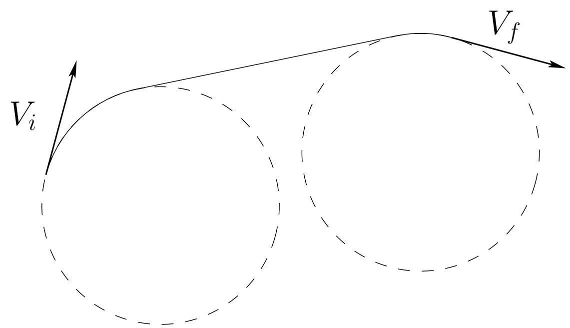
$\dot{x} = v \cos\theta$
$\dot{y} = v \sin\theta$
$\dot{\theta} = vu/l$
$v = 1$
$u \in [-\tan\delta_{max}, \tan\delta_{max}]$

where the objective function is time and $\tan\delta_{max} = l/R_{min}$
This problem was studied in the classical paper

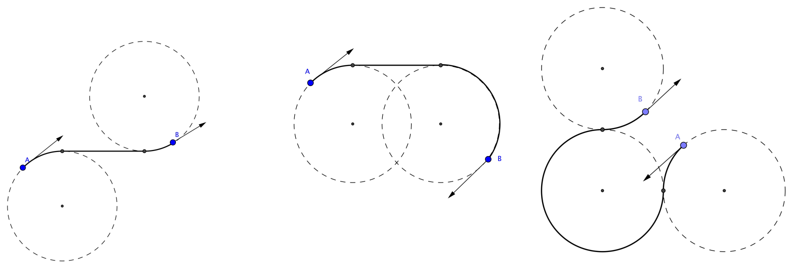
Dubins, L.E. (July 1957). "On Curves of Minimal Length with a Constraint on Average Curvature, and with Prescribed Initial and Terminal Positions and Tangents". *American Journal of Mathematics*. **79** (3): 497–516 doi:10.2307/2372560.

In the paper it was shown that any optimal solution will consist of segments with minimal curvature R_{min} and straight lines.

For example, the solution in the example above is



Three examples of solutions: RSL (right-straight-left), RSR, and LRL



It was shown that there exists three more types of solutions LSR, LSL, and RLR.

Reeds-Shepp Paths

The results were extended to the case where the car is allowed to move backward and forwards in the paper:

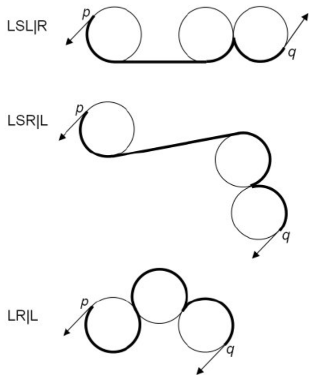
Reeds, J.A. and L.A. Shepp, "Optimal paths for a car that goes both forwards and backwards", Pacific J. Math., 145 (1990), pp. 367–393.

Can be formulated as an optimal control problem with the model

$\dot{x} = v \cos\theta$
$\dot{y} = v \sin\theta$
$\dot{\theta} = vu/l$
$v \in \{-1, 0, +1\}$
$u \in [-\tan\delta_{max}, \tan\delta_{max}]$

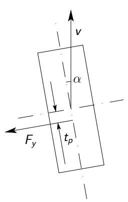
and time as objective function.

Some examples of optimal Reeds-Shepp paths:



Dynamic models

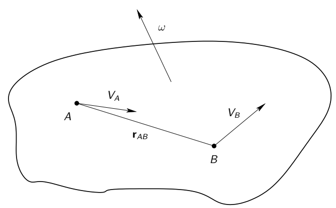
In the kinematic single track model, it is assumed that wheels are rolling without slipping, i.e., the velocity vector is parallel to the direction of the wheels. This model is valid if the speed is low. At higher speeds the deformation of the wheels has to taken into account and the slip angle α is defined as the angle between direction of the wheels and velocity vector:



A simple linear model is to assume that the later force F_y is proportional to the slip angle: $F_y = C_\alpha \alpha$

A Useful formula

Consider a rigid body:

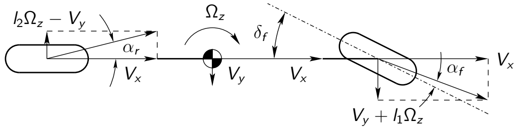


The relation between the velocities at the two points A and B is

$$\mathbf{v}_B = \mathbf{v}_A = \omega \times \mathbf{r}_{AB}$$

Dynamic Single Track Model

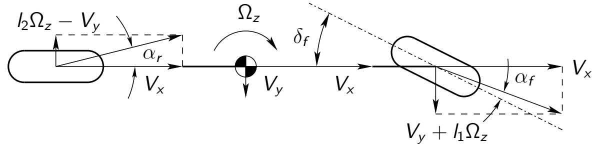
With A at center of gravity and B at the front/rear wheel:



$$\text{Front wheel:} \quad \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_z \end{pmatrix} \times \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} V_x \\ V_y + l_1\Omega_z \\ 0 \end{pmatrix}$$

$$\text{Rear wheel:} \quad \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_z \end{pmatrix} \times \begin{pmatrix} -l_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} V_x \\ V_y - l_2\Omega_z \\ 0 \end{pmatrix}$$

Dynamic Single Track Model



Kinematics from the figure

$$\alpha_f = \delta_f - \frac{l_1\Omega_z + V_y}{V_x}$$

$$\alpha_r = \frac{l_2\Omega_z - V_y}{V_x}$$

Model for tire forces

$$F_{yf} = 2C_{\alpha f}\alpha_f$$

$$F_{yr} = 2C_{\alpha r}\alpha_r$$

Conclusion: The lateral forces F_{yf} (front) and F_{yr} (rear) are functions of the states V_x, V_y, and Ω_z. Using the equations of motions it is possible to derive a model in the form $\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \mathbf{u})$. For more information, see TSFS02 Vehicle Dynamics and Control.