DISCRETE MATHEMATICS

Propositional Calculus

Propositional Calculus

1.1. Logic is a science of the necessary laws of thought, without which no employment of the understanding and the reason takes place.

Consider the following argument:

All mathematicians wear sandals

Anyone who wears sandals is an algebraist

Therefore, all mathematicians are algebraist.

Technically, logic is of no use in determining whether any of these statements is true. However, if the first two statements are true, logic assures us that the statement.

All mathematicians are algebraists is also true.

Example:- which of sentences are true or false (but not both)?

(a) The only positive integers that divide 7 are 1 and 7 itself.
(b) For every positive integer n, there is a prime number larger than n.
(c) Earth is the only planet in the universe that has life.

Solution:- (a) We call an integer n prime if $n>1$ and the only positive integers that divide n are 1 and n itself. Sentence (a) is another way say that 7 is a prime. Hence sentence (a) is true.
(b) Sentence (b) is another way to say that there are an infinite number of prime. Hence (b) is true.
(d) Sentence (c) is either true or false (but not both) but no one knows which at this time.

Definition:- A declarative sentence that is **either true** or false, but not both is called a **Proposition** (or **statement**).

For example, sentences (a) to (c) in the above example are propositions.

But the sentence
$x + y > 0$

is **not** a statement because for some values of x and y the sentence is true whereas for other values of x and y it is false. For example, if $x = 1$, $y = 3$, the sentence is true, but for $x = -2$, $y = 0$, it is false.
Similarly, the sentence
Take two crocins is not a statement. It is a command.

The propositions are represented by **lower case letters** such as p, q and r. We use the notation p: 1+1=3 to define p to be the proposition 1+1=3.

Many propositions are composite, that is, composed of subpropositions and various connectives. The "Composite propositions are called **compound propositions."** A proposition which is not compound is said to be **primitive.** Thus, a primitive proposition cannot be broken into simpler propositions.

Example:- The sun is shining and it is cold. This is a compound proposition composed of two propositions
The sun is shining
and
It is cold.
Connected by the connective "and".
On the other hand, the proposition
London is in Denmark
is **primitive** statement.

Definition:- The truth values of a compound statement in terms of its component parts, is called a **truth table.**

1.2. Basic Logical Operations

The three basic logical operations are

1. Conjunction
2. Disjunction
3. Negation

which correspond, respectively, to "and", "or" and "not".

Definition:- The conjunction of two propositions p and q is the proposition

p and q.

It is denoted by $p \wedge q$.
Example:- Let

p : This child is a boy
q : This child is intelligent

Then

$p \wedge q$: This child is a boy and intelligent.

Thus p ∧ q is true, if the child is a boy and intelligent both.
Even if one of the component is false, p ∧ q is false. Thus

"the proposition p ∧ q is true if and only if the proposition p and q are both true".

The truth value of the compound proposition p ∧ q is defined by the truth table:

P	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

Example:- If

$$p : 1 + 1 = 3$$

q : A decade is 10 years,

then p is false, q is true and the conjunction

p ∧ q : 1 +1 = 3 and a decade is 10 years

is **false**.

Definition:- The **disjunction** of two proposition p and q is the proposition

p or q

It is denoted by p ∨ q.

The compound statement p ∨ q is true if at least one of p or q is true. **It is false when both p and q are false.**

The truth values of the compound proposition p ∨ q is defined by the truth table:

P	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

For example, if

p : 1 + 1 = 3

q : A decade is 10 years,

then p is false, q is true. The disjunction

p ∨ q : 1 + 1 = 3 or a decade is 10 years is **true**.

Definition:- If p is a statement, the **negation** of p is the statement not p, denoted by ~p.
Thus ~p is the statement "it is not the case that p".
Hence if p is true than ~p is false and if p is false, then ~p is true.

The truth table for negation is

p	~p
T	F
F	T

Example:- Give the negation of the following statements :

(a) p : 2 + 3 > 1 (b) q : It is cold

Solution:-
(a) ~p : 2 + 3 is not greater than 1. That is, ~p : $2 + 3 \le 1$.
Since p is true in this case, ~p is false.

(b) ~q : It is not the case that it is cold. More simply, ~q : It is not cold.

Translating from English to Symbols :- We consider

Example:- Write each of the following sentences symbolically, letting p : "It is hot" and q : " It is sunny":

(a) It is not hot but it is sunny

(b) It is neither hot nor sunny.

Solution:- (a) The convention in logic is that the words "but" and "and" **mean the same thing.** Generally, but is used in place of and when the part of the sentence that follows is in some way unexpected.
The given sentence is equivalent to " It is not hot and it is sunny" which can be written symbolically as $\sim p \wedge q$.

(c) The phrase neither A nor B means the same as not A and not B. Thus to say " IT is neither hot nor sunny" means that it is not hot and it is not sunny. Therefore the given sentence can be written symbolically as $\sim p \wedge \sim q$.

Definition:- A "Statement form" or "Propositional form" is an expression made up of **statement variables** (such as ~, ∧, ∨) that becomes a statement when actual statements are substituted for the component statement variable. The **truth table** for a given statement form displays the truth values that correspond to the different combinations of truth values for the variables.
Example:- Construct a truth table for the statement form:

$$(p \wedge q) \vee \sim r.$$

solution:-The truth table for the given statement form is

p	q	r	P∧q	~r	(p∧q) ∨ ~r
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

Definition:- Two different compound propositions(or statement forms) are said to **logically equivalent** if they have the same truth value no matter what truth values their constituent propositions have.
We use the symbol ≡ for logical equivalent.

Example:- Consider the statements forms

(a) Dogs bark and cats meow
(b) Cats meow and dogs bark

If we take

p : Dogs bark
q : Cats meow,

then (a) and (b) are in logical expression

(a) $p \wedge q$

(b) $q \wedge p$

If we construct the truth tables for $p \wedge q$ and $q \wedge p$, we observe that $p \wedge q$ and $q \wedge p$ have same truth values.

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

p	q	q ∧ p
T	T	T
T	F	F
F	T	F
F	F	F

Thus $p \wedge q$ and $q \wedge p$ are logically equivalent. That is

$$p \wedge q \equiv q \wedge p$$

Example:- Negation of the negation of a statement is equal to the statement. Thus

$\sim(\sim p) \equiv p$.

Solution:- The truth table of $\sim(\sim p)$ is

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Thus truth values for p and $\sim(\sim p)$ are same and hence p and $\sim(\sim p)$ are logically equivalent. The logical equivalence $\sim(\sim p) \equiv p$ is called **Involution Law.**

Example:- Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

Solution:- Construct the truth table for both statement forms:

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

Thus we have different truth values in rows 2 and 3 and so $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not topologically equivalent.

Remark:- If we consider $\sim p \vee \sim q$, then its truth values shall be

F
T
T
T

and hence $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are logically equivalent. Symbolically

$$\sim(p \wedge q) \equiv \sim p \vee \sim q \tag{1}$$

Analogously,

$$\sim(p \vee q) \equiv \sim p \wedge \sim q \tag{2}$$

The above two logical equivalence are known as **De Morgan's Laws of Logic.**

Example:- Use De Morgan's Laws to write the negation of

p : Jim is tall and Jim is thin.

Solution:-The negation of p is

~p : Jim is not tall or Jim is not thin.

Definition:- A compound proposition which is **always true** regardless of truth values assigned to its component propositions is called a **Tautology.**

Definition:- A compound proposition which is **always false** regardless of truth values assigned to its component propositions is called a **Contradiction**.

Definition:- A compound proposition which can be either true or false depending on the truth values of its component propositions is called a **Contingency.**

Example:- Consider the statement form

$$p \vee \sim p.$$

The truth table for this statement form is

P	~p	p ∨ ~p
T	F	T
F	T	T

↑ all T's

Hence p ∨ ~p is a tautology.

Exercise :- Show that p ∧ ~p is a contradiction.

Remark:- If τ and c denote tautology and contradictions respectively, then we notice that

$$\sim\tau \equiv c \qquad (1)$$

and

$$\sim c \equiv \tau \qquad (2)$$

Also from the above two examples

$$p \vee \sim p \equiv \tau \qquad (3)$$

and

$$p \wedge \sim p \equiv c \qquad (4)$$

the logical equivalence (1), (2), (3) and (4) are known as **Complement Laws.**

Logical Equivalence involving Tautologies and Contradictions

If t is a tautology and c is a contradiction, then the truth tables for p ∧ τ and p ∧ c are :

p	τ	$p \wedge \tau$
T	T	T
F	T	F

Same truth values and so $p \wedge \tau \equiv p$

p	c	$p \wedge c$
T	F	F
F	F	F

Same truth values and so $p \wedge c \equiv c$

Similarly, the truth tables for $p \vee \tau$ and $p \vee c$ are

p	τ	$p \vee \tau$	p	c	$p \vee c$
T	T	T	T	F	F
F	T	T	F	F	F

Same truth values So $p \vee \tau \equiv \tau$

Same truth value and so $p \vee c \equiv p$

Thus we have the following logical equivalence:

$p \wedge \tau \equiv p$ $\qquad$ $p \wedge c \equiv c$

$p \vee \tau \equiv \tau$ $\qquad$ $p \vee c \equiv p$ (**universal bound laws**)

These four logical equivalence are known as **Identity Law.**

Example:- (**Idempotent Laws**): Consider the truth tables for $p \wedge p$ and $p \vee p$ given below:

p	p	$p \wedge p$
T	T	T
F	F	F

p	p	$p \vee p$
T	T	T
F	F	F

We note that

(i) $p \wedge p$ and p have same truth values

(ii) $p \vee p$ and p have same truth values

Hence

$$p \wedge p \equiv p \quad \text{and } p \vee p \equiv p$$

These two logical equivalence are known as **Idempotent Laws.**

Exercise :- Show that $p \wedge q \equiv q \wedge p$ and $p \vee q \equiv q \vee p$ (these logical equivalences are known as **Commutative Laws).**

Exercise :- Prove that

$$p \wedge (p \vee q) \equiv p$$

and

$$p \vee (p \wedge q) \equiv p .$$

(These logical equivalence are known as **Absorption Laws).**

Exercise :- Show that

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$, $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (**Associative Laws**)
and

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ (**Distributive Laws**)

1.3. Conditional Propositions

Definition:- If p and q are propositions, the compound proposition

if p then q or p implies q

is called a **conditional proposition** or **implication** and is denoted by

$$p \rightarrow q \ .$$

The proposition p is called the hypothesis or antecedent whereas the proposition q is called the **conclusion** or **consequent.**

The connective if…then is denoted by the symbol $\rightarrow$.

It is false when p is true and q is false, otherwise it is true. In particular, if p is false, then $p \rightarrow q$ is true for any q.

Definition:- A conditional statement that is true by virtue of the fact that its hypothesis is false is called **true by default** or **vacuously true**.
For example, the conditional statement
" If 3 + 3 = 7, then I am the king of Japan" is **true** simply because p : 3 + 3 = 7 is false. So it is not the case that p is true and q is false simultaneously.
Thus the truth values of the conditional proposition $p \rightarrow q$ are defined by the truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Each of the following expressions is an equivalent form of the conditional statement $p \rightarrow q$:

p implies q
q if p
p only if q

p is sufficient condition for q
q is necessary condition for p.

Example:- Restate each proposition in the form of a conditional proposition:

(a) I will eat if I and hungry

(b) 3 + 5 = 8 if it is snowing

(c) when you sing, my ears hurt

(d) Ram will be a good teacher if he teaches well.

(e) A necessary condition for English to win the world series is that they sign a right handed relief pitcher.

(f) A sufficient condition for Sohan to visit Calcutta is that he goes to Disney land.

Solution:-

(a) If I am hungry, then I will eat

(b) If it is snowing, then 3 + 5 = 8

(c) If you sing, then my ears hurt

(d) If Ram teaches well, then he will be a good teacher

(e) If English win the world series , then they sign a right handed relief pitcher

(f) If Sohan visit Calcutta, then he goes to Disney land.

Representation of "If …..then" as OR.

Lemma:- Show that for proposition p and q,

$$p \rightarrow q \equiv \sim p \vee q$$

Proof:- The truth values for $p \rightarrow q$ and $\sim p \vee q$ are given below:

P	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Same truth values

Hence

$$p \rightarrow q \equiv \sim p \vee q$$

Example:- Rewrite the statement in "If….then" form:

Either you get to work on time or you are fired.

Solution:- Let

$\sim p$: you get to work on time

and

q : you are fired

then the given statement is $\sim p \vee q$. But

p : you do not get to work on time.

Hence according to above lemma, the equivalent "If….then" version of the given statement is

If you do not get to work on time, then you are fired.

Negation of a conditional statement:- We know that $p \to q$ is false if and only if p is true and its conclusion q is false. Also, we have shown above that

$$p \to q \equiv \sim p \vee q$$

Taking negation of both sides, we have

$$\begin{aligned} \sim(p \to q) &\equiv \sim(\sim p \vee q) \\ &\equiv \sim(\sim p) \wedge (\sim q) \qquad \text{(De-Morgan's Law)} \\ &\equiv p \wedge \sim q \qquad \text{(Double negative Law or Involution Law)} \end{aligned}$$

(This can also be obtained by constructing the truth tables for $\sim(p \to q)$ and $p \wedge \sim q$; the truth tables would have the same truth values proving the logical equivalence)

Thus

The negation of "If p then q" is **logically equivalent to** "p and not q".

Example:- Write negations for each of the following statements:

(a) If I am ill, then I cannot go to university

(b) If my car is in the repair shop, then I cannot attend the class.

Solution:- We know that negation of " If p then q" is logically equivalent to "p and not q". Using this fact, the negations of (a) and (b) are respectively

(1) I am ill and I can go to university
(2) My car is in the repair shop and I can attend the class.

Remark:- The negation of a "if…..then" proposition does not start with the word if.

Definition:- If $p \rightarrow q$ is an implication, then the **converse** of $p \rightarrow q$ is the implication $q \rightarrow p$.

Definition:- The **contrapositive** of a conditional statement "If p then q" is "If ~q then ~pf".
In symbols,

$$\text{The contrapositive of } p \rightarrow q \text{ is } \sim q \rightarrow \sim p.$$

Lemma:- A conditional statement is logically equivalent to its contrapositive.

Solution:- The truth tables of $p \rightarrow q$ and $\sim q \rightarrow \sim p$ are:

$p \rightarrow q$			$\sim q \rightarrow \sim p$				
P	q	$p \rightarrow q$	p	q	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	F	F	T
T	F	F	T	F	F	T	F
F	T	T	F	T	T	F	T
F	F	T	F	F	T	T	T

Same truth values

Hence

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

Example:- Give the converse and contrapositive of the implications

(a) If it is raining, then I use my umbrella.

(b) If today is Monday, then tomorrow is Tuesday.

Solution:- (a) we have

p: It is raining

q : I use my umbrella

The converse is $q \rightarrow p$: If I use my umbrella, then it is raining.
The contrapositive is $\sim q \rightarrow \sim p$: If I do not use my umbrella, then it is not raining.

(b) we have

p : Today is Monday

q : Tomorrow is Tuesday

The converse is $q \rightarrow p$: If Tomorrow is Tuesday, then today is Monday.
The contrapositive is $\sim q \rightarrow \sim p$: If tomorrow is not Tuesday, then today is not Monday.

Definition:- The **inverse** of the conditional statement $p \rightarrow q$ is $\sim p \rightarrow \sim q$.
For example, the inverse of "If today is Easter, then tomorrow is Monday" is "If today is not Easter, then tomorrow is not Monday".

Remark:- If a conditional statement is true, then its converse and inverse may or may not be true. For example, on any Sunday except Ester, the conditional statement is true in the above example yet its inverse is false.

Only if:- " p only if q " means that p can take place only if q takes place also. That is, if q does not take place, then p cannot take place, i.e. $\sim q \rightarrow \sim p$. Therefore equivalence between a statement and its contrapositive imply that " if p occurs, then q must also occur". Hence

If p and q are statements, "p only if" means "if not q, then not p" or equivalently "if p then q".

Remark:- "p only if q" does not mean "p if q".

Example:- Use contrapositive to rewrite the following statement I n" if ….then" form:

"Ram will stand first in the class only if he works twelve hours a day."

Solution:- Version 1: We have

p : Ram will stand first in the class

q: he works twelve hours a day

The contrapositive is $\sim q \rightarrow \sim p$: If Ram does not works twelve hours a day, then he will not stand first in the class.

Version 2 : If Ram stands first in the class, then he will work twelve hours a day.

Definition:- If p and q are statements, the compound statement "p if and only if q" is called a **Biconditional statement** or an **equivalence**. It is denoted by $p \leftrightarrow q$. Observe that $p \leftrightarrow q$ is true only when both p and q are true or when both p and q are false.(i.e. if both p and q have same truth values) and is false if p and q have opposite truth values.

The biconditional statement has the following truth table:

$p \leftrightarrow q$

P	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Lemma:- Show that

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Solution:- We know that " p if and only if q" means that both " p if q" and " p only if q" hold. This means $p \leftrightarrow q$ should be logically equivalent to $(p \to q) \wedge (q \to p)$. We verify it using the truth table:

P	q	$p\to q$	$q\to p$	$p\leftrightarrow q$	$(p\to q)\wedge(q\to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Same truth values

Hence

$$p\leftrightarrow q \equiv (p\to q) \wedge (q\to p)$$

Remark:- It follows there for that biconditional statement can be written as the **conjunction** of two "if……then" statement namely $p \to q$ and $q \to p$. Also we know that

$$p \to q \equiv \sim p \vee q$$

and so

$$q \to p \equiv \sim q \vee p$$

Hence

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$$

Thus the statements having $\to$ or $\leftrightarrow$ symbol are logically equivalent to statement having $\sim$, $\wedge$ and $\vee$.

Definition:- Let p and q be statements. Then p is a sufficient condition for q means "if p then q" p is a necessary condition for q means " if not p then not q".

The hierarchy of operations of logical connectives : The order of operations of connectives are

$$\sim,\ \wedge,\ \vee,\ \to,\ \leftrightarrow$$

1.4. Arguments and Their Validity

Definition:- An **argument** is a sequence of statements. **All statements but the final one** are called premises (or assumptions or hypothesis). The **final statement** is called the **conclusion**.

The symbol $\therefore$, read “therefore”, is generally placed just before the conclusion.

Logical form of an argument : The logical form of an argument can be obtained from the contents of the given argument. For example, consider the argument:

If a man is a bachelor, he is unhappy

If a man is unhappy, he dies young

$\therefore$ Bachelors die young.

This argument has the abstract form

If p then q

If q then r

$\therefore p \rightarrow r$,

where

p : He is bachelor

q : He is unhappy

r : He dies young

Consider another example:

If Socrates is a human being, then Socrates is mortal

Socrates is a human being

$\therefore$ Socrates is mortal.

The abstract form of this argument is

If p then q

p

$\therefore$ q,

where

p : Socrates is human being

q : he is mortal

Definition:- An argument is said to be **valid** if the conclusion is true whenever all the premises are true.

Definition:- An **argument** which is **not true** is called a **fallacy**.

Method to Test Validity of an Argument

1. Identify the premises and conclusion of the argument

2. Construct a truth table showing the truth values of all the premises and conclusion

3. Find the rows (called **critical rows**) in which all the **premises are true.**

4. In each critical row, determine whether the conclusion of the argument is also true.

(a) If in each critical row the conclusion is also true, then the **argument form is valid**.

(b) If **there is at least one critical row** in which conclusion is **false**, the argument form is **fallacy** (invalid).

Example:- Show that the argument

$$\begin{array}{l} p \\ p \rightarrow q \\ \therefore q \end{array}$$

is valid.

Solution:- The premises are p and p → q. The conclusion is q. The truth table is

		Premises		Conclusion	
p	q	p	p→q	q	
T	T	T	T	T	→Critical row
T	F	T	F	F	
F	T	F	T	T	
F	F	F	T	F	

In the first row, all the premises are true. Therefore the first row is critical row. The conclusion in this critical row is also true. Hence the argument is valid.

The argument (discussed above)

$$\begin{array}{l} p \\ p \rightarrow q \\ \therefore q \end{array}$$

is known as **Law of Detachment.**

Example:- Consider the following argument form

$$p \rightarrow q$$

$$p$$
$$\therefore q$$

An argument of this type is

$p \rightarrow q$: If the last digit of this number is a 0, then this is divisible by 10

p : The last digit of this number is a 0

$\therefore$ This number is divisible by 10.

The truth table for the premises and conclusion is

		Premises		Conclusion	
P	q	p→q	p	Q	
T	T	T	T	T	→Critical row
T	F	F	T	F	
F	T	T	F	T	
F	F	T	F	F	

The first row is critical row and the conclusion I the critical row is true. Hence the given argument form is **Valid**.

The fact that this argument form is valid is called **Modus ponens**. This Latin term means "**Method of affirming**" (since the conclusion is an affirmation).

Example:- Consider the argument form

$$p \rightarrow q$$
$$\sim q$$
$$\therefore \sim p$$

An example of this type of argument form is

If Zeus is human, then Zeus is mortal

Zeus is not mortal

$\therefore$ Zeus is not human.

The truth table for the premises and conclusion is

		Premises		Conclusion	
p	q	p→q	~q	~p	
T	T	T	F	F	
T	F	F	T	F	
F	T	T	F	T	
F	F	T	T	T	→Critical row

The last row is critical row and conclusion in this row is also true. Hence the argument form is valid.

The fact that this argument is valid is called **Modus Tollens** which means (**Method of denying**) since the conclusion is denial.
The above example can be solved by "Method of contradiction" also in the following way : Suppose that the conclusion is false, i.e, Zeus is human. Then by the given statement (If…..then) Zeus is mortal. But this contradicts the premises "Zeus is not mortal". Hence the argument is valid and so Zeus is not human.

Exercise :- Using truth table or critical row method, show that the argument

$$p \to q$$
$$q \to r$$
$$\therefore p \to r$$

is universally valid. This argument is known as **Rule of Inference** or **Law of Syllogism.**

Example:- Consider the argument

Smoking is healthy

If smoking is healthy, then cigarettes are prescribed by physicians

∴ Cigarettes are prescribed by physicians.

Solution:- In symbols, the argument is

$$p$$
$$p \to q$$
$$\therefore q$$

The argument is of the form **Modus Ponens** (or Law of Detachment) and so is valid. However, the **conclusion is false**. Observe that the first premises, p : " Smoking is healthy", is **false**. The second premises, $p \to q$ is then true and conjunction of the two premises $(p \wedge (p \to q))$ is false.

Example:- Fill in the blanks of the following arguments so that they become valid inferences :

(a) If there are more pigeons than there are pigeonholes, then two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes
∴---
(b) If this number is divisible by 6, then it is divisible by 2
This number is not divisible by 2
∴ ----------------------------------

Solution:- (a) In logical symbols, the argument is

$$p \to q$$
$$p$$
$$\therefore \text{-----------}.$$

Hence, by **Modus ponens**, the answer is q, that is,

Two pigeons roost in the same hole.

(b) In logical symbols, the given premises and conclusion are

$$p \to q$$
$$\sim q$$
$$\therefore \text{------------}.$$

Hence, by **Modus tollen**, the answer is ~p, that is,

This number is not divisible by 6.

Example:- Using rules of valid inference solve the problem:

(a) If my glasses are on the kitchen table, then I saw them at breakfast
(b) I was reading the newspaper in the living room or I was reading in the kitchen
(c) If I was reading the newspaper in the living room. Then my glasses are on the coffee table.
(d) I did not see my glasses at breakfast
(e) If I was reading my book in bed, then my glasses are on the bed table.
(f) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

Where are the glasses?
Solution:-Let

p : my glasses are on the kitchen table

q : I saw them at breakfast

r : I was reading the newspaper in the living room

s : I was reading the newspaper in the kitchen

t : my glasses are on the coffee table

u : I was reading my book in bed

v : my glasses are on the bed table.

Then the given statements are

(a) $p \to q$ (b) $r \vee s$ (c) $r \to t$

(d) ~q (e) $u \to v$ (f) $s \to p$

The following deductions can be made:(1)

$p \to q$	by (a)
~q	by (d)
$\therefore$ ~p	by Modus Tollen (2)
$s \to p$	by (f)
~p	by the conclusion of (1)
$\therefore$ ~ s	by Modus Tollen (3)
$r \vee s$	by (b)
~s	by the conclusion of (2)
$\therefore$r	by disjunctive syllogism(4)
$r \to t$	by (c)
r	by the conclusion of (3)
$\therefore$ t	by Modus Ponens

Hence t is true and the glasses are on the coffee table.

Contradiction Rule:- If the supposition that the statement p is false leads logically to a contradiction, then you can conclude that p is true.
In symbols,
$\sim p \to c$, where c is a contradiction
$\therefore$ p
The truth table for the premise and the conclusion of this argument is given below:

p	~p	c	$\sim p \to c$	p	
T	F	F	T	T	→Critical row
F	T	F	F	F	

The premises and conclusion are both true in the critical row and hence the argument is valid.

Example:- Knights and Knaves (Raymond Smullyan's Description of an island containing two types of people):

This island contains two types of people: knights who always tell the truth and Knaves who always lie. A visitor visits the island and approached two natives who spoke to the visitor as follows:

A says : B is a knight
B says : A and I are of opposite type.

What are A and B?

Solution:- Suppose A is a knight. Because A always tells the truth, it follows that B is a knight.

Therefore what B says is true (by the definition of Knight). Therefore A and B are of opposite type. Thus we arrive at a contradiction: A and B are both Knights and A and B are of opposite type. Therefore supposition is wrong. Hence A is not a Knight. So A is a Knave. Therefore what A says is false. Hence B is not a Knight and so is a Knave. Hence A and B **are both Knaves**.

1.5. Quantifiers

So far we have studied the compound statements which were made of simple statements joined by the connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$. That study cannot be used to determine validity in the majority of everyday and mathematical situations. For example, the argument

All human being are mortal

Socrates is a human being

$\therefore$ Socrates is mortal

is intuitively correct. Yet its validity cannot be derived using the methods studied so far. To check the validity of such argument it is necessary to separate the statements into parts-subjects and predicates. Also we must analyse and understand the special role played by words denoting quantities such as All or Some.

Definition:- The symbolic analysis of predicates and quantified statements is called the **predicate calculus** whereas the symbolic analysis of ordinary compound statements is called the **Statement Calculus** (or **prepositional calculus**).

In English grammar, the predicate is the part of a sentence that gives information about the subject. For example, in the sentence "Ram is a resident of Karnal", the word Ram is the subject and the phrase "is a resident of Karnal" is the predicate. Thus, **predicate is the part of the sentence from which the subject has been removed**.

In logic, predicates can be obtained by removing any nouns from a statement. For example, if P stands for "is a resident of Karnal" and Q stands for "is a resident of", then both P and Q are predicate symbols. The sentences "x is a resident of Karnal" and "x is a resident of y" are denoted as P(x) and Q(x, y) respectively, where x and y are predicate variables that take values in appropriate sets.

Definition:- A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

The domain of a predicate variable is the set of all values that may be substituted in place of the variables. The predicates are also known as "**propositional functions** or **open sentences**".

Definition:- Let P(x) be a predicate and x has domain D. Then the set

$$\{ x \in D : P(x) \text{ is true}\}$$

is called the **truth set of P(x)**.

For example, let P(x) be " x is an integer less than 8" and suppose the domain os x is the set of all positive integers. Then the truth set of P(x) is {1, 2, 3, 4, 5, 6, 7}

Let P(x) and Q(x) be predicates with common domain D of x. The notation P(x)   Q(x) means that every element in the truth set of P(x) is in the truth set of Q(x).

Similarly $P(x) \Leftrightarrow Q(x)$ means that P(x) and Q(x) have **Identical truth sets**.
For example, let

P(x) be "x is a factor of 8"

Q(x) be "x is a factor of 4"

R(x) be " $x < 5$ and $x \neq 3$"

and let the domain of x be set of positive integers (Zahlen).
Then

Truth set of P(x) is {1, 2, 4, 8}

Truth set of Q(x) is {1, 2, 4}

Since every element in the truth set of Q(x) is in the truth set of P(x), Q(x) P(x).

Further, truth set of R(x) is {1, 2, 4}, which is identical to the truth set of Q(x). Hence $R(x) \Leftrightarrow Q(x)$.

Definition:- The words that refer to quantities such as "All", or "some" and tell for how many elements a given predicate is true are called **quantifiers**.
By adding quantifier, we can obtain statements from a predicate.

1.6. Universal Quantifiers and Existence Of Quantifiers

Definition:- The symbol $\forall$ denotes " for all" and is called the **Universal quantifier**.

Thus the sentence

All human beings are mortal

Can he written as

$$\forall\, x \in S, \text{ x is mortal},$$

where S denotes the set of all human being.

Definition:- Let P(x) be a predicate and D the domain of x. A statement of the form "$\forall\, x \in D, P(x)$" is called a **universal statement**.

A universal statement P(x) is true if and only if P(x) is true **for every** x in D and a universal statement P(x) is false if and only if P(x) is false **for at least one** $x \in D$.

A value for x for which P(x) is **false** is called a **Counterexample** to the universal statement.]

Example:- Let D = {1, 2, 3, 4} and consider the universal statement

$$P(x) : \forall\, x \in D, x^3 \geq x$$

This is true for all values of $x \in D$ since $1^3 \geq 1$, $2^3 \geq 2$ and so on.

But the universal statement

$$Q(x) : \forall\, n \in N, n + 2 > 8$$

is not true because if we take n = 6, then 8 > 8 which is absurd.

Definition:- The symbol $\exists$ denotes "there exists" and is called the **existential quantifier**.

For example, the sentence " There is a University in Kurukshetra" can be expressed as

$\exists$ a university u such that u is in Kurukshetra.

or, we can write

$\exists\, u \in U$ such that u is in Kurukshetra, where U is the set of universities.

The words **such that** are inserted just before the predicate.

Definition:- Let P(x) be a predicate and D is the domain of x. a statement of the form "$\exists\, x \in D$ such that P(x)" is called an **Existential Statement**. It is defined to be true if and only if P(x) is true for at least one x in D.

It is false if and only if P(x) is false for all x in D.

For example the existential statement

$$\exists\, n \in N : n + 3 < 9$$

is **true** since the set

$$\{n : n + 3 < 9\} = \{1, 2, 3, 4, 5\}$$

is not empty.

Example:- Let A = {2, 3, 4, 5}, then the existence statement

$$\exists\ n \in A : n^2 = n$$

is false because there is no element in A whose square is equal to itself.

Definition:- A statement of the form

$$\forall\ x, \text{ if } P(x) \text{ then } Q(x)$$

is called **universal conditional statement**.
Consider the statement

$$\forall\ x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4$$

can be written in any of the form

(i) If a real number is greater than 2, then its square is greater than 4
(ii) Whenever a real number is greater than 2, its square is greater than 4
(iii) The square of any real number that is greater than 2 is greater than 4.
(iv) The squares of all real numbers greater than 2 are greater than 4.

On the other hand, consider the statements

(i) All bytes have eight bits

(ii) No fire trucks are green.

These can be written as

(i) ∀ x, if x is a byte, then x has eight bits

(ii) ∀ x, if x is a fire truck , then x is not green.

Example:- Consider the statement

(i) ∀ Polygons p, if p is a square, then p is a rectangle.

This is equivalent to the universal statement

" ∀ squares p, p is a rectangle".

(ii) ∃ a number n such that n is prime and n is even.

This is equivalent to

"∃ a prime number n such that n is even".

Remark:- Existential quantification can also be implicit. For example, the statement

"The number 24 can be written as a sum of two even integers"

can be expressed as

"∃ even integers m and n such that 24 = m + n".

1. Universal quantification can also be implicit. For example the statement

"If a number is an integer, then it is rational number"

is equivalent to

"$\forall$ real number x, if x is an integer, then it is a rational number."

1.7. Negation of University Statement

Definition:- The negation of a universal statement

$\forall$ x in D, P(x)

is logically equivalent to a statement of the form

$\exists$ x in D such that ~P(x)

Thus

$$\sim(\forall x \in D, P(x)) \equiv \exists x \in D, \sim P(x)$$

Hence

The negation of a universal statement "all are" is logically equivalent to an existential statement "some are not".
For example, the negation of

(i) " For all positive integer n, we have n + 2 > 9"

is

" There exists a positive integer n such that $n + 2 \not> 0$".

(ii) The negation of

" All students are intelligent"

is

"Some students are **not** intelligent"

or

"$\exists$ a student who is **not** intelligent".

(iii) the negation of

" No politicians are honest"

is

"$\exists$ a politician x such that x is honest."

or

"Some politicians are honest".

Definition:- The **negation of a universal conditional statement** is defined by

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists \text{ x such that } \sim(P(x) \rightarrow Q(x)).$$

Also we know that the negation of if-then statement is

$$\sim (P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x).$$

Hence

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists\ x \text{ such that } P(x) \wedge \sim Q(x) ,$$

that is,

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists\ x \text{ such that } P(x) \text{ and } \sim Q(x).$$

Example:- The negation of

$\forall$ people p, if p is blond then p has blue eyes

is

$\exists$ a person p such that p is blond and p does not have blue eyes.

Example:- Suppose there is a bowl and we have no ball in the bowl. Then the statement

"All the balls in the bowl are blue"

is true "by default" or " Vacuously true" because there is no ball in the bowl which is not blue.

If P(x) is a predicate and the domain of x is $D = \{x_1, x_2, \ldots, x_n)$, then the statement

$$\forall\ x \in D, P(x)$$

and

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

Are logically equivalent.
For example, let P(x) be

"$x . x = x$"

and let $D = \{0, 1\}$. Then

$$\forall\ x \in D, P(x)$$

can be written as

$\forall$ binary digits x , $\quad x . x = x.$

This is equivalent to

$$0 . 0 = 0 \text{ and } 1 . 1 = 1$$

which can be written as

$$P(0) \wedge P(1)$$

Similarly, if P(x) is a predicate and $D = (x_1, x_2, \ldots, x_n\}$ then the statements

$$\exists\ x \in D, P(x)$$

and

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

are logically equivalent.

Definition:- Let

$$\forall\ x \in D, \text{ if } P(x) \text{ then } Q(x)$$

be a statement. Then

(i) **Contrapositive** of this statement is

$$\forall\ x \in D, \text{ if } \sim Q(x) \text{ then } \sim P(x)$$

(ii) **Converse** of this statement is

$$\forall\ x \in D, \text{ if } Q(x) \text{ then } P(x)$$

(iii) **Inverse** of this statement is

$$\forall\ x \in D, \text{ if } \sim P(x) \text{ then } \sim Q(x)$$

1.8. Universal Modus Ponens

The following argument form is valid

Formal Version	**Informal Version**
$\forall$ x if P(x) then Q(x)	If x makes P(x) true, then x makes Q(x) true
P(a) for a particular a	a makes P(x) true
$\therefore$ Q(a)	$\therefore$ a makes Q(x) true.

An argument of this form is called a **Syllogism**. The first and second premises are called its **major premises** and **minor premises** respectively.

Example:- Consider the argument:

If a number is even, then its square is even

K is a particular number that is even

$\therefore$ K^2 is even

The major premises of this argument can be written as

$$\forall\ x, \text{ if x is even then } x^2 \text{ is even}$$

Let

P(x) : "x is even"

Q(x) : "x^2 is even"

and let k be an even number. Then the argument is

$$\forall\ x, \text{ if } P(x) \text{ then } Q(x)$$

$P(k)$ for k

$\therefore Q(k)$

This form of argument is valid by universal Modus Ponens.

1.9. Universal Modus Tollens

The following argument form is valid

Formal Version	**Informal Version**
$\forall$ x if P(x) then Q(x)	If x makes P(x) true, then x makes Q(x) true
~Q(a) for a particular a	a does not makes Q(x) true
$\therefore$ ~P(a)	$\therefore$ a does not makes P(x) true.

Example:-

All human being are mortal

Zeus is not mortal

$\therefore$ Zeus is not human

The major premise of this argument can be rewritten as

$\forall$ x, if x is human, then x is mortal

Let

P(x) : x is human

Q(x) : x is mortal

let Z = Zeus

Then we have

$\forall$ x, if P(x) then Q(x)

~Q(Z)

$\therefore$ ~P(Z)

which is valid by Universal Modus Tollens.

Example:- The argument

All professors are absent minded

Tom is not absent minded

$\therefore$ Tom is not a professor.

The major premise can be written as

$\forall$ x, if x is professor, then x is absent minded.

Let

$P(x)$: x is professor.

$Q(x)$: x is absent minded.

Z = Tom

Then we have

$\forall$ x, if $P(x)$ then $Q(x)$

$\sim Q(Z)$

$\therefore \sim P(Z)$.

Hence, by Universal Modus Tollens, Tom is not a professor.

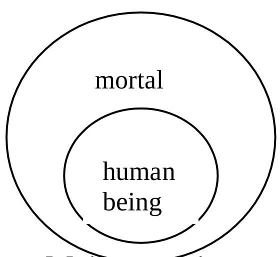
1.10. Use of Diagrams For Validity of Arguments

Consider the argument:

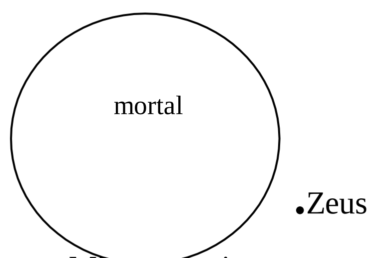
All human beings are mortal

Zeus is not mortal

$\therefore$ Zeus is not a human being.

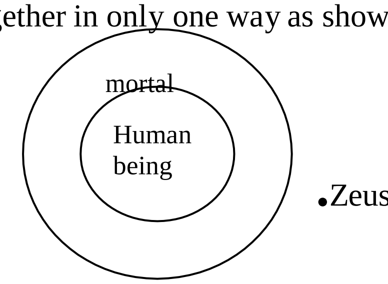


Major premise



Minor premise

The two diagrams fit together in only one way as shown below:



Since Zeus is outside the mortal disc it is necessarily outside the human beings disk. Hence the Conclusion is true.

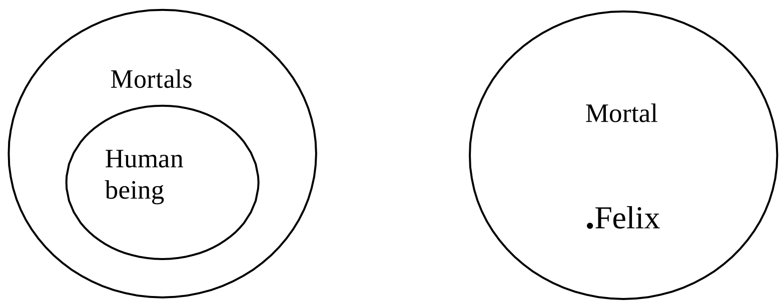
Example:- Use a diagram to show the invalidity of the arguments

All human being are mortal

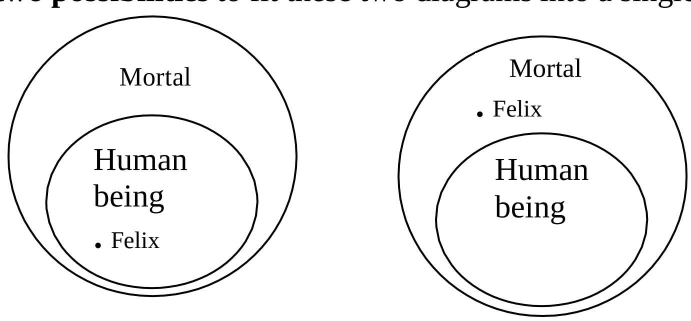
Felix is mortal

∴ Felix is a human being.

Solution:- The major premise and a minor premise of the arguments are shown in the diagrams below :



There are **two possibilities** to fit these two diagrams into a single one.



(1) (2)

The conclusion “Felix is a human being” is true in the first case but not in the second. Hence the argument is invalid.

is ambiguous.