1

Digital and Analog Communication

DAC

Section-A

Fourier Series: → Fourier Series is a tool used to analyze any periodic signal. After the analysis we obtain the fallowing information about the signal

1. what all frequency component are present in the signal.?
2. Their Amplitude
3. The relative phase difference between these frequency components.

Types of Fourier Series: →

(i) Trigonometric or quadrature fourier Series

(2) Polar Fourier Series

(3) Exponential F. S.

here we are going to discuss only trigonometric Fourier Series

A periodic signal $x(t)$ with a period of T_0 is given the trigonometric Fourier Series as.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \quad \text{(I)}$$

where a_0, a_n and b_n are known as Fourier Coefficient and $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

The coefficients are as fallows:-

1. The value of $\boxed{a_0 = \frac{1}{T_0}\int_t^{t+T_0} x(t)\,dt}$ — (II)

It is called d.c component of $x(t)$

2. The value of $\boxed{a_n = \frac{2}{T_0}\int_t^{t+T_0} x(t).\cos(n\omega_0 t)\,dt}$ — (III)

3. $\boxed{b_n = \frac{2}{T_0}\int_t^{t+T_0} x(t).\sin(n\omega_0 t)\,dt}$ — (IV)

now eq (I) becomes.

$$x(t) = \underbrace{\boxed{a_0}}_{\text{D.C value of } x(t)} + \underbrace{a_1 \cos\omega_0 t}_{\text{fundamental}} + \underbrace{a_2 \cos 2\omega_0 t}_{\text{Second harmonic}} + \cdots + b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + \cdots \quad \text{(V)}$$

Conclusion :→ The eq (v) is suitable to Plot the line spectrum.

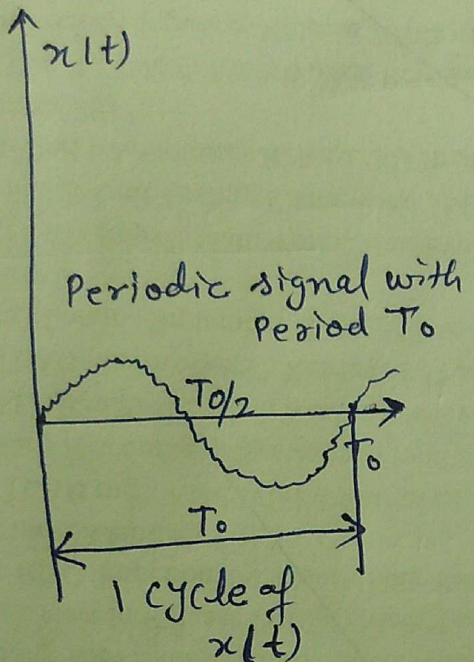

fig (a). Time domain representation of $x(t)$

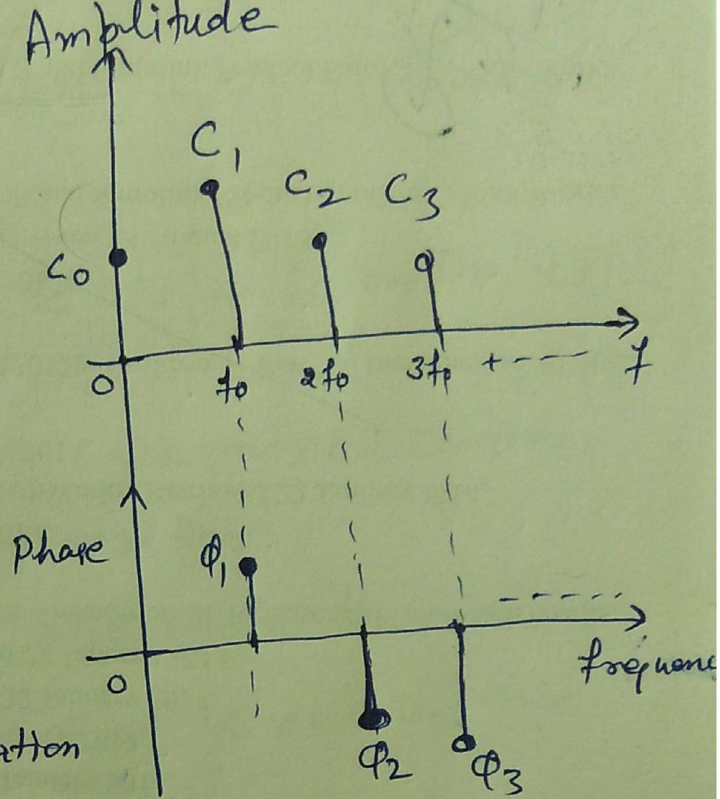

fig (b). Line spectrum of $x(t)$ after solving the Fourier Series

Example 1. Numerical Problem based on F.S.

Q1 Obtain Fourier Series for the rectangular pulse train shown in figure (i)

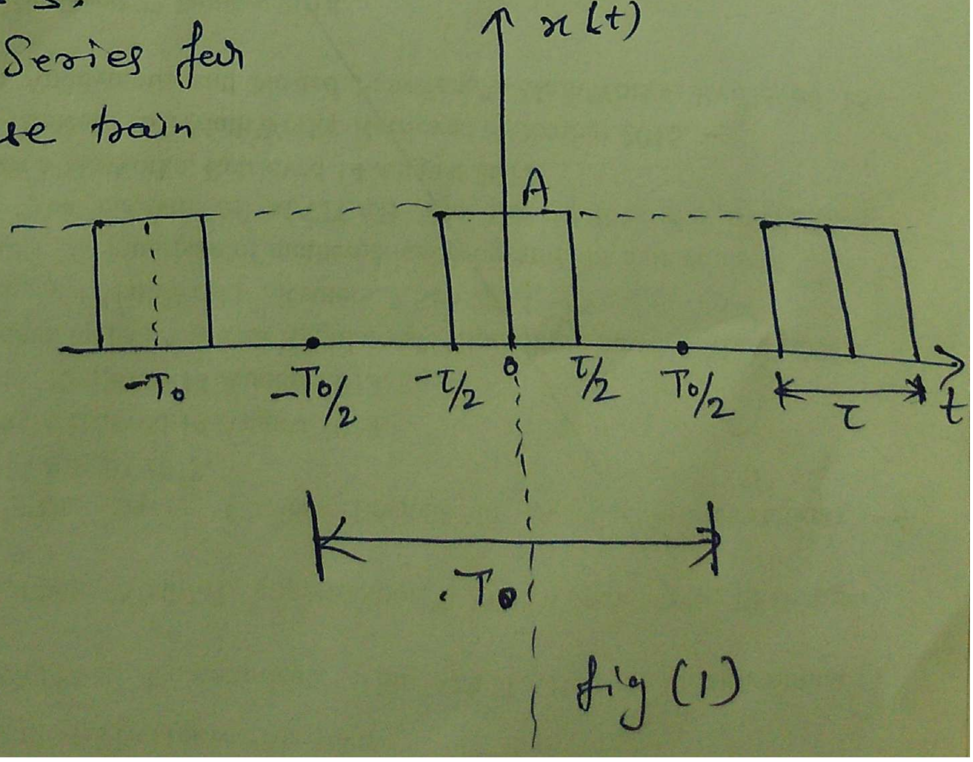

fig (1)

Solution The Fourier Series is given by (4)

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \quad \text{--- (1)}$$

substituting $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$ in eq (1) we get

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left[\frac{2\pi n t}{T_0}\right] + \sum_{n=1}^{\infty} b_n \sin\left[\frac{2\pi n t}{T_0}\right] \quad \text{--- (2)}$$

To find Fourier Coefficients, we must consider one cycle of $x(t)$ for integration.

Here we will consider one cycle from $t = -T_0/2$ to $t = T_0/2$. Let us obtain the Fourier coefficients now

1. To obtain the value of a_0 :-

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\,dt$$

$$= \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} x(t)\,dt \quad \text{--- As } x(t) \text{ exists from } -\tau/2 \text{ to } \tau/2 \text{ only}$$

$$= \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A \cdot dt \qquad \text{As } x(t) = A, \text{ for } -\tau/2 \text{ to } \tau/2$$

$$\therefore \boxed{a_0 = \frac{A\tau}{T_0}} \quad \text{--- (3)}$$

2. To obtain the value of a_n:

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cdot\cos\left[\frac{2\pi n t}{T_0}\right]dt = \frac{2}{T_0}\int_{-\tau/2}^{\tau/2} A\cdot\cos\left[\frac{2\pi n t}{T_0}\right]dt$$

$$= \frac{\cancel{2}A}{\cancel{T_0}}\cdot\frac{1}{\frac{\cancel{2}\pi n}{\cancel{T_0}}}\left[\sin\left(\frac{2\pi n t}{T_0}\right)\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{n\pi}\left[\sin\left(\frac{\cancel{2}\pi n\tau}{\cancel{2}T_0}\right) - \sin\left(-\frac{\cancel{2}\pi n\tau}{\cancel{2}T_0}\right)\right]$$

$$= \frac{A}{n\pi}\left[\sin\left(\frac{\pi n\tau}{T_0}\right) + \sin\left(\frac{\pi n\tau}{T_0}\right)\right]$$

$$\boxed{a_n = \frac{2A}{n\pi}\sin\left[\frac{n\pi\tau}{T_0}\right]} \quad (4)$$

3. To obtain the value of b_n:

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t).\sin\left[\frac{2\pi n t}{T_0}\right]dt$$

$$= \frac{2}{T_0}\int_{-\tau/2}^{\tau/2} A.\sin\left[\frac{2\pi n t}{T_0}\right]dt$$

$$b_n = -\frac{\cancel{2}A}{\cancel{T_0}} \cdot \frac{1}{\frac{\cancel{2}\pi n}{\cancel{T_0}}} \left[\cos\left(\frac{2\pi n t}{T_0}\right) \right]_{-\tau/2}^{\tau/2}$$

$$= -\frac{A}{n\pi} \left[\cos\left(\frac{\cancel{2}\pi n \tau}{\cancel{2}T_0}\right) - \cos\left(-\frac{\cancel{2}\pi n \tau}{\cancel{2}T_0}\right) \right]$$

$$= -\frac{A}{n\pi} \left[\cancel{\cos\left(\frac{n\pi\tau}{T_0}\right) - \cos\left(\frac{n\pi\tau}{T_0}\right)} \right]$$

$$\boxed{b_n = 0} \quad \text{——} (5)$$

now Putting the value of a_0, a_n and b_n in eq (2) we get the trigonometric Fourier Series for $x(t)$ as.

$$\boxed{x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left[\frac{n\pi\tau}{T_0}\right] \cos\left[\frac{2\pi n t}{T_0}\right]}$$

Ans

Fourier Transform :→

The Fourier transform of a signal $x(t)$ is defined as

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt} \quad \text{— I}$$

or

$$\boxed{X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt} \quad \text{— II}$$

Inverse Fourier Transform :→

The signal $x(t)$ can be obtained back from Fourier transform $X(f)$ by using the inverse fourier transform. It is defined as

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) . e^{j\omega t}\, d\omega} \quad \text{— III}$$

OR

$$\boxed{x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df} \quad \text{— IV}$$

Conditions for the existence of F.T :→

For a Periodic signals the integration is obtained over one Period however for the non-Periodic signals, it will be obtained over a range $-\infty$ to ∞

The signal $x(t)$ will have to satify the following conditions so that its F.T can be obtained.

1. The function $x(t)$ should be single valued in any finite interval T. 8.
2. It should have a finite number of discontinuities in any finite interval T.
3. The function $x(t)$ should have a finite number of maxima and minima in any finite interval of time T.
4. The function $x(t)$ should be an absolutely integral function i.e $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

The conditions stated above are sufficient conditions, but they are not the necessary conditions.

Merits of F.T :→

(i) It is Possible to uniquely recover the original time function $x(t)$.

(2) We can evaluate the convolutional integrals using F.T

(3) F.T is very useful in comm. Systems

Limitations of F.T :→ The most important limitation of F.T is that there are many time functions for which the F.T does not exist, because such functions are not absolutely integrable.

Properties of F.T :→

(i) Linearity or superposition :→

if $x_1(t) \xleftrightarrow{F} X_1(f)$ and $x_2(t) \xleftrightarrow{F} X_2(f)$ then all constants such as a_1 and a_2 we can write

$$[a_1 x_1(t) + a_2 x_2(t)] \xleftrightarrow{F} [a_1 X_1(f) + a_2 X_2(f)] \quad \text{(I)}$$

That means the linear combination of inputs gets transformed into linear combination of their Fourier transform.

(2) Time scaling :→

let $x(t) \xleftrightarrow{F} X(f)$ and let α be a constant. Then the time scaling Property states that.

$$\boxed{x(\alpha t) \xleftrightarrow{F} \frac{1}{|\alpha|} X(f/\alpha)} \quad \text{(II)}$$

(i) For $\alpha < 1$, $x(\alpha t)$ will be a compressed signal but $X(f/\alpha)$ will be an expanded version of $X(f)$.

(2) For $\alpha > 1$, $x(\alpha t)$ will be expanded signal in the time domain. But the F.T $X(f/\alpha)$ represents a compressed version of $X(f)$.

(3) Duality or Symmetry Property :→

It states that

if $x(t) \xleftrightarrow{F} X(f)$ then.

$$\boxed{X(t) \xleftrightarrow{F} x(-f)} \quad \text{— (3)}$$

i.e t and f can be interchanged.

The duality theorem tell us that if $x(t) \xleftrightarrow{F} X(-f)$ then the shape of the signal in time domain and the shape of the spectrum can be interchanged.

(4) Time shifting :→ It states that if $x(t)$ and $X(f)$ form a fourier transform pair then

$$\boxed{x(t-t_d) \xleftrightarrow{F} e^{-j2\pi f t} X(f)} \quad \text{— (4)}$$

here the signal $x(t-t_d)$ is a time shifted signal. It is the same signal $x(t)$ only shifted in time.

(5) Frequency Shifting :→ The frequency shifting characteristics states that if $x(t)$ and $X(f)$ form a Fourier transform Pair then

$$\boxed{e^{j2\pi f_c t} x(t) \xleftrightarrow{F} X(f-f_c)} \quad \text{— (5)}$$

Here f_c is a real constant.

(6) Differentiation in time domain :→

Some Processing techniques invalves differentiation and integration of the signal $x(t)$. This Property is applicable if and only if the derivative of $x(t)$ is Fourier transformable.

Let $x(t) \xleftrightarrow{F} X(f)$ and let the derivative of $x(t)$ be fourier transformable. Then,

$$\boxed{\frac{d}{dt} x(t) \xleftrightarrow{F} j2\pi f \, X(f)} \quad \text{—— (6)}$$

That means differentiating the signal in time domain is equivalent to multiplying its F.T by $(j2\pi f)$.

(7) Integration in time domain :→

Integration in time domain is equivalent to dividing the Fourier transform by $(j2\pi f)$.

i.e if $x(t) \xleftrightarrow{F} X(f)$ and Provided that $X(0) = 0$ then

$$\boxed{\int_{-\infty}^{t} x(\lambda)\, d\lambda \xleftrightarrow{F} \frac{1}{2j\pi f} X(f)} \quad \text{—— (7)}$$