The Probability Distribution of a Continuous Random Variable

It focuses on how probabilities are assigned to intervals of continuous values rather than specific values. For instance, if a commuter waits for a bus, the waiting time could be any value between 0 and 30 minutes, making it impractical to assign a meaningful probability to a specific waiting time like exactly 7.211916 minutes

Definition of Probability Distribution

The probability distribution of a continuous random variable X is described using a density function $f(x)$. The probability that X falls within an interval $[a, b]$ is given by the area under the curve of $f(x)$ between a and b.

Key properties of Density functions

1) $f(x) \geq 0$ for all values of x (the graph never goes below the x-axis).
2) The total area under the curve is 1 (total Probability).
3) The probability of an exact value is zero
   $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$
   (End Points do not change the probability of an interval).

Example: Waiting Time for a Bus

- A man arrives randomly at a bus stop. Buses arrive every 30 minutes.
- The waiting time X follows a uniform distribution on $[0, 30]$

The density function is

$$f(x) = \frac{1}{30}, \quad 0 \le x \le 30$$

Finding probability that a Bus arrives in 10 minutes

$$P(0 \le X \le 10) = 10 \times \frac{1}{30} = 1/3$$

∴ The probability that the Bus comes within next 10 minutes is 1/3

Normal Distribution (The Bell Curve)

The Normal distribution describes the behavior of many natural phenomena, such as :-

Heights of people, product weights in manufacturing

The density function (bell curve) is defined as:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where, μ = mean (center of the curve)

σ = Standard deviation (spread of the curve)

Properties of Normal Distribution

1) Symmetric about the mean μ
2) Different values of σ determine the shape:
   smaller σ → Taller and narrower curve
   larger σ → Shorter and wider curve
3) The total area under the curve is 1

Example Height of a 25 year old Men

Heights of 25 years old men follow a normal distribution

Mean μ = 69.75 inches

Standard deviation: σ = 2.59 inches

Finding Probability that a Random Man is taller than 69.75 inches

Since the normal curve is symmetric, the probability that a man is taller than 69.75 inches is

$$P(X > 69.75) = 0.5$$

because half of the total area lies above the mean

Summary

1) Continuous random variables do not have probabilities for exact values, only for intervals.
2) The probability distribution is given by a density function
3) Uniform distributions have constant probabilities within a given range
4) Normal distribution (bell curves) describe many real-world variables, characterized by their mean (μ) and standard deviation (σ)
5) The area under the curve represents probabilities, and for a normal distribution, the total area is 1

Probability Computations for General Normal Random Variables

Key formula

If X is a normally distributed random variable with mean μ and standard deviation σ, then the probability of the dev

$p(a < X < b)$ is computed as

$$P(a<X<b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

where Z is the standard normal variable

- a and b can be any decimal number or extend to ~~$-\infty$ or~~ $-\infty$ or ∞
- The values $\frac{a-\mu}{\sigma}$ and $\frac{b-\mu}{\sigma}$ are called Zscores
- The probability is determined using the standard normal table (or cumulative probability chart)

Example: compute $P(8 < X < 14)$:

$$P(8 < X < 14) = P\left(\frac{8-10}{2.5} < Z < \frac{14-10}{2.5}\right)$$

$$P(-0.80 \le Z < 1.60) = P(Z<1.60) - P(Z<-0.8)$$

using normal table
$$= 0.9452 - 0.2119$$
$$= 0.7333$$

Example: College Entrance Exam Scores;

Given

- CEE scores follow $X \sim N(510, 60)$
- A selective university considers only applicants with scores above 650
- Find the percentage of test-takers meeting the university's requirement.

Solution

$$P(X > 650) = P\left(Z > \frac{650-510}{60}\right) = P(Z > 2.33)$$

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using the normal table: $P(Z < 2.33) = 0.9901$

$P(Z > 2.33) = 1 - 0.9901$

$= 0.0099$

Thus, only 0.99% or (about 1%) of test takers qualify for admission

Key Takeaways

1) General normal probabilities are computed by converting x-values into z-scores: $Z = \frac{X - \mu}{\sigma}$

2) Use the standard normal table to find cumulative probabilities
3) Subtract cumulative probabilities to compute interval probabilities
4) for upper/lower tail probabilities, subtract from 1.