MAEG 5720: Computer Vision in Practice

Lecture 10:

Projective Geometry

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Camera Image

Size preserving?

lines remains lines?

Angle preserving?

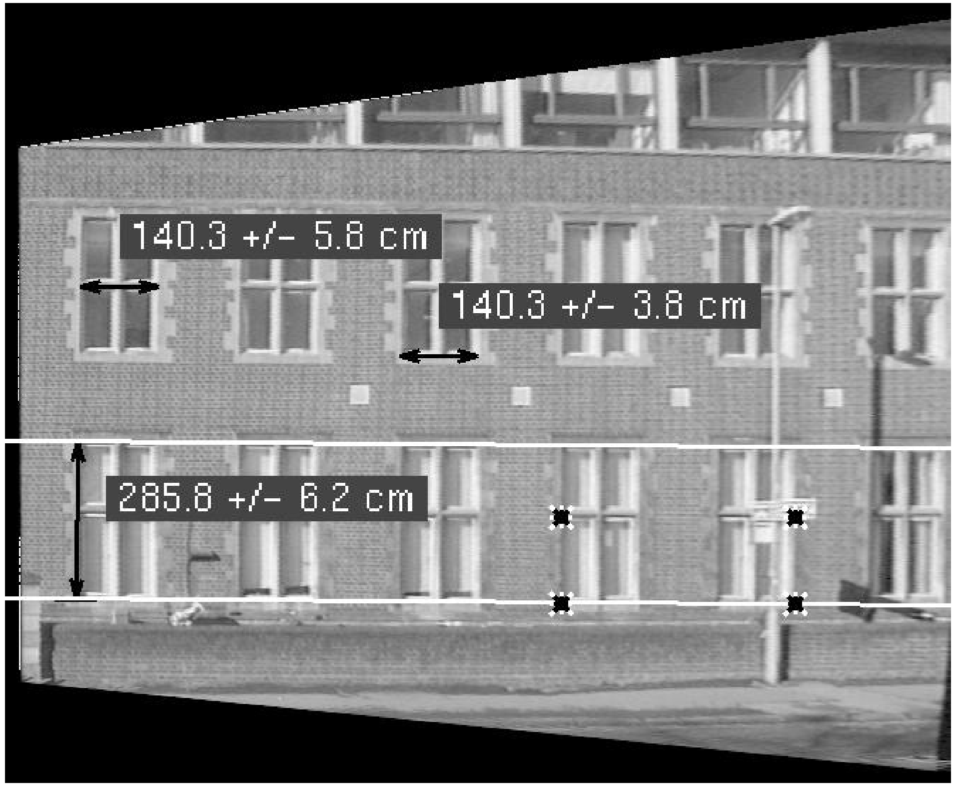
- What property you observe in the image?

Image: http://www.pixels.com

Camera properties

- Properties:
  - Line-preserving
    - Straight lines map to straight lines
  - Length-Not-preserving
    - Size of objects inversely proportional to distance
  - Angle-Not-preserving
    - Angle between lines changes
  - Circle is-Not-Circle

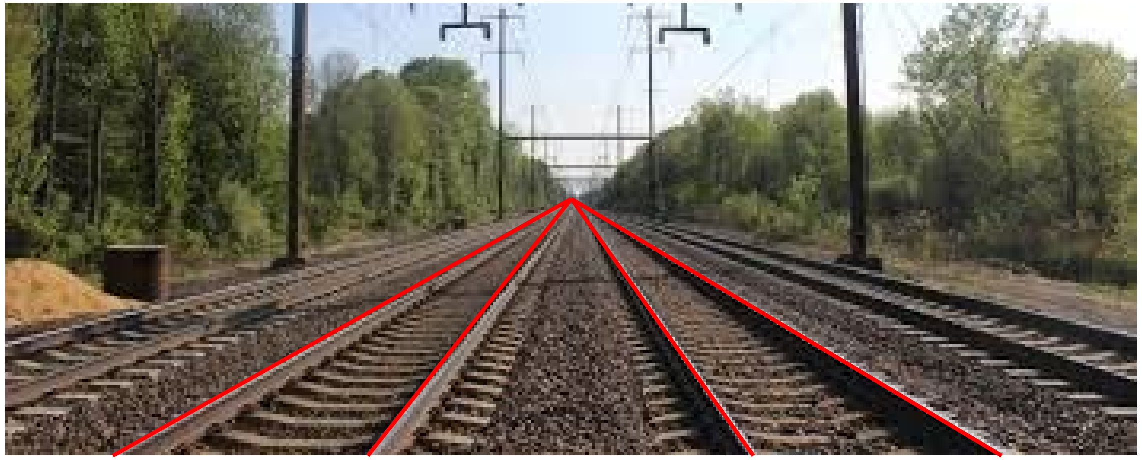
Usage: Image Rectification for measurement



Another important property

- In projective image
  - Parallel lines are *not* parallel anymore
  - Parallel lines intercept at a *vanishing point*
  - Vanishing point is "*point at infinity*" for parallel lines
  - *Every direction* has it's *vanishing point*

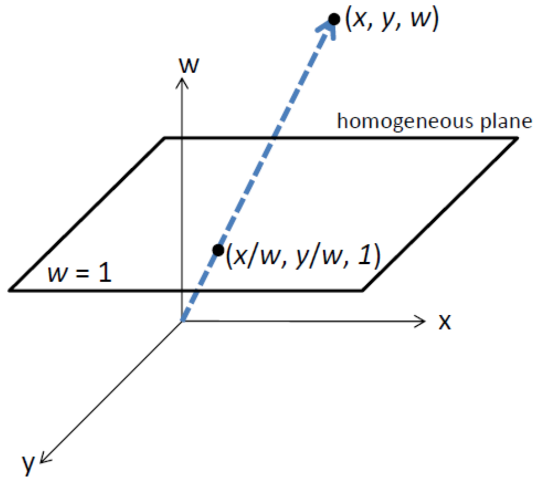
- How can we describe "points at infinity"?
  - Can we represent in Euclidean Coordinates



Parallel Lines intercept at "point at infinity"

Homogenous Coordinates

- *Homogenous coordinates* are a system used in *projective geometry*
- A single transformation matrix can be used to represent *affine* and *projective* transformations
- *Points at infinity* can be represented using *finite coordinates*
- A single matrix can represent *affine* and *projective transformations*



Notations for Homogenous Representation

- Name and basic geometric entities

element	2D	3D
planes		$\mathcal{A}$, $\mathcal{B}$, ...
lines	l, m, ...	$\mathcal{L}$, $\mathcal{M}$, ...
points	x, y, ...	$\mathcal{X}$, $\mathcal{Y}$, ...

- Notation for in homogenous and homogenous vectors and matrices

	2D	3D	transformations
inhomogeneous	$\boldsymbol{x}$	$\boldsymbol{X}$	R
homogeneous	$\mathbf{l}$, $\mathbf{x}$	$\mathbf{A}$, $\mathbf{L}$, $\mathbf{X}$	H

Source: Fronster

Homogenous Coordinates

- Definition

The representation of x of a geometric object is homogenous if

$$\mathbf{x} = \lambda\mathbf{x} \qquad \lambda \neq 0$$

Example

$$\mathbf{x} = \lambda\mathbf{x} \qquad\qquad \boldsymbol{x} \neq \lambda\boldsymbol{x}$$

Homogenous   Euclidian

Homogenous Coordinates

- *Homogenous Coordinates* add one extra dimension to the vector.
- i.e. use n+1 dimensional vector to represents n dimensions point

- Transforming Euclidean coordinates to Homogenous Coordinates

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix} = \begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix}$$

$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ in Euclidean coordinates

Homogenous coordinates

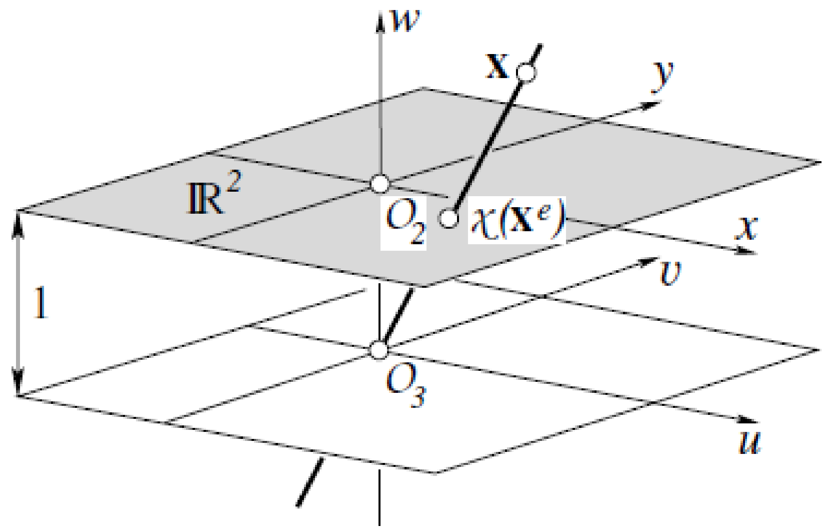
- *Homogeneous Coordinates* of a point x in the plane in $\mathbb{R}^2$ is a 3-dimensional vector

x: $\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ where

$$u^2 + v^2 + w^2 \neq 0$$

- In Euclidian coordinates

x: $\boldsymbol{x} = \begin{bmatrix} u/w \\ v/w \end{bmatrix}$ where $w \neq 0$



Example

- All point χ of Euclidian plane $\mathbb{R}^2$

$$\boldsymbol{x} = [x, y]^T$$

- Represented in homogenous coordinates

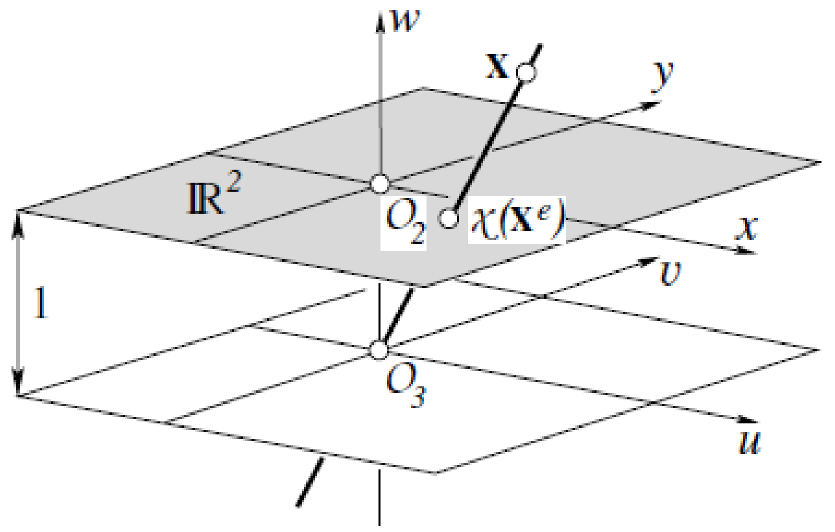
$$\mathbf{x} = [x, y, 1]^T$$

How about point at infinity?

$$\mathrm{x} = [x, y, 0]^T$$

except

$$\mathrm{x} = [0,0,1]^T$$



Example

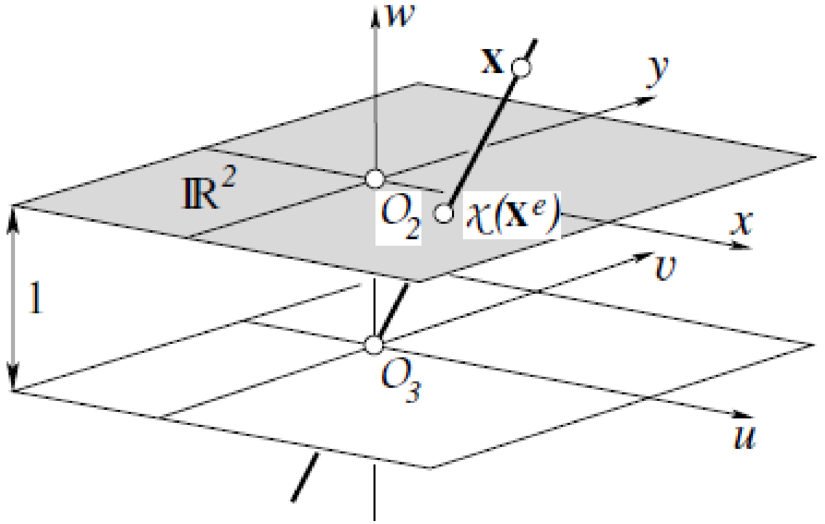
Point **x** in Homogenous Coordinates

$$\mathbf{x} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in Euclidean Coordinate

In General:

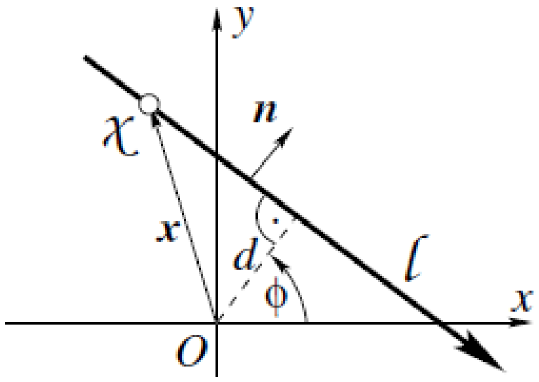
$$(x, y, 1) = (kx, ky, k)$$



Representation of lines in Homogenous coordinates?

- Hessian normal form

$$x cos\emptyset + y sin\emptyset - d = 0$$



- Intercept form

$$\frac{x}{x_0} + \frac{y}{y_0} = 1 \text{ or } \qquad \frac{x}{x_0} + \frac{y}{y_0} - 1 = 0$$

- Standard form

$$ax + by + c = 0$$

Representation of lines in Homogenous coordinates?

- Hessian normal form

$$x cos\emptyset + y sin\emptyset - d = 0$$

$$\begin{bmatrix} cos\emptyset & sin\emptyset & -d \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

- Intercept form

$$\frac{x}{x_0} + \frac{y}{y_0} = 1 \text{ or } \quad \frac{x}{x_0} + \frac{y}{y_0} - 1 = 0$$

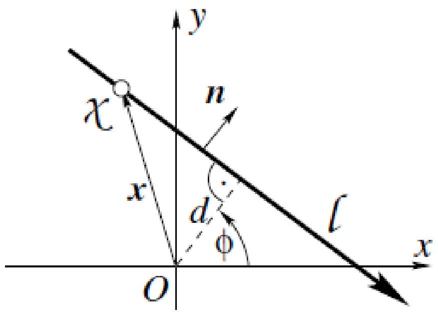
$$\begin{bmatrix} \frac{1}{x_0} & \frac{1}{y_0} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

- Standard form

$$ax + by + c = 0$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

Representations of Lines in H.C.



Point $\mathrm{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

- Hesse: $\mathrm{l} = \begin{bmatrix} cos\emptyset \\ sin\emptyset \\ -d \end{bmatrix}$

Intercept: $\mathrm{l} = \begin{bmatrix} \frac{1}{x_0} \\ \frac{1}{y_0} \\ -1 \end{bmatrix}$

- standard: $\mathrm{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Representation of Line

- Homogenous Coordinate of a line L in the plane is represented as 3-dimensional vector

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \quad \text{with } |L|^2 = l_1^2 + l_2^2 + l_3^2 \neq 0$$

This corresponds to Euclidian representation

$$l_1 x + l_2 y + l_3 = 0$$

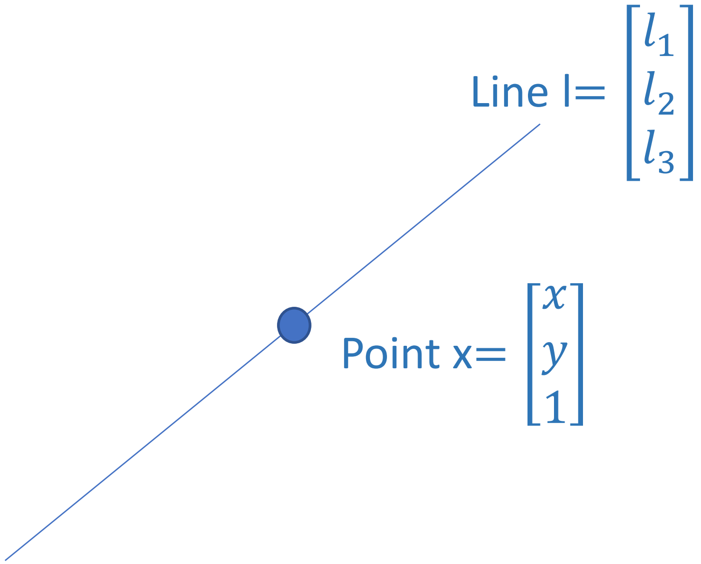
If a point lies on a Line

- A Point

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Lies on a line

$$\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$$



If

$$x \cdot L = 0$$

If a point x lies on a line l, then $x^{T} \cdot l = 0$

Intersecting of Lines

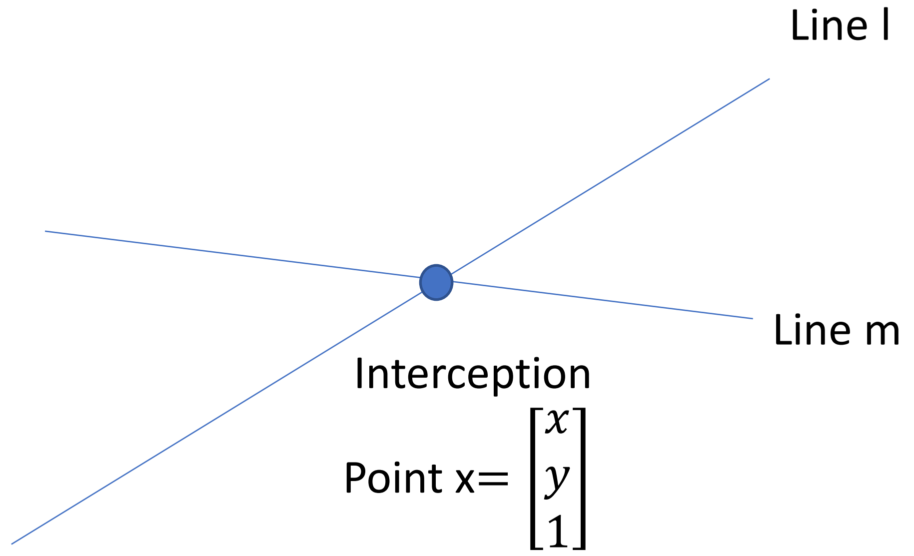
- Given two lines l, m expressed in homogenous coordinates, how can we find the intersection **x**=l∩m

- **x**= $[x, y]^T$ is the interception point, x must lies on both line l, m. Then we have the linear system

$$\begin{bmatrix} \mathrm{l} \cdot \mathrm{x} \\ \mathrm{m} \cdot \mathrm{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}$$

- We can solve this by Cramer's rule!



Cramer's Rule

- Considering a system

$$Ax = b$$

  - Where A is *nxn* matrix and $\det(\mathrm{A}) \neq 0$
  - vector $x = (x_1 \dots x_n)^{T}$

$$x_i = \frac{\det(A_i)}{\det(A)}, i = 1, \dots n$$

  - Where $A_i$is the matrix formed by replacing the *i-th* column of A by the column of b

Example for Cramer's Rule

- Consider the linear system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

- In Matrix form

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - b_1a_2}$$

Intersecting Lines

- We want to find solutions of

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}$$ Then

By Cramer's rule

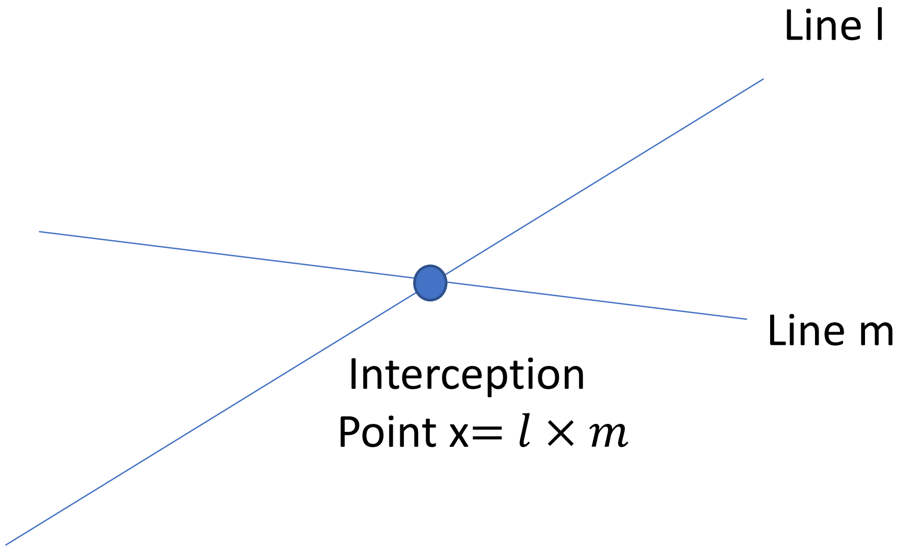
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} l_2 m_3 - l_3 m_2 \\ l_3 m_1 - l_1 m_3 \\ l_1 m_2 - l_2 m_1 \end{bmatrix}$$ Let $$\begin{aligned} D_1 &= l_2 m_3 - l_3 m_2 \\ D_2 &= l_3 m_1 - l_1 m_3 \\ D_3 &= l_1 m_2 - l_2 m_1 \end{aligned}$$

$$x = \frac{D_1}{D_3} \qquad y = \frac{D_2}{D_3}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{D_3} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

In vector form

$$l \times m = D = \frac{1}{D_3} D = \mathrm{x}$$



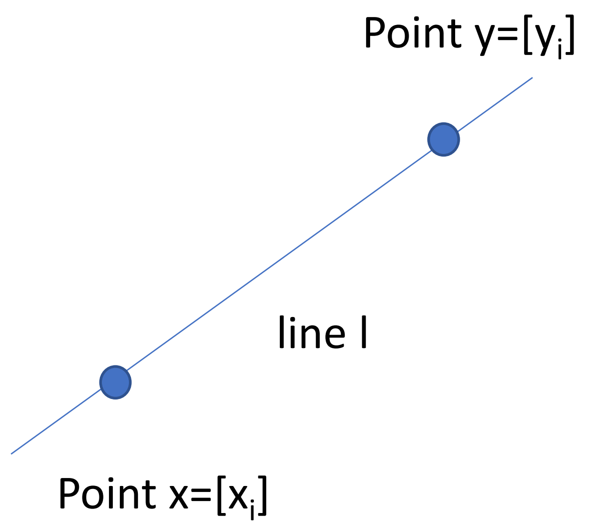
Interception of two lines x= $\mathrm{l} \times m$

Line passing through two points

- Given two points x=[x_i], y=[y_i], find the line l=[l_i]
- The point x and y must lie on Line l
- Then we have

$$\begin{bmatrix} \mathrm{x} \cdot \mathrm{l} \\ \mathrm{y} \cdot \mathrm{l} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -x_3 l_3 \\ -y_3 y_3 \end{bmatrix}$$



Line passing through two points

- Cramer's Rule again

$$l_1 = \frac{D_1}{D_3} \qquad l_2 = \frac{D_2}{D_3} \qquad \begin{aligned} D_1 &= l_3(\,x_2y_3 - x_3y_2) \\ D_2 &= l_3(x_3y_1 - x_1y_3) \\ D_3 &= (x_1y_2 - x_2y_1) \end{aligned}$$

Putting into l

$$\mathrm{l} = \begin{bmatrix} \frac{D_1}{D_3} & \frac{D_2}{D_3} & l_3\frac{D_3}{D_3} \end{bmatrix}^T \rightarrow \mathrm{l} = \frac{l_3}{D_3}\begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

Line passing through two points

- We again have the cross product

$$\mathrm{n} = \mathrm{x} \times \mathrm{y}$$

- We have

$$\mathrm{l} = \frac{l_3}{D_3}\begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} = \frac{l_3}{D_3}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \mathrm{x} \times \mathrm{y}$$

- $\mathrm{l} = \mathrm{x} \times \mathrm{y}$

If line l pass through x and y, then $\mathrm{l} = \mathrm{x} \times \mathrm{y}$

Summary

- A point lies on a line if

$$x^T \cdot \mathrm{l} = 0$$

- Two lines l and m intercept x

$$x = \mathrm{l} \times m$$

- Lines l Passing through two points x and y

$$\mathrm{l} = \mathrm{x} \times \mathrm{y}$$

Points at Infinity a.k.a Ideal Points

- Point at infinity with Euclidean direction vector $[u, v]^T$

$$\lim_{w \to 0} \begin{bmatrix} u/w \\ v/w \end{bmatrix}$$

- Possible to model infinitively distance points with ***finite coordinates***

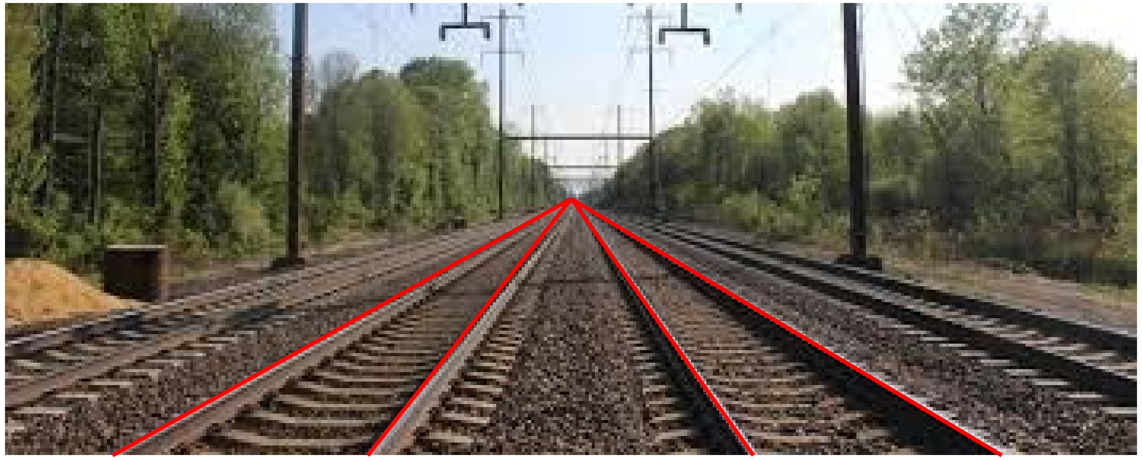
$$\boldsymbol{x}_\infty = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{0} \end{bmatrix}$$

- The direction of line u,v to infinity is maintained

Intersection of parallel lines at Point of Infinity

- All line l with $l \cdot x_\infty = 0$ pass through

$$x_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$



- i.e.
  - $l \cdot x_\infty = [\cos\emptyset, sin\emptyset\ , d] \cdot [u, v, 0]^T = 0$

- Therefore

- Any line $\mathrm{m} = [\cos\emptyset, sin\emptyset, a\ ]$ parallel to l pass through x_∞

Example

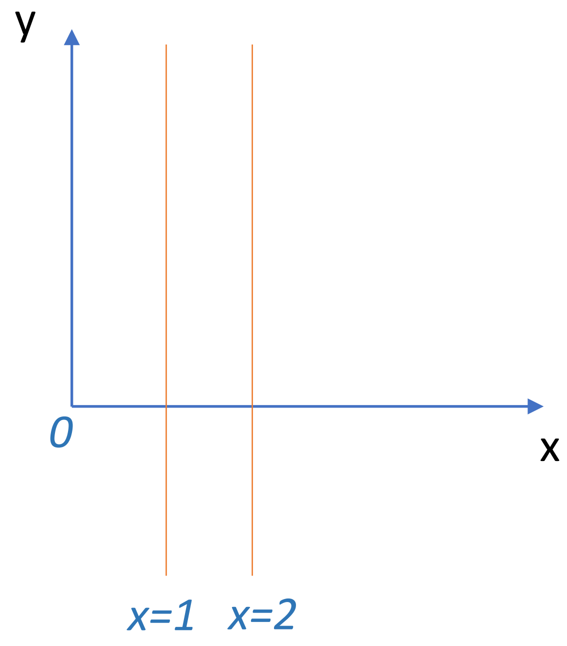
- Consider *two lines x=1 and x=2*

- In homogeneous notation

$$\mathrm{l} = (-1,0,1)^T \;\&\; \mathrm{l}' = (-1,0,2)^T$$

$$\mathrm{x} = \mathrm{l} \times \mathrm{l}' = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ -1 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Which is the point at infinity in the direction of y-axis

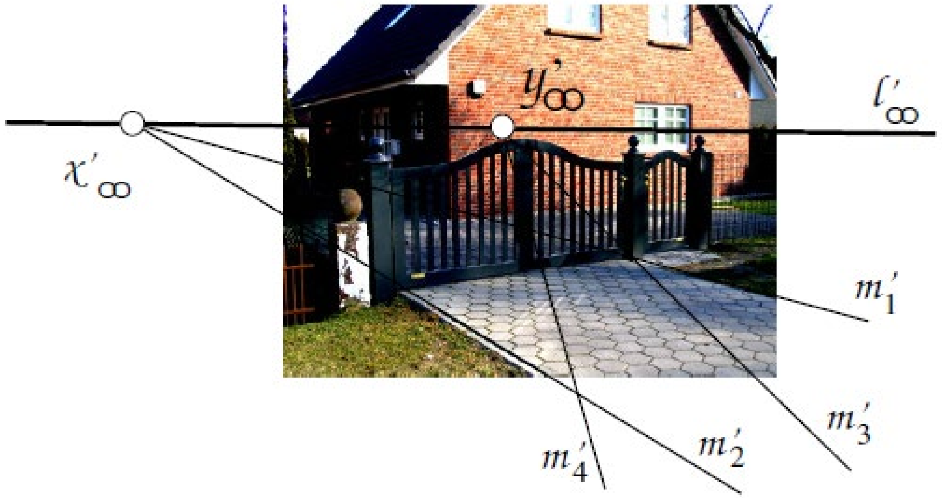


Line at Infinity

- The line at infinity is the ideal line

$$l_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Where all *points of infinity* intercept. This is also called the *horizon.*



Line at infinity

- All points at infinity meet the line of infinity, we have

$$\mathrm{x}_\infty \cdot l_\infty = 0$$

$$[u \quad v \quad 0] \cdot [0 \quad 0 \quad 1] = 0$$

- All x_∞ lies on $l_\infty \ = [0 \quad 0 \quad 1]^T$

A Model of Projective Plane

- Points and Lines in $\mathbb{P}^2$ are represented by rays and planes respectively, through the origins in $\mathbb{R}^3$

- Lines lying in $x_1x_2 - plane$ represent *ideal points*, and

- $x_1x_2 - plane$ represents *line at infinity* l_∞

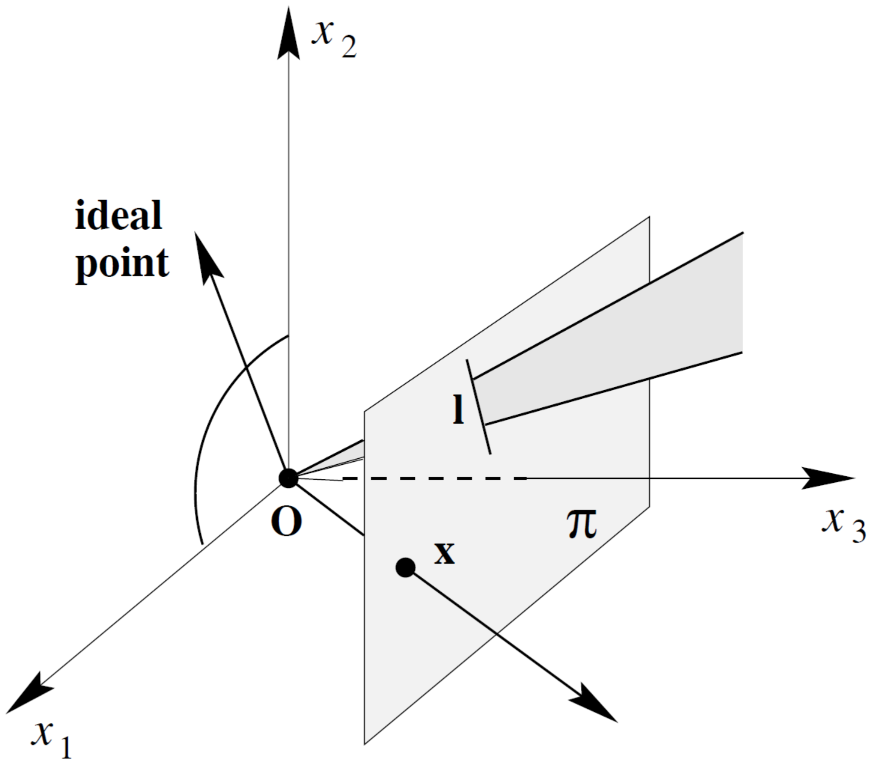


Image Credit: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

Duality principle

- The role of points and lines may be *interchanged* in the statement
- Line passing through a point can be expressed as

$$\mathrm{x}^{\mathrm{T}}\mathrm{l} = 0 \Rightarrow \mathrm{l}^{\mathrm{T}}\mathrm{x} = 0$$

- *Dual Principle*: To any theorem of 2-dimensional projective geometry, there corresponds a *dual theorem* which maybe derived by *interchanging the roles of points and lines* in the original theorem

- For example: A line passing through 2 points is dual to a point is intersection of 2 lines

$\mathrm{l} = \mathrm{x} \times \mathrm{y}$ dual to $\mathrm{x} = \mathrm{l}_1 \times \mathrm{l}_2$

Conics

- A *Conic* is a curve described by a second-degree equations in a plane
- There are three main types of conics, *parabola*, *ellipse*(including circle) and hyperbola

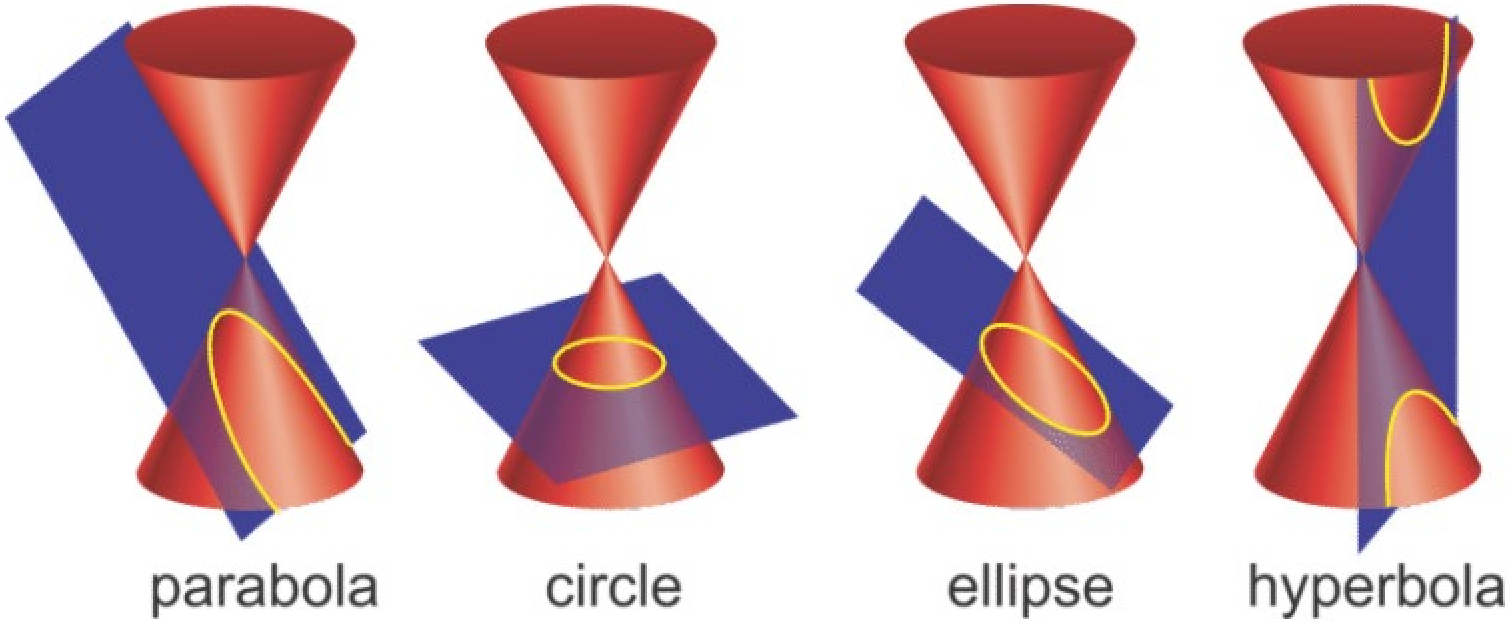


Image Credit: Conic Sections | CK-12 Foundation

Degenerated Conics

- Degenerated conics arises from planes which contain the vertex

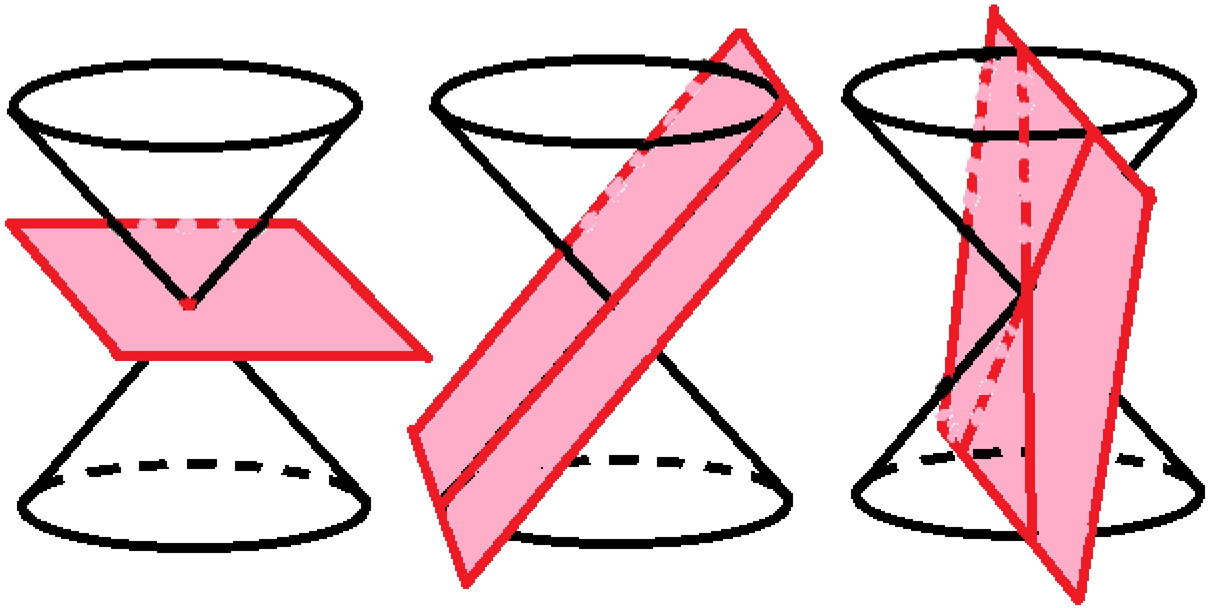


Image Credit: The line at infinity: conics - Making Your Own Sense (adelaide.edu.au)

Equation of Conics

- The equation of a conic in inhomogeneous coordinates is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

- Putting into homogenous coordinates, replacing $x \mapsto x_1/x_3$ and $y \mapsto x_2/x_3$,the equation becomes

$$a{x_1}^2 + bx_1x_2 + c{x_2}^2 + dx_1x_3 + ex_2x_3 + fx_3 = 0$$

- In matrix form

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ or } \mathrm{x}^{\mathrm{T}}\mathrm{C}\mathrm{x} = 0 \text{ where } \mathrm{C} = \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix}$$

- Note: conic coefficient matrix is symmetric. Since C is *homogenous*. It has *five degree of freedom*

Defining a Conic

- Suppose we wish to compute the conic which passes through a set of points, x_i, then

$$a{x_i}^2 + bx_iy_i + c{y_i}^2 + dx_i + ey_i + f = 0$$

The constraint can be written as

$$({x_i}^2 + x_iy_i + {y_i}^2 + x_i + y_i + 1)\mathbf{c} = 0$$

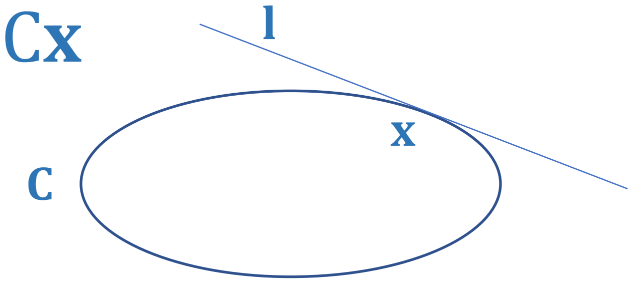
Where $\mathbf{c} = (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f})^{\mathrm{T}}$ and stacking the constraints

$$\begin{bmatrix} {x_1}^2 + x_1y_1 + {y_1}^2 + x_1 + y_1 + 1 \\ {x_2}^2 + x_2y_2 + {y_2}^2 + x_2 + y_2 + 1 \\ {x_3}^2 + x_3y_3 + {y_3}^2 + x_3 + y_3 + 1 \\ {x_4}^2 + x_4y_4 + {y_4}^2 + x_4 + y_4 + 1 \\ {x_5}^2 + x_5y_5 + {y_5}^2 + x_5 + y_5 + 1 \end{bmatrix} \mathbf{c} = 0$$

Conic is the null vector of 5x6 matrix, it can be uniquely(up to scale) defined by five points.

Tangent lines to conics

- The line l *tangent* to C at a point $\mathbf{x}$ on C is given by $\mathbf{l} = \mathrm{C}\mathbf{x}$



- Proof: The line $\mathbf{l} = \mathrm{C}\mathbf{x}$ passes through $\mathbf{x}$

  The point $\mathbf{x}$ on C

$\mathbf{l}^{\mathbf{T}}\mathbf{x}=0$ **and** $\mathbf{x}^{\mathbf{T}}\mathbf{C}\mathbf{x} = 0$

For single contact point, we have $\mathbf{l} = \mathrm{C}\mathbf{x}$

Tangent to conics (Cont'd)

- Suppose that **l** meets another point **y** on C, then

$$\mathbf{y}^{\mathrm{T}}\mathbf{C}\mathbf{y} = 0 \text{ and } \mathbf{l}^{\mathrm{T}}\mathbf{y}=0$$

- Since $\mathbf{l} = \mathrm{C}\mathbf{x}$ is also true, we have $\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{y} = 0$

- From this it follows that $(\mathbf{x} + \alpha\mathbf{y})^{\mathrm{T}}\mathbf{C}(\mathbf{x} + \alpha\mathbf{y}) = 0$ for all α, which means that the *whole line* $\mathbf{l} = \mathrm{C}\mathbf{x}$ joining $\mathbf{x}, \mathbf{y}$ *lies on* the conic C which is *degenerated*.

Dual Conics

- The conic C is defined using points is termed a *point* conic. By the *Duality principle*, we can also define *conics* with *lines*.

- This *dual*(or *line*) *conic* is represented by a 3x3 matrix denoted as C^*. A line l *tangent* to the conic C satisfies $\mathrm{l}^{\mathrm{T}}\mathrm{C}^*\mathrm{l} = 0$

- For a non-singular symmetric matrix $\mathrm{C}^* = \mathrm{C}^{-1}$

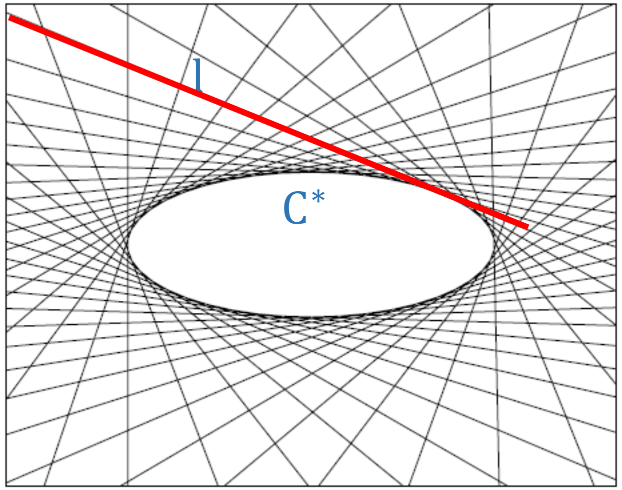


Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Dual Conics

- Proof: In case C has full rank. At a point $\mathbf{x}$ on C the tangent is $\mathbf{l} = \mathrm{C}\mathbf{x}$. Inverting, we have $\mathbf{x} = \mathrm{C}^{-1}\mathbf{l}$, Since $\mathrm{x}^{\mathrm{T}}\mathrm{C}\mathrm{x}$, we have

  $(\mathrm{C}^{-1}\mathbf{l})^{\mathrm{T}}\mathrm{C}(\mathrm{C}^{-1}\mathbf{l}) = 0$ hence $\mathbf{l}^{\mathrm{T}}\mathrm{C}^{-\mathrm{T}}\mathrm{C}\mathrm{C}^{-1}\mathbf{l} = 0$

- Therefore

  $\mathbf{l}^{\mathrm{T}}\mathrm{C}^{-\mathrm{T}}\mathbf{l} = 0$

- As C is symmetric, $\mathrm{C}^{-\mathrm{T}} = \mathrm{C}^{-1} = \mathrm{C}^{*}$
- $\mathbf{l}^{\mathrm{T}}\mathrm{C}^{*}\mathbf{l} = 0$

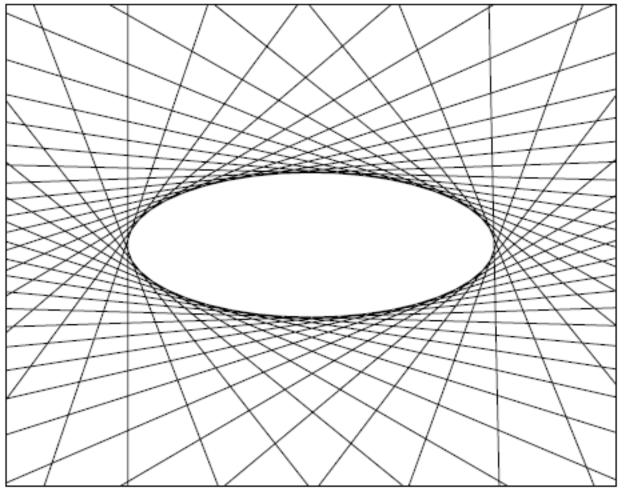
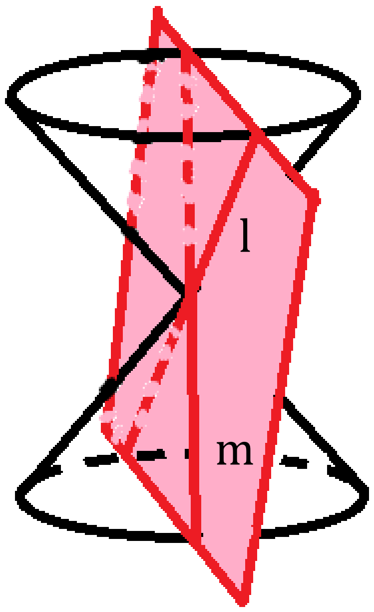


Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

The Degenerated Conics

- The conic $\mathrm{C} = \mathrm{l}\mathrm{m}^{\mathrm{T}} + \mathrm{m}\mathrm{l}^{\mathrm{T}}$ is composed of two lines l and m. Points on l satisfy $\mathrm{l}^{\mathrm{T}}\mathrm{x} = 0$ and are on the conic since

$$\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} = (\mathrm{x}^{\mathrm{T}}\mathrm{l})(\mathrm{m}^{\mathrm{T}}\mathrm{x}) + (\mathrm{x}^{\mathrm{T}}\mathrm{m})(\mathrm{l}^{\mathrm{T}}\mathrm{x}) = 0$$



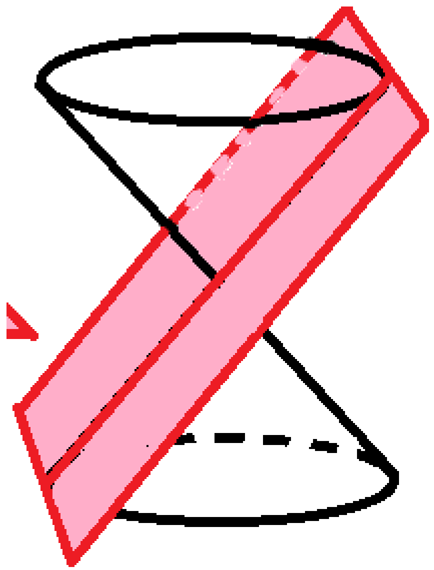
C is rank 2, this consist of two straight lines

$$\mathrm{x}^{\mathrm{T}}\mathrm{l} = x_1 l_1 + x_2 l_2 + l_3$$
$$\mathrm{x}^{\mathrm{T}}\mathrm{l} = x_1 m_1 + x_2 m_2 + m_3$$

The Degenerated Conics

- The conic $\mathrm{C} = \mathrm{l}\mathrm{l}^\mathrm{T} + \mathrm{l}\mathrm{l}^\mathrm{T}$ is composed of one line l. Points on l satisfy $\mathrm{l}^\mathrm{T}\mathrm{x} = 0$ and are on the conic since

$$\mathbf{x}^\mathbf{T}\mathbf{C}\mathbf{x} = (\mathrm{x}^\mathrm{T}\mathrm{l})(\mathrm{l}^\mathrm{T}\mathrm{x}) + (\mathrm{x}^\mathrm{T}\mathrm{l})(\mathrm{l}^\mathrm{T}\mathrm{x}) = 0$$



C is rank 1, this consist of one straight lines

$$\mathrm{x}^\mathrm{T}\mathrm{l} = x_1 l_1 + x_2 l_2 + l_3$$

Projective Transformation

- *Projective geometry* is the study of *properties of the projective plane*

$\mathbb{P}^2$ that are invariant under a group of transformations known as *projectivities*

- A *projectivities* is an i*nvertible* mapping from points in $\mathbb{P}^2$ (that is homogeneous 3-vectors) to points in $\mathbb{P}^2$ that maps line to lines.

- A *projectivities* is an i*nvertible* mapping h *from* $\mathbb{P}^2$ to itself such that three points $\mathrm{x}_1, \mathrm{x}_2$ and x_3 lie on the *same line* if and only if $h(\mathrm{x}_1)$, $h(\mathrm{x}_2)$ and $h(\mathrm{x}_3)$ do.

- *Inverse* of a projectivities and *composition* of two projectivities are also called projectivity, also *collineation*, a *projective transformation* or a *homography*.

Collineation

- A *projectivities* is an *invertible* mapping h *from* $\mathbb{P}^2$ to itself such that three points $\mathrm{x}_1, \mathrm{x}_2$ and x_3 lie on the *same line* if and only if $h(\mathrm{x}_1)$, $h(\mathrm{x}_2)$ and $h(\mathrm{x}_3)$ do.

- Let $\mathrm{x}_1, \mathrm{x}_2$ and x_3 lie on a line l, thus $\mathrm{l}^{\mathrm{T}}\mathrm{x}_i = 0$ for i=1,…3. Let H be a non-singular 3x3 matrix, we have $\mathrm{l}^{\mathrm{T}}\mathrm{H}^{-1}\mathrm{H}\mathrm{x}_i = 0$. Thus the points $\mathrm{H}\mathrm{x}_i$ lies on the line $\mathrm{H}^{-\mathrm{T}}\mathrm{l}$,

- The *collinearity* is preserved by the transformation.

Projective transformation

- A *planar projective transformation* is a linear transformation on homogeneous 3-vectors represented by a *non-singular 3x3 matrix*

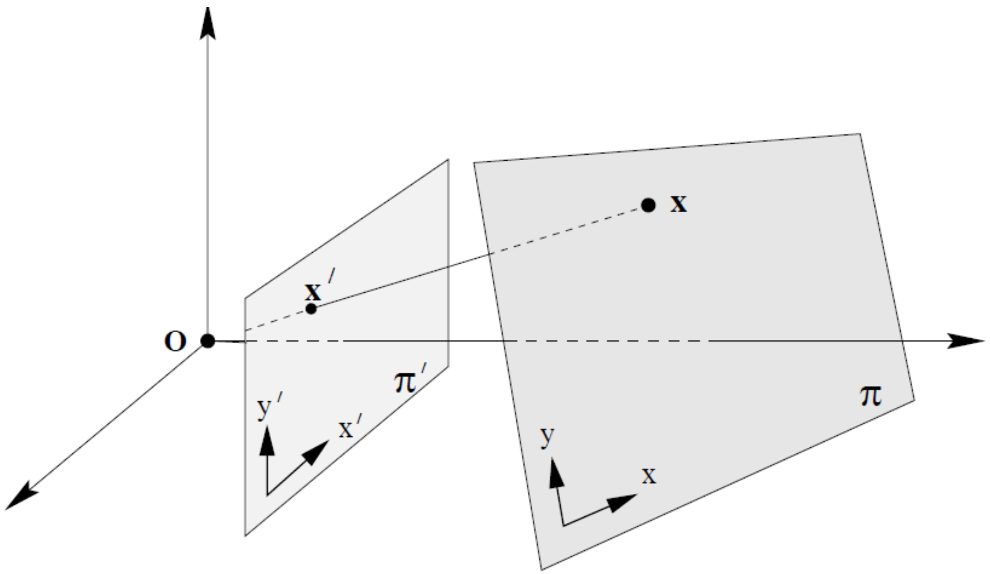
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Or more briefly, $\mathrm{x}' = \mathrm{Hx}$

- Note that the matrix H may be changed by multiplication by an arbitrary non-zero scale factor without altering the projective transformation.

- H is a *homogenous matrix*. There are eight independent ratios amongst the nine elements. *Projective Transformation has 8-DOF.*

Mapping points on one plane to another plane



- The *central projection* maps points on one plane to points on another plane.
- Since lines are mapped to lines, *central projection* is a *projectivity* and may be represented by a linear mapping of homogenous coordinates $\mathrm{x}' = \mathrm{H}\mathrm{x}$.

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Application of homography

- *Removing the projective distortion* from a perspective image of a plane

- The original image with perspective distortion – the lines of windows clearly converge at a finite point. The image is related via a *projective transformation* to the true geometry.
- Synthesized frontal view – the *inverse of the transformation* is computed by mapping the four image window corners to corners of an appropriate sized rectangle.

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

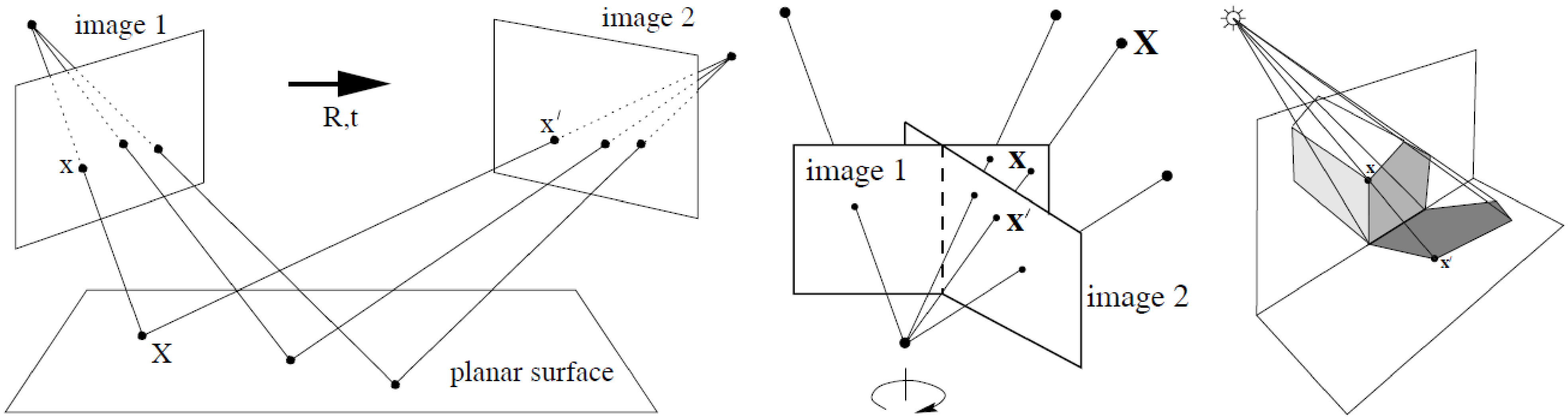
Removing the perspective distortion

- Let the inhomogeneous coordinates of a pair of matching points x and x' in the world and image plane be (x, y) and (x', y') respectively.
- The projective transformation can be written in inhomogeneous form.

$$x' = \frac{x_1'}{x_3'} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x_2'}{x_3'} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

- Each point correspondence generates two equations for elements of H, therefore *4 points are sufficient* to solve for H up to insignificant multiplicative factor.
- Note: No three points are collinear

Examples of Projective transformation



- The transformation $\mathrm{x}' = \mathrm{Hx}$ examples
  - Two images induced by a world plane(concatenation of two projective transformations is a projective transformation)
  - Two images with same camera centres
  - Image of a plane and image of its shadow onto another plane.

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Transformations of lines

- From the collineation example, if points x_i lie on a line l, then $\mathrm{x}_i' = \mathrm{H}\mathrm{x}_i$ lies on the line $\mathrm{l}' = \mathrm{H}^{-\mathrm{T}}\mathrm{l}$. This gives $\mathrm{l}^{\mathrm{T}'}\mathrm{x}_i' = \mathrm{l}^{\mathrm{T}}\mathrm{H}^{-1}\mathrm{H}\mathrm{x}_i = 0$ and

$\mathrm{l}' = \mathrm{H}^{-\mathrm{T}}\mathrm{l}$ or $\mathrm{l}'^{\mathrm{T}} = \mathrm{l}^{\mathrm{T}}\mathrm{H}^{-1}$

- fundamentally different way in *point* and *lines transformation*
  - *Point transforms according to* H
  - *Line transforms according to* H^{-1}

Transformation of conics

- Under a point transformation $\mathrm{x}' = \mathrm{Hx}$. The conic is transformed as follow:

$$\begin{aligned}\mathrm{x}^{\mathrm{T}}\mathrm{Cx} &= \mathrm{x}'^{\mathrm{T}}[\mathrm{H}^{-1}]^{\mathrm{T}}\mathrm{CH}^{-1}\mathrm{x}' \\ &= \mathrm{x}'^{\mathrm{T}}\mathrm{H}^{-\mathrm{T}}\mathrm{CH}^{-1}\mathrm{x}' \\ &= \mathrm{x}'^{\mathrm{T}}\mathrm{C}'\mathrm{x}' \text{ where } \mathrm{C}' = \mathrm{H}^{-\mathrm{T}}\mathrm{CH}^{-1}\end{aligned}$$

- This gives the transformation of *conic*: Under a point transformation $\mathrm{x}' = \mathrm{Hx}$, a *conic* C transforms to $\mathrm{C}' = \mathrm{H}^{-\mathrm{T}}\mathrm{CH}^{-1}$
- Also, under the same point transformation $\mathrm{x}' = \mathrm{Hx}$, the *dual conic* C^*, the transforms to $\mathrm{C}^* = \mathrm{HC}^*\mathrm{H}^{-\mathrm{T}}$

A Hierarchy of Transformation

- Distortions arising under different transformations

Similarity

Affine

Projective

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Class I: Isometries

- *Isometries* are transformation of the plane $\mathbb{R}^2$ that preserve the *Euclidean distance*. An isometry is represented as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- where $\epsilon = \pm 1$

- In block form

$$\mathrm{x}' = \mathrm{H_E}\mathrm{x} = \begin{bmatrix} \mathrm{R} & \mathrm{t} \\ 0^\mathrm{T} & 1 \end{bmatrix} \mathrm{x}$$

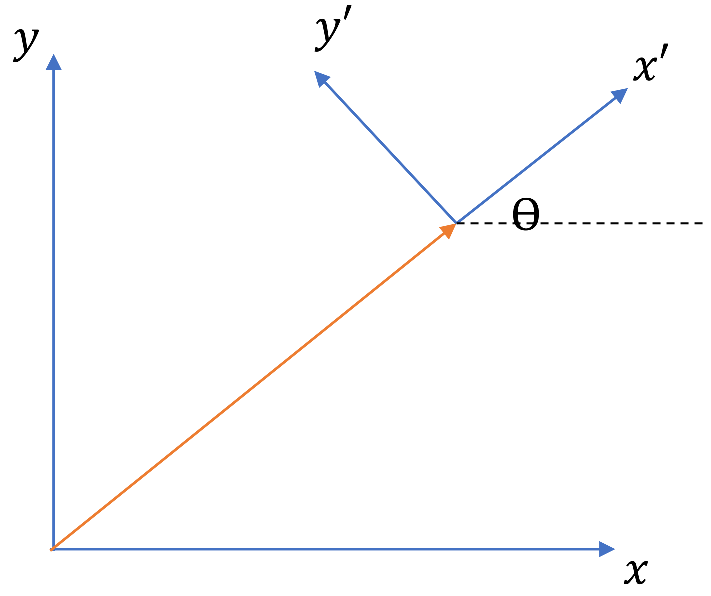
where R is 2x2 rotation matrix such at $\mathrm{R^T R} = \mathrm{R R^T} = \mathrm{I}$

Class I: Isometries

- When $\epsilon = 1$, the isometry is *orientation-preserving* and is a *Euclidean transformation* which compose of a *translation* and *rotation*.

- When $\epsilon = -1$, the isometry is *reverse-orientation* and is compose of *reflection* which compose of a *translation* and *rotation*.

- Invariant: *angle*, *length* and *area*



Class II: Similarity

- A *similarity transformation* (or simply *similarity)* is an *isometry* composed with *isotropic scaling.* In the case of a Euclidean transformation composed with a scaling(i.e. no reflection) the similarity is represented as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- In block form

$$\mathrm{x}' = \mathrm{H_s}\mathrm{x} = \begin{bmatrix} \mathrm{sR} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix} \mathrm{x}$$

Class II: Similarity

- A similarity transformation is also known as an *equi-form transformation*, because it preserves ‘Shape’

- A planar similarity transformation has *four degrees of freedom* – rotation, translation and also scaling

- Invariant: *Angles* between lines, *ratio* of two lengths and ratio or areas are invariant.

Class III: Affine transformations

- An *affine transformation* (or simply *affinity*) is a non-singular linear transformation flowed by a translation. It has the matrix representation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Or in block form

$$\mathrm{x}' = \mathrm{H_A x} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix} \mathrm{x}$$

Class III: Affine transformations

- A affine matrix A can always be decomposed as

$$A = R(\theta)R(-\emptyset)DR(\emptyset)$$

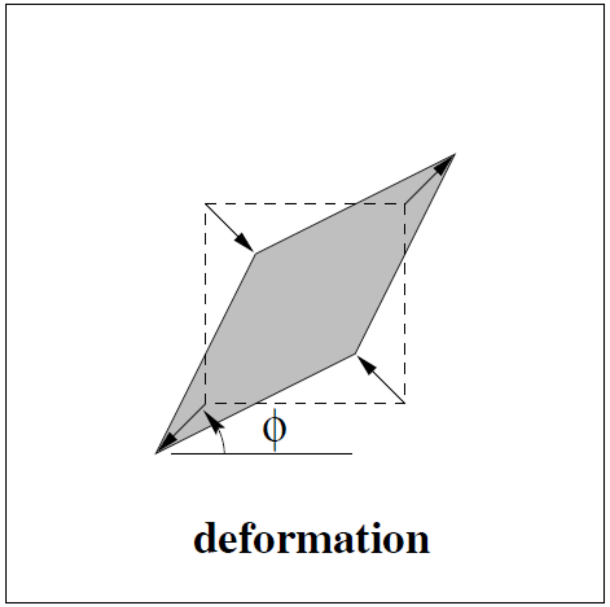
- Where $R(\theta)$ and $R(\emptyset)$ are rotations by θ and $\emptyset$ respectively. D is a diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Class III: Affine transformations

- Steps for Affine transformation
  - Rotate by $\emptyset$
  - Stretch by λ_1 and λ_2
  - Rotate back by $-\emptyset$
  - Rotate finally another rotation θ

- Invariant: *Parallel lines*, *Ratio of Area*



- An Affinity is *orientation-preserving* or *reversing* according to $\det(A) = \lambda_1\lambda_2$ is *positive* or *negative*.

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Class IV: Projective transformations

- A projective transformation is a general non-singular linear transformation of homogenous coordinates of the form

$$\mathrm{x}' = \mathrm{H}_{\mathrm{p}}\mathrm{x} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathrm{T}} & 1 \end{bmatrix} \mathrm{x}$$

- where $\mathrm{v} = (v_1, v_2)^{\mathrm{T}}$ and v might be zero
- A *projective transformation* has *8-DOF*

Class IV: Projective transformations

- A *projective transformation* between two planes can be computed by *four point correspondences*.

- It is not possible to distinguish between orientation preserving and orientation reversing projectivities in $\mathbb{P}^2$

- Invariants. Cross ratio of four collinear points.

Comparison of Affine and Projective Transformation

- The *key difference* between a projective and affine transformation is the vector v is not *null* for a projectivity.
- Mapping of an ideal point $(x_1, x_2, 0)$ under affinity and projectivity are as follow:
- Affinity: Ideal point remains ideal

$$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

- Projectivity: Ideal point maps to finite point

$$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathrm{T}} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Decomposition of a projective transformation

- A projective transformation can be decomposed into a chain of transformations.

$$\mathrm{H} = \mathrm{H_S H_A H_P} = \begin{bmatrix} \mathrm{sR} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathrm{K} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathrm{I} & \mathrm{t} \\ \mathrm{v}^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathrm{T}} & 1 \end{bmatrix}$$

- A is a *non-singular* matrix given by $\mathrm{A} = \mathrm{sRK} + \mathrm{tv}^{\mathrm{T}}$
- K is an *upper-triangular* matrix and det K = 1
- The decomposition is valid provided $v \neq 0$ and s is positive

Example

- The projective transformation

$$H = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

May be decomposed as

$$H = \begin{bmatrix} 2\cos 45^o & -2\sin 45^o & 1 \\ 2\sin 45^o & 2\cos 45^o & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Summary of Invariant properties of planar transformation

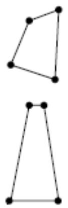
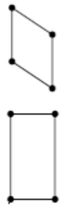
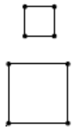
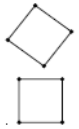
Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

The projective geometry of 1D

- A point x on the line is represented by homogenous coordinates $(x_1, x_2)^{\mathrm{T}}$. A point $x_2 = 0$ is an *ideal point* of the line

- A *projective transformation* of a line is represented by 2x2 homogeneous matrix

$$\bar{\mathrm{x}}' = \mathrm{H}_{2\times 2}\bar{\mathrm{x}}$$

- It has 3-DOF. A projective transformation of a line may be determined from three corresponding points

The cross ratio

- The *cross ratio* is the *basic projective invariant* of $\mathbb{P}^1$. Given 4 points $\bar{\mathrm{x}}_\mathrm{i}$ the cross ratio is defined as

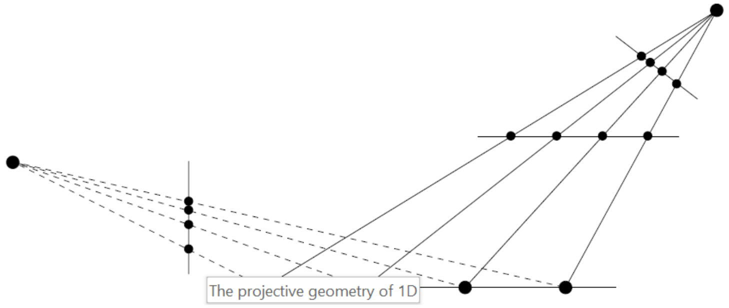
$$\mathrm{Cross}(\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2, \bar{\mathrm{x}}_3, \bar{\mathrm{x}}_4) = \frac{|\bar{\mathrm{x}}_1\bar{\mathrm{x}}_2||\bar{\mathrm{x}}_3\bar{\mathrm{x}}_4|}{|\bar{\mathrm{x}}_1\bar{\mathrm{x}}_3||\bar{\mathrm{x}}_2\bar{\mathrm{x}}_4|}$$

Where

$$|\bar{\mathrm{x}}_1\bar{\mathrm{x}}_2| = \det\begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$$

The cross ratio

- If each point $\bar{x}_i$ is a finite point, $|\bar{x}_i\bar{x}_j|$ represents the signed distance from $\bar{x}_i$ to $\bar{x}_j$
- The definition of the cross ratio is also valid if one of the point $\bar{x}_i$ is an ideal point.
- The value of the cross ratio is invariant under any projective transformation of the line. If $\bar{x}' = H_{2\times 2}\bar{x}$

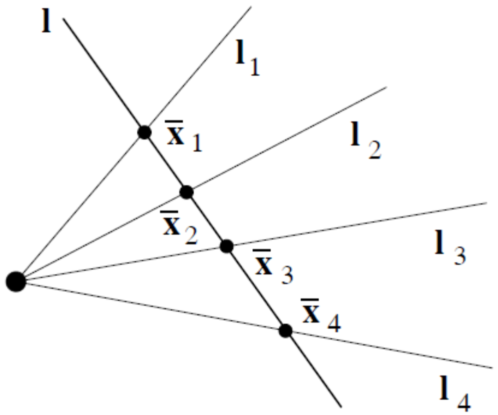


- $\mathrm{Cross}(\bar{x}_1', \bar{x}_2', \bar{x}_3', \bar{x}_4') = \mathrm{Cross}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Concurrent lines

- The *cross ratio* of concurrent *lines* is *dual to* collinear *points* on a line
- The cross ratio of these lines is invariant to projective transformations of plane.
- The *cross ratio* is given by $\mathrm{Cross}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$
- The *cross ratio* is invariant to choice of line



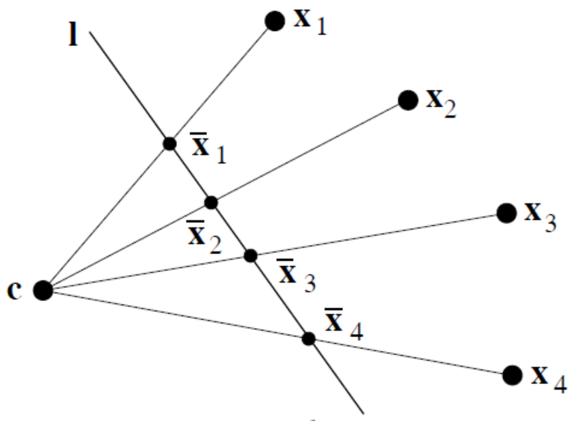


Image Credit: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

The line a infinity

- Under a *projective transformation*, *ideal points* may be mapped to *finite points* and l_∞ is mapped to a *finite line*.

- If transformation is an *affinity* $\mathrm{H_A} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{0^T} & 1 \end{bmatrix}$, l_∞ remains at infinity

$$\mathrm{l}'_\infty = \mathrm{H_A^{-T}} \mathrm{l}_\infty = \begin{bmatrix} \mathrm{A^{-T}} & 0 \\ -\mathrm{t^T A^{-T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathrm{l}_\infty$$

- The converse is also true. If we require point at infinity say $\mathrm{x} = (1,0,0)^\mathrm{T}$ be mapped to a point at infinity, $h_{31} = h_{32} = 0$

The line a infinity

- However l_∞ is not fixed pointwise under an affine transformation.
- Under an affinity, a point on l_∞ (an ideal point) is mapped to another point on l_∞

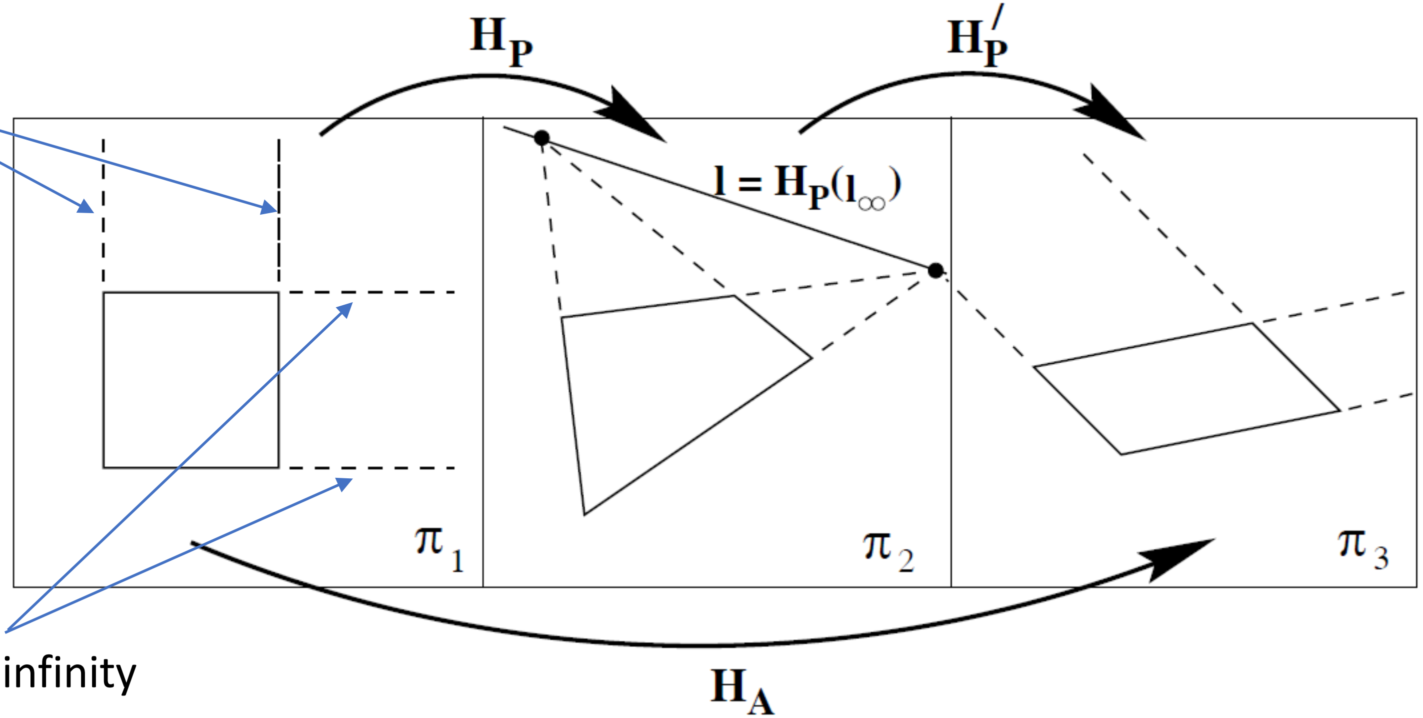
$$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

- Except $A(x_1, x_2)^T - k(x_1, x_2)^T$

Recovery of affine properties from images

Map $l_\infty = (0,0,1)^T$ to finite point

Meet at infinity



Meet at infinity

- *Affine rectification*: A projective transformation maps $l_\infty = (0,0,1)^T$ from $\pi_1 \rightarrow \pi_2$. We can use this mapping to *remove* the projective distortion

Image Credit: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

Recovery of affine properties from images

- The *projective transformation* $\mathrm{H_P}$ maps the $\mathrm{l}_\infty = (0,0,1)^\mathrm{T}$ to the imaged line $\mathrm{l} = (l_1, l_2, l_3)^\mathrm{T}$, provided that $l_3 \neq 0$.

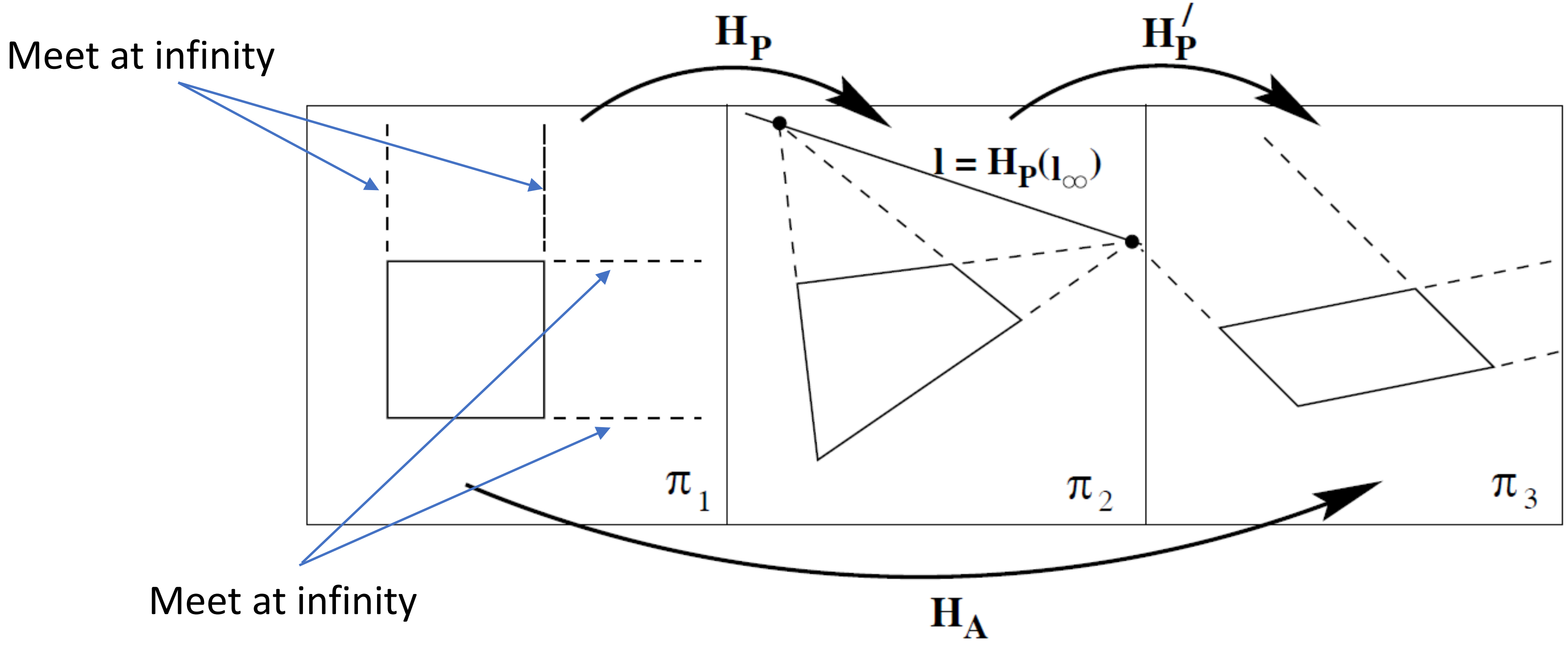


Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Recovery of affine properties from images

- We have $\mathrm{l} = \mathrm{H_P}^{-\mathrm{T}}\mathrm{l}_\infty \Rightarrow \mathrm{l}_\infty = \mathrm{H_P}^{\mathrm{T}}\mathrm{l}$, where $\mathrm{l} = (l_1, l_2, l_3)^{\mathrm{T}}$ and $\mathrm{l}_\infty = (0,0,1)^{\mathrm{T}}$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & l_3 \end{bmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \Rightarrow \qquad \mathrm{H_P}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & l_3 \end{bmatrix}$$

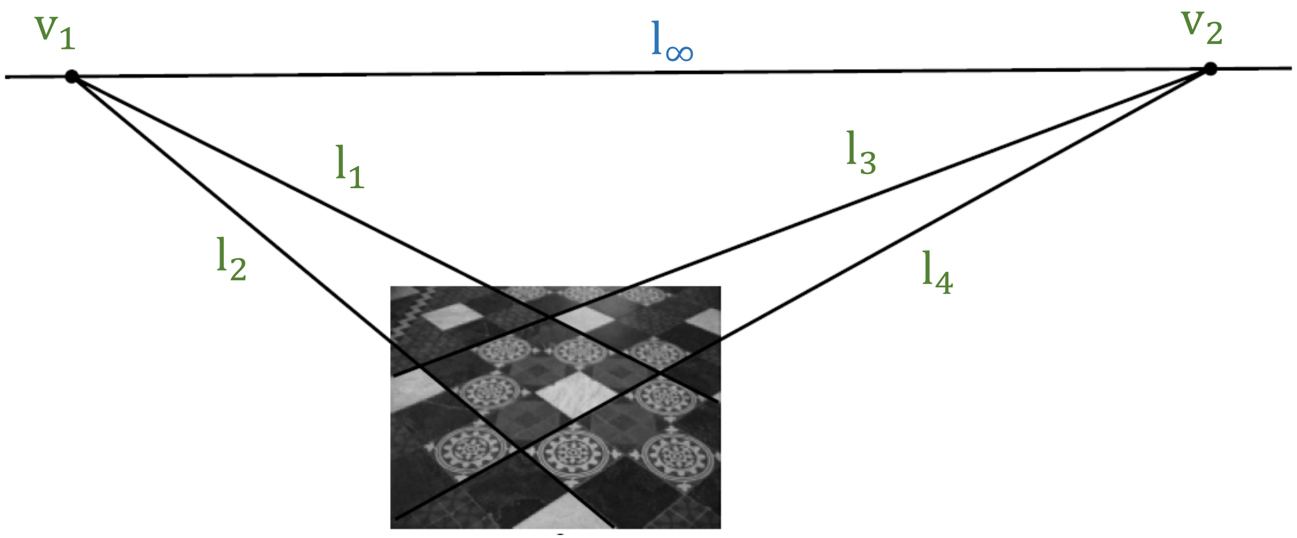
- Since $\mathrm{H_A} = \mathrm{H_P}'\mathrm{H_P}$

$$\mathrm{H}'_\mathrm{P} = \mathrm{H_A}\mathrm{H}_\mathrm{P}^{-1} = \mathrm{H_A} \begin{bmatrix} 1 & 0 & -\frac{l_1}{l_3} \\ 0 & 1 & -\frac{l_2}{l_3} \\ 0 & 0 & l_3 \end{bmatrix}^{-\mathrm{T}} \Rightarrow \mathrm{H_A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

- Where $\mathrm{H_A}$ is any affine transform.

Affine Rectification via the vanishing line

- The vanishing line of the plane is computed from intersection of two sets of imaged parallel lines.
- We can compute the $\mathrm{H}'_{\mathrm{P}} = \mathrm{H}_{\mathrm{A}}\mathrm{H}_{\mathrm{P}}^{-1}$ with arbitrary affinity H_{A}



$$\mathrm{v}_1 = \mathrm{l}_1 \times \mathrm{l}_2$$
$$\mathrm{v}_2 = \mathrm{l}_3 \times \mathrm{l}_4$$
$$\mathrm{l}_\infty = \mathrm{v}_1 \times \mathrm{v}_2$$

Image Credit: R. Hartley and A. Zisserman, "Multiple View Geometry in Computer Vision"

Results of Affine Rectification

- Compute H'_{P}
- Using H'_{P} to protectively warp the image to produce affinely rectified image
- Parallel lines are now parallel but angle is not preserved.

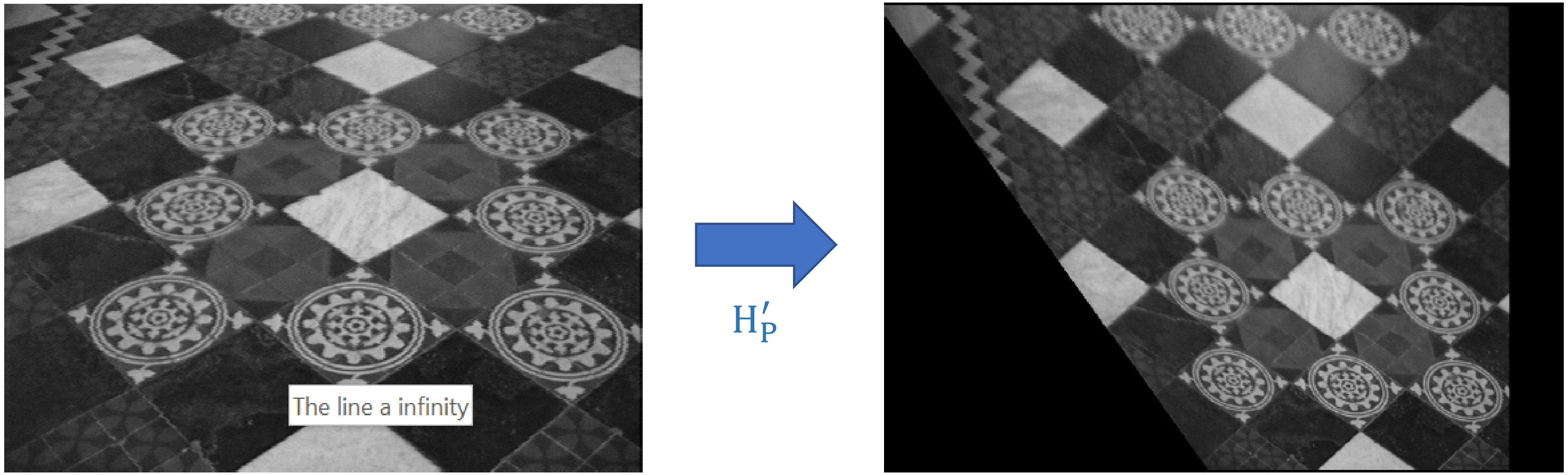


Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

The circular points and their dual

- Under any similarity transformation, two points on l_∞ which are fixed.

- These are the *circular points* (also called *absolute points*), I, J with canonical coordinates

$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathrm{I} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

- The circular points are a pair of *complex conjugate ideal points*.

The circular points and their dual

- The circular points are fixed under an orientation-preserving similarity:

$$\mathrm{I}' = \mathrm{H_S I}$$

$$= \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & \mathrm{s}\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

$$= se^{-i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$ where $e^{i\theta} = \cos\theta - i\sin\theta$

- With an analogues proof for J
- The *converse* is also true, i.e. if the circular points are fixed then the linear transformation is a *similarity*

The circular points and their dual

- The name “circular points” arises because every circle intersects l_∞ at circular points.

- We start from equation of conic.

$$ax_1{}^2 + bx_1x_2 + cx_2{}^2 + dx_1x_3 + ex_2x_3 + fx_3 = 0$$

- In case it is a circle: $a = c$, $b = 0$, then

$$x_1{}^2 + x_2{}^2 + dx_1 + ex_2 + fx_3 = 0$$

- This conic intersect l_∞ in the ideal point for which $x_3 = 0$, namely

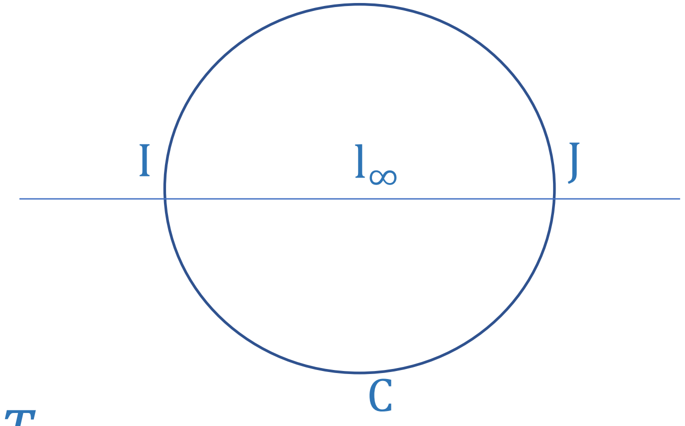
$$x_1{}^2 + x_2{}^2 = 0$$

The circular points and their dual

- When $x_3 = 0$, we have

$$x_1{}^2 + x_2{}^2 = 0$$

$$\Rightarrow (x_1 + ix_2)(x_1 - ix_2) = 0$$



- With solution $\mathrm{I} = (1, i, 0)^T$, and $\mathrm{J} = (1, -i, 0)^T$

- Hence any circle intersect l_∞ in circular points.

The conic dual to the circular points

- The *conic dual to the circular points* is defined as

$$\mathrm{C}_{\infty}^{*} = \mathrm{I}\mathrm{J}^{\mathrm{T}} + \mathrm{J}\mathrm{I}^{\mathrm{T}}$$

- The conic C_{∞}^{*} is a degenerated (rank 2) line conic which consists of the two circular points.

- In a Euclidean coordinate system, it is given by

$$\mathrm{C}_{\infty}^{*} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Fixed under similarity transform

- The conic C_∞^* is fixed under similarity transform. Since C_∞^* is a dual conic, it follows $C^{*\prime} = HC^*H^T$.
- Under the point transformation $x' = H_s x$, we have

$$C_\infty^{*\prime} = H_s C_\infty^* {H_s}^T$$

$$C_\infty^{*\prime} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$C_\infty^{*\prime} = \begin{bmatrix} s\cos\theta & -s\sin\theta & 0 \\ s\sin\theta & s\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s\cos\theta & s\sin\theta & 0 \\ -s\sin\theta & s\cos\theta & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$$C_\infty^{*\prime} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Properties

- The properties of C^*_∞ in any projective frame:

- C^*_∞ has 4 degree of freedom: a 3x3 homogenous symmetric matrix has 5 degrees of freedom, but constraint $\det C^*_\infty = 0$ reduces the degree by 1

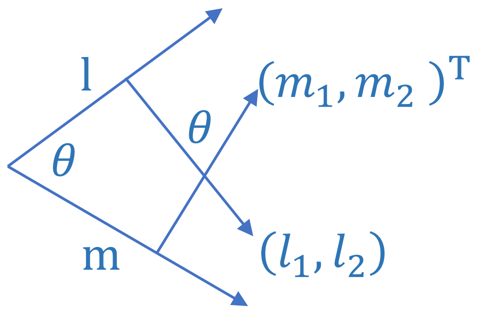
- l_∞ is the null vector of C^*_∞. This is clear from the definition: the circular points lie on l_∞, so that $I^T l_\infty = J^T l_\infty = 0$ then

$$C^*_\infty l_\infty = \left(IJ^T + JI^T\right) l_\infty = I(J^T l_\infty) + J(I^T l_\infty) = 0$$

Angles on the projective plane

- In Euclidean Geometry, the angle between two lines $\mathrm{l} = (l_1, l_2, l_3)^{\mathrm{T}}$ and $\mathrm{m} = (m_1, m_2, m_3)^{\mathrm{T}}$ with normal $(l_1, l_2)^{\mathrm{T}}$ and $(m_1, m_2\ )^{\mathrm{T}}$ is

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$



- The problem is the first two components of l and m is not defined under projective transformations

- This cannot be applied after affine or projective transformation.

Angles on the projective plane

- We can put C_∞^* into the following expression to make it *invariant to projective transformation*:

$$\cos\theta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

- Under the transformation, ($C^{*\prime} = HC^*H^T$ and $l' = H^{-T}l$). Then the numerator transforms as

$$l^T C_\infty^* m \longmapsto (l^T H^{-1})(H C_\infty^* H^T)(H^{-T} m)$$

- It may also be verified for the denominator and hence the equation is *invariant to the projective frame*

Angles on the projective plane

- Once the conic C_{∞}^{*} is identified on the projective plane then Euclidean angles may be measure.

- Special Case: Line l and m are orthogonal if $\mathrm{l}^{\mathrm{T}}\mathrm{C}_{\infty}^{*}\mathrm{m} = 0$

$$\cos\left(\frac{\pi}{2}\right) = \frac{\mathrm{l}^{\mathrm{T}}\mathrm{C}_{\infty}^{*}\mathrm{m}}{\sqrt{(\mathrm{l}^{\mathrm{T}}\mathrm{C}_{\infty}^{*}\mathrm{l})(\mathrm{m}^{\mathrm{T}}\mathrm{C}_{\infty}^{*}\mathrm{m})}} = 0$$

Recovery of metric properties from images

- Metric rectification using C_∞^*: The dual conic C_∞^* enable both the projective and affine components to be determined leaving only similarity distortions.
- If point transformation is $x' = Hx$, the projective transformation of $C^{*\prime} = HC^* H^T$
- $C_\infty^{*\prime} = (H_P H_A H_S) C_\infty^* (H_P H_A H_S)^T = (H_P H_A)\big(H_S C_\infty^* H_S{}^T\big)(H_A{}^T H_P{}^T)$

$$= (H_P H_A) C_\infty^* (H_A{}^T H_P{}^T)$$

$$= \begin{bmatrix} KK^T & KK^T v \\ v^T KK^T v & v^T KK^T v \end{bmatrix}$$

- It is clear that the projective (v) and affine (K) are determined directly from the image of C_∞^* but *similarity is undetermined*.

Recovery of metric properties from images

- Once the conic C_∞^* is identified on the projective plane, then *projective distortion* may be rectified up to a *similarity*

- A suitable rectifying homography may be obtained directly from identified $C_\infty^{*\prime}$ in image using SVD

$$SVD(C_\infty^*) = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T$$

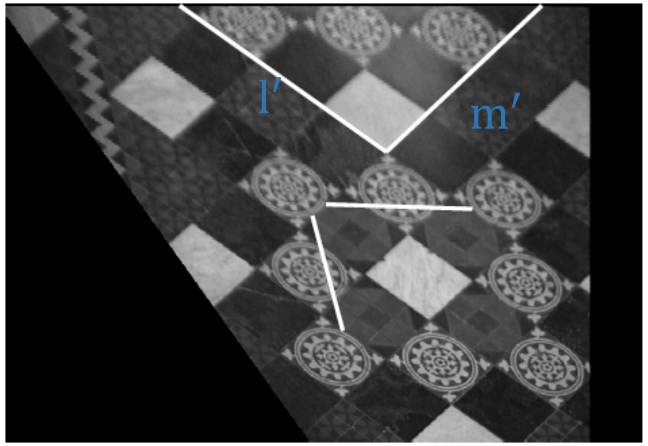
- The *rectifying projectivity* is $H = U$ up to a similarity

Metric Rectification

- Example 1: Metric rectification from affinely rectified image

- Suppose the lines l' and m' in the affinely rectified image correspond to an orthogonal line pair l and m,
- we have

$$\mathrm{l}^{\mathrm{T}}\mathrm{H}_{\mathrm{A}}^{-1}(\mathrm{H}_{\mathrm{A}}\mathrm{C}_{\infty}^{*}\mathrm{H}_{\mathrm{A}}^{\mathrm{T}})\mathrm{H}_{\mathrm{A}}^{-\mathrm{T}}\mathrm{m} = 0$$



$$\mathrm{l}'(\mathrm{H}_{\mathrm{A}}\mathrm{C}_{\infty}^{*}\mathrm{H}_{\mathrm{A}}^{\mathrm{T}})\mathrm{m}' = 0 \qquad \mathrm{H}_{\mathrm{A}} = \begin{bmatrix} \mathrm{K} & 0 \\ 0^{\mathrm{T}} & 1 \end{bmatrix}, \mathrm{C}_{\infty}^{*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(l_1', l_2', l_3') \begin{bmatrix} \mathrm{KK}^{\mathrm{T}} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix} \begin{pmatrix} m_1' \\ m_2' \\ m_3' \end{pmatrix} = 0$$

Which is a linear constraint on 2x2 matrix $\mathrm{S} = \mathrm{KK}^{\mathrm{T}}$ is symmetric with three in independent elements

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Metric Rectification

- The *orthogonality* condition reduce the equation to

$$(l_1', l_2')S\begin{pmatrix} m_1' \\ m_2' \end{pmatrix} = 0 \qquad S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \text{ since } s_{12} = s_{21}$$

- Which may be written as

$$(l_1' m_1', l_1' m_2' + l_2' m_1', l_2' m_2')\mathrm{s} = 0$$

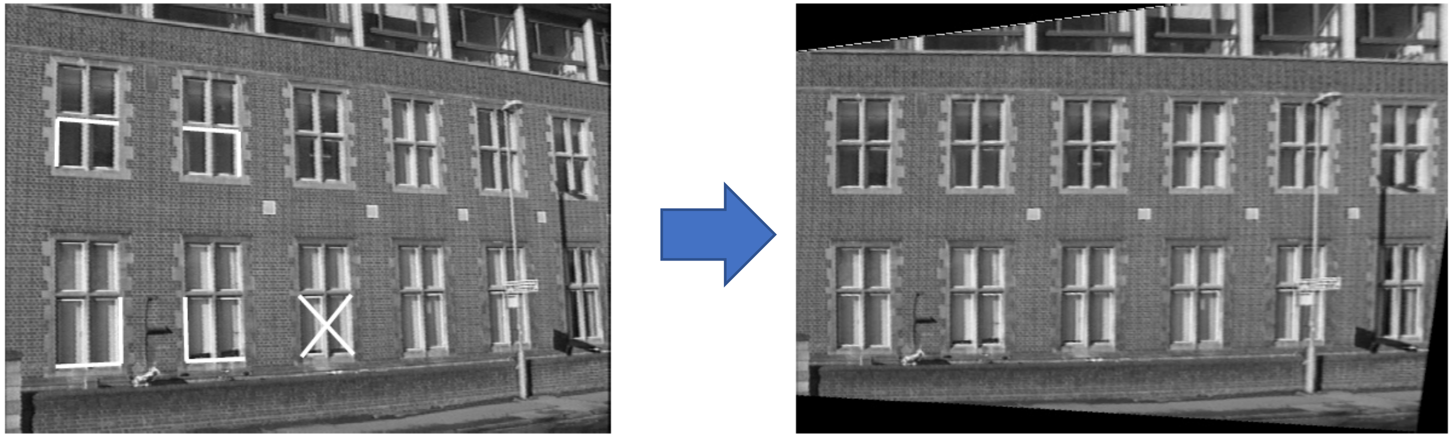
- where $\mathrm{s} = (s_{11}, s_{12}, s_{22})^{\mathrm{T}}$ is S written as a *3-vector*

- 2 *orthogonal line* pairs provide two constraints and may be stacked to give 2x3 matrix. S is determined as the *null vector*.

- S and K is then obtained up to scale by Cholesky decomposition.

Metric Rectification

- Example 2: *Metric rectification* from *original perspective image* of the plane



- Suppose the lines l and m are image of the *orthogonal lines* on the world plane such at $\mathrm{l}^{\mathrm{T}} \mathrm{C}_{\infty}^{*} \mathrm{m} = 0$

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Metric Rectification

If point transformation is $\mathrm{x}' = \mathrm{Hx}$, the projective transformation of $\mathrm{C}^{*\prime} = \mathrm{HC}^{*}\,\mathrm{H}^{\mathrm{T}}$

$$\mathrm{C}_{\infty}^{*\prime} = (\mathrm{H_P H_A})\mathrm{C}_{\infty}^{*}(\mathrm{H_A}^{\mathrm{T}}\mathrm{H_P}^{\mathrm{T}}) = \begin{bmatrix} \mathrm{KK^T} & \mathrm{KK^T v} \\ \mathrm{v^T KK^T v} & \mathrm{v^T KK^T v} \end{bmatrix}$$

- Since $\mathrm{l}^{\mathrm{T}}\mathrm{C}_{\infty}^{*}\mathrm{m} = 0$

$$\left(l_1 m_1, \frac{(l_1 m_2 + l_2 m_1)}{2}, l_2 m_2, \frac{(l_1 m_3 + l_3 m_1)}{2}, \frac{(l_2 m_3 + l_3 m_2)}{2}, l_3 m_3\right)\mathrm{c}=0$$

- Where $\mathrm{c} = (a, b, c, d, e, f)^{\mathrm{T}}$ is the conic matrix of C_{∞}^{*} written in 6-vector.
- Each pair of orthogonal lines l_i and m_i gives 2 constraints. We need 3 pairs on orthogonal lines to solve for c

Discussion

- The First method uses two steps to rectify the distorted image. i.e. Projective->Affine and Affine->Similarity.
  - This method is more robust by separating into two steps
  - Involves solving a 2-dimensional functions.
  - This method is termed *Stratification*.

- The second method is one-step approach. i.e. *Projective->Similarity*
  - This method involve solving *5-dimensional* functions.
  - More *sensitive* to choice of orthogonal lines.

Example


Original Image showing parallel lines


Image with Affine Distortion Removed, showing perpedicular lines


Rectified Image

[GitHub - cdbunker/Image-Rectification: Remove projective and affine distortion from images]

Projective Geometry in 3D

- Many of the properties of projective geometry in 3-space $\mathbb{P}^3$ are *straightforward generalization* of those of the projective plane in $\mathbb{P}^2$
- In $\mathbb{P}^3$ Euclidean 3-space is augmented with a set of *ideal points* which are on a *plane at infinity* π_∞ (analogue of l_∞ in $\mathbb{P}^2$). Parallel lines, and now parallel planes intersect at π_∞
- *Homogenous coordinate* with all dimensions increased by *one*.

Points and projective transformations in 3D

- A point $\mathbf{X}$ in 3-space is represented in homogenous coordinate as a 4-vector $\mathbf{X} = (X_1, X_2, X_3, X_4)^T$ with $X_4 \neq 0$ represents the point $(X, Y, Z)^T$ of $\mathbb{R}^3$ with inhomogeneous coordinates

$$X = X_1/X_4\ , Y = X_2/X_4, Z = X_3/X_4$$

- For example, a homogeneous representation of $(X, Y, Z)^T$ is $\mathbf{X} = (X, Y, Z, 1)^T$. Homogenous points with $X_4 = 0$ represents *points at infinity*

Points and projective transformations in 3D

- A *projective transformation* acting on $\mathbb{P}^3$ is linear transformation on homogenous 4-vectors.

- It is represented by non-singular 4x4 matrix:

$$\mathrm{X}' = \mathrm{HX}$$

- The matrix H has *16 elements* less one for overall scaling. Therefore it has *15-DOF*

- In $\mathbb{P}^3$, points and planes are dual. The representation is analogous to point-line duality in $\mathbb{P}^2$

Geometric properties invariant to common transformation of 3-space

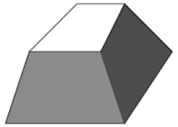
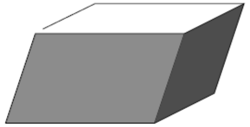
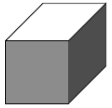
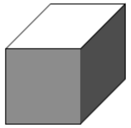
Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, $\boldsymbol{\pi}_\infty$, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Volume.

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

Planes Representation

- A *plane* in 3-space may be written as

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

- The equation is unaffected by multiplication by a non-zero scalar. Only the *three independent ratios* $\{\pi_1 : \pi_2 : \pi_3 : \pi_4\}$ of the *plane coefficient* are significant. It has *3-DOF*.

- The homogenous representation is *4-vector* $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)^T$. Replacing $X \mapsto X_1/X_4$, $Y \mapsto X_2/X_4$, $Z \mapsto X_3/X_4$ gives

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

Planes Representation

A more concise representation

$$\boldsymbol{\pi}^{\mathrm{T}}\mathrm{X} = 0$$

- which expresses that the point X is on the plane π.
- The first *3* components of $\boldsymbol{\pi}$ correspond to the *plane normal* of Euclidean Geometry. Using the inhomogeneous notation, where $\mathbf{n} = (\pi_1, \pi_2, \pi_3)^{\mathrm{T}}, \widetilde{\mathrm{X}} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z})^{\mathrm{T}}, \mathrm{X}_4 = 1, d = \pi_4$, the equation is written as

$$n \cdot \tilde{X} + d = 0$$

- Where $d/\|n\|$ is the distance of the plane from the origin

Plane Transformation

- Under the point transformation $\mathrm{X}' = \mathrm{HX}$, a plane transform as

$$\pi' = \mathrm{H}^{-\mathrm{T}}\pi$$

Proof: Suppose $\boldsymbol{\pi}^{\mathrm{T}}\mathrm{X} = 0 \Rightarrow \left(\boldsymbol{\pi}^{\mathrm{T}}\mathrm{H}^{-1}\right)(\mathrm{HX})$, then $\pi' = \mathrm{H}^{-\mathrm{T}}\pi$

The plane at infinity

- In planar projective geometry identifying the *line at infinity*, l_∞, allowed *affine properties* of the plane to be measured.

- Identifying the *circular points* on l_∞ then allowed the measurement of *metric properties*.

- In projective geometry of 3-space the corresponding geometric entities are π_∞ and the absolute conic Ω_∞

The plane at infinity

- The *plane at infinity* has the canonical position

$$\pi_\infty = (0,0,0,1)^T$$

- It contains the directions $D = (X_1, X_2, X_3, 0)^T$. It enables the identification of *affine properties* such as *parallelism*.

- Two planes are *parallel* if, and only if, their *line of intersection* is on π_∞

- A line is *parallel* to another line, if the *point of intersection* is on π_∞

The plane at infinity

- The *plane at infinity*, π_∞, is a fixed plane under the *projective transformation* H if, and only if, H is an *affinity*

- *Proof:*

$$\pi'_\infty = H_A^{-T}\pi_\infty = \begin{bmatrix} A^{-T} & 0 \\ -t^T A^{-T} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Note: The plane π_∞ is, in general, only *fixed as a set* under an *affinity*; it is *not fixed pointwise*.
- Under a particular affinity (for example a Euclidean motion) there may be planes in addition to π_∞ which are fixed. However, only π_∞ is fixed under *any affinity*.

Example

- Consider the Euclidean transformation represented by the matrix

- $H_E = \begin{bmatrix} R & 0 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- This is a rotation by θ about the z-axis with a *zero translation*. This means there is a *pencil of fixed planes* orthogonal to the z-axis.

- The planes are fixed as sets, but not pointwise as any(finite) point (not on the axis) is rotated in horizontal circles by this Euclidean action.

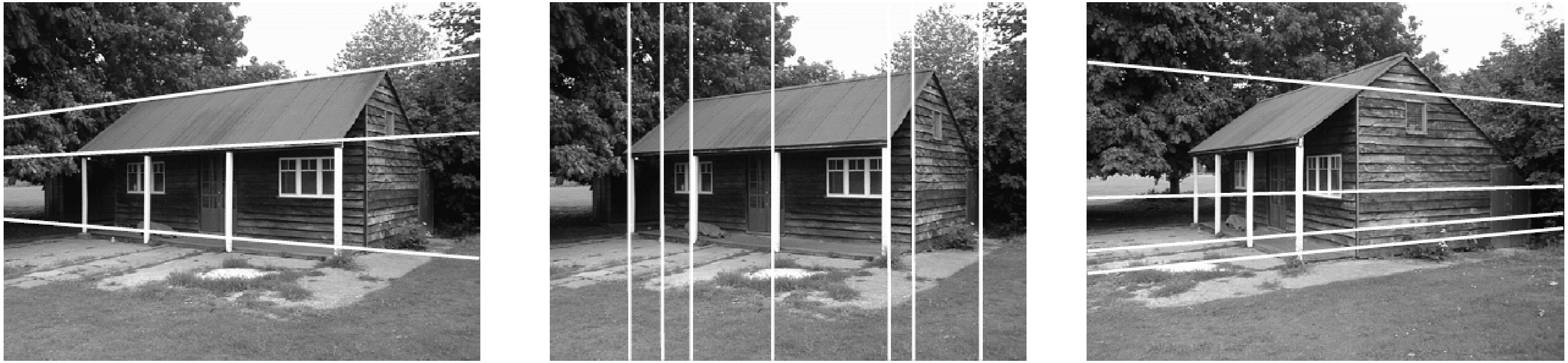
Affine properties of a reconstruction

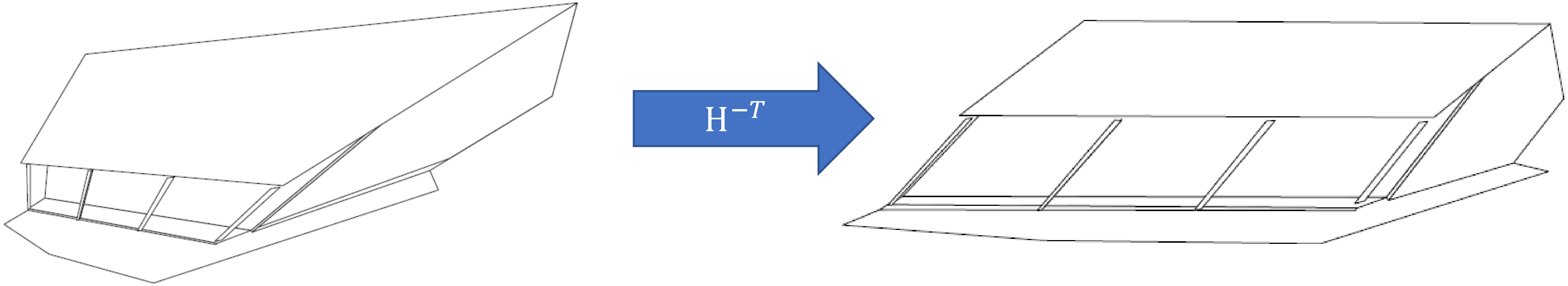
- We have seen how to restore the affine properties in image early in this lecture.
- Once we have π_∞ is identified in projective 3-space, it is then possible to determine affine properties of the reconstruction.
- A more algorithmic approach is to transform $\mathbb{P}^3$ so that the identified π_∞ is moved to its canonical position at

$$\pi_\infty = (0,0,0,1)^\mathrm{T}$$

- We will talk more in reconstruction section.

From Projective to Affinity





The Absolute Conic

- The *absolute conic*, Ω_∞, is a (point) conic on π_∞.

- In a metric frame $\pi_\infty = (0,0,01)^\mathrm{T}$ and points on Ω_∞ satisfy

$$\left.\begin{matrix} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{matrix}\right\} = 0$$

- Note the two equations are required to define Ω_∞

The Absolute Conic

- For directions on π_∞ (i.e. points with $X_4 = 0$) the defining equation can be written

$$(X_1, X_2, X_3)I(X_1, X_2, X_3)^T = 0$$

- So that Ω_∞ corresponds to a conic C with matrix $C = I$
- It is a conic of purely imaginary point on π_∞
- The conic Ω_∞ is a geometric representation of the 5 additional degrees of freedom that are required to specify metric properties in an affine coordinate frame.

The Absolute Conic

- A key property of Ω_∞ is that it is a fixed conic under any similarity transformation

- Poof: Since the *absolute conic* lies in the *plane at infinity*, a transformation fixing it must fix the *plane at infinity*, hence must be *affine*. *Such a transformation is of the form*

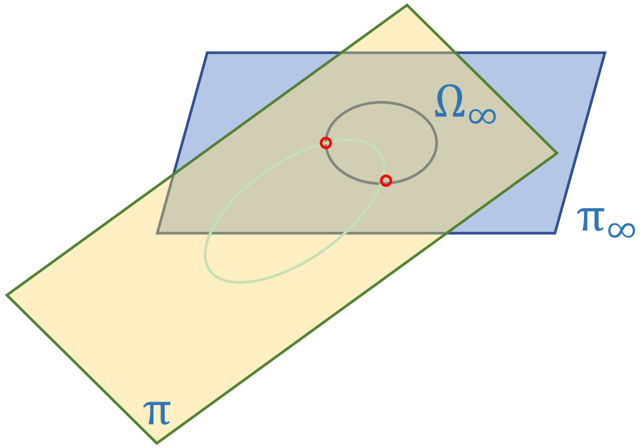
$$H_A = \begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$$

- Restricting to the plane at infinity, the *absolute conic* is represented b the matrix $I_{3\times 3}$, one has $A^T I A^{-1} = I$, taking inverses gives $AA^T = I$.

- A is orthogonal, and scaled rotation, or scaled rotation with reflection. This complete the proof.

Properties of the Absolute Conic

- Even though Ω_∞ does not have real points, it shares the properties of any conic – such as that a line intersect a conic in two points
- Ω_∞ is only fixed as a set by a general similarity; it is not fixed pointwise.
- All circles intersect Ω_∞ in two points.
- All spheres intersect π_∞ in Ω_∞



Metric properties

- Once the Ω_∞ (and its support plane π_∞) have been identified in projective 3-space, the metric properties such as angles and relative lengths can be measured.
- Consider two lines with directions (3-vector) $\mathbf{d}_1$ and $\mathbf{d}_2$. The angle between these direction in a Euclidean world frame is given by

$$\cos\theta = \frac{(\mathbf{d}_1^\mathrm{T}\mathbf{d}_2)}{\sqrt{(\mathbf{d}_1^\mathrm{T}\mathbf{d}_1)(\mathbf{d}_2^\mathrm{T}\mathbf{d}_2)}}$$

- This may be written as

$$\cos\theta = \frac{(\mathbf{d}_1^\mathrm{T}\Omega_\infty\mathbf{d}_2)}{\sqrt{(\mathbf{d}_1^\mathrm{T}\Omega_\infty\mathbf{d}_1)(\mathbf{d}_2^\mathrm{T}\Omega_\infty\mathbf{d}_2)}}$$ where $\mathbf{d}_1$ and $\mathbf{d}_2$ are points intersection of lines with π_∞, Ω_∞

- $\mathbf{d}_1^\mathrm{T}\Omega_\infty\mathbf{d}_2 = 0$ if $\mathbf{d}_1$ and $\mathbf{d}_2$ are orthogonal

Metric Reconstruction Result.

- Once the Ω_∞ (and its support plane π_∞) have been identified in projective 3-space, we can remove the affine distortion of the reconstructed scene.

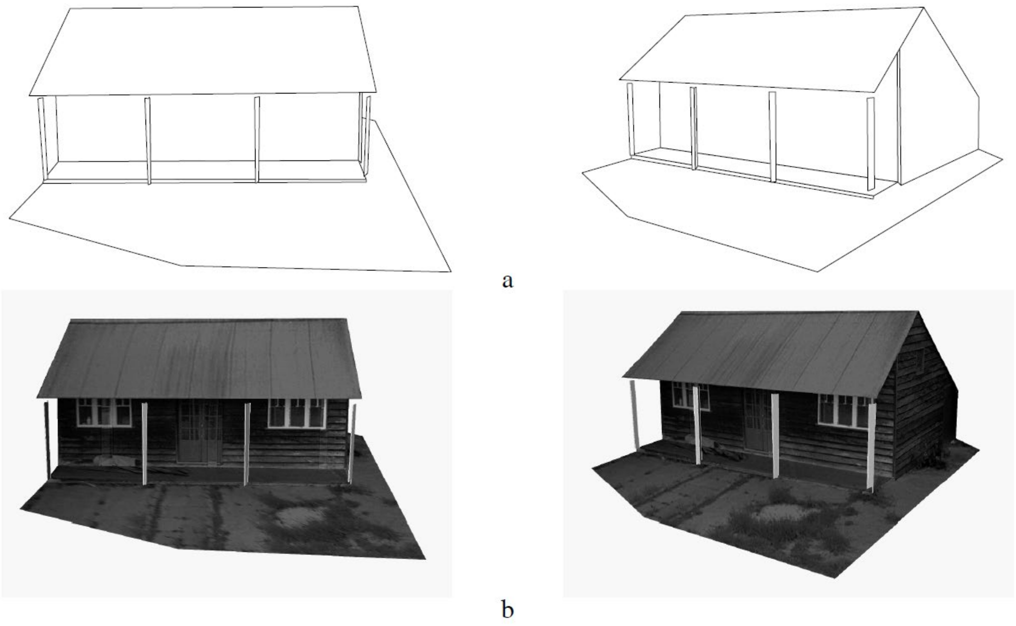


Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

The absolute dual quadric

- Ω_∞ is defined by two equations – it is a conic on the plane at infinity.
- The dual of the absolute conic Ω_∞ is called absolute dual quadric, and denoted Q_∞^*.
- Geometrically Q_∞^* consists of the plane tangent to Ω_∞.
- Algebraically Q_∞^* is represented by a 4×4 homogenous matrix of rank 3. In metric 3-space, it has the canonical form

$$Q_\infty^* = \begin{bmatrix} \mathrm{I} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix}$$

The absolute dual quadric

- The *absolute dual quadric*, Q_∞^*, is fixed under the projective transformation$\mathrm{H}$ *if, and only if* , H *is a similarity*

- Proof: Since Q_∞^* is a dual quadric, it is fixed under H if and only if $Q_\infty^* = \mathrm{H}Q_\infty^*\mathrm{H}^\mathrm{T}$. Applying this with arbitrary transform.

$$\mathrm{H}_\mathrm{A} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^\mathrm{T} & k \end{bmatrix}$$

- we find

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0^\mathrm{T} & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{A} & \mathbf{t} \\ \mathbf{v}^\mathrm{T} & k \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 \\ 0^\mathrm{T} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{A} & \mathbf{v} \\ \mathbf{t}^\mathrm{T} & k \end{bmatrix} = \begin{bmatrix} \mathrm{A}\mathrm{A}^\mathrm{T} & \mathbf{A}\mathbf{v} \\ \mathbf{v}^\mathrm{T}\mathrm{A}^\mathrm{T} & \mathbf{v}^\mathrm{T}\mathbf{v} \end{bmatrix}$$

- which must be true up to scale if and only if $\mathbf{v} = 0$ and A is a *scaled orthogonal matrix*. In other words, H is a *similarity transform*.

The absolute dual quadric

- The plane at infinity π_∞ is the null-vector of Q^*_∞ .

- This is easily verified when Q^*_∞ has its canonical form in metric frame with $\pi_\infty = (0,0,0,1)^T$

$$Q^*_\infty \pi_\infty = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

- This property holds in any frame from the transformation properties of planes and dual quadrics: if $X' = HX$, then $Q^{*\prime}_\infty = HQ^*_\infty H^T$, $\pi'_\infty = H^{-T}\pi_\infty$

$$Q^{*\prime}_\infty \pi'_\infty = \left(HQ^*_\infty H^T\right)H^{-T}\pi_\infty = H\,Q^*_\infty \pi_\infty = 0$$

The angle between two planes

- The *angle* between two planes π_1 and π_2 is given by

$$\cos\theta = \frac{(\pi_1^T Q_\infty^* \pi_2)}{\sqrt{(\pi_1^T Q_\infty^* \pi_1)(\pi_2^T Q_\infty^* \pi_2)}}$$

- Proof: Consider two planes with Euclidean coordinates $\pi_1 = (n_1^T, d_1)^T$, $\pi_2 = (n_2^T, d_2)^T$, Q_∞^* has the form $\begin{bmatrix} I & 0 \\ 0^T & 1 \end{bmatrix}$

- we have

$$\cos\theta = \frac{(\mathbf{n}_1^T \mathbf{n}_2)}{\sqrt{(\mathbf{n}_1^T \mathbf{n}_1)(\mathbf{n}_2^T \mathbf{n}_2)}}$$

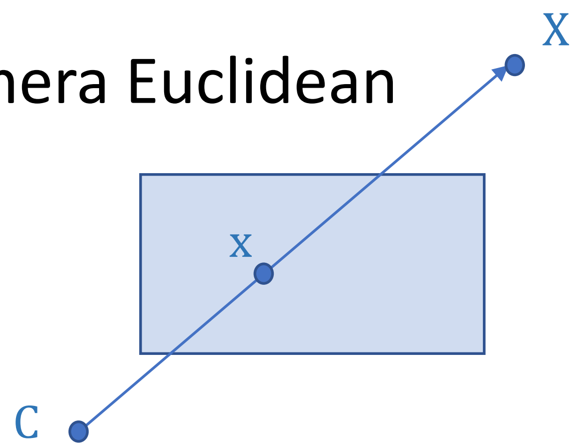
- Note: The angle θ can *still be determined* even the planes and Q_∞^* are *protectively transformed*, due to the (covariant) transformation properties of planes and dual quadrics.

Camera Calibration and Image of Absolute Conic

- Suppose a point on the ray are written as $\tilde{\mathrm{X}} = \lambda\mathbf{d}$ in the camera Euclidean coordinate frame, then these points map to

$$\mathrm{x} = \mathrm{K}[\mathrm{I}\,|\mathbf{0}](\lambda\mathbf{d}^{\mathrm{T}}, 1)^{\mathrm{T}} = \lambda\mathbf{d}$$

up to scale



- Conversely the direction $\mathbf{d}$ is obtained from the image point x as $\mathbf{d} = K^{-1}\mathrm{x}$

- Note, $\mathbf{d} = K^{-1}\mathrm{x}$ is in general *not a unit vector*

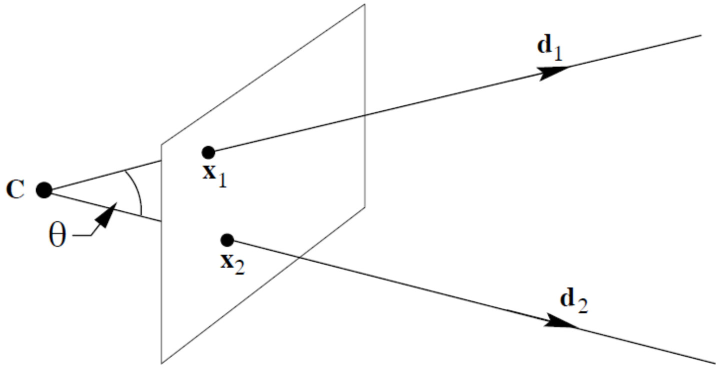
- Result: The camera calibration matrix K is the (affine) transformation between x and the ray's direction $\mathbf{d} = K^{-1}\mathrm{x}$

Angle between two rays

- The angle between two rays with direction $\mathbf{d_1}$, $\mathbf{d_2}$ corresponding to image points $\mathbf{x_1}$, $\mathbf{x_2}$ respectively, may be obtained below

$$\cos\theta = \frac{\mathrm{d}_1^{\mathrm{T}}\mathrm{d}_2}{\sqrt{\mathrm{d}_1^{\mathrm{T}}\mathrm{d}_1}\sqrt{\mathrm{d}_2^{\mathrm{T}}\mathrm{d}_2}} = \frac{\left(\mathrm{K}^{-1}\mathrm{x}_1\right)^{\mathrm{T}}\left(\mathrm{K}^{-1}\mathrm{x}_2\right)}{\sqrt{(\mathrm{K}^{-1}\mathrm{x}_1)^{\mathrm{T}}(\mathrm{K}^{-1}\mathrm{x}_1)}\sqrt{(\mathrm{K}^{-1}\mathrm{x}_2)^{\mathrm{T}}(\mathrm{K}^{-1}\mathrm{x}_2)}}$$

$$\cos\theta = \frac{\mathrm{x}_1^{\mathrm{T}}(\mathrm{K}^{-\mathrm{T}}\mathrm{K}^{-1})\mathrm{x}_2}{\sqrt{\mathrm{x}_1^{\mathrm{T}}(\mathrm{K}^{-\mathrm{T}}\mathrm{K}^{-1})\mathrm{x}_1}\sqrt{\mathrm{x}_2^{\mathrm{T}}(\mathrm{K}^{-\mathrm{T}}\mathrm{K}^{-1})\mathrm{x}_2}}$$



- The formula shows if K and consequently the matrix $K^{-T}K^{-1}$ is known, the angle between rays can be measured.

Image Credit: R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”

The Image of the Absolute Conic

- Points on π_∞ may be written as $X_\infty = (\mathbf{d}^T, 0)^T$, and are imaged by a general camera $P = KR[I \mid -\tilde{C}]$ as

$$x = PX_\infty = KR[I \mid -\tilde{C}] \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} = KR\mathbf{d}$$

- This shows that the mapping between π_∞ and an image is given by *the planar homography* $x = H\mathbf{d}$

$$H = KR$$

- The mapping is *independent of the position* of the camera C, and depends only on the camera internal calibration and orientation with respect to the world coordinate frame.

The image of the Absolute Conic (IAC)

- Since the absolute conic Ω_∞ is on π_∞, we can compute its image under H

- The image of the ***absolute conic (IAC)*** is the conic $\omega = (KK^T)^{-1} = \mathrm{K}^{-\mathrm{T}}\mathrm{K}^{-1}$

- Like Ω_∞ , ω is an *imaginary point conic* with *no real points*. It can thought of as a convenient algebraic device for camera calibration.

Absolute Conic and IAC

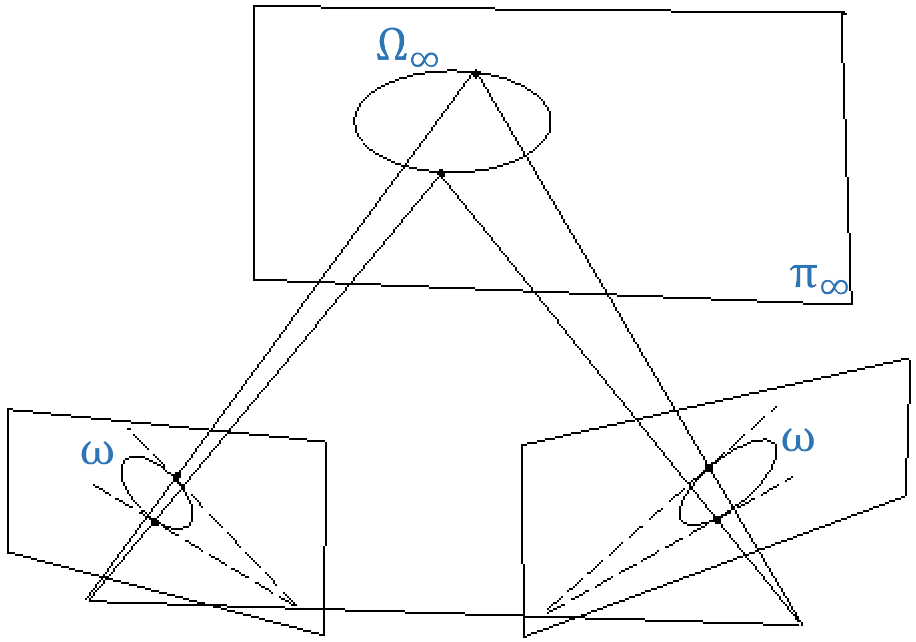


Image Source : http://lear.inrialpes.fr/

The image of the Absolute Conic (IAC)

- Under a point homograhy $\mathrm{x} \longmapsto \mathrm{Hx}$, a conic C maps as $\mathrm{C} \longmapsto \mathrm{H}^{\mathrm{T}}\mathrm{C}\mathrm{H}^{-1}$

- It follows that Ω_∞, which is the conic $\mathrm{C} = \Omega_\infty = \mathrm{I}$ on π_∞, maps to

$$\omega = (\mathrm{KR})^{-\mathrm{T}}\mathrm{I}(\mathrm{KR})^{-1} = \mathrm{K}^{-\mathrm{T}}(\mathrm{R}\mathrm{R}^{-1})\mathrm{K}^{-1} = \left(\mathrm{K}\mathrm{K}^{\mathrm{T}}\right)^{-1}$$

- So the *IAC* $\omega = \left(\mathrm{K}\mathrm{K}^{\mathrm{T}}\right)^{-1}$

A few remarks of IAC

- The *image of absolute conic*, ω, depends only on the *internal parameters* K of the matrix P, it *does not depend* on the *camera orientation or position*
- The angle between two rays is given by the simple expression below and it is *unchanged under projective transformation* of the image.

$$\cos\theta = \frac{x_1^T \omega x_2}{\sqrt{x_1^T \omega x_1}\sqrt{x_2^T \omega x_2}}$$

- If two images points $\mathbf{x_1}$, $\mathbf{x_2}$ correspond to *orthogonal directions*, then $x_1^T \omega x_2 = 0$
- Once ω is identified in an image, then K is also determined by *Cholesky factorization*

Summary

- Use the *line at infinity* to remove the *projective distortions*
- Use the *circular points* to remove the *affine distortions*
- Discussion of *plane at infinity* which is *invariant* under *affinity* transform.
- Describe the *absolute conic* and *absolute dual quadrics* which is invariant under similarity transform.

Reference

- R. Hartley and A. Zisserman, “Multiple View Geometry in Computer Vision”, Chapter 2 & 3