矩阵函数

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定义 $f: M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$

$\quad A \longrightarrow f(A)$

(1) 多项式函数

$f(z) = a_0 + a_1 z + \cdots + a_k z^k$

$\downarrow$

$f(A) = a_0 I_n + a_1 A + \cdots + a_k A^k$

性质(1) 若 $P^{-1}AP = B$，则 $P^{-1}f(A)P = f(B)$

若 $P^{-1}AP = J$，则 $A = PJP^{-1}$.

$\Rightarrow f(A) = P\begin{pmatrix} f(J_1) & & \\ & \ddots & \\ & & f(J_s) \end{pmatrix}P^{-1}$

定理：设 $f(x) = a_m x^m + \cdots + a_1 x + a_0$，　$J_0 = \begin{pmatrix} \lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{pmatrix}$

则 $f(J_0) = \begin{pmatrix} f(\lambda_0) & f'(\lambda_0) & \frac{1}{2!}f''(\lambda_0) & \cdots & \frac{1}{(r-1)!}f^{(r-1)}(\lambda_0) \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{1}{2!}f''(\lambda_0) \\ & & & \ddots & f'(\lambda_0) \\ & & & & f(\lambda_0) \end{pmatrix}_{r\times r}$.

证明：$J = (\lambda_0 I_r + N)$ ～

(2) 一般矩阵函数定义

$f(x) = e^x \to e^A$

幂级数收敛半径：R

定理：存在 $0 \le R \le +\infty$，使得对于 $0 \le |z| < R$，级数收敛，对于 $|z| > R$，级数发散

Taylor定理：

$f(z) = f(0) + f'(0) + \frac{1}{2!}f''(0) + \frac{1}{3!}f'''(0) + \cdots + \frac{1}{k!}f^{(k)}(0)$

例 $e^z = \sum \frac{z^n}{n!}$ $(R = +\infty)$

$\frac{1}{1-z} = \sum z^n$ $(R = 1)$

$\frac{1}{1+z} = \sum (-1)^n z^n$ $(R = 1)$

$\sin z = \sum_{0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ $(R = \infty)$

$\cos z = \sum_{0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ $(R = \infty)$

$\ln(1+z) = \sum (-1)^n \frac{z^{n+1}}{n+1}$ $(R = 1)$

$\sin z = \frac{e^{iz} - e^{-iz}}{2}$　　$\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$S(z) \to S(A)$?

$S(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \cdots$

$S(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

性质(1) 设 $P^{-1}AP = B$，则

$S(A)$ 收敛 $\Leftrightarrow$ $S(B)$ 收敛.

(2) $S(A)$ 收敛 $\Leftrightarrow$ $S(A_1) \cdots S(A_n)$ 收敛

(3). $S(z)$ 收敛半径 r，$J_0 = \begin{pmatrix} \lambda_0 & 1 & \\ & \ddots & 1 \\ & & \lambda_0 \end{pmatrix}$

当 $\lambda_0 < r$，$S(J_0)$ 收敛.

$S(J_0) = \begin{pmatrix} S(\lambda_0) & S'(\lambda_0) & \frac{1}{2!}S''(\lambda_0) & \cdots \\ & \ddots & \ddots & \frac{1}{2!}S''(\lambda_0) \\ & & S(\lambda_0) & \vdots \\ & & & S(\lambda_0) \end{pmatrix}$

定义：$A \in M_n(\mathbb{C})$

$m_A(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_s)^{m_s}$

$\begin{cases} S(\lambda_1) \cdots S^{(m_1)}(\lambda_1) \\ \vdots \\ S(\lambda_k) \cdots S^{(m_k-1)}(\lambda_k) \end{cases}$

存在则称 $S(z)$ 在 A 的谱上取值

例 $A = \begin{pmatrix} 3 & 0 & -1 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ 求 e^A，$\sin A$.

解：$|\lambda I_3 - A| = (\lambda - 2)^3$.

$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$　$P^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

$P^{-1}AP = \begin{pmatrix} 2 & & \\ & 2 & 1 \\ & & 2 \end{pmatrix} = \begin{pmatrix} J_1 & \\ & J_2 \end{pmatrix} = J$.

$S(A) = P S(J) P^{-1}$　　$S(J_1) = S(2)$　　$S(J_2) = \begin{pmatrix} S(2) & S'(2) \\ & S(2) \end{pmatrix}$

定理. 设 $A \in M_n(\mathbb{C})$　$\deg m_A(x) = m \ge 2$，

则 e^A 是 A 的次数 $\le m-1$ 的多项式.

证 $e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{1}{k!}A^k \cdots$

$m_A(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$

$\Rightarrow$　$A^m = -[a_{m-1}A^{m-1} + a_{m-2}A^{m-2} + \cdots + a_0]$

$A^{m+1} = \cdots$

则次数 $\ge m$ 的项都可以用 $\le m-1$ 的项线性表示.

(维度为 m ?)

性质：(1) A, B 交换 $\Rightarrow$ $e^A \cdot e^B = e^{A+B}$

(2) e^A 可逆，逆为 e^{-A}，

(3). A 为 Hermite 阵，则 iA 为 skew-Hermite 阵.

$\Rightarrow e^{iA}$ 是酉阵

(类比 $\theta \to i\theta \to e^{i\theta}$)

一阶常系数微分方程组

设 $A \in M_n(\mathbb{C})$　$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$

求解：$\frac{dx(t)}{dt} = Ax(t)$

其中 $\frac{dx(t)}{dt} = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$

例，$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

定理：$\begin{cases} \frac{dx(t)}{dt} = Ax(t) \\ x(0) = \alpha \in \mathbb{C}^n \text{（初值）} \end{cases}$

的解存在且唯一

$x(t) = e^{At}\alpha$，$\Longleftrightarrow$　$x(t) = e^{At}PP^{-1}\alpha = (e^{At}P)(P^{-1}\alpha)$

$= (Pe^{Jt})(P^{-1}\alpha)$

$e^{At} = Pe^{Jt}P^{-1} = P\begin{pmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_s t} \end{pmatrix}P^{-1}$

$J_i = \lambda_i I_{r_i} + N_i$

$e^{J_i t} = e^{(\lambda_i t)I_{r_i} + N_i t}$

$= e^{\lambda_i t I_{r_i}} \cdot e^{N_i t}$

$= e^{\lambda_i t I_{r_i}} \cdot (I_{r_i} + N_i t + \frac{1}{2!}N_i^2 t^2 + \cdots + \frac{N_i^{r_i-1}}{(r_i-1)!}t^{r_i-1})$

$= e^{\lambda_i t}\begin{pmatrix} 1 & t & \cdots & \frac{1}{(r_i-1)!}t^{r_i-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{pmatrix}$

应用：求 $y'' - 2y' + y = 0$，求 $y(t) = ?$

$x(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$　$x'(t) = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ 2y' - y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$