Wildfire Intervention

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June 27, 2022

1 Wildfire Intervention Problem

1.1 Wildfire Model: An Example

We will illustrate our algorithms on a fictional landscape consisting of different vegetation types, a city, water, grassland, eucalypti forest and desert. We represent this landscape as a graph with $n = 1000$ (25 rows, 40 columns) nodes. Each node represents a square with side length $l = 10$. Fire can spread from one node to 8 neighborhood nodes, i.e., fire transmission in horizon, vertical and diagonal bi-directions, see Fig.1.

The spreading rates are determined by the vegetation type, wind and land elevation, see [?!] Specifically, the spreading rate $\beta \in [0, 1]$ between neighborhood nodes can be expressed as

$$\beta = \beta_b \beta_{veg} \beta_w \beta_s. \tag{1}$$

Here $\beta_b = 0.5$ is the baseline spreading rate. The vegetation correction β_{veg} is listed in Table 1. β_w is used to model the effect of wind, that is, large β_w if fire goes the same direction as wind and small β_w if it goes the opposite direction. In this example, it is calculated for a northeasterly wind ($\theta_w = 5\pi/4$) of $V = 4$ m/s:

$$\beta_w = s \exp(c_1 V) \exp\big(c_2 V (\cos(\theta - \theta_w) - 1)\big) \tag{2}$$

where $c_1 = 0.045, c_2 = 0.131$ are constants, s_1 represents the correction for diagonal transmission (diagonal $s = 2\big(\sqrt{2} - 1\big)$, otherwise $s = 1$), θ is the angle of the fire propagation. β_s is used to model the effect of ground elevation (slope), that is, it is easy for fire to go uphill ($\beta_s > 1$) then downhill ($\beta_s < 1$). We can model this effect by

$$\beta_s = \exp(a\theta_s), \quad \theta_s = tan^{-1}\left(\frac{E_1 - E_2}{lb}\right) \tag{3}$$

where E_1, E_2 are the altitude of the two cells, l is the length of the square side, constants a, b are taken as $a = 0.078$, $b = \sqrt{2}$ for diagonal link and $b = 1$ for non-diagonal link.

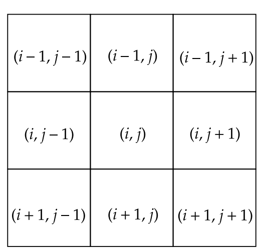

Figure 1: Caption

Table 1: Vegetation type and spreading rate correction

Type	forest	grassland	desert	city	water
Enum	1	2	3	4	5
β_{veg}	1.4	1.0	0.1	0.5	0.0

1.2 Spreading Process Model

We model the wildfire dynamics as a spreading process over a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with n nodes in the node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and edge set $\mathcal{E} = \{(i, j) \mid i \neq j\}$. Note that different from Section 1.1 and code implementation, we will use i, j to represent the node indices throughout the rest report.

Let $x_i \in [0, 1]$ be the probability of an outbreak at the ith node. The dynamics of spreading process can be described by the following nonlinear discrete-time system:

$$x_i(k+1) = (1 - h\delta_i(k))x_i(k) + h(1 - x_i(k)) \sum_{j=1}^{n} \beta_{ij}(k)x_j(k) \tag{4}$$

where $x(k)$ is the state variable, $\beta_{ij}(k) \in [0, \overline{\beta}_{ij}]$ and $\delta_i(k) \in [0, \overline{\Delta}]$ denote the manipulatable spreading and recover rates respectively, $0 < h < 1/\overline{\Delta}$ is the time interval. System (4) can be written in a compact form

$$x(k+1) = [A(k) - B(x, k)]x(k) \tag{5}$$

where the matrices $A, B \in \mathbb{R}^{n\times n}_{\geq 0}$ are given by

$$\begin{aligned} A_{ij}(k) &= \begin{cases} 1 - h\delta_i(k) > 0 & \text{if } i = j \\ h\beta_{ij}(k) > 0 & \text{if } i \neq j,\ (i, j) \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases} \\ B_{ij}(x, k) &= \begin{cases} hx_i(k)\beta_{ij}(k) \geq 0 & \text{if } i \neq j,\ (i, j) \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6}$$

2 Model Predictive Control

2.1 Notation

- $A \geq B \implies A_{ij} \geq B_{ij}$.
- $A \gneq B \implies A \geq B,\ A \neq B$.
- $A \in \mathbb{R}^{n\times m}_{\geq 0} \implies A \geq 0$.
- Hadamard product $[A \odot B]_{ij} = A_{ij}B_{ij}$
- $\{\beta(k|t)\}_{0\leq k\leq K-1}$ is a predictive sequence for the optimal control problem at time t. For simplicity we will also use the following indices:
  - $k < 0$ means the past recorded information, i.e., $\beta(k|t) = \beta(t + k)$;
  - $k \geq K$ means a copy of the last element, i.e., $\beta(k|t) = \beta(K - 1|t)$.

2.2 Intervention Problem

At each time step k we can allocate certain amount of resources to reduce the spreading rates $\beta_{ij}(k)$ and increase recovery rates $\delta_i(k)$ to reduce the overall risk. We assume bounded ranges of possible spreading and recover rates:

$$0 < \underline{\beta}_{ij} \leq \beta_{ij}(k) \leq \beta_{ij}(k-1), \quad 0 < \delta_i(k-1) \leq \delta_i(k) \leq \overline{\delta}_i < \overline{\Delta}, \quad \forall k \geq 0, \tag{7}$$

where $\beta_{ij}(-1)$ and $\delta_i(-1)$ are the unmodified rates of the system (5).

$$A(-1) \geq A(0) \geq A(1) \geq \cdots \geq \underline{A} \tag{8}$$

where

$$\underline{A}_{ij} = \begin{cases} 1 - h\underline{\delta}_i > 0 & \text{if } i = j \\ h\underline{\beta}_{ij} > 0 & \text{if } i \neq j,\ (i, j) \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

The resource model we consider is defined by

$$
\begin{aligned}
f_{ij}(\beta_{ij}(k-1), \beta_{ij}(k)) &= w_{ij} \log \left(\frac{\beta_{ij}(k-1)}{\beta_{ij}(k)} \right), \\
g_i(\delta_i(k-1), \delta_i(k)) &= w_{ii} \log \left(\frac{\overline{\Delta} - \delta_i(k-1)}{\overline{\Delta} - \delta_i(k)} \right).
\end{aligned} \tag{10}
$$

We assume bounded resource budget for each time step, i.e.,

$$
\Gamma(A(k-1), A(k)) = \sum_{ij} f_{ij} + \sum_i g_i \leq \overline{\Gamma}, \quad \forall k \geq 0. \tag{11}
$$

A natural choice of cost function is

$$
J(x) = \sum_{k=0}^{\infty} \alpha^k C x(k) \quad \text{s.t.} \quad (5) \text{ with } x(0) = x \tag{12}
$$

where $C = [c_1, \ldots, c_n] > 0$ is the row vector defining the cost associated with each node, $\alpha \in (0, 1)$ is the discount rate. A natural model predictive control problem formulation can be described by

$$
\begin{aligned}
\min_{\beta(k|t), \delta(k|t)} \quad & \sum_{k=0}^{K} \alpha^k C x(k|t) \\
\text{s.t.} \quad & x(k+1|t) = [A(k|t) - B(x, k, t)] x(k|t), \quad x(0|t) = x(t) \\
& \underline{\beta}_{ij} \leq \beta_{ij}(k|t) \leq \beta_{ij}(k-1|t), \; \delta_i(k-1|t) \leq \delta_i(k|t) \leq \overline{\delta}_i, \\
& \Gamma(A(k-1|t), A(k|t)) \leq \overline{\Gamma}, \; k = 0, \ldots, K-1.
\end{aligned} \tag{13}
$$

The above controller involves solving large-scale nonlinear programming problems online. To address the scalability issue, we will take an alternative way. The basic idea is to use the monotone properties of the system to construct bounds of $J(x)$ via a dynamic-programming-like formulation, where these bounds can be minimized via linear programming.

Proposition 1. *Suppose that there exists a sequence of row vectors $p(k) \in \mathbb{R}^{1 \times n}_{\geq 0}$ with $k = 0, \ldots, K-1$ such that the following inequalities hold:*

$$
p(k) \geq C + \alpha p(k+1) A(k), \; k = 0, \ldots, K-1 \tag{14}
$$

where $p(K) = p(K-1)$. Then $J(x) \leq \widehat{J}(x) := p(0)x$.

The closed-loop controller for the spreading process is obtained by solving the following control problem in a receding horizon fashion:

1) At time step t, measure the state $x(t)$ and solve the following resource-constrained risk minimization problem:

$$
\begin{aligned}
\min_{p(k|t), \beta(k|t), \delta(k|t)} \quad & R(p, x(t)) = p(0|t) x(t) + \lambda \|p(0|t)\|_1 \\
\text{s.t.} \quad & p(k|t) \geq 0, \quad p(k|t) \geq C + \alpha p(k+1|t) A(k|t), \\
& \underline{\beta}_{ij} \leq \beta_{ij}(k|t) \leq \beta_{ij}(k-1|t), \\
& \delta_i(k-1|t) \leq \delta_i(k|t) \leq \overline{\delta}_i, \\
& \Gamma(A(k-1|t), A(k|t)) \leq \overline{\Gamma}, \\
& k = 0, \ldots, K-1
\end{aligned} \tag{15}
$$

   where $\lambda > 0$ is a small regularization factor. Let (p^*, β^*, δ^*) be the optimal solution.

2) Implement the first step, i.e., $\beta(t) = \beta^*(0|t)$ and $\delta(t) = \delta^*(0|t)$.

3) When the $(t+1)$th sampling event comes, go to 1) and solve (15) with warm start of $p(k|t+1) = p^*(k+1|t), \beta(k|t+1) = \beta^*(k+1|t), \delta(k|t+1) = \delta^*(k+1|t)$ for $k = 0, \ldots, K-1$ where $p^*(K|t) = p^*(K-1|t), \beta^*(K|t) = \beta^*(K-1|t), \delta^*(K|t) = \delta^*(K-1|t)$.

2.3 Recursive Feasibility and Stability

Fact 1: The optimal solution lies on the boundary of (p, A)-inequalities if the regularized risk is minimized.

Proposition 2. *The optimal solution satisfies*

$$p^*(k|t) = C + \alpha p^*(k+1|t)A^*(k|t),\ k = 0, \dots, K-1 \tag{16}$$

where $p^*(K|t) = p^*(K-1|t)$.

Fact 2: Both A^* and p^* is strictly deceasing before hitting the lower bound.

Proposition 3. *If* $A^*(-1|t) \gneqq \underline{A}$*, then we have*

$$A^*(k-1|t) \gneqq A^*(k|t), \quad p^*(k-1|t) \gneqq p^*(k|t) \tag{17}$$

for $k = 0, \dots, l$*, where* $0 \le l < K$ *is the largest integer such that* $A^*(l|t) \gneqq \underline{A}$.

Fact 3: The risk minimization solution is an "all-in" policy.

Proposition 4. *If* $\Gamma(A^*(-1|t), \underline{A}) > \overline{\Gamma}$*, then* $\Gamma(A^*(-1|t), A^*(0|t)) = \overline{\Gamma}$.

Assumption 1. *The discount rate* α *satisfies* $\alpha < 1/\rho(A(-1))$ *where* $\rho(\cdot)$ *denotes the matrix spectrum radius.*

Assumption 2. K *is large enough such that* $C - (1-\alpha)p^*(K-1) > 0$.

Theorem 1. *If Assumptions 1 and 2 hold, then the resource-constrained minimization problem* (15) *is recursively feasible. Moreover,* $V(x_t) > V(x_{t+1})$ *for all* $t \ge K-1$ *and* $x_t \gneqq 0$.

Proof.

Recursive feasibility. Let $\overline{A} = A(-1) \gneqq 0$, Assumption 1 implies that $I - \alpha\overline{A}$ has a nonnegative inverse [1, p. 113]. For time point t and budget bound $\overline{\Gamma} > 0$, it is easy to find a sequence (e.g., one-step shift of the optimal solution at time $t-1$) such that

$$\overline{A} \ge A(-1|t) \ge A(0|t) \ge \cdots \ge A(K-1|t) \ge \underline{A} \gneqq 0 \tag{18}$$

and $\Gamma(A(k-1|t), A(k|t)) \le \overline{\Gamma}$ with $k = 0, \dots, K-1$, we have $\rho(A(K-1|t)) \le \rho(\overline{A})$ ([1, p. 15]) and thus $(I - \alpha A(K-1|t))^{-1} \gneqq 0$. A sequence p can be recursively computed via (16) with $p(K|t) = C(I - \alpha A(K-1|t))^{-1}$ and it is easy to verify that $p(k|t) \ge 0$ for $k = 0, \dots, K$. Thus, (p, A) is a feasible solution. Moreover, it is easy to verify that if Assumption 2 holds for $t = 0$, then it also holds for all $t > 0$.

Stability. From Propositions 3 and Assumption 2 we have

$$p^*(0|K-1) \lneqq p^*(1|K-2) \lneqq \cdots \lneqq p^*(K-1|0) < \frac{1}{1-\alpha}C. \tag{19}$$

Therefore, $C > (1-\alpha)p^*(0|t)$ for all $t \ge K-1$ and

$$\begin{aligned}
&V(x,t) - V(x,t+1) \\
&\quad = p^*(0|t)x(t) - p^*(0|t+1)x(t+1) \\
&\quad \ge p^*(0|t)x(t) - p^*(1|t)x(t+1) \quad \text{(ignore the effect of small } \lambda) \\
&\quad = (p^*(0|t) - p^*(1|t)A^*(0|t))x(t) + p^*(1|t)B(x,t)x(t) \\
&\quad = \frac{1}{\alpha}[C - (1-\alpha)p^*(0|t)]x(t) + p^*(1|t)B(x,t)x(t) \\
&\quad \ge \frac{1}{\alpha}[C - (1-\alpha)p^*(0|t)]x(t) > 0.
\end{aligned} \tag{20}$$

□

2.4 Geometric Programming

We show that the online optimal control problem (15) can be reformulated as a convex optimization problem, in particular an exponential cone program. We first introduce the coordinate transformation

$$y_i(k) = \log(p_i(k)),\ u_{ij}(k) = \log\left(\frac{\beta_{ij}(k-1)}{\beta_{ij}(k)}\right),\ u_{ii} = \log\left(\frac{\overline{\Delta} - \delta_i(k-1)}{\overline{\Delta} - \delta_i(k)}\right) \tag{21}$$

where $k = 0, \dots, K-1$ and $y_i(K-1) = y_i(K)$. The budget constraint (11) can be rewritten as

$$\Gamma(A(k-1), A(k)) = w_{ij}u_{ij}(k) + w_{ii}u_{ii}(k) = \sum_{ij} W \odot U(k) \leq \overline{\Gamma} \tag{22}$$

where $W = [w_{ij}]$ and $U(k) = [u_{ij}(k)]$. The bounded rate condition becomes

$$U(k) \geq 0, \quad \sum_{k=0}^{K} U(k) \leq \overline{U} \tag{23}$$

where

$$\overline{u}_{ij} = \log\left(\frac{\beta_{ij}(t-1)}{\underline{\beta}_{ij}}\right), \quad \overline{u}_{ii} = \log\left(\frac{\overline{\delta} - \delta_i(t-1)}{\overline{\Delta} - \overline{\delta}_i}\right). \tag{24}$$

The dynamic coupling constraint for p can be rewritten as

$$\sum_{i=1}^{n} \alpha p_i(k+1)a_{ij}(k) - p_j(k) \leq -c_j(k) \tag{25}$$

From (6) we have

$$\sum_{i\neq j} p_i(k+1)\alpha h\beta_{ij}(k) + p_j(k+1)\alpha(1-h\delta_j(k)) - p_j(k) \leq -c_j \tag{26}$$

which is equivalent to

$$\begin{aligned}\sum_{i\neq j} \frac{p_i(k+1)\alpha h\beta_{ij}(k)}{p_j(k)} + \frac{p_j(k+1)\alpha(1-h\Delta)}{p_j(k)} + \frac{p_j(k+1)\alpha h(\Delta - \delta_j(k))}{p_j(k)} \\ + \frac{c_j}{p_j(k)} \leq 1.\end{aligned} \tag{27}$$

In the new coordinate, the above inequality can be transformed into

$$\begin{aligned}\log\Bigg(&\sum_{(i,j)\in\mathcal{E}} \exp\left(y_i(k+1) - y_j(k) + \log(\alpha h\beta_{ij}(t-1)) - \sum_{l=0}^{k} u_{ij}(l)\right) \\ &+ \exp\left(y_j(k+1) - y_j(k) + \log(\alpha(1-h\overline{\Delta}))\right) \\ &+ \exp\left(y_j(k+1) - y_j(k) + \log(\alpha h(\overline{\Delta} - \delta_j(t-1)) - \sum_{l=0}^{k} u_{jj}(l)\right) \\ &+ \exp(\log c_j - y_j(k))\Bigg) \leq 0\end{aligned} \tag{28}$$

The cost minimization can be represented by $\min J_y$ with

$$\log\left(\sum_i \exp[\log(x_i(t)+\lambda) + y_i(0) - J_y]\right) \leq 0 \tag{29}$$

3 Proofs

3.1 Proof of Proposition 1

We first extend the sequence of p to any $k \geq 0$, that is, $p(k) = p(K-1)$ for $k \geq K$. Eq. (14) implies

$$p(k) \geq C + \alpha p(k+1)A(k) \tag{30}$$

for $k = 0, \ldots, K-1$. For $k \geq K$ we have

$$\begin{aligned} p(k) = p(K-1) &\geq C + \alpha p(K)A(K-1) \\ &= C + \alpha p(k+1)A(K-1) \geq C + \alpha p(k+1)A(k), \end{aligned} \tag{31}$$

where the last inequality follows by (8). By defining $V(x,k) := p(k)x(k)$ we obtain

$$\begin{aligned} V(x,k) &\geq Cx(k) + \alpha p(k+1)A(k)x(k) \\ &\geq Cx(k) + \alpha p(k+1)[A(k) - B(x,k)]x(k) \\ &= Cx(k) + \alpha V(x,k+1) \end{aligned} \tag{32}$$

for all $k \geq 0$. Telescoping sum gives

$$V(x,0) \geq \sum_{k=0}^{N-1} \alpha^k Cx(k) + \alpha^N V(x,N+1). \tag{33}$$

Note that $\lim_{N\to\infty} \alpha^N V(x,N+1) = 0$ as both $p(N+1)$ and $x(N+1)$ are bounded. Thus, for any time k, $V(x,k)$ provides an upper bound on the cost-to-go from state $x(k)$, i.e., $J(x(k)) \leq V(x,k)$.

3.2 Proof of Proposition 2

Suppose that $0 \leq l < K-1$ is the smallest integer such that

$$p^*(l) \gneqq C + \alpha p^*(l+1)A^*(l). \tag{34}$$

If $l = 0$, we can construct a new vector sequence $\tilde{p}$ with $\tilde{p}(k|t) = p^*(k|t)$ for $k > 0$ and $\tilde{p}(0|t) = C + \alpha p^*(1|t)A^*(0|t)$. Note that $\sum(p_i^*(0|t) - \tilde{p}_i(0|t)) > 0$ as $p^*(0|t) \gneqq \tilde{p}(0|t)$. Thus, we have

$$R(p^*,x) - R(\tilde{p},x) = (p^*(0|t) - \tilde{p}(0|t))(x + \lambda\mathbf{1}) \geq \lambda \sum(p_i^*(0|t) - \tilde{p}_i(0|t)) > 0 \tag{35}$$

which contradicts with the optimality of p^*.

For the case of $l \geq 1$, we construct a sequence $\tilde{p}$ with $\tilde{p}(k|t) = p^*(k|t)$ for $l < k \leq K-1$ and

$$\tilde{p}(k|t) = C + \alpha\tilde{p}(k+1|t)A^*(k|t),\ k = l, l-1, \ldots, 0. \tag{36}$$

By comparing p^* and $\tilde{p}$ we have

$$p^*(k|t) - \tilde{p}(k|t) = \alpha[p^*(k+1|t) - \tilde{p}(k+1|t)]A(k|t),\ k = 0, \ldots, l-1. \tag{37}$$

Recursively applying the above relationship yields

$$p^*(0|t) - \tilde{p}(0|t) = \alpha^l[p^*(l|t) - \tilde{p}(l|t)]A(l-1|t)\cdots A(0|t). \tag{38}$$

By the construction of $\tilde{p}$ we have $p^*(l|t) \gneqq \tilde{p}(l|t)$. Moreover, since $A(k|t) \geq 0$ has positive diagonal, we can conclude that $p^*(0|t) \gneqq \tilde{p}(0|t)$. The rest is same as the case of $l = 0$.

3.3 Proof of Proposition 3

Main steps:

(1) **Assume that there is no resource allocated at time step s, i.e., $A^*(s-1|t) = A^*(s|t)$**

We first prove $A^*(k-1|t) \gneqq A^*(k|t)$ by contradiction. Suppose that there exist $0 \le k \le l$ such that $A^*(k-1|t) = A^*(k|t)$, i.e., there is no resource allocated at time k. Let s be the largest k satisfying $A^*(k-1|t) = A^*(k|t)$. The basic idea is to construct feasible sequences $\tilde{A}$ and $\tilde{p}$ which yield smaller cost.

(2) **Construct a new sequence $\tilde{A}$ by allocating a bit of resources on certain link/node, then show that $\tilde{A}$ satisfies Constraints (7), (11).**

We consider sequence constructions for the case where there exists an edge $(i,j) \in \mathcal{E}$ such that $\beta^*_{ij}(l) > \underline{\beta}_{ij}$. Similar construction can also be applied to the case with $\delta_i(l) < \overline{\delta}_i$. First, we define a new sequence

$$\tilde{\beta}_{ij}(k|t) = \begin{cases} \eta\beta^*_{ij}(k|t), & s \le k \le l \\ \beta^*_{ij}(k|t), & 0 \le k < s \end{cases} \tag{39}$$

where $\eta = \max\left(e^{-\Gamma/w_{ij}}, \underline{\beta}_{ij}/\beta^*_{ij}(l)\right) \in (0,1)$. Note that (7) holds as $\tilde{\beta}_{ij}(l) \ge \underline{\beta}_{ij}$. And the bounded budget condition (11) holds for $s+1 \le k \le l$ as

$$f_{ij}(\tilde{\beta}_{ij}(k-1|t), \tilde{\beta}_{ij}(k|t)) = w_{ij}\log\left(\frac{\eta\beta^*_{ij}(k-1|t)}{\beta^*_{ij}(k|t)}\right) = f_{ij}(\beta^*_{ij}(k-1|t), \beta^*_{ij}(k|t)). \tag{40}$$

Moreover, it also holds for time $k = s$ since

$$\Gamma(\tilde{A}(s-1|t), \tilde{A}(s|t)) = w_{ij}\log\left(\frac{\beta^*_{ij}(s-1|t)}{\eta\beta^*_{ij}(s|t)}\right) = -w_{ij}\log(\eta) \le \Gamma. \tag{41}$$

For $k < s$, Inequality (11) holds as $\tilde{A}(k|t) = A^*(k|t)$. Therefore, $\tilde{A}$ is a feasible solution.

(3) **Construct a new sequence $\tilde{p}$ based on $\tilde{A}$ and (16), then show that $p^*(0|t) \gneqq \tilde{p}(0|t)$ leading to the contradiction of optimality.**

We now construct a feasible $\tilde{p}$ based on $\tilde{A}$:

$$\tilde{p}(k|t) = C + \alpha\tilde{p}(k+1|t)\tilde{A}(k|t), \quad k = K-2, \dots, 0 \tag{42}$$

where $\tilde{p}(K-1|t) = C\big(I - \alpha\tilde{A}(K-1|t)\big)^{-1}$.

If $l = K-1$ we have

$$\begin{aligned} p^*(l|t) - \tilde{p}(l|t) &= C(I - \alpha A^*(l|t))^{-1} - C(I - \alpha\tilde{A}(l|t))^{-1} \\ &= \alpha C(I - \alpha A^*(l|t))^{-1}[A^*(l|t) - \tilde{A}(l|t)](I - \alpha\tilde{A}(l|t))^{-1} \gneqq 0. \end{aligned} \tag{43}$$

When $l < K-1$ we have

$$p^*(l|t) - \tilde{p}(l|t) = \alpha p^*(l+1|t)[A^*(l|t) - \tilde{A}(l|t)] \gneqq 0 \tag{44}$$

as $p^*(l+1|t) > 0$ and $A^*(l|t) - \tilde{A}(l|t) \gneqq 0$. Similarly, since $p^*(k+1|t) > 0$ and $A^*(k|t) - \tilde{A}(k|t) \gneqq 0$ for $s \le k < l$, we can have

$$\begin{aligned} &p^*(k|t) - \tilde{p}(k|t) \\ &\quad = \alpha[p^*(k+1|t)A^*(k|t) - \tilde{p}(k+1|t)\tilde{A}(k|t)] \\ &\quad = \alpha[p^*(k+1|t)(A^*(k|t) - \tilde{A}(k|t)) + (p^*(k+1|t) - \tilde{p}(k+1|t))\tilde{A}(k|t)] \\ &\quad \gneqq \alpha(p^*(k+1|t) - \tilde{p}(k+1|t))\tilde{A}(k|t). \end{aligned} \tag{45}$$

Recursively applying the above inequality yields

$$p^*(s|t) - \tilde{p}(s|t) \gneqq \alpha^{l-s}(p^*(l|t) - \tilde{p}(l|t))\tilde{A}(l-1|t)\cdots\tilde{A}(s|t) \gneqq 0, \tag{46}$$

which leads to

$$\begin{aligned} p^*(0|t) - \tilde{p}(0|t) &= \alpha(p^*(1|t) - \tilde{p}(1|t))A^*(0|t) \\ &= \alpha^s(p^*(s|t) - \tilde{p}(s|t))A^*(s-1|t)\cdots A^*(0|t) \gneqq 0. \end{aligned} \tag{47}$$

This also implies that $R(p^*, x) - R(\tilde{p}, x) \geq \lambda \sum_i (p_i^*(0|t) - \tilde{p}_i(0|t)) > 0$ which contradicts with the optimality of p^*.

- **The second part relies on the fact $A^*(k-1|t) \geq A^*(k|t)$.**

Now we prove $p^*(k-1|t) \gneqq p^*(k|t)$. First, we consider the case of $1 \leq k \leq l$. From (16) we have

$$\begin{aligned} p^*(k-1|t) - p^*(k|t) &= \alpha[p^*(k|t)A^*(k-1|t) - p^*(k+1|t)A^*(k+1|t)] \\ &= \alpha[p^*(k|t)(A^*(k-1|t) - A^*(k|t)) + \\ &\qquad (p^*(k|t) - p^*(k+1|t))A^*(k|t)] \\ &\gneqq \alpha(p^*(k|t) - p^*(k+1|t))A^*(k|t). \end{aligned} \tag{48}$$

Recursively applying the above inequality yields

$$p^*(k-1|t) - p^*(k|t) \gneqq \alpha^{l-k+1}(p^*(l|t) - p^*(l+1|t))A^*(l|t). \tag{49}$$

If $l = K-1$, then $p^*(k-1|t) - p^*(k|t) \gneqq 0$ as $p^*(l|t) = p^*(l+1|t)$. For $l \leq K-2$, we have $A^*(k|t) = \underline{A}$ for $l < k \leq K-1$. From (16) we can obtain $p^*(l|t) = p^*(l+1|t)$ since

$$\begin{aligned} p^*(K-1|t) &= C + \alpha p^*(K-1|t)\underline{A}, \\ p^*(K-2|t) &= C + \alpha p^*(K-1|t)\underline{A} = p^*(K-1|t), \\ &\vdots \\ p^*(l+1|t) &= C + \alpha p^*(l+2|t)\underline{A} = \cdots = p^*(K-1|t), \\ p^*(l|t) &= C + \alpha p^*(l+1|t)\underline{A} = \cdots = p^*(K-1|t). \end{aligned} \tag{50}$$

Thus, we have $p^*(k-1|t) \gneqq p^*(k|t)$ for $k = 1, \ldots, l$.

Now we consider the case of $k = 0$. First, applying the above conclusion to the optimal solution at time $(t-1)$ yields $p^*(0|t-1) \gneqq p^*(1|t-1)$. Also, $\{p^*(1|t-1), \ldots, p^*(K-1|t-1), p^*(K-1|t-1)\}$ is a feasible solution to the optimal control problem at t. The principle of optimality implies that $p^*(1|t-1) \geq p^*(0|t)$, which further leads to $p^*(-1|t) = p^*(0|t-1) \gneqq p^*(0|t)$.

4 Proof of Proposition 4

We first introduce the new decision variables

$$U_{ij}(k|t) = \begin{cases} f_{ij}(\beta_{ij}(k-1|t), \beta_{ij}(k|t)), & \text{if } i \neq j,\ (i,j) \in \mathcal{E} \\ g_i(\delta_i(k-1|t), \delta_i(k|t)), & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{51}$$

where $k = 0, \ldots, K-1$. Given $A(k-1|t)$ and $U(k|t)$, the resource model (10) defines a mapping $\mathcal{F}$ such that $A(k|t) = \mathcal{F}(A(k-1|t), U(k|t))$. Note that the map $\mathcal{F}$ satisfies $\mathcal{F}(A, U_0 + U_1) = \mathcal{F}(\mathcal{F}(A, U_0), U_1) = \mathcal{F}(\mathcal{F}(A, U_1), U_0)$.

Let $\gamma := \overline{\Gamma} - \sum W \odot U^*(0|t) \in [0, \overline{\Gamma}]$ where W is the weighting matrix. We will show that $\gamma = 0$ by contradiction. Since $\Gamma(A^*(-1|t), \underline{A}) > \overline{\Gamma}$, then there exists a $0 \leq k \leq K-1$ such that $A^*(k|t) \gneqq \underline{A}$. Otherwise, $\Gamma(A^*(-1|t), A^*(0|t)) = \Gamma(A^*(-1|t), \underline{A}) > \overline{\Gamma}$ which violates the budget condition (11). Let $0 \leq l \leq K-1$ be the largest integer such that $A^*(l|t) \gneqq \underline{A}$. We have $\gamma' := \min\{\Gamma(A^*(l|t), \underline{A}), \gamma\}$ satisfying $0 < \gamma' \leq \overline{\Gamma}$, which means that we can find a nonnegative matrix U with $U_{ij} \geq 0$ for $i = j$ or $(i,j) \in \mathcal{E}$ satisfying

$$\mathcal{F}(A^*(l|t), U) \geq \overline{A} \quad \text{and} \quad \sum W \odot U \leq \gamma'. \tag{52}$$

Now we construct a new sequence $\tilde{U}$ with $\tilde{U}(0|t) = U^*(0|t) + U$ and $\tilde{U}(k|t) = U^*(k|t)$ for $k = 1, \ldots, K-1$. Then, a sequence $\tilde{A}$ can be computed recursively:

$$\tilde{A}(k|t) = \mathcal{F}(\tilde{A}(k-1|t), \tilde{U}(k|t)), \quad k = 0, \ldots, K-1 \tag{53}$$

where $\tilde{A}(-1|t) = A^*(-1|t)$. It is easy to verify that $\tilde{A}$ is a feasible solution since

$$\begin{aligned}\tilde{A}(l|t) &= \mathcal{F}\left(A^*(-1|t), U + \sum_{k=0}^{l} U(k|t)\right) \\ &= \mathcal{F}\left(\mathcal{F}\left(A^*(-1|t), \sum_{k=0}^{l} U(k|t)\right), U\right) = \mathcal{F}(A^*(l|t), U) \geq \underline{A}\end{aligned} \tag{54}$$

and

$$\begin{aligned}\Gamma(\tilde{A}(-1|t), \tilde{A}(0|t)) &= \sum W \odot (U^*(0|t) + U) = \overline{\Gamma} - \gamma + \gamma' \leq \overline{\Gamma}, \\ \Gamma(\tilde{A}(k-1|t), \tilde{A}(k|t)) &= \Gamma(A^*(k-1|t), A^*(k|t)) \leq \overline{\Gamma}, \quad k = 1, \ldots, K-1.\end{aligned} \tag{55}$$

Since $A^*(k|t) \gneqq \tilde{A}(k|t)$ for $0 \leq k \leq l$, similar technique in the proof of Proposition 3 can be used to obtain a contradiction.

References

[1] A. Berman and R. J. Plemmons, *Nonnegative matrices in the mathematical sciences.* SIAM, 1994.