Econometrics HW8

陈子睿 15220212202842

December 17, 2023

EX 1

Solution We consider the simple regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$ under assumptions of strict exogeneity, $\{(Y_t, X_t')'\}_{t=1}^n$ are IID and $\mathbb{E}X_i^4, \mathbb{E}Y_i^4 < \infty$.

(a) We need to check the two conditions for valid instruments. For **instrument relevance**, since there is only one regressor X, and the coefficient of the first stage regression of regressing X_t on Z_t is given by

$$\gamma = [\mathbb{E}(Z_t Z_t')]^{-1}\mathbb{E}(Z_t X_t') = [\mathbb{E}(X_t X_t')]^{-1}\mathbb{E}(X_t X_t') = 1.$$

And for **instrument exogeneity**, under assumption of strict exogeneity, we know that $\mathbb{E}(u_t|X_t) = 0$. Which implies that $corr(X_t, u_t) = 0$. Hence X_t is a valid instrument.

(b) In the given simple regression model, there's actually no exogeneity exists, hence condition 1 is obviously satisfied since

$$\mathbb{E}(u_t|W_{1t}, ..., W_{rt}) = \mathbb{E}(u_t) = \mathbb{E}[\mathbb{E}(u_t|X_t)] = 0.$$

And condition 2 is satisfied by our IID assumption that $\{(Y_t, X_t')'\}_{t=1}^n$ are IID. Condition 3 is satisfied by applying $\mathbb{E}X_i^4, \mathbb{E}Y_i^4 < \infty$. Condition 4 is satisfied because of the result obtained in (a).

(c) For the $2SLS$ method, we have

$$\hat{\beta}_{2SLS} = (\hat{X}_t'\hat{X}_t)^{-1}\hat{X}_t'Y_t = (X_t'X_t)^{-1}X_t'Y_t = \hat{\beta}_{OLS}.$$

EX 2

Solution We consider a product market with a supply function $Q_i^s = \beta_0 + \beta_1 P_i + u_i^s$, a demand function $Q_i^d = \gamma_0 + u_i^d$, and a market equilibrium condition $Q_i^s = Q_i^d$, where u_i^s and u_i^d are mutually independent IID r.v.s with zero mean.

(a) We combine

$$\begin{cases} Q_i^s = \beta_0 + \beta_1 P_i + u_i^s \\ Q_i^d = \gamma_0 + u_i^d \\ Q_i^s = Q_i^d \end{cases}$$

obtaining $P_i = \frac{\gamma_0 - \beta_0}{\beta_1} + \frac{u_i^d - u_i^s}{\beta_1}$. Therefore we have

$$\begin{aligned}
cov(P, u^s) &= \mathbb{E}(P_i u_i^s) - \mathbb{E}(P_i)\mathbb{E}(u_i^s) \\
&= \mathbb{E}(\frac{\gamma_0 - \beta_0}{\beta_1} u_i^s) + \mathbb{E}(\frac{u_i^d - u_i^s}{\beta_1} u_i^s) - \mathbb{E}(\frac{\gamma_0 - \beta_0}{\beta_1} + \frac{u_i^d - u_i^s}{\beta_1})\mathbb{E}(u_i^s) \\
&= 0 + \frac{1}{\beta_1}[\mathbb{E}(u_i^d u_i^s) - \mathbb{E}(u_i^s)^2] - 0 \\
&= \frac{-\mathbb{E}(u_i^s)^2}{\beta_1} \\
&= \frac{-\sigma_{u^s}^2}{\beta_1},
\end{aligned}$$

which indicates that P_i and u_i^s are correlated.

(b) For OLS estimator, we have

$$\begin{aligned}
\hat{\beta}_1^{OLS} - \beta_1 &= (P'P)^{-1}P'u^s \\
&= (\frac{1}{n}\sum_{i=1}^n P_i^2)^{-1}(\frac{1}{n}\sum_{i=1}^n P_i u_i^s) \\
&\xrightarrow{p} \mathbb{E}(P_i^2)^{-1}\mathbb{E}(P_i u_i^s) \\
&\neq 0,
\end{aligned}$$

by WLLN. Hence the OLS estimator for β_1 is inconsistent.

(c) We notice that the demand Q_i^d is completely inelastic, which means that Q_i will not be affected by the shifts in supply. Therefore γ_0 can be estimated simply by using OLS, yielding

$$\hat{\gamma}_0^{OLS} = \frac{1}{n} Q_i^d.$$

We have discussed in (b) that the OLS estimator for β_1 is inconsistent, therefore we need an IV. In this case, the observed $\{Q_i\}_{i=1}^n$ can serve as the instrument.

First we consider the auxiliary linear regression model

$$P_i = \delta_0 + \delta_1 Q_i^d + \nu_i,$$

we obtain by applying OLS to the auxiliary regression,

$$\hat{\delta}_{OLS} = (Q^{d'} Q^d)^{-1} Q^{d'} P,$$

with

$$\hat{P} = \hat{\delta}^{OLS'} Q^d.$$

Then we have

$$\hat{\beta}_{2SLS} = (\hat{P}'\hat{P})^{-1}\hat{P}'Q^s.$$

EX 3

Solution

(a) There are some other factors that could affect the choice to serve in the military and the annual earnings. For example, the **level of education** is obviously endogenous. Another important variable is **comprehensive ability** which is hard to measure, but it is indeed endogenous.

Therefore the OLS estimates are much likely to be unreliable.

(b) Since the draft was determined by a national lottery, namely it was selected stochasticly. Thus the lottery number is uncorrelated with any factors that may affect earning and hence the instrument is exogenous. And since it had a effect on the individual's probability of serving in the military, the lottery number is somehow correlated with X_i.

EX 4

Solution We consider a simple regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$ with measurement errors $X_i^* = X_i + e_i$ and another set of independent variables with different measurement errors $X_i^{**} = X_i + \epsilon_i$, and we assume that e_i and ϵ_i are both IID with zero mean and mutually independent.

(a) Because we only observe (Y_i, X_i^*, X_i^{**}), we are forced to estimate the following regression model

$$Y_i = \beta_0 + \beta_1 X_i^* + \nu_i,$$

where ν_i is some unobservable disturbance different from the true disturbance u_i. Even though the regression model is correctly specified, we no longer have $\mathbb{E}(\nu_i|X_i^*) = 0$ due to the existence of measurement error. We have

$$\begin{aligned}
\nu_i &= Y_i - \beta_0 - \beta_1 X_i^* \\
&= (\beta_0 + \beta_1 X_i + u_i) - \beta_0 - \beta_1 (X_i + e_i) \\
&= u_i - \beta_1 e_i,
\end{aligned}$$

which indicates that the regression error ν_i contains the true disturbance u_i and a linear combination of measurement error e_i.

Now the expectation

$$\begin{aligned}
\mathbb{E}(X_i^* \nu_i) &= \mathbb{E}[(X_i + e_i)\nu_i] \\
&= \mathbb{E}(X_i \nu_i) + \mathbb{E}(e_i \nu_i) \\
&= 0 - \beta_1 \mathbb{E}(e_i^2) \\
&= -\beta_1 \sigma_e^2 \\
&\neq 0.
\end{aligned}$$

Now we want to show that the new observed independent variables X_i^{**} can be a valid instrument. First we check the exogeneity of X_i^{**}, yielding

$$\begin{aligned}
\mathbb{E}(X_i^{**} \nu_i) &= \mathbb{E}[(X_i + \epsilon_i)\nu_i] \\
&= \mathbb{E}(X_i \nu_i) + \mathbb{E}(\epsilon_i \nu_i) \\
&= 0 + \mathbb{E}(\epsilon_i u_i) - \beta_1 \mathbb{E}(\epsilon_i e_i) \\
&= 0 + 0 - 0 \\
&= 0.
\end{aligned}$$

Then we check the first stage regression estimated coefficient is nonzero.Namely for the auxiliary regression model $X_i^* = \gamma_0 + \gamma_1 X_i^{**} + \omega_i$, we need to check that $\hat{\gamma}_1 \neq 0$.We have

$$\begin{aligned}\hat{\gamma}_1 &= (X_i^{**'} X_i^{**})^{-1} X_i^{**'} X_i^* \\ &\xrightarrow{p} \mathbb{E}(X_i^{**2})^{-1} \mathbb{E}(X_i^{**} X_i^*) \\ &= 1 \\ &\neq 0.\end{aligned}$$

Therefore, the new observed independent variables X_i^{**} can be a valid instrument.

(b) We directly show the result

$$\begin{aligned}\beta_1^{2\hat{S}LS} &= (\hat{X}^{*'} \hat{X}^*)^{-1} \hat{X}^{*'} Y \\ &= [(X^{**}\hat{\gamma})'(X^{**}\hat{\gamma})]^{-1}(X^{**}\hat{\gamma})'Y \\ &= \{[X^{**}(X^{**'}X^{**})^{-1}X^{**'}X^*]'[X^{**}(X^{**'}X^{**})^{-1}X^{**'}X^*]\}^{-1}[X^{**}(X^{**'}X^{**})^{-1}X^{**'}X^*]'Y \\ &= [X^{*'}X^{**}(X^{**'}X^{**})^{-1}X^{**'}X^{**}(X^{**'}X^{**})^{-1}X^{**'}X^*]^{-1}X^{*'}X^{**}(X^{**'}X^{**})^{-1}X^{**'}Y \\ &= [X^{*'}X^{**}(X^{**'}X^{**})^{-1}X^{**'}X^*]^{-1}X^{*'}X^{**}(X^{**'}X^{**})^{-1}X^{**'}Y \\ &= [\frac{X^{*'}X^{**}}{n}(\frac{X^{**'}X^{**}}{n})^{-1}\frac{X^{**'}X^*}{n}]^{-1}\frac{X^{*'}X^{**}}{n}(\frac{X^{**'}X^{**}}{n})^{-1}\frac{X^{**'}Y}{n}.\end{aligned}$$

Using our expression of $Y = X^*\beta + \nu$, we have

$$\begin{aligned}\beta_1^{2\hat{S}LS} - \beta_1 &= [\frac{X^{*'}X^{**}}{n}(\frac{X^{**'}X^{**}}{n})^{-1}\frac{X^{**'}X^*}{n}]^{-1}\frac{X^{*'}X^{**}}{n}(\frac{X^{**'}X^{**}}{n})^{-1}\frac{X^{**'}\nu}{n} \\ &\xrightarrow{p} [\mathbb{E}(X^{*'}X^{**})\mathbb{E}(X^{**'}X^{**})^{-1}\mathbb{E}(X^{**'}X^*)]^{-1}\mathbb{E}(X^{*'}X^{**})\mathbb{E}(X^{**'}X^{**})^{-1}\mathbb{E}(X^{**'}\nu) \\ &= 0.\end{aligned}$$

By WLLN.