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Econometrics HW2 陈韬 15220212202842

1.(a) Derive the $\hat{\beta}_0^{OLS}$ and $\hat{\beta}_1^{OLS}$ by minimizing the SSE $\sum_i (Y_i - b_0 - b_1 X_i)^2$

Solution. Consider the FOC:

$$\begin{cases} \frac{\partial SSE}{\partial b_0} = -2\sum_i (Y_i - b_0 - b_1 X_i) = 0 \\ \frac{\partial SSE}{\partial b_1} = -2\sum_i X_i (Y_i - b_0 - b_1 X_i) = 0 \end{cases}$$

We can rearrange and solve for b_1 which gives the OLS estimator for b_1:

$$\hat{\beta}_1^{OLS} = \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2} \xrightarrow{p} \frac{cov(Y_i, X_1)}{var(X_1)}.$$

and for b_0:

$$\hat{\beta}_0^{OLS} = \bar{Y} - \hat{\beta}_1^{OLS} \bar{X}_1$$

(b) Show that

$$\hat{\beta}_1^{OLS} \xrightarrow{p} \beta_1 + \frac{cov(X_i, u_i)}{var(X_i)}.$$

Proof. By our assumption, we have

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

From (a) we've shown that

$$\begin{aligned} \hat{\beta}_1^{OLS} &= \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2} \\ &= \frac{\sum_i (X_i - \bar{X})\left[\beta_0 + \beta_1 X_i + u_i - \beta_0 + \beta_1 \bar{X} + \bar{u}\right)}{\sum_i (X_i - \bar{X})^2} \\ &= \frac{\sum_i \left[(X_i - \bar{X}) \cdot \beta_1 (X_i - \bar{X}) + (X_i - \bar{X})(u_i - \bar{u})\right]}{\sum_i (X_i - \bar{X})^2} \\ &= \beta_1 + \frac{\sum_i (X_i - \bar{X})(u_i - \bar{u})}{\sum_i (X_i - \bar{X})} \xrightarrow{p} \beta_1 + \frac{cov(X_i, u_i)}{var(X_i)} \end{aligned}$$

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(c) If $E[u_iX_i]=0$ and $Eu_i=0$, is the $\hat{\beta}_1^{OLS}$ consistent?

Claim: $\hat{\beta}_1^{OLS}$ is consistent.

Proof. Since $cov(u_i, X_i)=E(u_iX_i)-Eu_iEX_i=0-0=0$, therefore by previous proof, we have

$$\hat{\beta}_1^{OLS} \xrightarrow{P} \beta_1+\frac{cov(u_i, X_i)}{var(X_i)}=\beta_1 .$$

(d) If $E[u_iX_i]=0$, but $Eu_i\neq 0$, is the $\hat{\beta}_1^{OLS}$ consistent?

Claim: $\hat{\beta}_1^{OLS}$ is NOT consistent.

Proof. Obvious without the zero mean assumption, the error term will have a systematic component that would bias the OLS estimates.

(e) If $E[u_i|X_i]=0$, is the $\hat{\beta}_1^{OLS}$ consistent?

Claim: $\hat{\beta}_1^{OLS}$ is consistent.

Proof. Since

$$E(u_i)=E[E(u_i|X_i)]=0$$

and

$$E(u_iX_i)=E[X_iE(u_i|X_i)]=E(X_i\cdot 0)=0 .$$

Then by the conclusion of (c), we finish the proof.

2. Let $Y_i=\beta_0+\beta_1X_i+u_i$ but $Eu_i\neq 0$, reparametrize the model s.t. $Y_i=\beta_0^*+\beta_1X_i+e_i$ s.t. $Ee_i=0$

Proof. Assume that $Eu_i=\mu\neq 0$, let $\beta_0=\beta^*-\mu$, i.e.

$$\begin{aligned}Y_i&=\beta_0^*+\beta_1X_i+(u_i-\mu)\\&=\beta_0^*+\beta_1X_i+e_i\end{aligned}$$

with

$$Ee_i=E(u_i-\mu)=0 .$$

3. For a model without the intercept:

$$Y_i = \beta_1 X_i + u_i$$

Derive $\hat{\beta}_1^{OLS}$ for β_1.

Solution. We have:

$$SSE = \sum_i (Y_i - b_1 X_i)^2$$
$$= \sum_{i=1}^n Y_i^2 - 2b_1 \sum_{i=1}^n Y_i X_i + b_1^2 \sum_{i=1}^n X_i^2$$

Differentiating w.r.t b_1:

$$\frac{\partial SSE}{\partial b_1} = -2\sum_{i=1}^n Y_i X_i + 2b_1 \sum_{i=1}^n X_i^2$$

Letting $\frac{\partial SSE}{\partial b_1} = 0$ to solve for $\hat{\beta}_1^{OLS}$, we have:

$$\hat{\beta}_1^{OLS} = \frac{\sum_{i=1}^n Y_i X_i}{\sum_{i=1}^n X_i^2}$$

4. (a) EX 4.7. Show that $\hat{\beta}_0$ is unbiased for β_0

Proof. We've shown that in simple regression model,

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

Therefore

$$E\hat{\beta}_0 = E\bar{Y} - E(\hat{\beta}_1 \bar{X})$$
$$= E(\beta_0 + \beta_1 \bar{X} + \bar{u}) - E(\hat{\beta}_1 \bar{X})$$
$$= \beta_0 + \beta_1 \bar{X} + E\bar{u} - \bar{X} E\hat{\beta}_1$$
$$= \beta_0 + \beta_1 \bar{X} + 0 - \beta_1 \bar{X}$$
$$= \beta_0$$

(b) EX 4.9.

(i) $\hat{\beta}_1 = 0 \Rightarrow R^2 = 0$

(ii) $R^2 = 0 \overset{?}{\Rightarrow} \hat{\beta}_1 = 0$

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(i) Proof. If $\hat{\beta}_1 = 0$, then

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i = \hat{\beta}_0$$

The sum of squared error (residual) is:

$$\begin{aligned} SSR &= \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 \\ &= \sum_{i=1}^{n} (Y_i - \hat{\beta}_0)^2 \end{aligned}$$

Under the assumption of strict exogeinity, $\hat{\beta}_1$ is an unbiased estimates for β_1, therefore

$$Y_i = \beta_0 + u_i$$

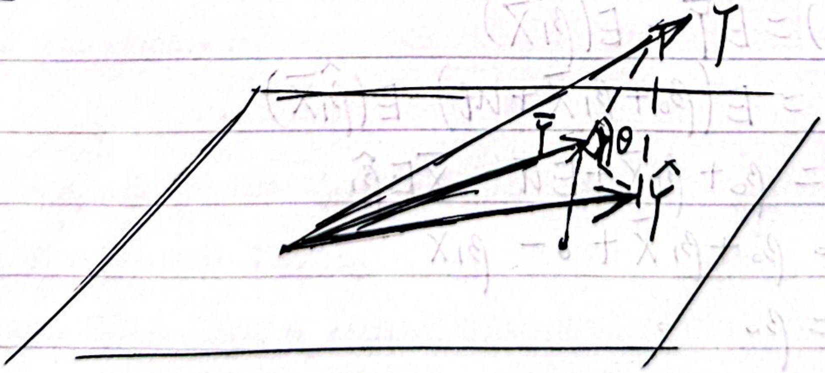
Therefore the best OLS prediction of any X_i is $\hat{\beta}_0^{OLS} = \bar{Y}$, i.e.

$$SSR = \sum_{i=1}^{n} (Y_i - \bar{Y}) = SST$$

Therefore

$$R^2 = 1 - \frac{SSR}{SST} = 0.$$

(ii) Proof. Claim: R^2 doesn't mean $\hat{\beta}_1 = 0$. It may occur that the variability explained by the model is counteracted by the variability introduced due to the model's errors



$R^2 = \cos^2\theta$, θ can be $\frac{\pi}{2}$, i.e. $R^2 = 0$.

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(c) EX 4.10, b.

Sol. By the previous lemma, we have

$$\sigma^2_{\hat\beta_1} = \frac{1}{n}\frac{var[(X_i-\mu_x)u_i]}{[var X_i]^2}$$

$$= \frac{1}{n}\cdot\frac{var(X_iu_i)-var(\mu_x u_i)}{(Var X_i)^2}$$

$$= \frac{1}{n}\cdot\frac{E[(X_iu_i)^2]-E(X_iu_i)^2-\mu_x var u_i}{(var X_i)^2}$$

$$= \frac{1}{n}\cdot\frac{E\left[E[X_i^2u_i^2|X_i]\right]-\mu_x\cdot E[var u_i|X_i]}{(var X_i)^2}$$

$$= \frac{1}{n}\cdot\frac{E(u_i^2|X_i=1)-0.2\times 1.6}{0.16^2}$$

$$= \frac{1}{n}\cdot\frac{4-0.32}{0.16^2} = \frac{575}{4n}$$