Econometrics HW7

陈子睿 15220212202842

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EX 1

Solution We consider the fixed effects regression model,

$$Y_{it} = \beta_1 X_{it} + \alpha_i + u_{it} \tag{1}$$

And we use the binary variables to develop the fix effects regression model, which is given by:

$$Y_{it} = \beta_0 + \beta_1 X_{it} + \gamma_2 D2_i + \gamma_3 D3_i + ... + \gamma_n Dn_i + u_{it}, \tag{2}$$

where the binary variable $D1_i$ is omitted.

Therefore we have:

(a) For entity 1 in time period 1, namely $i = 1, t = 1$, the regression model becomes

$$Y_{11} = \beta_0 + \beta_1 X_{11},$$

where $slope = \beta_1$ and $intercept = \beta_0$.

(b) For entity 1 in time period 3, namely $i = 1, t = 3$, the regression model becomes

$$Y_{13} = \beta_0 + \beta_1 X_{13},$$

where $slope = \beta_1$ and $intercept = \beta_0$.

(c) For entity 3 in time period 1, namely $i = 3, t = 1$, the regression model becomes

$$Y_{31} = \beta_0 + \beta_1 X_{31} + \gamma_3,$$

where $slope = \beta_1$ and $intercept = \beta_0 + \gamma_3$.

(d) For entity 3 in time period 3, namely $i = 3, t = 3$, the regression model becomes

$$Y_{33} = \beta_0 + \beta_1 X_{33} + \gamma_3,$$

where $slope = \beta_1$ and $intercept = \beta_0 + \gamma_3$.

Next we consider the time fixed effects regression model:

$$Y_{it} = \beta_0 + \beta_1 X_{it} + \delta_B 2_t + ... + \delta_T BT_t + u_{it}, \tag{3}$$

where the binary variable $B1_t$ is omitted.

Therefore we have:

(a) For entity 1 in time period 1, namely $i = 1, t = 1$, the regression model becomes

$$Y_{11} = \beta_0 + \beta_1 X_{11},$$

where $slope = \beta_1$ and $intercept = \beta_0$.
(b) For entity 1 in time period 3, namely $i = 1, t = 3$, the regression model becomes

$$Y_{13} = \beta_0 + \beta_1 X_{13} + \delta_3,$$

where $slope = \beta_1$ and $intercept = \beta_0 + \delta_3$.
(c) For entity 3 in time period 1, namely $i = 3, t = 1$, the regression model becomes

$$Y_{31} = \beta_0 + \beta_1 X_{31},$$

where $slope = \beta_1$ and $intercept = \beta_0$.
(d) For entity 3 in time period 3, namely $i = 3, t = 3$, the regression model becomes

$$Y_{33} = \beta_0 + \beta_1 X_{33} + \delta_3,$$

where $slope = \beta_1$ and $intercept = \beta_0 + \delta_3$.

Finally we consider the entity and time fixed effects regression model:

$$Y_{it} = \beta_0 + \beta_1 X_{it} + \gamma_2 D2_i + ... + \gamma_n Dn_i + \delta_B 2_t + ... + \delta_T BT_t + u_{it}. \tag{4}$$

(a) For entity 1 in time period 1, namely $i = 1, t = 1$, the regression model becomes

$$Y_{11} = \beta_0 + \beta_1 X_{11},$$

where $slope = \beta_1$ and $intercept = \beta_0$.
(b) For entity 1 in time period 3, namely $i = 1, t = 3$, the regression model becomes

$$Y_{13} = \beta_0 + \beta_1 X_{13} + \delta_3,$$

where $slope = \beta_1$ and $intercept = \beta_0 + \delta_3$.
(c) For entity 3 in time period 1, namely $i = 3, t = 1$, the regression model becomes

$$Y_{31} = \beta_0 + \beta_1 X_{31} + \gamma_3,$$

where $slope = \beta_1$ and $intercept = \beta_0 + \gamma_3$.
(d) For entity 3 in time period 3, namely $i = 3, t = 3$, the regression model becomes

$$Y_{33} = \beta_0 + \beta_1 X_{33} + \gamma_3 + \delta_3,$$

where $slope = \beta_1$ and $intercept = \beta_0 + \gamma_3 + \delta_3$.

EX 2

Solution (a) For the fixed entity effects model (1), we consider the least square estimator of α_i, we minimize the objective function:

$$\sum_t \sum_i u_{it}^2 = \sum_t \sum_i (Y_{it} - \alpha_i D1_i - \alpha_2 D2_i - ... - \alpha_n Dn_i)^2.$$

By the FOC condition we have,

$$\frac{\partial}{\partial \alpha_j} \sum_t \sum_i (Y_{it} - \alpha_i D1_i - \alpha_2 D2_i - ... - \alpha_n Dn_i)^2 = 0, \quad j = 1, 2, ..., n.$$

Specially for α_1, we have

$$
\begin{aligned}
&\frac{\partial}{\partial \alpha_1}\sum_t\sum_i (Y_{it}-\alpha_1 D1_i-\alpha_2 D2_i-...-\alpha_n Dn_i)^2\\
&=\sum_t\sum_i \frac{\partial}{\partial \alpha_1}(Y_{it}-\alpha_1 D1_i-\alpha_2 D2_i-...-\alpha_n Dn_i)^2\\
&=\sum_t\sum_i -2D1_i(Y_{it}-\alpha_1 D1_i-\alpha_2 D2_i-...-\alpha_n Dn_i)\\
&=\sum_t -2(Y_{1t}-\alpha_1)=0.
\end{aligned}
$$

Therefore we have

$$\hat{\alpha}_1=\frac{1}{T}\sum_{t=1}^{T} Y_{1t}.$$

Generally,

$$\hat{\alpha}_j=\frac{1}{T}\sum_{t=1}^{T} Y_{jt},\quad j=1,2,...n.$$

Given that T is fixed, the statistical property of α_j won't change, that is to say WLLN is invalid. Therefore, α_j is not consistent as $n\to\infty$ for all j.

(b) From (a), we know that

$$\hat{\alpha}_j=\frac{1}{T}\sum_{t=1}^{T} Y_{jt},\quad j=1,2,...n.$$

In the case that $T=4$, the normal approximation from CLT in not likely to be valid, hence $\hat{\alpha}_j, j=1,2,...,n$ are not likely to be approximately normally distributed.

EX 3

Solution (a) For the fixed effects regression model

$$Y_{it}=\alpha_i+X_{it}'\beta+u_{it}, \tag{5}$$

we assume that the individual-specific effect α_i is correlated with X_{it}, which may cause the omitted variables which do not change over time. By letting $\bar{Y}_{i\cdot}=T^{-1}\sum_{t=1}^{T} Y_{it}$ and similarly for $\bar{X}_{i\cdot}$ and $\bar{u}_{i\cdot}$. Then we define $\tilde{Y}_{it}=Y_{it}-\bar{Y}_{i\cdot}$ and similarly for $\tilde{X}_{it}$ and $\tilde{u}_{it}$. We have

$$
\begin{aligned}
\bar{Y}_{i\cdot}&=\frac{1}{T}\sum_{t=1}^{T} Y_{it}\\
&=\frac{1}{T}\sum_{t=1}^{T}(\alpha_i+X_{it}'\beta+u_{it})\\
&=\frac{1}{T}(\sum_{t=1}^{T}\alpha_i+\sum_{t=1}^{T}X_{it}'\beta+\sum_{t=1}^{T}u_{it})\\
&=\alpha_i+\bar{X}_{i\cdot}'\beta+\bar{u}_{i\cdot}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\tilde{Y}_{it}&=Y_{it}-\bar{Y}_{i\cdot}\\
&=\alpha_i+X_{it}'\beta+u_{it}-(\alpha_i+\bar{X}_{i\cdot}'\beta+\bar{u}_{i\cdot})\\
&=\tilde{X}_{it}'\beta+\tilde{u}_{it}
\end{aligned}
$$

(b) Then we consider the OLS estimator for the demeaned model by minimizing

$$\frac{1}{N}\frac{1}{T}\sum_i\sum_t \tilde{u}_{it}^2 = \frac{1}{N}\frac{1}{T}\sum_i\sum_t(\tilde{Y}_{it} - \tilde{X}_{it}'\beta)^2.$$

We have

$$\begin{aligned}
\frac{d}{d\beta}(\frac{1}{N}\frac{1}{T}\sum_i\sum_t \tilde{u}_{it}^2) &= \frac{d}{d\beta}[\frac{1}{N}\frac{1}{T}\sum_i\sum_t(\tilde{Y}_{it} - \tilde{X}_{it}'\beta)^2] \\
&= \frac{1}{N}\frac{1}{T}\sum_i\sum_t \frac{d}{d\beta}(\tilde{Y}_{it} - \tilde{X}_{it}'\beta)^2 \\
&= \frac{1}{N}\frac{1}{T}\sum_i\sum_t 2(\tilde{Y}_{it} - \tilde{X}_{it}'\beta)\frac{d}{d\beta}(\tilde{Y}_{it} - \tilde{X}_{it}'\beta) \\
&= -2\frac{1}{N}\frac{1}{T}\sum_i\sum_t \tilde{X}_{it}(\tilde{Y}_{it} - \tilde{X}_{it}'\beta) \\
&= -2\frac{1}{N}\frac{1}{T}\tilde{X}'(\tilde{Y} - \tilde{X}\beta),
\end{aligned}$$

where we have

$$\tilde{X} = \begin{pmatrix} \tilde{X}_{11} & \cdots & \tilde{X}_{1T} \\ \vdots & \ddots & \vdots \\ \tilde{X}_{N1} & \cdots & \tilde{X}_{NT} \end{pmatrix}, \tilde{Y} = \begin{pmatrix} \tilde{Y}_{11} & \cdots & \tilde{Y}_{1T} \\ \vdots & \ddots & \vdots \\ \tilde{Y}_{N1} & \cdots & \tilde{Y}_{NT}. \end{pmatrix}$$

For FOC, namely $\frac{dSSR(\beta)}{d\beta} = 0$,

$$\begin{aligned}
&\tilde{X}'(\tilde{Y} - \tilde{X}\hat{\beta}^{FE}) = 0 \\
\Rightarrow\ &\tilde{X}'\tilde{Y} - (\tilde{X}'\tilde{X})\hat{\beta}^{FE} = 0 \\
\Rightarrow\ &\hat{\beta}^{FE} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{Y}.
\end{aligned}$$

And we have

$$\begin{aligned}
\hat{\beta}^{FE} &= (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{Y} \\
&= (\tilde{X}'\tilde{X})^{-1}\tilde{X}'(\tilde{X}\beta + \tilde{u}) \\
&= \beta + (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{u} \\
&= \beta + (\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{X}_{it}')^{-1}(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{u}_{it}).
\end{aligned}$$

(c) We notice that

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{u}_{it} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(X_{it} - \frac{1}{T}\sum_{t=1}^{T}X_{it})(u_{it} - \frac{1}{T}\sum_{t=1}^{T}u_{it}).$$

We assume that $\{Y_{it}, X_{it}'\}_{i=1}^{'N}$ is an observable IID random sample. By WLLN for IID samples, we have as $N \to \infty$

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(X_{it} - \frac{1}{T}\sum_{t=1}^{T}X_{it})(u_{it} - \frac{1}{T}\sum_{t=1}^{T}u_{it}) \\
&\xrightarrow{p} \mathbb{E}[\frac{1}{T}\sum_{t=1}^{T}(X_{it} - \frac{1}{T}\sum_{t=1}^{T}X_{it})(u_{it} - \frac{1}{T}\sum_{t=1}^{T}u_{it})] \\
&= \mathbb{E}[\mathbb{E}[\frac{1}{T}\sum_{t=1}^{T}(X_{it} - \frac{1}{T}\sum_{t=1}^{T}X_{it})(u_{it} - \frac{1}{T}\sum_{t=1}^{T}u_{it})]|X_{i1}, ..., X_{iT}] \\
&= \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[(X_{it} - \frac{1}{T}\sum_{t=1}^{T}X_{it})\mathbb{E}[u_{it} - \frac{1}{T}\sum_{t=1}^{T}u_{it}|X_{i1}, ..., X_{iT}]] \neq 0.
\end{aligned}$$

We can see that without the assumption of strict exogeneity, the FE estimator $\hat{\beta}^{FE}$ is not consistent.

(d) Under assumption of strict exogeneity, we have that as $N \to \infty$

$$
\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(X_{it}-\frac{1}{T}\sum_{t=1}^{T}X_{it})(u_{it}-\frac{1}{T}\sum_{t=1}^{T}u_{it})\\
&\xrightarrow{p}\mathbb{E}[\frac{1}{T}\sum_{t=1}^{T}(X_{it}-\frac{1}{T}\sum_{t=1}^{T}X_{it})(u_{it}-\frac{1}{T}\sum_{t=1}^{T}u_{it})]\\
&=\mathbb{E}[\mathbb{E}[\frac{1}{T}\sum_{t=1}^{T}(X_{it}-\frac{1}{T}\sum_{t=1}^{T}X_{it})(u_{it}-\frac{1}{T}\sum_{t=1}^{T}u_{it})]|X_{i1},...,X_{iT}]\\
&=\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[(X_{it}-\frac{1}{T}\sum_{t=1}^{T}X_{it})\mathbb{E}[u_{it}-\frac{1}{T}\sum_{t=1}^{T}u_{it}|X_{i1},...,X_{iT}]]\\
&=\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[(X_{it}-\frac{1}{T}\sum_{t=1}^{T}X_{it})\cdot 0]=0.
\end{aligned}
$$

Therefore, under assumption of strict exogeneity, the FE estimator $\hat{\beta}^{FE}$ is consistent.

(e) By the WLLN of IID samples and ergodic stationary processes (We assume that the observable stochastic process $\{Y_{it}, X_{it}'\}_{t=1}^{'T}$ is jointly stationary and ergodic for all i), we have as $N, T \to \infty$

$$
\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(X_{it}-\frac{1}{T}\sum_{t=1}^{T}X_{it})(u_{it}-\frac{1}{T}\sum_{t=1}^{T}u_{it})\\
&\xrightarrow{p}\mathbb{E}[(X_{it}-\mathbb{E}[X_{it}])(u_{it}-\mathbb{E}[u_{it}])]\\
&=Cov(X_{it},u_{it})=0
\end{aligned}
$$