The MIU System and the Impossibility of MIII

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Goal

The goal is to determine whether it is possible to derive `MIII` starting from `MI` using the rules of the MIU system.

Rules

Rule I: If you possess a string whose last letter is `I`, you can add a `U` at the end.

Rule II: Suppose you have `Mx`. Then you may add `Mxx` to your collection.

Rule III: If `III` occurs in one of the strings in your collection, you may make a new string with `U` in place of `III`.

Rule IV: If `UU` occurs inside one of your strings, you can drop it.

Attempts

```
                            v-----------------------v
MI - MIU - MIUIU - MIUUIUU - MII - MIIU - MIIUU - MII *
                            MIIII - MUI - MUI - MUIIU
*THOUGHT* If possible, try to get MIII
```

For example: starting with one `I`, Rule II can double it, giving two total `I`'s with a `U` in between. Because the only way for the number of `I`'s to grow is doubling, every time Rule III might apply, there will always be one leftover. This means it is never possible to reach exactly three `I`'s.

Conclusion

None of the rules leads to the number of `I`'s being three.

- Rule I has no effect on the number of `I`'s.
- Rule II can only double the `I`'s, which means they are always even after the first step. Getting exactly three `I`'s is therefore impossible.
- Rule III is the important rule, because if we could get three `I`'s, this is how we would turn it into `MU`. But since we can't reach three `I`'s, the rule never helps.
- Rule IV has no effect on the number of `I`'s.

Therefore, it is impossible to derive `MIII` from `MI`.

Abstract Reduction Systems: Pictures and Properties

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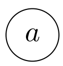
ARS pictures & properties

1. $A = \varnothing$

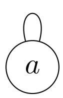
No nodes/edges. *Terminating:* True. *Confluent:* True (vacuous). *Unique NFs:* True.

2. $A = \{a\},\ R = \varnothing$



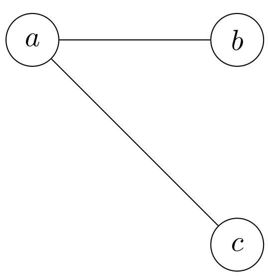
Normal forms: a. Terminating: True. Confluent: True. Unique NFs: True.

3. $A = \{a\},\ R = \{(a, a)\}$



Infinite $a \to a \to \cdots$: Terminating: **False**. Confluent: **True**. Unique NFs: **False**.

4. $A = \{a, b, c\},\ R = \{(a, b), (a, c)\}$

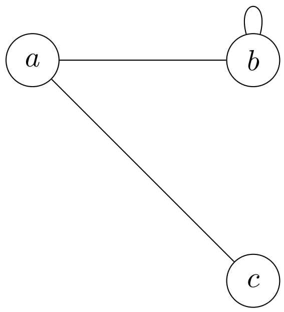


b, c are normal; from a you can reach two distinct NFs. Terminating: **True**. Confluent: **False**. Unique NFs: **False**.

5. $A = \{a, b\},\ R = \{(a, a), (a, b)\}$

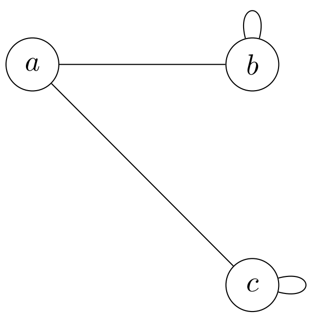
b is normal; every element has the (unique) NF b. Terminating: **False**. Confluent: **True**. Unique NFs: **True**.

6. $A = \{a, b, c\},\ R = \{(a,b), (b,b), (a,c)\}$



Non-terminating via $b \to b$; c is normal, b has no NF. Confluent: **False**. Terminating: **False**. Unique NFs: **False**.

7. $A = \{a, b, c\},\ R = \{(a,b), (b,b), (a,c), (c,c)\}$



Both b and c loop; no NFs reachable from a. Terminating: **False**. Confluent: **False**. Unique NFs: **False**.

Summary Table

#	(A, R)	confluent	terminating	has unique normal forms
1	$(\varnothing, \varnothing)$	True	True	True
2	$(\{a\}, \varnothing)$	True	True	True
3	$(\{a\}, \{(a,a)\})$	True	False	False
4	$(\{a,b,c\}, \{(a,b), (a,c)\})$	False	True	False
5	$(\{a,b\}, \{(a,a), (a,b)\})$	True	False	True
6	$(\{a,b,c\}, \{(a,b), (b,b), (a,c)\})$	False	False	False
7	$(\{a,b,c\}, \{(a,b), (b,b), (a,c), (c,c)\})$	False	False	False

All 8 combinations

confluent	terminating	has unique NFs	example
True	True	True	e.g. ARS 2 (or 1)
True	True	False	*Impossible*
True	False	True	ARS 5
True	False	False	ARS 3
False	True	True	*Impossible*
False	True	False	ARS 4
False	False	True	*Impossible*
False	False	False	ARS 6 (or 7)