

CPSC-354 Report

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September 2, 2025

Abstract

This report collects my weekly notes, homework, and reflections for CPSC-354.

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1 Introduction

This section intentionally left blank for now.

2 Week by Week

2.1 Week 1

2.1.1 Notes and Exploration

We studied the MIU system from Hofstadter's *Gödel, Escher, Bach*. The task was to decide whether the string MIII can be derived from MI using the system's rules.

2.1.2 Homework

Rules.

- I. If a string ends with I, you may append U.
- II. From Mx you may infer Mxx.
- III. Replace any occurrence of III by U.
- IV. Delete any occurrence of UU.

Reasoning. We begin with MI. Rule I allows MIU. Rule II doubles the sequence after M: $MI \Rightarrow MII$, then $MIIII$, etc. Rule III can only replace consecutive III with a U. But because doubling produces powers of two I's (1, 2, 4, 8, ...), we never get exactly three I's. Thus, no sequence of rules produces MIII.

Conclusion. It is impossible to derive MIII from MI. The parity of the number of I's (always even after the first doubling) prevents reaching 3.

2.1.3 Questions

What is a rule that could be implemented that, while still requiring many steps, makes the MU-Puzzle solvable?


2.2 Week 2


2.2.1 Notes and Exploration

We explored Abstract Reduction Systems (ARS), focusing on termination, confluence, and unique normal forms (UNFs). Each ARS was represented as a graph, and we determined its key properties.

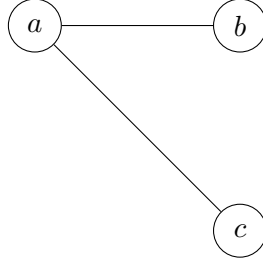
2.2.2 Homework

1. $A = \emptyset$ No nodes or edges. Terminating: True. Confluent: True. UNFs: True.

2. $A = \{a\}$, $R = \emptyset$ 
Normal forms: a . Terminating: True. Confluent: True. UNFs: True.


3. $A = \{a\}, R = \{(a, a)\}$ 

Infinite sequence $a \rightarrow a \rightarrow \dots$. Terminating: False. Confluent: True. UNFs: False.

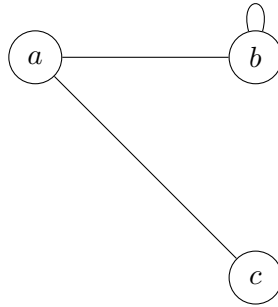


4. $A = \{a, b, c\}, R = \{(a, b), (a, c)\}$

b and c are normal forms, so from a two distinct endpoints are reachable. Terminating: True. Confluent: False. UNFs: False.

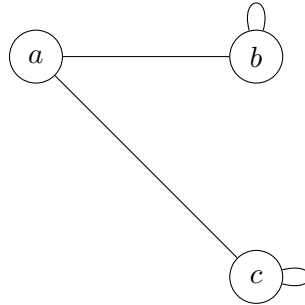
5. $A = \{a, b\}, R = \{(a, a), (a, b)\}$ 

b is normal; all paths from a can reach b . Terminating: False. Confluent: True. UNFs: True.



6. $A = \{a, b, c\}, R = \{(a, b), (b, b), (a, c)\}$

Non-terminating through $b \rightarrow b$; c is normal but unreachable from b . Terminating: False. Confluent: False. UNFs: False.



7. $A = \{a, b, c\}, R = \{(a, b), (b, b), (a, c), (c, c)\}$

Both b and c loop indefinitely. Terminating: False. Confluent: False. UNFs: False.

Summary Table:

#	(A, R)	confluent	terminating	unique NFs
1	(\emptyset, \emptyset)	True	True	True
2	$(\{a\}, \emptyset)$	True	True	True
3	$(\{a\}, \{(a, a)\})$	True	False	False
4	$(\{a, b, c\}, \{(a, b), (a, c)\})$	False	True	False
5	$(\{a, b\}, \{(a, a), (a, b)\})$	True	False	True
6	$(\{a, b, c\}, \{(a, b), (b, b), (a, c)\})$	False	False	False
7	$(\{a, b, c\}, \{(a, b), (b, b), (a, c), (c, c)\})$	False	False	False

All 8 Combinations:

confluent	terminating	unique NFs	example
True	True	True	ARS 2 (or 1)
True	True	False	<i>Impossible</i>
True	False	True	ARS 5
True	False	False	ARS 3
False	True	True	<i>Impossible</i>
False	True	False	ARS 4
False	False	True	<i>Impossible</i>
False	False	False	ARS 6 (or 7)

2.2.3 Questions

Is there an easy way to tell from the graph if an ARS will terminate, or do you always have to trace every path?

2.3 Week 3

2.3.1 Notes and Exploration

Exercises 5 and 5b extended ARS analysis with additional looping rules.

2.3.2 Homework

Exercise 5. $A = \{a, b\}$ with $a \rightarrow a$, $a \rightarrow b$. Loop $a \rightarrow a$ means not terminating, but since all paths eventually lead to b , the system is confluent with a unique NF.

Exercise 5b. Add c with $a \rightarrow c$, $c \rightarrow c$. Now one branch terminates (b) and the other loops, breaking confluence.

2.3.3 Questions

Could we make a version of 5b that still loops but somehow keeps unique normal forms?

2.4 Week 4

2.4.1 Notes and Exploration

We introduced termination proofs and measure functions.

2.4.2 Homework

HW 4.1 (Euclidean Algorithm).

```
while b != 0:
    temp = b
    b = a mod b
    a = temp
return a
```

This algorithm terminates when $b > 0$ because b decreases with each iteration. The measure function $m(a, b) = b$ is always non-negative and strictly decreases.

HW 4.2 (Merge Sort).

```
function merge_sort(arr, left, right):
    if left >= right: return
    mid = (left + right) / 2
    merge_sort(arr, left, mid)
    merge_sort(arr, mid+1, right)
    merge(arr, left, mid, right)
```

The measure $\varphi(left, right) = right - left + 1$ decreases with every recursive call. When it reaches 1, the recursion stops, proving termination.

2.4.3 Questions

What would it look like to design a sorting algorithm that checks to see if an input is terminating?

2.5 Week 5

2.5.1 Notes and Exploration

We practiced λ -calculus reduction and substitution.

2.5.2 Homework

Evaluate:

$$(\lambda f. \lambda x. f(f(x))) (\lambda f. \lambda x. f(f(f(x)))).$$

Step 1. Parentheses group left:

$$((\lambda f. \lambda x. f(f(x))) (\lambda f. \lambda x. f(f(f(x)))))$$

Step 2. Apply β -reduction:

$$\mapsto \lambda x. (\lambda f. \lambda x. f(f(f(x)))) ((\lambda f. \lambda x. f(f(f(x)))) x).$$

Step 3. Rename inner $x \rightarrow y$ and reduce:

$$(\lambda f. \lambda y. f(f(f(y)))) x \mapsto \lambda y. x(x(x(y))).$$

Step 4. Apply again:

$$\lambda x. \lambda y. x^9(y).$$

Conclusion. The function composes f twice and f^3 thrice, resulting in a ninefold application.

2.5.3 Questions

What would happen if we swapped the two functions in the workout?

2.6 Week 6

2.6.1 Notes and Exploration

We computed factorial using the fixed-point combinator `fix`.

2.6.2 Homework

Compute:

```
let rec fact = \n. if n=0 then 1 else n * fact (n-1) in 3
```

Using the rules:

$$\text{fix } F \mapsto F (\text{fix } F), \quad \text{let rec } f = e_1 \text{ in } e_2 \mapsto \text{let } f = (\text{fix } (\lambda f. e_1)) \text{ in } e_2.$$

We unfold recursively:

$$\text{fact } 3 \mapsto 3 * (\text{fix } F) \ 2 \mapsto 3 * 2 * 1 = 6.$$

Conclusion. Following only the allowed computation rules, `fact 3` reduces to `6`.

2.6.3 Questions

Would using α -conversion anywhere in the factorial example change the result, or just make it cleaner?

2.7 Week 7

2.7.1 Notes and Exploration

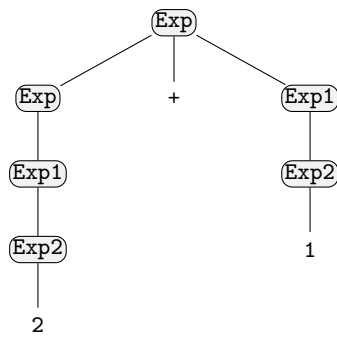
Parsing and context-free grammars. We use the grammar (nonterminals `Exp`, `Exp1`, `Exp2`):

```
Exp  -> Exp '+' Exp1
Exp1 -> Exp1 '*' Exp2
Exp2 -> Integer
Exp2 -> '(' Exp ') '
Exp  -> Exp1
Exp1 -> Exp2
```

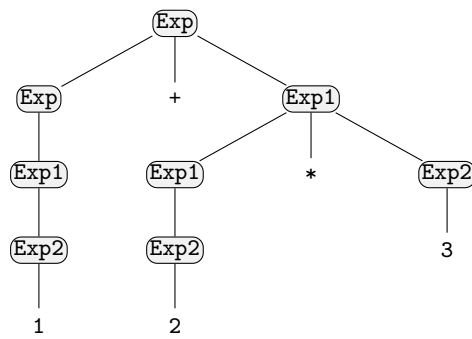
2.7.2 Homework: Parse Trees

Below are derivation trees (concrete syntax trees) for the required expressions. Nonterminals are boxed; terminals are leaves.

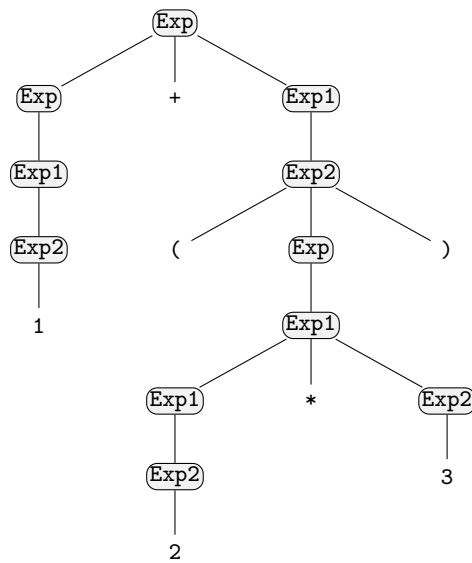
(a) $2+1$



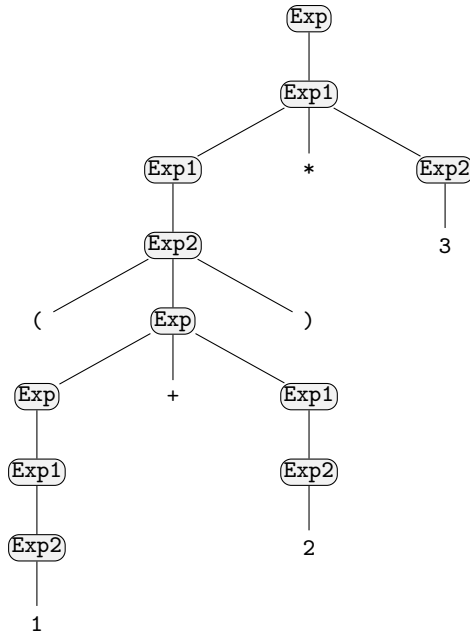
(b) $1+2*3$



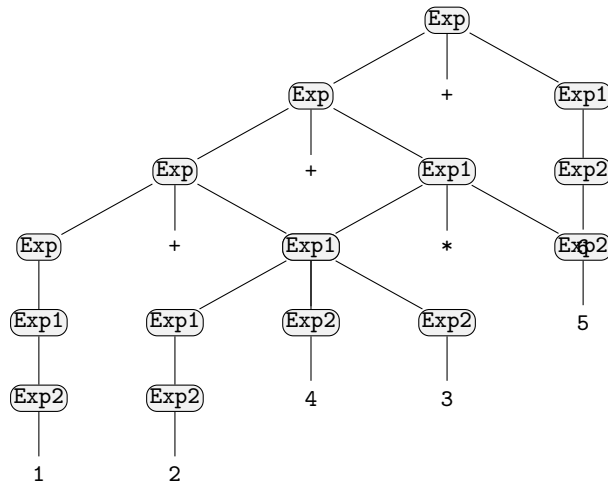
(c) $1+(2*3)$



(d) $(1+2)*3$



(e) $1+2*3+4*5+6$



2.7.3 Questions

Could this grammar still work if “+” and “*” had the same priority?

2.8 Week 8

2.8.1 Notes and Exploration

I completed Levels 5–8 of the *Natural Number Game* (NNG). The key tools were:

- **Rewriting** with `rw [lemma]` to substitute equals for equals.
- `rf1` to close a goal when both sides are definitionally identical.
- Peano-style equations for addition and numerals, including `add_zero`, `add_succ`, `add_one`, `one_eq_succ_zero`, and `two_eq_succ_one`.

2.8.2 Homework (NNG Levels 5–8)

5/8: Adding zero. *Goal:* $a + (b + 0) + (c + 0) = a + b + c$.

Lean steps: `rw [add_zero] (on b + 0), rw [add_zero] (on c + 0), rfl.`

6/8: Precision rewriting. *Goal:* $a + (b + 0) + (c + 0) = a + b + c$.

Lean steps: `rw [add_zero] (on b + 0), rw [add_zero] (on c + 0), rfl.`

7/8: add_succ. *Goal:* $\text{succ } n = n + 1$.

Lean steps: `rw [one_eq_succ_zero]` to get $n + 1 = n + \text{succ } 0$; `rw [add_succ]` to obtain $\text{succ}(n + 0)$; `rw [add_zero]; rfl.`

8/8: $2 + 2 = 4$. *Outline:* Rewrite 2 and 4 as successors (`two_eq_succ_one`, etc.), use `add_succ` to pull `succ` out of addition stepwise, and finish by `rfl` once both sides match.

2.8.3 Natural-Language Proof (English math proof)

Claim. $2 + 2 = 4$.

Assumptions/Definitions. Let 0 be the base natural number and $\text{succ}(n)$ the successor of n . Define $1 = \text{succ}(0)$, $2 = \text{succ}(1)$, and $4 = \text{succ}(\text{succ}(\text{succ}(\text{succ}(0))))$. Addition is defined by the Peano axioms:

$$\forall a, a + 0 = a \quad \text{and} \quad \forall a, b, a + \text{succ}(b) = \text{succ}(a + b).$$

Proof. We compute $2 + 2$ using the addition axioms. First rewrite $2 = \text{succ}(1)$, so

$$2 + 2 = 2 + \text{succ}(1) = \text{succ}(2 + 1) \quad (\text{by the second axiom}).$$

Again rewrite $1 = \text{succ}(0)$ to get

$$2 + 1 = 2 + \text{succ}(0) = \text{succ}(2 + 0) = \text{succ}(2) \quad (\text{using both axioms}).$$

Therefore

$$2 + 2 = \text{succ}(2 + 1) = \text{succ}(\text{succ}(2)).$$

Since $2 = \text{succ}(1)$, we have

$$\text{succ}(\text{succ}(2)) = \text{succ}(\text{succ}(\text{succ}(1))) = \text{succ}(\text{succ}(\text{succ}(\text{succ}(0)))) = 4.$$

Hence $2 + 2 = 4$. \square

2.8.4 Discord Question

When using `rw [add_zero]`, how does Lean know which part of the equation to rewrite first?

2.9 Week 9

2.9.1 Notes and Exploration

This week I worked through *Addition World* in the Natural Number Game. Key theorems proved in this world:

- **zero_add:** $0 + n = n$ (proved by induction).
- **succ_add:** $\text{succ}(a) + b = \text{succ}(a + b)$ (proved by induction).
- **add_comm:** $a + b = b + a$ (commutativity of $+$).
- **add_assoc:** $(a + b) + c = a + (b + c)$ (associativity of $+$).
- **add_right_comm:** $(a + b) + c = (a + c) + b$.

The HW 9 instruction is: For Level 5 of addition world (**add_right_comm**), give two solutions in the report: (1) a proof that uses induction and (2) a proof that does *not* use induction. For each version, also write the corresponding “pen-and-paper” math proof.

2.9.2 Homework (Addition World Level 5 / HW 9)

Theorem (Level 5). For all natural numbers a, b, c , we have

$$(a + b) + c = (a + c) + b.$$

In Lean, this theorem is called **add_right_comm**.

Solution A: With Induction. We prove **add_right_comm** by induction on c .

Lean tactic steps:

- induction c with d hd
- rw [add_zero]
- rw [add_zero]
- rfl
- rw [add_succ]
- rw [succ_add]
- rw [hd]
- rfl

Explanation of structure (what the tactics are doing, summarized in math terms):

Base case ($c = 0$). Goal:

$$(a + b) + 0 = (a + 0) + b.$$

By **add_zero**, $(a + b) + 0 = a + b$, and $(a + 0) + b = a + b$. So both sides are $a + b$. Closed by **rfl**.

Inductive step ($c = \text{succ}(d)$). Induction hypothesis (called **hd** in Lean):

$$(a + b) + d = (a + d) + b.$$

Goal:

$$(a + b) + \text{succ}(d) = (a + \text{succ}(d)) + b.$$

Using the Peano definition of $+$ on the right argument,

$$(a + b) + \text{succ}(d) = \text{succ}((a + b) + d) \quad \text{and} \quad a + \text{succ}(d) = \text{succ}(a + d).$$

So the right-hand side becomes

$$(a + \text{succ}(d)) + b = (\text{succ}(a + d)) + b = \text{succ}((a + d) + b)$$

using `succ_add`, which says $\text{succ}(x) + y = \text{succ}(x + y)$.

Thus the goal reduces to

$$\text{succ}((a + b) + d) = \text{succ}((a + d) + b).$$

By the induction hypothesis, $(a + b) + d = (a + d) + b$, so the two $\text{succ}(\cdot)$ terms are equal. This closes with `rfl`.

Conclusion. By induction on c , $(a + b) + c = (a + c) + b$ holds for all a, b, c .

Solution B: Without Induction. We can also prove `add_right_comm` using only associativity and commutativity of addition, without `induction`.

Lean tactic steps (no induction):

- `rw [add_assoc]`
- `rw [add_comm b c]`
- `rw [\leftarrow add_assoc]`
- `rfl`

Those steps correspond exactly to the following algebra moves:

$$\begin{aligned} (a + b) + c &= a + (b + c) \quad (\text{associativity, add_assoc}) \\ &= a + (c + b) \quad (\text{commutativity on } b + c, \text{ add_comm b c}) \\ &= (a + c) + b \quad (\text{associativity again, undoing add_assoc}). \end{aligned}$$

This matches the goal $(a + b) + c = (a + c) + b$.

English/Pen-and-Paper Proofs for HW 9

Solution A (Induction Proof in Math). We prove $(a + b) + c = (a + c) + b$ for all $a, b, c \in \mathbb{N}$ by induction on c .

Base case: $c = 0$. Then

$$(a + b) + 0 = a + b \quad \text{and} \quad (a + 0) + b = a + b,$$

because $x + 0 = x$ for any x . So $(a + b) + 0 = (a + 0) + b$.

Inductive step: Assume $(a + b) + d = (a + d) + b$ for some d . We must show $(a + b) + \text{succ}(d) = (a + \text{succ}(d)) + b$.

By the Peano definition of addition on the right argument,

$$(a + b) + \text{succ}(d) = \text{succ}((a + b) + d).$$

Also,

$$a + \text{succ}(d) = \text{succ}(a + d),$$

so

$$(a + \text{succ}(d)) + b = (\text{succ}(a + d)) + b = \text{succ}((a + d) + b),$$

using the lemma $\text{succ}(x) + y = \text{succ}(x + y)$.

So it suffices to show

$$\text{succ}((a + b) + d) = \text{succ}((a + d) + b),$$

which follows from the induction hypothesis $(a + b) + d = (a + d) + b$. Therefore the statement holds for $\text{succ}(d)$.

By induction on c , $(a + b) + c = (a + c) + b$ for all a, b, c .

Solution B (Algebraic / No Induction). We use associativity and commutativity of addition on \mathbb{N} .

Starting from the left-hand side,

$$\begin{aligned}(a + b) + c &= a + (b + c) \quad (\text{associativity of } +), \\ &= a + (c + b) \quad (\text{commutativity of } +), \\ &= (a + c) + b \quad (\text{associativity of } + \text{ again}).\end{aligned}$$

This is exactly the desired right-hand side $(a + c) + b$. No induction was needed.

2.9.3 Questions

When should I solve these problems using induction vs without?

3 Essay (Synthesis)

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4 Evidence of Participation

This section intentionally left blank.

5 Conclusion

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References