Simplifying A Logic Program Using Its Consequence

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Abstract

1 Introduction

A *consequence* of a program is a set of literals that are satisfied by every answer set of the program. The problem of program simplification considers how to simplify the program by a consequence, so that the answer sets of the program can be computed from the answer sets of the resulting program with the consequence.

2 Preliminaries

2.1 Logic Programs

In this paper, we consider only fully grounded finite disjunctive logic programs. A *disjunctive logic program* (DLP) is a finite set of (disjunctive) rules of the form

$$a_1 \vee \cdots \vee a_k \leftarrow a_{k+1}, \ldots, a_m, not\, a_{m+1}, \ldots, not\, a_n, \tag{1}$$

where $n \geq m \geq k \geq 1$ and $a_1, \ldots, a_n$ are atoms. If $k = 1$, it is a *normal rule*. In particular, a *normal logic program* (NLP) is a finite set of normal rules.

We will also write rule r of form (1) as

$$head(r) \leftarrow body(r), \tag{2}$$

where $head(r)$ is $a_1 \vee \cdots \vee a_k$, $body(r) = body^+(r) \wedge body^-(r)$, $body^+(r)$ is $a_{k+1} \wedge \cdots \wedge a_m$, and $body^-(r)$ is $\neg a_{m+1} \wedge \cdots \wedge \neg a_n$, and we identify $head(r)$, $body^+(r)$,

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$body^-(r)$ with their corresponding sets of atoms, and $body(r)$ the set $\{ a_{k+1}, \dots, a_m, \neg a_{m+1}, \dots, \neg a_n \}$ of literals obtained from the body of the rule with "*not*" replaced by "$\neg$". We denote $Atoms(r) = head(r) \cup body^+(r) \cup body^-(r)$. Let R be a set of rules, we denote $head(R) = \bigcup_{r \in R} head(r)$, $body^+(R) = \bigcup_{r \in R} body^+(r)$, and $Atoms(R) = \bigcup_{r \in R} Atoms(r)$.

A set S of atoms is said to *satisfy* a rule r, if $body^+(r) \subseteq S$ and $body^-(r) \cap S = \emptyset$ implies $head(r) \cap S \neq \emptyset$. S *satisfies* a program P, if S satisfies every rules in P. With a slight abuse of the notion, S *satisfies* the body of r, denoted by $S \models body(r)$, if $body^+(r) \subseteq S$ and $body^-(r) \cap S = \emptyset$.

The *answer sets* of a disjunctive logic program is defined as in [2]. Given a disjunctive logic program P and a set S of atoms, the Gelfond-Lifschitz transformation of P on S, written P^S, is obtained from P by deleting:

1. each rule that has a formula $not\, p$ in its body with $p \in S$, and
2. all formulas of the form $not\, p$ in the bodies of the remaining rules.

Clearly for any S, P^S is the set of rules without any negative literals, so that P^S has a set of minimal models, denoted by $\Gamma_P(S)$. Now a set S of atoms is an answer set of P iff $S \in \Gamma_P(S)$.

2.2 Loops and Loop Formulas

We now briefly review the notions of loops and loop formulas in disjunctive logic programs [3]. Given a disjunctive logic program P, the *positive dependency graph* of P, written G_P, is the directed graph whose vertices are atoms in P, and there is an arc from p to q if there is a rule $r \in P$ such that $p \in head(r)$ and $q \in body^+(r)$. A set L of atoms is said to be a *loop* of P if for any p and q in L, there is a path from p to q in G_P such that all the vertices in the path are in L, i.e. the L-induced subgraph of G_P is strongly connected. Note that, every singleton whose atom occurs in P is also a loop of P.

Given a loop L, a rule r is an *external support* of L if $head(r) \cap L \neq \emptyset$ and $L \cap body^+(r) = \emptyset$. In the following, let $R^-(L, P)$ be the set of external support rules of L. Note that, the function R^- can be defined for any set E of atoms in P, i.e.

$$R^-(E, P) = \{r \mid r \in P,\ head(r) \cap E \neq \emptyset \text{ and } E \cap body^+(r) = \emptyset\}.$$

Given a set X of atoms, the set of external support rules of L under X, denoted $R^-(L, P, X)$, is the set of rules r such that $r \in R^-(L, P)$ and $X \models body(r)$. If P is constant, we can simplify $R^-(L, P)$ to $R^-(L)$ and $R^-(L, P, X)$ to $R^-(L, X)$.

The (conjunctive) *loop formula* of L under P, written $LF(L, P)$, is the following implication

$$\bigwedge_{p \in L} p \supset \bigvee_{r \in R^-(L,P)} \left(body(r) \wedge \bigwedge_{q \in head(r) \setminus L} \neg q \right). \tag{3}$$

A set S of atoms *satisfies* $LF(L, P)$, if $L \subseteq S$ implies there exists a rule $r \in R^-(L, P)$ such that $S \models body(r)$ and $(head(r) \setminus L) \cap S = \emptyset$. Similarly, the notion of loop formulas can also be defined for any set E of atoms in P.

Theorem 1 (Theorem 1 in [3]) *Let P be a logic program and S a set of atoms. If S satisfies P, then following conditions are equivalent:*

1. *S is an answer set of P;*
2. *S satisfies $LF(L, P)$ for all loops L of P.*
3. *S satisfies $LF(C, P)$ for all sets E of atoms of P.*

2.3 Strong Equivalence

Program P_1 and P_2 are *strongly equivalent* if for any set R of rules, the programs $P_1 \cup R$ and $P_2 \cup R$ have the same set of answer sets.

Let P be a program and X, Y sets of atoms such that $X \subseteq Y$. The pair (X, Y) is an *SE-model* of P, if Y satisfies P and X satisfies P^Y. By $M_s(P)$ we denote the set of all SE-models of P.

Proposition 1 ([5]) *For any program P_1 and P_2, P_1 is strongly equivalent to P_2 if and only if $M_s(P) = M_s(Q)$.*

2.4 Well-Founded Model

Let P be a logic program and I a set of literals, a set X of atoms is an *unfounded set* of P w.r.t. I if for each atom $p \in X$ and each rule $r \in P$ such that $p \in head(r)$, at least one of the following conditions holds:

- $I \models \neg body(r)$,
- $X \cap body^+(r) \neq \emptyset$,
- $(head(r) \setminus X) \cap I \neq \emptyset$.

Proposition 2 *Let P be a logic program and I a set of literals, a set X of atoms is an unfounded set of P w.r.t. I if and only if for every rule $r \in R^-(X, P)$, $I \models \neg body(r) \vee \bigvee_{q \in head(r) \setminus X} q$.*

Proof: $\Rightarrow$: X is an unfounded set of P w.r.t. I, then for every $r \in R^-(X, P)$, $I \models \neg body(r)$ or $(head(r) \setminus X) \cap I \neq \emptyset$. So $I \models \neg body(r) \vee \bigvee_{q \in head(r) \setminus X} q$.

$\Leftarrow$: For every rule $r \in R^-(X, P)$, $I \models \neg body(r) \vee \bigvee_{q \in head(r) \setminus X} q$. For each rule r' such that $head(r') \cap X \neq \emptyset$, $r' \notin R^-(X, P)$ or $r' \in R^-(X, P)$. If $r' \notin R^-(X, P)$, then $X \cap body^+(r') \neq \emptyset$. If $r' \in R^-(X, P)$, then $I \models \neg body(r)$ or $head(r) \setminus X) \cap I \neq \emptyset$. So X is an unbounded set of P w.r.t. I. ■

If P is an NLP, the union of two unfounded sets is also an unfounded set of P, then there exists the greatest unfounded set of P. However, the property does not hold for DLPs in general. Let P be a logic program and I a set of literals, I is *unfounded-free* if $I \cap X = \emptyset$ for each unfounded set X of P w.r.t. I. From [4], if I is unfounded-free for a DLP P, then the union of two unfounded sets of P w.r.t. I is also an unfounded set, thus there exists the greatest unfounded set of P w.r.t. I. We use $U_P(I)$ to denote the greatest unfounded set of P w.r.t. I, if exists.

Now we define two operators for a logic program P w.r.t. a set I of literals:

- $T_P(I) = \{p \mid \text{there exists a rule } r \in P \text{ such that } p \in \mathit{head}(r) \text{ and } I \models \mathit{body}(r) \wedge \bigwedge_{q \in head(r) \setminus \{p\}} q\}$;
- $W_P(I) = T_P(I) \cup \neg U_P(I)$.

Note that, T_P, U_P, and W_P are monotonic operators. The *well-founded model* of an NLP P is the least fixed point of W_P. We use $\mathit{WFM}(P)$ to denote the well-founded models of an NLP P.

3 Background

Let set L of literals be a consequence of a program P, we define $tr_n(P, L)$ to be the program obtained from P by

1. deleting each rule r that has an atom $p \in \mathit{body}^+(r)$ with $\neg p \in L$, and
2. replacing each rule r that has an atom $p \in \mathit{head}(r)$ or $p \in \mathit{body}^-(r)$ with $\neg p \in L$ by the rule
$$\mathit{head}(r) \setminus \overline{L} \leftarrow \mathit{body}^+(r), \mathit{body}^-(r) \setminus \overline{L}.$$

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1. deleting each rule r that has an atom $p \in \mathit{head}(r)$ or $p \in \mathit{body}^-(r)$ with $p \in L$, and
2. replacing each rule r that has an atom $p \in \mathit{body}^+(r)$ with $p \in L$ by the rule
$$\mathit{head}(r) \leftarrow \mathit{body}^+(r) \setminus L, \mathit{body}^-(r).$$

Note that $tr_n(P, L)$ (resp. $tr_p(P, L)$) does not contain any atom p with $\neg p \in L$ (resp. $p \in L$) and $tr_p(tr_n(P, L), L)$ does not contain any atoms occurring in L.

Proposition 3 *Let L be a consequence of a logic program P, programs P and $tr_n(P, L)$ have the same set of answer sets.*

Proof: $\Rightarrow$: Let S be an answer set of P, then S satisfies L, thus S is a model of $tr_n(P, L)$. Assume that there exists another set S' of atoms such that $S' \subset S$ and S' is a model of $tr_n(P, L)^S$.

For each rule $r \in P$ such that there is an atom $p \in \mathit{body}^+(r)$ and $\neg p \in L$, as $S' \models \{\neg p \mid \neg p \in L\}$, then S' satisfies the rule r.

For each rule $r \in P$ such that there is an atom $p \in \mathit{head}(r)$ or $p \in \mathit{body}^-(r)$ with $\neg p \in L$, we denote r' to be the rule $\mathit{head}(r) \setminus \overline{L} \leftarrow \mathit{body}^+(r), \mathit{body}^-(r) \setminus \overline{L}$. As $S \models L$, then $r^S \in P^S$ iff $r'^S \in tr_n(P, L)^S$. Moreover, $S' \models \{\neg p \mid \neg p \in L\}$, then S' satisfies r^S iff S' satisfies r'^S.

So S' is also a model of P^S, which conflicts to the precondition that S is an answer set of P. Then such S' does not exist and S is an answer set of $tr_n(P, L)$.

$\Leftarrow$: Let S be an answer set of $tr_n(P, L)$, then $\{p \mid \neg p \in L\} \cap S = \emptyset$, i.e., $S \models \{\neg p \mid \neg p \in L\}$. So S is a model of $tr_n(P, L)$, then S is a model of P. Assume that there exists another set S' of atoms such that $S' \subset S$ and S' is a model of P^S. From the above proof, as $S' \models \{\neg p \mid \neg p \in L\}$, then S' is also a model of $tr_n(P, L)^S$, which conflicts to the precondition that S is an answer set of $tr_n(P, L)$. Then such S' does not exist and S is an answer set of P. ■

The well-founded model [6] of an NLP is a set of literals and it is also a consequence of the program.

Proposition 4 *Let L be a well-founded model of an NLP P, programs P and $tr_p(tr_n(P, L), L) \cup \{p \leftarrow \mid p \in L\}$ have the same set of answer sets.*

Proof: Let I be a set of literals, assume that P and $tr_p(tr_n(P, I), I) \cup \{p \leftarrow \mid p \in I\}$ have the same set of answer sets. We want to prove that $W_P(I)$ also satisfies the condition.

For $\neg U_P(I)$, from Proposition 3, P and $tr_n(P, \neg U_P(I))$ have the same set of answer sets.

For $T_P(I)$, we want to prove that P and $tr_p(P, T_P(I)) \cup \{p \leftarrow \mid p \in T_P(I)\}$ have the same set of answer sets. Let $p \in T_P(I)$, then there exists a rule $r \in P$ such that $p \in head(r)$ and $I \models body(r) \wedge \bigwedge_{q \in head(r) \setminus \{p\}} q$. So for every answer set S of P, if $S \models I$, then $S \models T_P(I)$.

Let S be an answer set of P, then $S \models T_P(I)$ and S is a model of $tr_p(P, T_P(I)) \cup \{p \leftarrow \mid p \in T_P(I)\}$. Assume that there exists a set S' of atoms such that $S' \subset S$ and S' is a model of $tr_p(P, T_P(I))^S \cup \{p \leftarrow \mid p \in T_P(I)\}$. Clearly, $T_P(I) \subseteq S'$. Then for each rule $r \in P^S$, if $head(r) \cap T_P(I) \neq \emptyset$, then S' satisfies r; if $body^+(r) \cap T_P(I) \neq \emptyset$, as S' satisfies the corresponding rule in $tr_p(P, T_P(I))^S$, then S' satisfies r. So S' is a model of P^S, which conflicts to the precondition that S is an answer set of P. So S is also an answer set of $tr_p(P, T_P(I)) \cup \{p \leftarrow \mid p \in T_P(I)\}$.

Let S be an answer set of $tr_p(P, T_P(I)) \cup \{p \leftarrow \mid p \in T_P(I)\}$, then $S \models T_P(I)$, S is a model of P, S is a model of $tr_p(tr_n(P, I), I) \cup \{p \leftarrow \mid p \in I\}$. Assume that there exists a set S' of atoms such that $S' \subset S$ and S' is a model of $tr_p(tr_n(P, I), I)^S \cup \{p \leftarrow \mid p \in I\}$.

Note that, $S' \models \{p \mid p \in I\}$ and $tr_p(tr_n(P, I), I)$ does not contain atoms appeared in I, then $S' \models I$. From the definition of T_P, then $S' \models T_P(I)$, so S' satisfies $tr_p(P, T_P(I)) \cup \{p \leftarrow \mid p \in T_P(I)\}$, which conflicts to the precondition that S is an answer set of $tr_p(P, T_P(I)) \cup \{p \leftarrow \mid p \in T_P(I)\}$. So S is also an answer set of $tr_p(tr_n(P, I), I) \cup \{p \leftarrow \mid p \in I\}$, and S is an answer set of P. ■

As a result, in all current ASP solvers, an NLP is first simplified by its well-founded model. Meanwhile, there are other consequences of a program that are larger than the well-founded model and can be computed efficiently. For instant, [1] computes such a consequence using loop formulas of loops with at most one external support rule. However, Proposition 4 does not hold in general for these consequences.

Proposition 5 *Let L be a well-founded model of an NLP P, $P \cup \{\leftarrow p \mid \neg p \in L\}$ is strongly equivalent to $tr_n(P, L) \cup \{\leftarrow p \mid p \in L\}$.*

Proof: Directly from Proposition 7. ■

Proposition 6 *Let L be a well-founded model of an NLP P, $P \cup \{\leftarrow p \mid \neg p \in L\}$ is strongly equivalent to $tr_p(P, L) \cup \{p \leftarrow \mid p \in L\} \cup \{\leftarrow p \mid \neg p \in L\}$.*

Proof: We can first use Proposition 7, then use Theorem 3, to get $P \cup \{\leftarrow p \mid \neg p \in L\}$ is strongly equivalent to $tr_p(tr_n(P, L), L) \cup \{p \leftarrow \mid p \in L\} \cup \{\leftarrow p \mid \neg p \in L\}$. Then we use Proposition 7 again, to get $P \cup \{\leftarrow p \mid \neg p \in L\}$ is strongly equivalent to $tr_p(P, L) \cup \{p \leftarrow \mid p \in L\} \cup \{\leftarrow p \mid \neg p \in L\}$. ■

Corollary 2 *Let L be a well-founded model of an NLP P, $P \cup \{\leftarrow p \mid \neg p \in L\}$ is strongly equivalent to $tr_p(tr_n(P, L), L) \cup \{p \leftarrow \mid p \in L\} \cup \{\leftarrow p \mid p \in L\}$.*

Proof: From above two propositions. ■

4 Negative Consequences

Proposition 7 *Let L be consequence of a program P, $P \cup \{\leftarrow p \mid \neg p \in L\}$ is strongly equivalent to $tr_n(P, L) \cup \{\leftarrow p \mid \neg p \in L\}$.*

Proof: From Proposition 1, we need to prove that $M_s(P \cup \{\leftarrow p \mid \neg p \in L\}) = M_s(tr_n(P, L) \cup \{\leftarrow p \mid \neg p \in L\})$.

Note that, for any possible SE-model (X, Y) of these sets, $\{p \mid \neg p \in L\} \cap X = \emptyset$ and $\{p \mid \neg p \in L\} \cap Y = \emptyset$. From the definition of $tr_n(P, L)$, $M_s(P \cup \{\leftarrow p \mid \neg p \in L\}) = M_s(tr_n(P, L) \cup \{\leftarrow p \mid \neg p \in L\})$. ■

5 Positive Consequences

Let P be a logic program, a set U of atoms is said to be a *reliable* set of P if for every nonempty subset E of U there exists a rule $r \in R^-(E, P)$ such that $head(r) \subseteq E$ and $U \models body(r)$.

Theorem 3 *Let U be a reliable set of a program P, P is strongly equivalent to $tr_p(P, U) \cup \{p \leftarrow \mid p \in U\}$.*

Proof: We need to prove that $M_s(P) = M_s(tr_p(P, U) \cup \{p \leftarrow \mid p \in U\}$.

$\Rightarrow$: Let (X, Y) be a SE-model of P.

Assume that $Y \not\models U$, then let $E = U \setminus Y$ and E is nonempty. Clearly, $U \setminus E \subseteq Y$ and $E \cap Y = \emptyset$. From the definition of the reliable set, there exists a rule $r \in R^-(E, P)$ such that $head(r) \subseteq E$ and $U \models body(r)$, which implies that $body^-(r) = \emptyset$ and $body^+(r) \subseteq U \setminus E$. Then $Y \models body(r)$, as Y satisfies P, so $head(r) \cap Y \neq \emptyset$, which conflicts to the precondition that $head(r) \subseteq E$ and $E \cap Y = \emptyset$. So for any SE-model (X, Y) of P, $U \subseteq Y$.

Assume that $X \not\models U$, then let $E = U \setminus X$ and E is nonempty. Similar to the above proof, there exists a rule $r \in P^Y$, as $body^-(r) = \emptyset$, such that $X \models body(r)$ and $head(r) \cap X \neq \emptyset$, which conflicts to the precondition. So for any SE-model (X, Y) of P, $U \subseteq X$.

As $U \subseteq X$ and $U \subseteq Y$, from the definition of $tr_p(P, U)$, (X, Y) is also a SE-model of $tr_p(P, U) \cup \{p \leftarrow \mid p \in U\}$.

$\Leftarrow$: Let (X, Y) be a SE-model of $tr_p(P, U) \cup \{p \leftarrow \mid p \in U\}$. Clearly, $U \subseteq X$ and $U \subseteq Y$. So Y satisfies P and X satisfies P^Y, then (X, Y) is a SE-model of P. ■

Corollary 4 *Let U be a reliable set of a program P, $\{r \mid r \in P, head(r) \cap U \neq \emptyset\}$ is strongly equivalent to $\{p \leftarrow \mid p \in U\}$.*

Proof: Directly from the above proof, by considering $\{r \mid r \in P, head(r) \cap U \neq \emptyset\}$ as the program. Note that, U is still a reliable set of the program $\{r \mid r \in P, head(r) \cap U \neq \emptyset\}$. ■

Corollary 5 *Let U be a reliable set of a program P, $P \setminus tr_p(P,U)$ is strongly equivalent to $(tr_p(P,U) \setminus P) \cup \{p \leftarrow \mid p \in U\}$.*

Proof: Similarly, U is still a reliable set of the program $P \setminus tr_p(P,U)$. ■

Theorem 6 *Let U be a set of atoms of a program P, $\{r \mid r \in P, head(r) \cap U \neq \emptyset\}$ is strongly equivalent to $\{p \leftarrow \mid p \in U\}$ if and only if U is a reliable set of P.*

Proof: $\Leftarrow$: U is a reliable set of P, then the programs are strongly equivalent.

$\Rightarrow$: Let P_1, P_2 stand for these programs respectively, P_1 and P_2 are strongly equivalent. Assume that, U is not a reliable set of P, then there exists a nonempty set $E \subseteq U$ such that for every rule $r \in R^-(E,P)$ with $head(r) \not\subseteq E$ or $U \not\models body(r)$. So U is not an answer set of P_1, as U does not satisfy the $LF(E,P_1)$. Then P_1 and P_2 do not have same set of answer sets, which conflicts to the precondition. So U must be a reliable set of P. ■

Corollary 7 *Let U be a set of atoms of a program P. $P \setminus tr_p(P,U)$ is strongly equivalent to $(tr_p(P,U) \setminus P) \cup \{p \leftarrow \mid p \in U\}$ if and only if U is a reliable set of P.*

Proof: Similarly to the above proof. ■

Note that, the reliable set is the necessary condition for simplifying a logic program using a set of atoms, while not considering the rest part of the program.

Let L be a consequence of a program P, a set U of atoms is said to be a *reliable* set of P *w.r.t.* L if for every nonempty subset E of $U \cup \{p \mid p \in L\}$ with $E \cap U \neq \emptyset$ there exists a rule $r \in R^-(E,P)$ such that $head(r) \setminus \{p \mid \neg p \in L\} \subseteq E$ and $U \cup L \models body(r)$. Note that, instead of U, we need to consider every nonempty subset of $U \cup \{p \mid p \in L\}$. Clearly, U is a reliable set of P if and only if U is a reliable set of P w.r.t. $\emptyset$.

Theorem 8 *Let U be a reliable set of a program P w.r.t. a consequence L, P and $tr_p(P,U) \cup \{p \leftarrow \mid p \in U\} \cup \{\leftarrow p \mid \neg p \in L\} \cup \{\leftarrow not\, p \mid p \in L\}$ have the same set of answer sets.*

Proof: Let P_2 stand for the second program.

$\Rightarrow$: Let S be an answer set of P. From the definition of the reliable set, L is a consequence of P, then U is also a consequence of P, so S satisfies P_2.

Assume that there is a set $S' \subset S$ such that S' satisfies P_2^S. Note that, $\{p \leftarrow \mid p \in U\} \subseteq P_2^S$, so $U \subseteq S'$, then S' also satisfies P^S, which conflicts to the precondition that S is an answer set of P. So S is also an answer set of P_2.

$\Leftarrow$: Let S be an answer set of P_2. Clearly, $S \models L$. U is a consequence of P, so S also satisfies P.

Assume that there is a set $S' \subset S$ such that S' satisfies P^S. If $U \not\subseteq S'$, let $E = (U \cup \{p \mid p \in L\}) \setminus S'$, E is nonempty, then there exists a rule $r \in R^-(E,P)$ such that $head(r) \subseteq E$ and $U \cup L \models body(r)$. Note that $S \models L \cup U$, so $S \models body(r)$. Moreover, $S' \subseteq S$, then $S' \models \{\neg p \mid \neg p \in L\}$ and $S' \models body^+(r) \cap (U \cup L)$, so $S' \models body(r)$. As S' satisfies P^S, then $S' \models head(r)$, which conflicts to the precondition that E is nonempty and $E \cap S' = \emptyset$. So $E = \emptyset$ and $U \subseteq S'$, then S' satisfies P_2^S, which conflicts to the precondition that S is an answer set of P_2. So S is also an answer set P. ■

Theorem 9 *Let U be a reliable set of a program P w.r.t. a set L of literals, $P \cup \{\leftarrow p \mid \neg p \in L\} \cup \{\leftarrow not\, p \mid p \in L\}$ is strongly equivalent to $tr_p(P,U) \cup \{p \leftarrow \mid p \in U\} \cup \{\leftarrow p \mid \neg p \in L\} \cup \{\leftarrow not\, p \mid p \in L\}$.*

Proof: Let P_1 and P_2 stand for these programs respectively. We want to prove that $M_s(P_1) = M_s(P_2)$.

$\Rightarrow$: Let (X,Y) be a SE-model of P_1. Clearly, $Y \models L$, Y satisfies P, $X \subseteq Y$, and X satisfies P^Y. From the definition of the reliable set, $Y \models U$, then Y satisfies P_2.

Clearly, $X \models \{\neg p \mid \neg p \in L\}$. If $X \models U$, then (X,Y) is also a SE-model of P_2.

If $X \not\models U$, let $E = (L \cup U) \setminus X$, then E is nonempty, there exists a rule $r \in R^-(E,P)$ such that $head(r) \subseteq E$ and $U \cup L \models body(r)$. As $X \models body(r)$, then $X \models head(r)$, which conflicts to the precondition $E \cap X = \emptyset$. So $E = \emptyset$ and $X \models U$, then (X,Y) is also a SE-model of P_2.

$\Leftarrow$: Let (X,Y) be a SE-model of P_2. Clearly, $Y \models L \cup U$ and $X \models U$. Then (X,Y) is also a SE-model of P_1. ■

Theorem 10 *Let U be a set of atoms, L a set of literals, and P a program, $\{r \mid r \in P,\ head(r) \cap U \neq \emptyset\} \cup \{\leftarrow p \mid \neg p \in L\} \cup \{\leftarrow not\, p \mid p \in L\}$ is strongly equivalent to $\{p \leftarrow \mid p \in U\} \cup \{\leftarrow p \mid \neg p \in L\} \cup \{\leftarrow not\, p \mid p \in L\}$ if and only if U is a reliable set of P w.r.t. L.*

Proof: $\Leftarrow$: U is a reliable set of P w.r.t. L, from Theorem 9, the programs are strongly equivalent.

$\Rightarrow$: Let P_1 and P_2 stand for these programs respectively. P_1 and P_2 are strongly equivalent. Assume that U is not a reliable set of P w.r.t. L, then there exists a nonempty set $E \subseteq U \cup L$ and $E \cap U \neq \emptyset$ such that for every rule $r \in R^-(E,P)$ with $head(r) \setminus \{p \mid \neg p \in L\} \not\subseteq E$ or $U \cup L \not\models body(r)$. So $U \cup \{p \mid p \in L\}$ is not an answer set of $P' = P_1 \cup \{p \leftarrow \mid p \in L \setminus U\}$, as $U \cup \{p \mid p \in L\}$ does not satisfy $LF(E,P')$, which conflicts to the precondition. So U must be a reliable set of P w.r.t. L. ■

Proposition 8 *Let L be a well-founded model of an NLP P, $\{p \mid p \in L\}$ is a reliable set of P w.r.t. L.*

Proof: Assume that $\{p \mid p \in I\}$ is a reliable set of P .w.r.t I, we only need to prove that $T_P(I)$ is also a reliable set of P w.r.t. I.

For each nonempty set $E \subseteq T_P(I)$, for each $p \in E$, there exists a rule $r \in P$ such that $p \in head(r)$ and $I \models body(r) \wedge \bigwedge_{q \in head(r) \setminus \{p\}} q$. Then $head(r) \setminus \{p \mid \neg p \in I\} \subseteq E$ and $I \models body(r)$. So $T_P(I)$ is a reliable set of P w.r.t. I. ■

Proposition 9 *Let U be a reliable set of P w.r.t. a set L_1 of literals and $L_1 \subseteq L_2$, then U is also a reliable set of P w.r.t. L_2.*

Proof: For any nonempty set $E \subseteq U \cup \{p \mid p \in L_2\}$ and $E \cap U \neq \emptyset$, $E' = E \setminus L_2$ is also nonempty. U is a reliable set of P w.r.t. L_1, then for E' there exists such a rule r. As $U \cup L_1 \models body(r)$, then r is also an external support of E and $U \cup L_2 \models body(r)$. So U is also a reliable set P w.r.t. L_2. ■

Proposition 10 *Let U_1 and U_2 be reliable sets of P w.r.t. a set L of literals. $U_1 \cup U_2$ is also a reliable set of P w.r.t. L.*

Proof: Given a set $E \subseteq U_1 \cup \{p \mid p \in L\}$ with $E \cap U \neq \emptyset$, there is a rule $r \in R^-(E, P)$ such that $U_1 \cup L \models body(r)$. Then for any set E' of atoms such that $E \subseteq E'$ and $E' \subseteq E \cup (U_2 \setminus \{p \mid p \in L\})$, $r \in R^-(E', P)$ and $U_1 \cup U_2 \cup L \models body(r)$. So $U_1 \cup U_2$ is also a reliable set of P w.r.t. L. ■

Then given a set L of literals and a program P, there exists a greatest reliable set U of P w.r.t. L, denoted by $GRS(P, L)$.

So given a consequence L of a program P, we can compute a greatest reliable set U of P w.r.t. L to simplify the program.

6 Computing The Greatest Reliable Set

In this section, we consider how to compute $GRS(P, L)$ for NLPs and DLPs.

6.1 NLP

Proposition 11 *A set U of atoms is the greatest reliable set of an NLP P if and only if U is the least fixed point of T_P.*

Proof: (1) Assume that U' is a reliable set of P and $U' \subseteq T_P(U')$, we want to prove that $T_P(U')$ is also a reliable set of P.

For any set $E \subseteq T_P(U')$, if $E \cap U' \neq \emptyset$, then for the set $E' = E \cap U'$ there exists a rule $r \in R^-(E', P)$ such that $head(r) \subseteq E'$ and $U' \models body(r)$. Clearly, $r \in R^-(E, P)$ and $T_P(U') \models body(r)$.

If $E \cap U' = \emptyset$, then for each $p \in E$ there exists a rule $r \in P$ such that $p \in head(r)$ and $U' \models body(r)$. Note that $E \cap body^+(r) = \emptyset$, then $r \in R^-(E, P)$ and $T_P(U') \models body(r)$.

So $T_P(U')$ is a reliable set of P, then the least fixed point of T_P is a reliable set of P.

(2) Let U be the greatest reliable set of P. Assume that there exists an atom $p \in U$ and p is in the least fixed point of T_P. Then for the set $\{p\}$, there exists a rule $r \in P$ such that $p \in head(r)$, $p \notin body^+(r)$ and $U \setminus \{p\} \models body(r)$. So the assumption implies that $U \setminus \{p\}$ is not belonged to the least fixed point of T_P. Then there exists another atom $q \in U \setminus \{p\}$ such that q is not in the least fixed point of T_P. The process can be continued, note that there is finite number of atoms appeared in P, then the assumption is impossible. So U is belonged to the least fixed point of T_P.

From (1) and (2), the proposition is proved. ■

Corollary 11 *Let L be the well-founded model of an NLP P, $\{p \mid p \in L\}$ is the greatest reliable set of P w.r.t. $\{\neg p \mid \neg p \in L\}$.*

Proof: Similarly to the above proof, as negative consequences do not affect the process of the proof. ■

Let L be a set of literals and P a program, we use T_P^L to denote the operator that:

$$T_P^L(X) = \{p \mid \text{there exists a rule } r \in P \text{ such that } p \in head(r) \text{ and } X \cup L \models body(r) \wedge \bigwedge_{q \in head(r) \setminus \{p\}} q\}.$$

Note that, T_P^L is monotonic. We use $lfp(T_P^L)$ to denote the least fixed point of T_P^L.

Let P be a logic program and I a set of literals, a set X of atoms is an *uncertain set* of P w.r.t. I if for each atom $p \in X$ and each rule $r \in P$ such that $p \in head(r)$, at least one of the following conditions holds:

- $I \not\models body(r)$,
- $X \cap body^+(r) \neq \emptyset$,
- $head(r) \setminus \{p \mid \neg p \in I\} \not\subseteq X$.

Proposition 12 *Let P be a logic program and I a consistent set of literals, if a set X atoms is an unfounded set of P w.r.t. I, then X is an uncertain set of P w.r.t. I.*

Proof: If $I \models \neg body(r)$, then $I \not\models body(r)$.

If $(head(r) \setminus X) \cap I \neq \emptyset$, then there exists an atom $p \in head(r) \setminus X$ with $p \in I$. I is consistent, then $\neg p \notin I$ and $head(r) \setminus \{p \mid \neg p \in I\} \not\subseteq X$. ■

Proposition 13 *Let P be an NLP and I a consistent set of literals, if X_1 and X_2 are uncertain sets of P w.r.t. I, then $X_1 \cup X_2$ is an uncertain set of P w.r.t. I.*

Proof: P is an NLP, then there does not exists a rule r such that $p \in X_1$, $p \in head(r)$, and $head(r) \not\subseteq X_1$. ■

So there exists the greatest uncertain set of an NLP P w.r.t. I, denoted by $UC_P(I)$. However, for a DLP P, X_1 and X_2 are uncertain sets of P w.r.t. I, then $X_1 \cup X_2$ may not be an uncertain set of P w.r.t. I. For example, consider the program $\{a \vee b \leftarrow\}$, $I = \emptyset$, the set $\{a\}$ and $\{b\}$ are uncertain sets of the program w.r.t. I, however, $\{a, b\}$ is not an uncertain set.

Proposition 14 *Let P be a logic program and I a set of literals, a set X of atoms is an uncertain set of P w.r.t. I if and only if for every rule $r \in R^-(X, P)$, $I \not\models body(r)$ or $head(r) \setminus \{p \mid \neg p \in L\} \not\subseteq X$.*

Proof: For any atom $p \in X$, any rule r with $p \in head(r)$, if $r \notin R^-(X, P)$, then $X \cap body^+(r) \neq \emptyset$. ■

Now we consider how to compute $UC_P(L)$ for an NLP P and a set L of literals.

Proposition 15 *Let L be a set of literals and P an NLP. The function $uc_P(L)$ returns $UC_P(L)$ in $O(n)$ time, where n is the number of atoms in P.*

Algorithm 1 $uc_P(L)$

```
1: X := Atoms(P);
2: do
3:    E := {head(r) | r ∈ R⁻(X, P) such that L ⊨ body(r)};
4:    X := X \ E;
5: while E ≠ ∅
6: return X
```

Proof: Note that, the algorithm delete at least one atom in each iterator, then the algorithm terminates in $O(n)$ time.

Clearly, $uc_P(L)$ is an uncertain set of P w.r.t. L. So $uc_P(L) \subseteq UC_P(L)$.

We want to prove that $UC_P(L) \subseteq uc_P(L)$. Assume that $UC_P(L) \subseteq X$ and we want to prove that $UC_P(L) \subseteq X \setminus E$ as defined in the algorithm.

Note that, if $r \in R^-(X, P)$ and $L \models body(r)$, then $r \in R^-(UC_P(L) \cup \{head(r)\})$. So $head(r) \notin UC_P(L)$. Then $UC_P(L) \subseteq uc_P(L)$ and $UC_P(L) = uc_P(L)$. ■

Theorem 12 *A set U of atoms is the greatest reliable set of an NLP P w.r.t. a set L of literals if and only if $U = lfp(T_P^L) \setminus UC_P(U \cup L)$.*

Proof: Let $U = GRS(P, L)$.

(1) $U \subseteq lfp(T_P^L) \setminus UC_P(U \cup L)$. We need to prove that $U \subseteq lfp(T_P^L)$ and $U \cap UC_P(U \cup L) = \emptyset$.

For each atom $p \in U$, there exists a rule $r \in R^-(\{p\}, P)$ such that $(U \setminus \{p\}) \cup L \models body(r)$. So $U \subseteq lfp(T_P^L)$.

Let $E = U \cap UC_P(U \cup L)$. Assume that $E \neq \emptyset$, then there exists a uncertain set $E^* \subseteq UC_P(U \cup L)$ with $E \subseteq E^*$ such that for any rule $r \in R^-(E^*, P)$, $U \cup L \not\models body(r)$.

Let $E' = E^* \cap (U \cup L)$, for any rule $r \in R^-(E, P)$, if $r \in R^-(E^*, P)$, then $U \cup L \not\models body(r)$; if $r \notin R^-(E^*, P)$, then $body^+(r) \cap (E^* \setminus E') \neq \emptyset$, thus $U \cup L \not\models body(r)$. So E' is also an uncertain set of P w.r.t. $U \cup L$.

As $E' \subseteq U \cup \{p \mid p \in L\}$ and $E' \cap U \neq \emptyset$, then there exists a rule $r \in R^-(E', P)$ and $U \cup L \models body(r)$, which conflicts to the above result that $U \cup L \not\models body(r)$. So $E = \emptyset$, and $U \cap UC_P(U \cup L) = \emptyset$.

(2) $U \supseteq lfp(T_P^L) \setminus UC_P(U \cup L)$. We need to prove that $lfp(T_P^L) \subseteq U \cup UC_P(U \cup L)$.

Assume that $X \subseteq lfp(T_P^L)$ and $X \subseteq U \cup UC_P(U \cup L)$. We want to prove that $T_P^L(X) \subseteq U \cup UC_P(U \cup L)$.

Let $E = T_P^L(X) \setminus (U \cup UC_P(U \cup L))$, assume that $E \neq \emptyset$.

$E \subseteq T_P^L(X)$, then for each $p \in E$, there exists a rule $r \in P$ such that $p \in head(r)$ and $X \cup L \models body(r)$.

$E \cap U = \emptyset$, then for each $p \in E$, there exists a set $E' \subseteq \{p\} \cup \{q \mid q \in L\}$ and $p \in E'$ such that for every $r \in R^-(E', P)$, $U \cup L \not\models body(r)$.

$E \cap UC_P(U \cup L) = \emptyset$, then for each $p \in E$, there does not exist a set E^* with $p \in E^*$ such that for every rule $r \in R^-(E^*, P)$, $U \cup L \not\models body(r)$.

Then both $E \cap U = \emptyset$ and $E \cap UC_P(U \cup L) = \emptyset$ is impossible. So $E = \emptyset$ and $U \supseteq lfp(T_P^L) \setminus UC_P(U \cup L)$. ■

Theorem 13 *A set U of atoms is the greatest reliable set of an NLP P w.r.t. a set L of literals if and only if $U = Atoms(P) \setminus UC_P(U \cup L)$.*

Proof: (1) From the above proof $U \cap UC_P(U \cup L) = \emptyset$.

(2) We want to prove that $Atoms(P) \setminus (U \cup UC_P(U \cup L)) = \emptyset$.

Let $E = Atoms(P) \setminus (U \cup UC_P(U \cup L))$, assume that $E \neq \emptyset$.

$E \cap U = \emptyset$, then for each $p \in E$, there exists a set $E' \subseteq \{p\} \cup \{q \mid q \in L\}$ and $p \in E'$ such that for every $r \in R^-(E', P)$, $U \cup L \not\models body(r)$.

$E \cap UC_P(U \cup L) = \emptyset$, then for each $p \in E$, there does not exist a set E^* with $p \in E^*$ such that for every rule $r \in R^-(E^*, P)$, $U \cup L \not\models body(r)$.

Both $E \cap U = \emptyset$ and $E \cap UC_P(U \cup L)$ is impossible. So $E = \emptyset$ and $Atoms(P) = U \cup UC_P(U \cup L)$. ■

Now we can define an operator RS_P^L for an NLP P and a set L of literals:

$$RS_P^L(X) = Atoms(P) \setminus UC_P(X \cup L).$$

Note that RS_P^L is monotonic, we use $gfp(RS_P^L)$ to denote the greatest fixed point of RS_P^L.

Theorem 14 *Let P be an NLP and L a set of literals, $GRS(P, L) = gfp(RS_P^L)$.*

Proof: (1) $gfp(RS_P^L) \subseteq GRS(P, L)$.

Let $E = Atoms(P) \setminus UC_P(gfp(RS_P^L) \cup L)$, we need to prove that E is a reliable set of P w.r.t. L. Clearly, for each $p \in E$, there does not exist a set E^* with $p \in E^*$ such that for every rule $r \in R^-(E^*, P)$, $gfp(RS_P^L) \cup L \not\models body(r)$. Then for every set E^* with $p \in E^*$ there exists a rule $r \in R^-(E^*, P)$, such that $gfp(RS_P^L) \cup L \models body(r)$.

So $gfp(RS_P^L)$ is a reliable set of P w.r.t. L and $gfp(RS_P^L) \subseteq GRS(P, L)$.

(2) $GRS(P, L) \subseteq gfp(RS_P^L)$.

Let $GRS(P, L) \subseteq X$, we need to prove that $GRS(P, L) \subseteq Atoms(P) \setminus UC_P(X \cup L)$.

Note that $GRS(P, L) = Atoms(P) \setminus UC_P(GRS(P, L) \cup L)$ and $UC_P(X \cup L) \subseteq UC_P(GRS(P, L) \cup L)$. So $GRS(P, L) \subseteq Atoms(P) \setminus UC_P(X \cup L)$ and $GRS(P, L) \subseteq gfp(RS_P^L)$. ■

6.2 DLP

Proposition 16 *Deciding whether a set U of atoms is a reliable set of a DLP P is coNP-complete.*

Proof: (1) The problem is a coNP problem, as U is not a reliable set of P, once we can guess a nonempty set $E \subseteq U$ that there does not exists a rule $r \in R^-(E, P)$ such that $head(r) \subseteq E$ and $U \models body(r)$.

(2) We want to prove that the 3SAT problem is equivalent to the problem that U is not a reliable set of P.

Let t and e be new atoms that do not appear in a set of clauses $\mathcal{C}$. We can construct a DLP P from $\mathcal{C}$ by:

- adding the rule $t \vee e \vee \bigvee_{p \in Atoms(\mathcal{C})} p \leftarrow$, and

- for each clause $C \in \mathcal{C}$, converting it to a DLP rule, like $e \vee \bigvee_{\neg p \in C} p \leftarrow$ $\mathbf{,}_{q \in C}\, q$,
- for each atom $p \in Atoms(\mathcal{C})$, adding rules $t \leftarrow p$ and $p \leftarrow e$,
- adding the rule $t \leftarrow e$.

Let $U = Atoms(\mathcal{C}) \cup \{t, e\}$ and $E \subseteq U$.

If $e \notin E$ and $E \neq \emptyset$, then there exists a rule r in the form of $p \leftarrow e$ or $t \leftarrow e$ such that $r \in R^-(E, P)$, $head(r) \subseteq E$ and $U \models body(r)$.

If $t \in E$ and $E \subset U$, then there exists a rule r in the form of $t \leftarrow p$ such that $r \in R^-(E, P)$, $head(r) \subseteq E$ and $U \models body(r)$. If $t \in E$ and $E = U$, then there exists such a rule in the form of $t \vee e \vee \bigvee_{p \in Atoms(\mathcal{C})} p \leftarrow$.

If $t \notin E$ and $e \in E$, then $E' = E \setminus \{e\} \subseteq Atoms(\mathcal{C})$. Clearly, the rule $r^* = t \vee e \vee \bigvee_{p \in Atoms(\mathcal{C})} p \leftarrow$ belongs to $R^-(E, P)$, however, $head(r^*) \not\subseteq E$.

If E' is a model of $\mathcal{C}$, then for each other rule $r \in R^-(E, P)$, $E' \not\models \bigvee_{q \in body^+(r)} q$. As E satisfies the corresponding clause C, then $E' \not\models \bigwedge_{\neg p \in C} p$, so $head(r) \not\subseteq E$.

If E' is not a model of $\mathcal{C}$, then there exists a clause C such that $E' \not\models C$, thus there exists a corresponding rule r such that $head(r) \subseteq E$ and $r \in R^-(E, P)$.

So U is a reliable set of P if and only if $\mathcal{C}$ is not satisfiable.

From (1) and (2), the problem of deciding whether U is a reliable set of P is coNP-complete. ■

Proposition 17 *Deciding whether a set U of atoms is a reliable set of a DLP P w.r.t. a set L of literals is coNP-complete.*

Proof: The problem is a coNP problem, as checking whether U is not a reliable set of P w.r.t. L, we can guess a nonempty set $E \subseteq U \cup \{p \in p \in L\}$ with $E \cap U \neq \emptyset$ that for each rule $r \in R^-(E, P)$, $head(r) \setminus \{p \mid \neg p \in L\} \not\subseteq E$ or $U \cup L \not\models body(r)$.

From the above proposition, the problem is coNP-complete. ■

Now we define uncertain sets for DLPs. Let P be a DLP, L a set of literals, and U a set of atoms, a set X is an uncertain set of P w.r.t. L under U, if for each atom $p \in X$ and each rule $r \in P$ such that $p \in head(r)$, at least one of the following conditions holds:

- $L \cup U \not\models body(r)$,
- $X \cap body^+(r) \neq \emptyset$,
- $head(r) \setminus \{p \mid \neg p \in L\} \not\subseteq X$.

Proposition 18 *Let P be a DLP, L a set of literals, and U a set of atoms, a set X is an uncertain set of P w.r.t. L under U if and only if for each rule $r \in R^-(X, P)$, $L \cup U \not\models body(r)$ or $head(r) \setminus \{p \mid \neg p \in L\} \not\subseteq X$.*

Proof: Directly , as $r \notin R^-(X, P)$ if and only if $head(r) \cap X \neq \emptyset$ or $X \cap body^+(r) \neq \emptyset$. ■

For a DLP P, X_1 and X_2 are uncertain sets of P w.r.t. L under U, then $X_1 \cup X_2$ may not be an uncertain set of P w.r.t. L under U.

However, we still use $UC_P^L(U)$ to denote the union of all possible uncertain sets of P w.r.t. L under U.

Theorem 15 *A set U of atoms is the greatest reliable set of a DLP P w.r.t. a set L of literals if and only if $U = \mathit{Atoms}(P) \setminus UC_P^L(U)$.*

Proof: (1) $U \cap UC_P^L(U) = \emptyset$.

Let $E^* = U \cap UC_P^L(U)$, assume that $E^* \neq \emptyset$. From the definition, for every set $E \subseteq E^* \cup \{p \mid p \in L\}$ with $E \cap E^* \neq \emptyset$, there exists a rule $r \in R^-(E, P)$ such that $\mathit{head}(r) \setminus \{p \mid \neg p \in L\} \subseteq E$ and $L \cup U \models \mathit{body}(r)$.

$E^* \subseteq UC_P^L(U)$, then there must exist a set E' such that $E' \cap E^* \neq \emptyset$ and for every $r \in R^-(E', P)$, $\mathit{head}(r) \setminus \{p \mid \neg p \in L\} \not\subseteq E'$ or $L \cup U \not\models \mathit{body}(r)$. Let $E'' = E' \cap (U \cup L)$, then for every $r \in R^-(E'', P)$, if $r \notin R^-(E', P)$, then $\mathit{body}^+(r) \cap (E' \setminus E'') \neq \emptyset$, so $U \cup L \not\models \mathit{body}(r)$. So for each $r \in R^-(E'', P)$, $\mathit{head}(r) \setminus \{p \mid \neg p \in L\} \subseteq E''$ or $L \cup U \not\models \mathit{body}(r)$.

So $E^* = \emptyset$.

(2) $\mathit{Atoms}(P) \setminus (U \cup UC_P^U(L)) = \emptyset$.

Note that for any $E \subseteq \mathit{Atoms}(P)$, both $E \cap U = \emptyset$ and $E \cap UC_P^U(L) = \emptyset$ is impossible.

So $\mathit{Atoms}(P) = U \cup UC_P^L(U)$. ■

Now we define an operator RS_P^L for a DLP P and a set L of literals:

$$RS_P^L(X) = \mathit{Atoms}(P) \setminus UC_P^L(X).$$

Note that RS_P^L is monotonic, we use $\mathit{gfp}(RS_P^L)$ to denote the greatest fixed point of RS_P^L.

Theorem 16 *Let P be a DLP and L a set of literals, $\mathit{GRS}(P, L) = \mathit{gfp}(RS_P^L)$.*

Proof: (1) $\mathit{gfp}(RS_P^L) \subseteq \mathit{GRS}(P, L)$.

Let $E = \mathit{Atoms}(P) \setminus UC_P^L(\mathit{gfp}(RS_P^L))$, we need to prove that E is a reliable set of P w.r.t. L under U. Clearly, for each $p \in E$, there does not exist a set E^* with $p \in E^*$ such that for every rule $r \in R^-(E^*, P)$, $\mathit{gfp}(RS_P^L) \cup L \not\models \mathit{body}(r)$ or $\mathit{head}(r) \setminus \{p \mid \neg p \in L\} \not\subseteq E^*$. Then for every set E^* with $p \in E^*$ there exists a rule $r \in R^-(E^*, P)$, such that $\mathit{head}(r) \setminus \{p \mid \neg p \in L\} \subseteq E^*$ and $\mathit{gfp}(RS_P^L) \cup L \not\models \mathit{body}(r)$.

So $\mathit{gfp}(RS_P^L)$ is a reliable set of P w.r.t. L under U and $\mathit{gfp}(RS_P^L) \subseteq \mathit{GRS}(P, L)$.

(2) $\mathit{GRS}(P, L) \subseteq \mathit{gfp}(RS_P^L)$.

Let $\mathit{GRS}(P, L) \subseteq X$, we need to prove that $\mathit{GRS}(P, L) \subseteq \mathit{Atoms}(P) \setminus UC_P^L(X)$.

Note that $\mathit{GRS}(P, L) = \mathit{Atoms}(P) \setminus UC_P^L(\mathit{GRS}(P, L))$ and $UC_P^L(X) \subseteq UC_P^L(\mathit{GRS}(P, L))$. So $\mathit{GRS}(P, L) \subseteq \mathit{Atoms}(P) \setminus UC_P^L(X)$ and $\mathit{GRS}(P, L) \subseteq \mathit{gfp}(RS_P^L)$. ■

Now the problem is how to compute $UC_P^L(X)$ for DLPs.

Proposition 19 *Let P be a DLP, L a set of literals and X a set of atoms, deciding whether an atom p belongs to $RS_P^L(X)$ is coNP-complete.*

Proof: We can guess a set E that prevents an atom p belongs to $RS_P^L(X)$, so the problem is a coNP problem.

Consider the proof for Proposition 16, deciding whether t belongs to $RS_P^L(\mathit{Atoms}(\mathcal{C} \cup \{t, e\}))$ is equivalent to whether $\mathcal{C}$ is not satisfiable. ■

Now we consider how to compute $UC_P^L(U)$ for an NLP P w.r.t. a set L of literals under a set U of atoms.

Algorithm 2 $uc^L_P(U, X)$

do
  $E := \{head(r) \mid r \in R^-(X, P)$ such that $U \cup L \models body(r)$ and $head(r) \setminus \{p \mid \neg p \in L\} \subseteq X\}$;
  for each $C \in E$:
    if $|C| = 1$ **then** $X := X \setminus C$ **else**
      $Y := \bigcup_{p \in C}(C \cap uc^L_P(U, X \setminus \{p\}))$;
      $X := X \setminus (C \setminus Y)$;
until X is not changed;
return X;

Lemma 1 *Let P be a DLP, L a set of literals, U, X sets of atoms, and an atom $p \in X$. If $p \in uc^L_P(U, X)$, then there exists a nonempty set $X' \subseteq X$ with $p \in X'$ such that X' is an uncertain set of P w.r.t. I under U.*

Proof: Assume that there does not exist such an under set X', then for every nonempty set $X' \subseteq X$ with $p \in X'$, there exists a rule $r \in R^-(X', P)$ such that $L \cup U \models body(r)$ and $head(r) \setminus \{p \mid \neg p \in L\} \subseteq X'$.

From Algorithm 2, $head(r) \in E$, note that $|head(r)| > 1$, if no, then $p \notin uc^L_P(U, X)$.

Note that $p \notin uc^L_P(U, X \setminus \{p\})$, then there should exist some $q \in head(r)$ different from p such that $p \in uc^L_P(U, X \setminus \{q\})$. However, $p \in X \setminus \{q\}$, there exists a rule $r' \in R^-(X \setminus \{q\}, P)$ such that $L \cup U \models body(r')$ and $head(r') \setminus \{p \mid \neg p \in L\} \subseteq X \setminus \{q\}$.

Then there should exist some $q' \in head(r')$ different from p such that $p \in uc^L_P(U, X \setminus \{q, q'\})$.

The process can be continued for ever, which conflicts to the fact that there are only finite number of atoms. So if $p \in uc^L_P(U, X)$, then there exists a nonempty set $X' \subseteq X$ with $p \in X'$ such that X' is an uncertain set of P w.r.t. I under U. ■

Lemma 2 *Let P be a DLP, L a set of literals, U, X sets of atoms, and an atom $p \in X$. If there exists a nonempty set $X' \subseteq X$ with $p \in X'$ such that X' is an uncertain set of P w.r.t. I under U, then $p \in uc^L_P(U, X)$.*

Proof: Assume that $p \notin uc^L_P(U, X)$, note that $p \in X$, then there exists a rule $r \in R^-(X, P)$ such that $U \cup L \models body(r)$, $head(r) \setminus \{p \mid \neg p \in L\} \subseteq X'$, and there exists an atom $q \in head(r)$ such that $q \notin uc^L_P(U, X \setminus s\{q'\})$ for every $q' \in head(r)$.

If p is such an atom, $p \notin uc^L_P(U, X \setminus \{q\})$ for every $q \in head(r)$, then for each set $X \setminus \{q\}$ there exists a rule r' such that $r' \in R^-(X \setminus \{q\}, P)$, $U \cup L \models body(r)$ and $head(r) \setminus \{p \mid \neg p \in L\} \subseteq X \setminus \{q\}$.

If p is not such an atom, then $p \notin uc^L_P(U, X \setminus \{q\})$ for every $q \in head(r)$.

The process can be continued until resulting a set $X^* \subseteq X$, $p \in X^*$, and there exists a rule $r^* \in R^-(X^*, P)$ satisfies the condition and $head(r^*) = \{p\}$. Then for every nonempty set $X' \subseteq X^*$ with $p \in X'$, X' is not an uncertain set of P w.r.t. I under U.

Moreover, for every nonempty set $X' \subseteq X$ with $p \in X'$, an atom $q \in X'$ and $q \notin X^*$, there exists such a rule r'. So X' is not an uncertain set of P w.r.t. I under U.

So for every set $X' \subseteq X$ with $p \in X'$, X' is not an uncertain set of P w.r.t. I under U, which conflicts to the precondition. So $p \in uc_P^L(U, X)$. ■

Theorem 17 *Let P be a DLP, L a set of literals, U and X sets of atoms, $uc_P^L(U, X)$ returns the union of uncertain set E of P w.r.t. L under U such that $E \subseteq X$.*

Proof: Directly from Lemma 1 and 2. ■

Proposition 20 *Let P be a DLP, L a set of literals, and U a set of atoms, $UC_P^L(U) = uc_P^L(U, Atoms(P))$.*

Proof: Directly from the above proposition. ■

7 Experiment

8 Conclusion

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