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THE DEAD LEAVES MODEL: A GENERAL TESSELLATION MODELING OCCLUSION

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Abstract

In this article, we study a particular example of general random tessellation, the *dead leaves model*. This model, first studied by the mathematical morphology school, is defined as a sequential superimposition of random closed sets, and provides the natural tool to study the occlusion phenomenon, an essential ingredient in the formation of visual images. We generalize certain results of G. Matheron and, in particular, compute the probability of n compact sets being included in *visible parts*. This result characterizes the distribution of the boundary of the dead leaves tessellation.

Keywords: General tessellation; dead leaves model; typical cell; image modeling

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1. Introduction

The dead leaves model was introduced by Matheron in [18], and results from the sequential superimposition of random sets. As such, it provides the natural tool for studying nonlinear *occlusion* phenomena, which are of great importance in image modeling and processing. However, to the best of the authors' knowledge, this model has not been systematically investigated, and even its definition lacks some precision. Our purpose in this paper is twofold: first, to provide a rigorous definition of the model as a random tessellation; second, to give new proofs or extensions of Matheron's results in the framework of Palm calculus.

A first motivation to study this model comes from applications. Among existing stochastic models for natural images, the dead leaves model is the only one whose definition agrees with their physical formation. Several recent studies have demonstrated the ability of specific dead leaves models to reproduce most known statistics of natural images; see [23], [1], and [16]. The model has also been proposed as a tool to resample random fields for texture synthesis; see [10]. Other examples of applications come from materials sciences; see [14] and [8].

As a second motivation, let us stress that the dead leaves model provides nontrivial examples of general random tessellations, in the sense that their cells are general closed sets. In particular, they are not necessarily polygonal, connected, or convex, as is the case for the most popular tessellation models, such as Poisson flats or Voronoi or Delaunay tessellations. Note that

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nonconvex and nonpolygonal cells are encountered in the case of Johnson–Mehl tessellation (see, e.g. [26, pp. 313, 333]), but that there are relatively few such examples. Therefore, there are few studies of 'general' tessellations, even though classical formulae originally proved in the convex and polygonal case have been shown to hold in more general contexts; see [25], [27], and [6].

In Section 2 we first recall some facts on random closed sets and slightly reformulate notions from [21] and [25] to define random tessellations and typical cell distributions. In Section 3 we define the dead leaves model as a random tessellation obtained from an initial Poisson process, and give some of its elementary properties. Then, in Section 4, we generalize the results of Matheron. In order to do so in a rigorous way, we make use of point process theory through the systematic use of Palm calculus. We first give the probability of n compact sets being included in n different visible parts, a result that completely characterizes the distribution of the boundary of our model as a random closed set. Then we compute the distribution of 'objects' that remain completely visible. Eventually, we recover, in the Palm calculus framework, a nice result of Matheron giving the length distribution of the intersection of objects with a line of fixed direction, stating in particular that its expectation is divided by two as a result of occlusion.

1.1. Previous work

The dead leaves model was introduced in [18], a technical report written in an informal style, yet which contained all the relevant basic ideas. The model was defined as the superimposition of infinitesimal Boolean models, and formulae for the probability of a compact set to be included in a visible part and for the distribution of completely visible parts, among other things, were derived. Most of these definitions and results are stated in the book by Serra [24]. Jeulin further studied this model in [13], still using the same infinitesimal formalism, and gave an explicit formula for the joint probability of two compact sets to be included in visible parts. In [12] he generalized the model to the case of random functions and extended to this setting formulae both for the distribution of visible parts and for inclusion probabilities. Cowan and Tsang, in a very interesting paper [5], made use of mean value formulae for tessellations to derive the expectations of quantities such as the number of connected components of visible parts and the length of their boundaries per surface unit.

2. Basic definitions

2.1. Closed sets and tessellations

Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{K}$ respectively be the sets of all closed, open, and compact sets of $\mathbb{R}^d$, $d \geq 1$. For any $A \subset \mathbb{R}^d$, we write

$$\mathcal{F}^A = \{F \in \mathcal{F} : F \cap A = \varnothing\} \quad \text{and} \quad \mathcal{F}_A = \{F \in \mathcal{F} : F \cap A \neq \varnothing\}.$$

The Borel σ-field, $\mathcal{B}_{\mathcal{F}}$, on $\mathcal{F}$ is generated by the basis of open sets $\{\mathcal{F}^K,\ K \in \mathcal{K};\ \mathcal{F}_G,\ G \in \mathcal{G}\}$. Borel sets are defined on $\mathcal{G}$ and $\mathcal{K}$ in a way similar to those of $\mathcal{F}$; see [19, p. 11]. A random closed set of $\mathbb{R}^d$ is a measurable function from a probability space $(\Omega, \mathcal{S}, \mathrm{P})$ into $(\mathcal{F}, \mathcal{B}_{\mathcal{F}})$. For any sets A and B, we write

$$A \ominus B = \{x \in \mathbb{R}^d : x + \check{B} \subset A\} \quad \text{and} \quad A \oplus B = \{x + y : x \in A,\ y \in B\},$$

where $\check{B} = \{-x,\ x \in B\}$. The set $A \ominus \check{B}$ is called the erosion of A by B, and $A \oplus \check{B}$ the dilation of A by B. Measurability properties of the operators '$\ominus$' and '$\oplus$' were established in [19, pp. 19–20].

A σ-finite measure on $\mathcal{F}' := \mathcal{F} \setminus \{\varnothing\}$ (endowed with its Borel σ-algebra, $\mathcal{B}_{\mathcal{F}'}$) is a measure taking finite values on $\mathcal{F}_K$ for all $K \in \mathcal{K}$; see [19, p. 57]. We denote by $\mathcal{N}_{\mathcal{F}'}$ the set of σ-finite counting measures on $(\mathcal{F}', \mathcal{B}_{\mathcal{F}'})$. For all $M \in \mathcal{N}_{\mathcal{F}'}$, we write $M = \sum_i \delta_{F_i}$, where δ_{F_i} is the unit-mass measure at point F_i. The *boundary* of M is defined as $\partial M = \bigcup_i \partial F_i$, where ∂F_i denotes the topological boundary of F_i. A point process on $\mathcal{F}'$ is a measurable function from a probabilistic space to $(\mathcal{N}_{\mathcal{F}'}, \mathcal{B}_{\mathcal{N}_{\mathcal{F}'}})$, where $\mathcal{B}_{\mathcal{N}_{\mathcal{F}'}}$ is the usual σ-field on $\mathcal{N}_{\mathcal{F}'}$; see, e.g. [7, Chapter 6].

Following Stoyan [25], a tessellation of $\mathbb{R}^d$ is defined as follows.

Definition 1. Let $T = \sum_i \delta_{F_i} \in \mathcal{N}_{\mathcal{F}'}$. We say that T is a tessellation of $\mathbb{R}^d$ if

(i) $\bigcup_i F_i = \mathbb{R}^d$ and,

(ii) for all $i \neq j$, $\operatorname{Int} F_i \cap F_j = \varnothing$, where $\operatorname{Int} F$ denotes the interior of F,

or, equivalently, if $\{(\operatorname{Int} F_i)_i, \partial T\}$ is a partition of $\mathbb{R}^d$.

Note that $T \in \mathcal{N}_{\mathcal{F}'}$ implies that the number of cells F_i hitting a compact set is finite. This condition was added in the original definition in [25], where the F_i are marks of a point process, $N = \sum_i \delta_{x_i}$, on $\mathbb{R}^d$ (where x_i is called the *centroid* of F_i). The centroids are unimportant in the definition of a tessellation, but they are quite useful in defining the *typical cell distribution*, as we will recall below.

Let $\mathcal{T}$ be the set of all tessellations in $\mathcal{N}_{\mathcal{F}'}$. By expressing assertions (i) and (ii) of Definition 1 as limits of the elementary set operations $(F, F') \mapsto F \cup F'$, $(F, F') \to F \cap F'$, and $F \mapsto \partial F$, which are measurable as a consequence of [19, Sections 1–2], we easily find that $\mathcal{T} \in \mathcal{B}_{\mathcal{N}_{\mathcal{F}'}}$. A random tessellation of $\mathbb{R}^d$ is then defined as a point process, T, on $\mathcal{F}'$, such that $T \in \mathcal{T}$ almost surely (a.s.). Classical examples of random tessellations (see the references in [26, Chapter 10] and [22]) include Poisson hyperplane processes and Delaunay, Voronoi, and Johnson–Mehl tessellations. A standard approach (see, e.g. [2], [4], [20], [21], or [26, Chapter 10]), which applies in these examples, is to define ∂T directly as a random closed set without considering the underlying random tessellation. However, it is not always possible to recover the F_i from ∂T (as they may not be connected; see [6] and Remark 2, below, for a precise example).

2.2. Typical cell distribution

In [21] a typical cell was defined using the Palm distribution of a simple marked point process, $N = \sum_i \delta_{x_i, F_i}$, of points in $\mathbb{R}^n$ with marks in $\mathcal{F}'$ that is stationary with respect to shifts $N \mapsto \sum \delta_{x_i - x, F_i - x}$, $x \in \mathbb{R}^d$. More precisely, let us denote by μ the intensity of N, which we assume to be finite, and by $\mathbb{P}_N^0$ its Palm distribution. Let x_0 be the point nearest to the origin and F_0 its corresponding cell. Then the *typical cell distribution* is defined on the σ-field, $\mathcal{I}$, of all translation-invariant events in $\mathcal{B}_{\mathcal{F}'}$ by $\chi \mapsto \mathbb{P}_N^0(F_0 \in \chi)$, $\chi \in \mathcal{I}$. A result of [21], proved for tessellations whose cells are bounded polytopes, can easily be extended as follows.

Proposition 1. *Let B be a Borel set in $\mathbb{R}^d$ such that*

$$0 < \nu(F_i \oplus B) < \infty \quad \textit{for all } i \textit{, a.s.,} \tag{1}$$

where ν is the Lebesgue measure on $\mathbb{R}^n$. Then $\mu = \mathbb{E}[\sum_i \mathbf{1}(0 \in F_i \oplus B)/\nu(F_i \oplus B)]$ and

$$\mathbb{P}_N^0(F_0 \in \chi) = \frac{1}{\mu}\, \mathbb{E}\bigg[\sum_i \frac{\mathbf{1}(0 \in F_i \oplus B)\, \mathbf{1}(F_i \in \chi)}{\nu(F_i \oplus B)}\bigg], \qquad \chi \in \mathcal{I}.$$

When starting from a stationary point process, $M = \sum_i \delta_{F_i}$, on $\mathcal{F}'$, a marked point process, N, can be obtained by constructing points $x_i = \Delta(F_i)$, where Δ is such that $\Delta(F_i - x) = \Delta(F_i) - x$. Classical examples for Δ include the set-centroid, the median point, and the extremal point in a given direction. Observe that, under condition (1), it is always possible to define such a set-centroid by taking for each coordinate the median of the marginal measure of ν restricted to $F_i \oplus B$; for instance, the first coordinate is then defined as the smallest x such that $\nu((F_i \oplus B) \cap (-\infty, x] \times \mathbb{R}^{d-1}) \geq \nu(F_i \oplus B)/2$. As was noticed in [21], the typical cell distribution should not depend on the choice of x_i, which is ensured by Proposition 1, provided that a Borel set B can be found for which (1) is fulfilled. This will be the case for the dead leaves model considered below.

In order to define the typical cell of a tessellation, assume that

$$0 < \nu(F_i) < \infty \quad \text{and} \quad \nu(\partial F_i) = 0, \qquad \text{for all } i \text{, a.s.} \tag{2}$$

Note that the first of these conditions is (1) with $B = \{0\}$. The second condition enables us to define $F_{\{x\}}$ almost everywhere as the cell to which the point x belongs. By the stationarity of N, $F_{\{0\}}$ is a.s. defined. Applying Proposition 1 then yields

$$\mu = \mathbb{E}\left[\frac{1}{\nu(F_{\{0\}})}\right] \quad \text{and} \quad \mathbb{P}^0_N(F_0 \in \chi) := \frac{1}{\mu}\,\mathbb{E}\left[\frac{\mathbf{1}(F_{\{0\}} \in \chi)}{\nu(F_{\{0\}})}\right], \qquad \chi \in \mathcal{I}.$$

We thus obtain the formula of the typical cell distribution derived in [20], [21] (when the F_i are bounded polytopes), and [4] (when the F_i are uniformly bounded polytopes).

We end this section with a limit theorem. Let $B_n = B(0, r_n)$ be the ball of radius r_n, $r_n \to \infty$, centered at 0. Let $(A_n)_{n\in\mathbb{N}}$ be any increasing sequence of compact convex sets such that, for all n, $B_n \subset A_n$. The individual ergodic theorem (Proposition 10.2.II of [7]) easily yields the following result.

Proposition 2. *If N is ergodic and satisfies (2), then, for all* $\chi \in \mathcal{I}$,

$$\lim_n \frac{\sum_i \mathbf{1}(F_i \in \chi)\nu(F_i \cap A_n)/\nu(F_i)}{\sum_i \nu(F_i \cap A_n)/\nu(F_i)} = \mathbb{P}^0_N(F_0 \in \chi) \quad \textit{a.s.} \tag{3}$$

Equation (3) is a weighted average in which each F_i has a weight equal to its proportion included in A_n. From a statistical point of view, (3) can be used in deriving a strongly consistent estimator of $\mathbb{P}^0_N(F_0 \in \chi)$, for a given $\chi \in \mathcal{I}$. Under a stronger hypothesis on the cells, there may be different sequences having the same limit as in (3). For example, if the cells are uniformly bounded (as in [4]), (3) implies that

$$\mathbb{P}^0_N(F_0 \in \chi) = \lim_n \frac{\sum_i \mathbf{1}(F_i \in \chi)\,\mathbf{1}(F_i \subset A_n)}{\sum_i \mathbf{1}(F_i \subset A_n)} = \lim_n \frac{\sum_i \mathbf{1}(F_i \in \chi)\,\mathbf{1}(F_i \cap A_n \neq \varnothing)}{\sum_i \mathbf{1}(F_i \cap A_n \neq \varnothing)} \quad \text{a.s.}$$

Sufficient conditions under which these equalities hold were studied in [6].

3. The dead leaves model

3.1. Definition

The dead leaves model is obtained through sequential superimposition of random objects 'falling' on $\mathbb{R}^d$. Let $\sum_{i\in\mathbb{N}} \delta_{x_i,t_i}$ be a homogeneous Poisson point process on the half-space $\mathbb{R}^d \times (-\infty, 0]$, with intensity 1. Let P be a probability measure on $(\mathcal{F}, \mathcal{B}_{\mathcal{F}})$, and let $(X_i)_{i\in\mathbb{N}}$

be independent, identically distributed random variables on $\mathcal{F}$ that have distribution P and are independent of the Poisson point process above. Equivalently, $\Phi = \sum_i \delta_{x_i, t_i, X_i}$ is a Poisson point process on $\mathbb{R}^d \times (-\infty, 0] \times \mathcal{F}$ with intensity measure $\nu(\mathrm{d}x)\, \mathrm{d}t\, \mathrm{P}(\mathrm{d}X)$.

We write $(\Omega, \mathcal{S}, \mathbb{P})$ for the probabilistic space on which Φ is defined and $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$. From now on, X will always denote a random variable on $\mathcal{F}$ that has distribution P and is independent of all other variables, and E will denote the expectation with respect to P.

Definition 2. For all $i \in \mathbb{N}$, the random closed set $x_i + X_i$ is called a leaf and

$$V_i = (x_i + X_i) \setminus \bigcup_{t_j \in (t_i, 0)} (x_j + \operatorname{Int} X_j) \tag{4}$$

is called a visible part.

From now on we assume that X satisfies the following three conditions.

(C1) For all $K \in \mathcal{K}$, $\mathrm{E}[\nu(X \oplus K)] < \infty$.

(C2) There exists a ball, B, with strictly positive radius, such that $\mathrm{E}[\nu(X \ominus B)] > 0$.

(C3) X is a regular closed set, i.e. X is the closure of its interior, P-a.s.

Proposition 3. *We denote by M the point process on $\mathcal{F}'$ obtained by removing from the collection $\{V_i\}$ all sets with empty interiors, that is,*

$$M = \sum_i \mathbf{1}\{\operatorname{Int} V_i \neq \varnothing\}\, \delta_{V_i}.$$

Then M is a random tessellation of $\mathbb{R}^d$. Moreover, $N = \sum_i \mathbf{1}(\operatorname{Int} V_i \neq \varnothing)\delta_{x_i, V_i}$ is stationary, mixing, and has finite intensity.

Remark 1. The condition $\operatorname{Int} V_i \neq \varnothing$ in the definitions of M and N is adopted for convenience as it eliminates visible parts with zero d-dimensional Lebesgue measures. The question arises as to whether $M' := \sum_i \mathbf{1}(V_i \neq \varnothing)\delta_{V_i}$ also has such a property. For simple examples of X, it is easily shown that $M = M'$ a.s. but we do not know whether this equality is true under the general assumptions (C1)–(C3). In any case, because (4) implies that $\partial V_i \subset \bigcup_{t_j > t_i} \partial\{\operatorname{Int} V_j\}$, we always have $\partial M = \partial M'$.

In order to prove Proposition 3 we will make use of the following two lemmas. The first one, which is easy to prove by referring to the definition of the intensity of the Poisson point process Φ, will be needed repeatedly in the sequel.

Lemma 1. *Let K be a bounded Borel set, let s_1 and s_2 satisfy $-\infty < s_1 < s_2 < 0$, and define*

$$\Phi_K(s_1, s_2) := \sum_i \mathbf{1}(t_i \subset (s_1, s_2] \textit{ and } K \subset x_i + X_i),$$

$$\Phi^K(s_1, s_2) := \sum_i \mathbf{1}(t_i \subset (s_1, s_2] \textit{ and } K \cap x_i + X_i \neq \varnothing).$$

These functions are Poisson random variables with respective means $(t_2 - t_1)\, \mathrm{E}[\nu(X \ominus \check{K})]$ and $(t_2 - t_1)\, \mathrm{E}[\nu(X \oplus \check{K})]$.

Lemma 2. *If K is a Borel set of $\mathbb{R}^d$ such that* $\mathrm{E}[\nu(X \ominus \check{K})] > 0$, *then K is almost surely covered by some leaf $x_i + X_i$. As a consequence, any bounded set is a.s. covered by a finite number of leaves.*

Proof. Let us fix $t < 0$. By Lemma 1, the probability, $\mathbb{P}(\Phi_K(t, 0) = 0)$, that none of the leaves $x_i + X_i$, with $t < t_i < 0$, satisfies $K \subset x_i + X_i$ is $\exp(t\,\mathrm{E}[\nu(X \ominus \check{K})])$, which yields the first assertion. Now let B be a ball such that (C2) is satisfied, that is, $\mathrm{E}[\nu(X \ominus B)] > 0$. Since any bounded set K is covered by a finite number of balls with the same radius as B, it follows that K is also covered by $\bigcup_{t_i > T}(x_i + X_i)$ for some $T < 0$.

Proof of Proposition 3. Let us now show that $M \in \mathcal{N}_{\mathcal{F}'}$ $\mathbb{P}$-a.s. In fact, we will show that

$$M' := \sum_i \mathbf{1}(V_i \neq \varnothing)\delta_{V_i} \in \mathcal{N}_{\mathcal{F}'} \quad \mathbb{P}\text{-a.s.}$$

(which implies that $M \in \mathcal{N}_{\mathcal{F}'}$), that is, that only a finite number of visible parts V_i may intersect a given compact set K. By Lemma 2, there $\mathbb{P}$-a.s. exists a negative T such that K is covered by leaves, $x_i + X_i$, satisfying $t_i > T$. It follows that the visible parts intersecting K correspond to leaves falling after time T. The number of such leaves is thus $\Phi^K(T, 0)$, which is $\mathbb{P}$-a.s. finite, by Lemma 1 under condition (C2). To show that M is a random tessellation, we now verify that it satisfies conditions (i) and (ii) of Definition 1. Let $T < 0$. Since $\bigcup_{t_i > T} V_i \subseteq \bigcup_{t_i > T}(x_i + X_i)$ and since a point in $x_i + X_i$ either belongs to V_i or to $x_j + \operatorname{Int} X_j$ for some $t_j > t_i$, we have $\bigcup_{t_i > T}(x_i + X_i) = \bigcup_{t_i > T} V_i$. Therefore by Lemma 2 we have $\bigcup_i V_i = \mathbb{R}^d$ $\mathbb{P}$-a.s. We observe from (C3) that $\operatorname{Int} V_i = (x_i + \operatorname{Int} X_i) \cap \bigcap_{t_j > t_i}(x_j + X_j)^{\mathrm{c}}$. It follows that $\operatorname{Int} V_i = \varnothing$ if and only if $V_i \subset \bigcup_{t_j > t_i}(x_j + X_j) = \bigcup_{t_j > t_i} V_j$. Indeed, the 'if' part is obvious, while the 'only if' part is obtained by observing that $(x_i + \operatorname{Int} X_i) \subseteq \bigcap_{t_j > t_i}(x_j + X_j)$ implies the same inclusion for $\overline{x_i + \operatorname{Int} X_i} = x_i + X_i \supseteq V_i$.

Finally, consider a realization of Φ such that $M' \in \mathcal{N}_{\mathcal{F}'}$ and $\bigcup_i V_i = \mathbb{R}^d$, which is $\mathbb{P}$-a.s. true, as we have shown above. Pick any point $x \in \mathbb{R}^d$. Since $M' \in \mathcal{N}_{\mathcal{F}'}$, there exists a positive, finite number of indices i such that $x \in V_i$, and hence one i such that $x \in V_i$ and $x \notin V_j$ for all $t_j > t_i$. By the above characterization, this implies that $\operatorname{Int} V_i \neq \varnothing$. Hence,

$$\bigcup_i \{V_i \colon \operatorname{Int} V_i \neq \varnothing\} = \mathbb{R}^d;$$

that is, M satisfies condition (i) of Definition 1. Condition (ii) of Definition 1 is easily obtained from (4) and (C3) by considering the cases $t_j > t_i$ and $t_i > t_j$ successively.

Next we demonstrate stationarity and the mixing property. Define

$$\Pi\colon \sum_i \delta_{x_i, t_i, X_i} \mapsto \sum_i \mathbf{1}(\operatorname{Int} V_i \neq \varnothing)\delta_{x_i, V_i}.$$

Recall that $\mathbb{P}$ denotes the distribution of the initial (homogeneous) Poisson point process Φ, meaning that $\mathbb{P}_\Pi := \mathbb{P} \circ \Pi^{-1}$ is the distribution of N. Furthermore, observe that translations of the x_i correspond to translations of the V_i through Π. It follows that the stationarity and the mixing property of N (with respect to shifts $N \mapsto \sum_i \delta_{x_i - x, V_i - x}$, $x \in \mathbb{R}^d$) are inherited from Φ.

It remains to prove that the intensity, μ, of N is finite. For all $T < 0$, let $N_T := \sum \delta_{x_i, V_i} \mathbf{1}(t_i > T, \operatorname{Int} V_i \neq \varnothing)$. Let μ_T denote the intensity of N_T; since

$$\mu_T \leq \mathbb{E}\biggl[\sum_i \mathbf{1}(x_i \in [0, 1]^n,\ t_i > T)\biggr] \leq -T,$$

μ_T is finite. By monotone convergence, since μ_T is nondecreasing as T decreases to $-\infty$, we have $\mu = \lim_{T\to-\infty}\mu_T$. Below we provide a uniform upper bound for μ_T, which will thus apply to μ and conclude the proof. Using Proposition 1 with B given by (C2), we obtain

$$\begin{aligned}\mu_T &= \mathbb{E}\bigg[\sum_i \frac{\mathbf{1}(0\in V_i\oplus B)}{\nu(V_i\oplus B)}\mathbf{1}(t_i>T,\ \mathrm{Int}\,V_i\neq\varnothing)\bigg]\\ &\le \nu(B)^{-1}\,\mathbb{E}\bigg[\sum_i \mathbf{1}\bigg(0\in x_i+X_i\oplus B,\ 0\notin\bigcup_{t_i>t}(x_i+\mathrm{Int}\,X_i\ominus B)\bigg)\bigg],\end{aligned}$$

where the inequality follows from both $\nu(V_i\oplus B)\ge\nu(B)$ and

$$V_i\oplus B\subset(x_i+X_i\oplus B)\setminus\bigcup_{t_i>t}(x_i+\mathrm{Int}\,X_i\ominus B),$$

the latter of which in turn follows from (4) and standard properties of morphological operations. Now, Campbell's formula and Slivnyak's theorem (see, e.g. [26, pp. 124, 125]) yield

$$\mu_T\le\frac{1}{\nu(B)}\int_{[T,0]\times\mathbb{R}^d\times\mathcal{F}}\mathbf{1}(0\in x+X\oplus B)\,\mathbb{P}\bigg(0\notin\bigcup_{t_i>t}(x_i+\mathrm{Int}\,X_i\ominus B)\bigg)\,\mathrm{d}t\,\nu(\mathrm{d}x)\,\mathrm{P}(\mathrm{d}X).$$

Noticing that $\bigcup_{t_i>t}(x_i+\mathrm{Int}\,X_i\ominus B)$ is a Boolean model with intensity t, we thus obtain

$$\mu_T\le\frac{1}{\nu(B)}\,\mathrm{E}[\nu(X\oplus B)]\int_T^0\exp(t\,\mathrm{E}[\nu(X\ominus B)])\,\mathrm{d}t\le\frac{1}{\nu(B)}\frac{\mathrm{E}[\nu(X\oplus B)]}{\mathrm{E}[\nu(X\ominus B)]},$$

which is finite under (C1) and (C2).

In the definition of M, we assume that $\sum_i\delta_{x_i,t_i}$ has intensity 1. However, rescaling the x_i is equivalent, up to a global rescaling of the model, to a rescaling of X, and any order-preserving modification of the t_i is unimportant, as seen from the definition.

Definition 3. The point process M, defined in Proposition 3, is called the dead leaves tessellation associated with the random closed set X.

Remark 2. The dead leaves model clearly shows the necessity of defining a tessellation through its cells, and not only its boundary. Indeed, visible parts, as defined by (4), are not necessarily connected; see Figure 1.

3.2. Perfect simulation

The term 'dead leaves model' originates from a more natural definition that consists in putting each new leaf *above* the previous ones and then considering the stationary distribution of this Markov process. Let K be a compact set of $\mathbb{R}^2$. A classical 'coupling from the past' argument enables perfect simulation of the stationary distribution restricted to K, by putting each new leaf *below* those already fallen, until K is completely covered (see the illustrative web applet by W. S. Kendall at http://www.warwick.ac.uk/statsdpt/staff/WSK/dead.html). This elegant argument was first introduced for the dead leaves model in [15]. In Figures 1 and 2 we show simulations of the model computed this way. To visualize the model, each grain is allocated a random gray level.

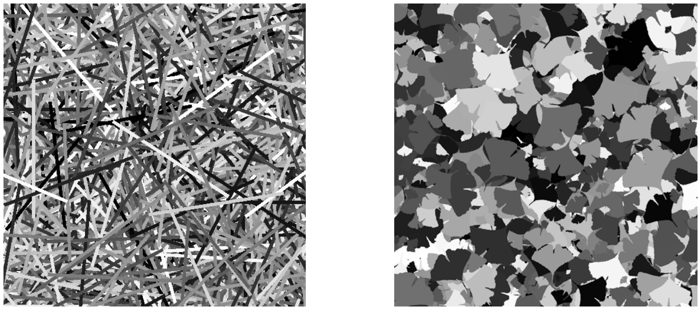

FIGURE 1: Simulations of dead leaves models. Left: the grain X is a rectangle with a direction uniformly distributed in $[0, \pi]$. Right: the grain is a more complicated shape, and its size distribution is uniform.

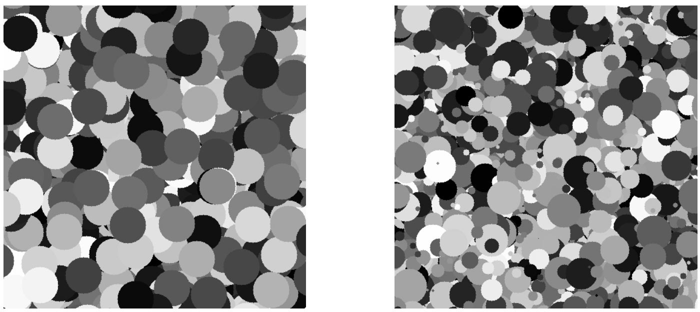

FIGURE 2: Left: simulation of a dead leaves model in which the grain X is a disk with constant radius. Right: simulation of a dead leaves model in which the grain X is a disk with a uniformly distributed radius.

3.3. Regularity properties of visible parts

Some almost-sure regularity results about visible parts are a consequence of the following remark. From Lemma 1, a visible part V_i is $\mathbb{P}$-a.s. equal to a leaf $x_i + X_i$ from which a *finite* number of other leaves have been removed. Now note that if A is a closed set and B is an open set, then $\partial(A \setminus B) = (\partial A \setminus B) \cup (\partial B \cap A)$. It follows that ∂V_i is a finite union of sets, each of which is included in $x_j + \partial X_j$ for some $t_j \geq t_i$; thus, some regularity properties of ∂X are inherited by the ∂V_i. Note, however, that the possible convexity of the grain X is not inherited by the V_i; see Figure 2.

Proposition 4. *We have $\nu(\partial M) = 0$ $\mathbb{P}$-a.s. if and only if $\nu(\partial X) = 0$ P-a.s.*

Proof. The discussion above implies that $\nu(\partial V_i) \leq \sum_{t_j \geq t_i} \nu(\partial X_i)$ $\mathbb{P}$-a.s. Since $\partial M = \bigcup_i \partial V_i$, if $\nu(\partial X) = 0$ P-a.s. then $\nu(\partial M) = 0$ $\mathbb{P}$-a.s. Now, if $\nu(\partial M) = 0$ $\mathbb{P}$-a.s. then $\nu(\partial V_i) = 0$ for all i; in particular for all cells such that $V_i = x_i + X_i$ (the so-called *relief cells*, to be studied in Section 4.2). We will see in Remark 4 that this in turn implies that $\nu(\partial X) = 0$ P-a.s.

If Int $V_i \neq \varnothing$ then $\nu(V_i) > 0$. Besides, $V_i \subset x_i + X_i$ is $\mathbb{P}$-a.s. bounded, by (C1). If, in addition, $\nu(\partial X) = 0$ P-a.s., then we are in the tessellations framework of Section 2.2. When $\nu(\partial X) = 0$, we say that X is ν-regular, a property that neither implies nor is implied by (C3). It is easy to find a set X that is ν-regular but not closed regular, for instance a set containing isolated points. To construct a closed regular set that is not ν-regular, we can proceed as follows (for $d \geq 2$). Let $\tilde{\nu}$ be the $(d-1)$-dimensional Lebesgue measure on the hyperplane $\{\mathbf{x} = (x_1, \ldots, x_d),\ x_1 = \frac{1}{2}\}$. Then there exists a homeomorphism, $h\colon [0,1]^d \to [0,1]^d$, such that $\nu + \tilde{\nu} = \nu \circ h$; see [9]. It follows that $X := h([0, \frac{1}{2}]^d)$ is not ν-regular, although it is closed regular.

4. Some characteristics of the dead leaves tessellation

4.1. Inclusion probabilities and boundary distribution

The main practical result from the original paper [18] introducing the dead leaves model is concerned with a functional that is defined on compact sets of the plane and equals the probability that a given compact set is included in a visible part of the model. It was shown that, for a non-empty set $K \in \mathcal{K}$,

$$\mathbb{P}(K \subset \operatorname{Int} V_i \text{ for some } i \in \mathbb{N}) = \frac{\mathrm{E}[\nu(\operatorname{Int} X \ominus \check{K})]}{\mathrm{E}[\nu(X \oplus \check{K})]}. \tag{5}$$

The consideration of simple examples for K, such as bipoints or segments, leads to valuable geometric information on the model.

In what follows, we generalize this result by studying the probability that n compact sets are included in n distinct visible parts. We define

$$Q^{(n)}(K_1, \ldots, K_n) = \mathbb{P}(K_1 \subset \operatorname{Int} V_{i_1}, \ldots, K_n \subset \operatorname{Int} V_{i_n} \text{ for some } t_{i_1} < \cdots < t_{i_n} < 0).$$

Proposition 5. *Let*

$$F^{(n)}(K_1, \ldots, K_n) = \mathrm{E}[\nu(\operatorname{Int} X \ominus \check{K}_1)] \prod_{j=2}^{n} \mathrm{E}[\nu((\operatorname{Int} X \ominus \check{K}_j) \cap (X \oplus \check{\underline{K}}_{j-1})^{\mathrm{c}})], \tag{6}$$

and

$$G^{(n)}(K_1, \ldots, K_n) = \prod_{j=1}^{n} \mathrm{E}[\nu(X \oplus \check{\underline{K}}_j)],$$

where, for all $j = 1, \ldots, n$,

$$\underline{K}_j = \bigcup_{k=1}^{j} K_k. \tag{7}$$

Then

$$Q^{(n)}(K_1, \ldots, K_n) = \frac{F^{(n)}(K_1, \ldots, K_n)}{G^{(n)}(K_1, \ldots, K_n)}. \tag{8}$$

Remark 3. Note that (C2) implies $\mathrm{E}[\nu(X)] > 0$ and, thus, that $G^{(n)}(K_1, \ldots, K_n)$ does not vanish for non-empty compact sets.

Proof of Proposition 5. Here we choose n fixed, non-empty compact sets, $K_1, \ldots, K_n$, and write $Q^{(n)}$ for $Q^{(n)}(K_1, \ldots, K_n)$. Summing over disjoint events yields

$$Q^{(n)} = \mathbb{E}\Big[\sum \mathbf{1}(t_{i_1} < \cdots < t_{i_n} < 0) \prod_{j=1}^{n} \mathbf{1}(K_j \subset \operatorname{Int} V_{i_j})\Big], \tag{9}$$

where the sum is taken over all n-tuples of points in Φ. First note that only n-tuples of distinct points may be considered in this sum and that, from the definition of visible parts in (4) and (C3), the summand in this equation may be written as

$$\mathbf{1}(t_{i_1} < \cdots < t_{i_n} < 0) \prod_{j=1}^{n} \mathbf{1}(K_j \subset (x_{i_j} + \operatorname{Int} X_{i_j})) \prod_{\{i\colon t_i > t_{i_j}\}} \mathbf{1}(K_j \cap (x_i + X_i) = \varnothing). \quad (10)$$

In the simplest case, $n = 1$, this amounts to saying that $Q^{(1)}$ is the probability that there exists a leaf, X_i, such that K_1 is included in $\operatorname{Int} X_i$ and is not hit by subsequent leaves. We will now apply the Campbell formula to compute this expectation, and therefore need the following notation. Let $\mathcal{E} := \mathbb{R}^2 \times (-\infty, 0] \times \mathcal{F}$. We write $\mathcal{N}^{(n)}$ (or $\mathcal{N}$ for $n = 1$) for the space of σ-finite counting measures on $\mathcal{E}^n$. For all $n \geq 1$, we define the point process

$$\Phi^{(n)} = \sum_{i_1,\dots,i_n} \delta_{z_{i_1},\dots,z_{i_n}},$$

on $\mathcal{E}^n$, where the sum is taken over all indices $(i_1, \dots, i_n)$ such that $z_{i_1}, \dots, z_{i_n}$ are distinct points of Φ. We define a function f from $\mathcal{E}^n \times \mathcal{N}^{(n)}$ to $\mathbb{R}$ such that (10) reads $f(\{z_{i_j}\}_{j=1}^n, \Phi^{(n)})$. By applying the refined Campbell theorem (see [7]) to compute the expectation in (9), we obtain

$$Q^{(n)} = \int_{Z \in \mathcal{E}^n} \int_{\phi \in \mathcal{N}^{(n)}} f(Z, \phi)\, \mathbb{P}^Z(\mathrm{d}\phi) \prod_{j=1}^{n} \mu_\Phi(\mathrm{d}\tilde{z}_j),$$

where $Z = \{\tilde{z}_j\}_{j=1}^n$, μ_Φ is the intensity measure of Φ, and $\mathbb{P}^Z$ is the Palm distribution of the process $\Phi^{(n)}$ at Z. Applying the generalized Slivnyak theorem (see [26, p. 124]) gives

$$Q^{(n)} = \int_{Z \in \mathcal{E}^n} \mathbb{E}[f(Z, (\Phi + \delta_{\tilde{z}_1} + \cdots + \delta_{\tilde{z}_n})^{(n)})] \prod_{j=1}^{n} \mu_\Phi(\mathrm{d}\tilde{z}_j), \quad (11)$$

where, as usual, $\mathbb{E}$ is the expectation associated to Φ. With $\tilde{z}_j = (\tilde{x}_j, \tilde{t}_j, \tilde{X}_j)$, $j = 1, \dots, n$, and $\tilde{t}_1 < \cdots < \tilde{t}_n < 0$, by definition of f we have

$$\begin{aligned} f(Z, (\Phi &+ \delta_{\tilde{z}_1} + \cdots + \delta_{\tilde{z}_n})^{(n)}) \\ &= f(Z, \Phi^{(n)}) \\ &= \Bigg(\prod_{j=1}^{n} \mathbf{1}(K_j \subset (\tilde{x}_j + \operatorname{Int} \tilde{X}_j))\Bigg)\Bigg(\prod_{j=2}^{n} \mathbf{1}(\underline{K}_{j-1} \cap (\tilde{x}_j + \tilde{X}_j) = \varnothing)\Bigg) \\ &\quad \times \prod_{j=1}^{n-1} \prod_{\{i\colon t_i \in (\tilde{t}_j, \tilde{t}_{j+1}]\}} \mathbf{1}(\underline{K}_j \cap (x_i + X_i) = \varnothing) \prod_{\{k\colon t_k \in (\tilde{t}_n, 0]\}} \mathbf{1}(\underline{K}_n \cap (x_k + X_k) = \varnothing), \end{aligned} \quad (12)$$

with $\underline{K}_j$ as defined in (7). The expectation in (11) is computed as follows. Since Φ is a Poisson process, the last line of (12) can be written as a product of independent terms whose expectations can be computed using the fact that, for fixed s and t, $s < t \leq 0$, and for K compact,

$$\mathbb{P}(K \cap (x_i + X_i) = \varnothing \text{ for all } t_i \in (s, t]) = \exp((s - t)\, \mathrm{E}[\nu(X \oplus \check{K})])$$

(see Lemma 1). Next, by integrating with respect to $\mathbf{1}(\tilde{t}_1 < \cdots < \tilde{t}_n < 0)\,\mathrm{d}\tilde{t}_1 \cdots \mathrm{d}\tilde{t}_n$ and making a change of variable $u_j = \tilde{t}_j - \tilde{t}_{j+1},\ j = 1, \dots, n-1$, we obtain

$$Q^{(n)} = \prod_{j=1}^{n} \mathrm{E}[\nu(X \oplus \check{\underline{K}}_j)]^{-1} \int_{(\mathbb{R}^2 \times \mathcal{F})^n} \prod_{j=1}^{n} \mathrm{d}\tilde{x}_j\, \mathrm{P}(\mathrm{d}\tilde{X}_j) \biggl(\prod_{j=1}^{n} \mathbf{1}(K_j \subset \tilde{(}x_j + \mathrm{Int}\, \tilde{X}_j)) \biggr) \times \biggl(\prod_{j=2}^{n} \mathbf{1}(\underline{K}_{j-1} \cap (\tilde{x}_j + \tilde{X}_j) = \varnothing) \biggr).$$

The first factor on the right-hand side of this equation is $(G^{(n)})^{-1}$, and the integral can be written

$$\prod_{j=1}^{n} \biggl(\int_{\mathbb{R}^2 \times \mathcal{F}} \mathbf{1}(K_j \subset (\tilde{x} + \mathrm{Int}\, \tilde{X}))\, \mathbf{1}(\underline{K}_{j-1} \cap (\tilde{x} + \tilde{X}) = \varnothing)\, \mathrm{d}\tilde{x}\, \mathrm{P}(\mathrm{d}\tilde{X}) \biggr),$$

with the convention $\underline{K}_0 = \varnothing$. Now, for two compact sets, A and B, we have

$$\int_{\mathbb{R}^2 \times \mathcal{F}} \mathbf{1}(A \subset (x + \mathrm{Int}\, X))\, \mathbf{1}(B \cap (x + X) = \varnothing) \nu(\mathrm{d}x)\, \mathrm{P}(\mathrm{d}X) = \mathrm{E}[\nu((\mathrm{Int}\, X \ominus \check{A}) \cap (X \oplus \check{B})^{\mathrm{c}})],$$

which, together with the last equation, yields $F^{(n)}$ (by 6) and, thus, (8).

For $n = 1$ we recover (5), the original result of Matheron. The case $n = 2$ was treated in [13]. Note that, from the $Q^{(n)}$, we can compute the probability

$$\mathbb{P}(K_1 \subset \mathrm{Int}\, V_{i_1}, \dots, K_n \subset \mathrm{Int}\, V_{i_n} \text{ for some } i_1, \dots, i_n \in \mathbb{N})$$

and, thus, the probability of n connected compact sets, $K_1, \dots, K_n$, avoiding the boundary of the dead leaves tessellation. For $n = 2$, for instance, this is

$$\mathbb{P}((K_1 \cup K_2) \cap \partial M = \varnothing) = Q^{(2)}(K_1, K_2) + Q^{(2)}(K_2, K_1) + Q^{(1)}(K_1 \cup K_2).$$

Moreover, it is easily seen that if we consider the random field obtained by independently coloring each visible part, then Proposition 5 enables us to compute the finite-dimensional distributions of this field. This is a useful result in the context of image modeling; see [11]. Next, we show that the knowledge of $Q^{(n)}$ for all n characterizes the distribution of ∂M in $(\mathcal{F}, \mathcal{B}_{\mathcal{F}})$.

Proposition 6. *The distribution of the boundary ∂M is uniquely determined by the functionals $Q^{(n)}$, $n \in \mathbb{N}$.*

Proof. The distribution of ∂M is characterized by its capacity functional, defined, for every compact set K, by $\mathbb{P}(F \cap K = \varnothing)$; see [19, p. 30]. Let $K \in \mathcal{K}$, let $(r_n) > 0$ be a sequence converging to 0, and, for each n, let $(x_i^{(n)})_{i=1,\dots,N_n}$ be a finite sequence in K such that $K \subset C_n = \bigcup_i B(x_i^n, r_n)$, where $B(x, r)$ is the (closed) ball of radius r centered at x. Note that, since each C_n is a finite union of connected compact sets, knowledge of the $Q^{(i)}$, $i \in \mathbb{N}$, uniquely determines $\mathbb{P}(C_n \cap \partial M = \varnothing)$. Now, since $C_n \downarrow K$, we have $\mathcal{F}^{C_n} \uparrow \mathcal{F}^K$ and, thus, $\mathbb{P}(C_n \cap \partial M = \varnothing) \uparrow \mathbb{P}(K \cap \partial M = \varnothing)$.

4.2. Typical relief cells

In this section, we consider the distribution of cells that remain completely visible. This problem was first addressed in [18]; also see [17], [24], and [12].

Definition 4. A cell V_i is a relief cell if $(x_i + X_i) = V_i$. Denote by

$$N_r = \sum_i \mathbf{1}(V_i = (x_i + X_i))\delta_{x_i, V_i}$$

the point process of relief cells.

As in the proof of Proposition 3, we can show that N_r is stationary and mixing. From condition (C3), if $V_i = (x_i + X_i)$ then $\operatorname{Int} V_i \neq \varnothing$. It follows that N_r is a thinning of N, and since N has finite intensity, so has N_r.

Proposition 7. *The typical relief cell distribution is absolutely continuous with respect to* P, *with Radon–Nikodým derivative* $F \mapsto (\mu_r \operatorname{E}[\nu(\operatorname{Int} X \oplus \check{F})])^{-1}$, *where*

$$\mu_r := \int_{\mathcal{F}} \frac{\operatorname{P}(\mathrm{d}F)}{\operatorname{E}[\nu(\operatorname{Int} X \oplus \check{F})]}$$

is the intensity of N_r.

Remark 4. As a consequence of this proposition, the typical relief cell distribution and the leaf distribution P are equivalent measures on $\mathcal{I}$. This remark completes the proof of the 'only if' part of Proposition 4.

Proof of Proposition 7. N_r is a simple point process with finite intensity. We denote by $\mathbb{P}^0_{N_r}$ the Palm distribution of N_r. With $N_r = \sum \delta_{x_i^r, V_i^r}$, for all $\chi \in \mathcal{I}$ we have

$$\begin{aligned}
\mathbb{P}^0_{N_r}(V_0^r \in \chi) &= \frac{1}{\mu_r}\mathbb{E}\bigg[\sum_i \mathbf{1}(V_i^r \in \chi)\,\mathbf{1}(x_i^r \in [0,1]^2)\bigg] \\
&= \frac{1}{\mu_r}\mathbb{E}\bigg[\sum_i \mathbf{1}\bigg(V_i \in \chi,\ x_i \in [0,1]^2,\ (x_i + X_i) \cap \bigcup_{\{j\colon t_j \in (t_i, 0]\}} (x_j + \operatorname{Int} X_j) = \varnothing\bigg)\bigg].
\end{aligned}$$

From Slivnyak's theorem and Campbell's formula, we obtain

$$\begin{aligned}
\mathbb{P}^0_{N_r}(V_0^r \in \chi) &= \frac{1}{\mu_r}\int_{\mathbb{R}^2 \times \mathbb{R}_- \times \chi} \mathbb{P}\bigg((x + F) \cap \bigcup_{\{j\colon t_j \in (t, 0]\}} (x_j + \operatorname{Int} X_j) = \varnothing\bigg)\nu(\mathrm{d}x)\,\mathrm{d}t\,\operatorname{P}(\mathrm{d}F) \\
&= \frac{1}{\mu_r}\int_{\mathbb{R}_- \times \chi} \exp(t\operatorname{E}[\nu(\operatorname{Int} X \oplus \check{F})])\,\mathrm{d}t\,\operatorname{P}(\mathrm{d}F) \\
&= \frac{1}{\mu_r}\int_{\chi} \operatorname{E}[\nu(\operatorname{Int} X \oplus \check{F})]^{-1}\operatorname{P}(\mathrm{d}F),
\end{aligned}$$

where the second equality follows from Lemma 1. Taking $\chi = \mathcal{F}'$, we also find the claimed formula for the intensity.

As an example, let us compute the area distribution of a typical relief cell: for $\chi_s = \{F \in \mathcal{F}'\colon \nu(F) > s\}$, we find that

$$\mathbb{E}^0_{N_r}[\nu(X_0^r)] = \frac{1}{\mu_r}\int_{\mathcal{F}'} \nu(F)\operatorname{E}[\nu(\operatorname{Int} X \oplus \check{F})]^{-1}\operatorname{P}(\mathrm{d}F).$$

Remark 5. For $d = 2$, if X is convex and isotropic a.s., we obtain the original result of Matheron by applying the Steiner formula (see [26, p. 13]) to compute μ_r. Let $l(K)$ denote the length of ∂K. Then, for K convex we have

$$\mu_r = \mathrm{E}[(\nu(X) + (2/\pi)l(X)\,\mathrm{E}[l(X)] + \mathrm{E}[\nu(X)])^{-1}].$$

4.3. Cells intersecting with a line

We now consider the intersection between the dead leaves model and a fixed line D. In this section we take $d \geq 2$ and assume, in addition to (C1)–(C3), that

(C4) $\nu(\partial X) = 0$ a.s. and, for any line D', $D' \cap \partial X$ is either empty, finite, or has positive $\nu_{D'}$-measure a.s.

Here $\nu_{D'}$ is the one-dimensional Lebesgue measure on D'. This assumption holds if X is a.s. a finite union of convex sets, for instance.

We will compute the Palm distribution of the point process $\partial M \cap D$ and, for X convex, prove a result of [18] in the Palm calculus framework.

Lemma 3. *$\partial M \cap D$ is a point process on D.*

Proof. Since ∂M is a.s. a locally finite union of sets ∂V_i, and since, for all i, ∂V_i is included in a finite union of sets $x_j + \partial X_j$, it is sufficient to show that, for any j, $(x_j + \partial X_j) \cap D$ is a.s. a finite or empty set. Let us suppose that this does not hold. Then, by (C4), with positive probability there exists a j such that $\nu_D(x_j + \partial X_j) > 0$. Thus, $\mathbb{E}[\nu_D(\bigcup_j (x_j + \partial X_j))] > 0$. Without loss of generality, we let D be the first coordinate axis. By Fubini's theorem and translation invariance, we obtain

$$\mathbb{E}\bigg[\nu\bigg\{\bigcup_j (x_j + \partial X_j)\bigg\}\bigg] = \int_{y \in \mathbb{R}^{d-1}} \mathbb{E}\bigg[\nu_{D_y}\bigg\{\bigcup_j (x_j + \partial X_j)\bigg\}\bigg]\,\mathrm{d}y > 0,$$

where, for any $y = (y_2, \dots, y_d)$, D_y is the line parallel to D passing through the point $(0, y_2, \dots, y_d)$. Thus, there a.s. exists a j such that $\mathbb{E}[\nu(\partial X_j)] > 0$, which contradicts (C4).

We let u be a unit vector collinear to D, denote by $[0, xu]$ the segment $\{\lambda xu,\ \lambda \in [0, 1]\}$, and define, for all $x \geq 0$,

$$L(x) = \mathbb{P}([0, xu] \subset \operatorname{Int} V_i \text{ for some } i \in \mathbb{N}) = Q^{(1)}([0, xu]) = \frac{\mathrm{E}[\nu(\operatorname{Int} X \ominus [0, -xu])]}{\mathrm{E}[\nu(X \oplus [0, -xu])]}, \tag{13}$$

where $Q^{(1)}$ is as defined above and the last equality follows from (5).

From now on we denote by $N_\ell = \sum_i \delta_{y_i}$ the simple point process defined in Lemma 3 with points in $\mathbb{R}$, and write $\mathbb{P}_{N_\ell}$ for its law and $\mathbb{P}^0_{N_\ell}$ for its associated Palm distribution. We index N_ℓ such that $\{y_i\}$ is increasing and $y_0 < 0 < y_1$. The following lemma links the Palm distribution of N_ℓ to L.

Lemma 4. *Let $N_\ell = \sum_i \delta_{y_i}$ be the simple stationary point process defined above. Then $L(x)$ is absolutely continuous, has a negative right derivative, $L'(0)$, at $x = 0$, and is such that, almost everywhere,*

$$\mathbb{P}^0_{N_\ell}(y_1 > x) = \frac{L'(x)}{L'(0)}. \tag{14}$$

Proof. Observe that $L(x) = \mathbb{P}_{N_\ell}(y_1 > x)$ for all nonnegative x. Let λ be the intensity of N_ℓ. For all $x \geq 0$, the inversion formula (see, e.g. [3, p. 20]) gives

$$L(x) = \mathbb{P}_{N_\ell}(y_1 > x) = \lambda \int_x^\infty \mathbb{P}^0_{N_\ell}(y_1 > t)\,\mathrm{d}t.$$

By differentiating, we find that $L'(x) = -\lambda\,\mathbb{P}^0_{N_\ell}(y_1 > x)$. Then, noting that $\mathbb{P}^0_{N_\ell}(y_1 = 0) = 0$, we obtain the differentiability of L at the origin and find that $L'(0) = -\lambda < 0$.

We end this section by considering the case of an a.s. convex X. First we introduce the *geometric covariogram*, γ_X, of X, defined for $x \geq 0$ by

$$\gamma_X(x) := \nu(X \cap (xu \oplus X)).$$

Note that the covariogram is usually defined on $\mathbb{R}^d$, but that here we consider it only on a half-line. Let $p_{u^\perp}$ denote the orthogonal projection on the hyperplane orthogonal to u, and $\nu_{u^\perp}$ the $(d-1)$-dimensional Lebesgue measure on this hyperplane. If X is convex then γ_X is a convex function on $[0, W_u)$, where W_u is the width of X in the u-direction, and is identically 0 outside this interval. Moreover, it is continuously differentiable on $[0, W_u)$, with derivative $\gamma'_X(x) = -\nu_{u^\perp}(p_{u^\perp}(X \cap (xu \oplus X))) \geq -\nu_{u^\perp}(p_{u^\perp}(X))$; see [19, p. 86]. From (C1) and (C2) we have $\mathrm{E}[\nu_{u^\perp}(p_{u^\perp}(X))] < \infty$. Hence, $\mathrm{E}[\gamma_X]$ is absolutely continuous with derivative $\mathrm{E}[\gamma'_X(x)]$ at almost every $x \geq 0$. Moreover $\gamma'_X(x)$ is right continuous at $x = 0$ and so is $\mathrm{E}[\gamma'_X(x)]$, by dominated convergence; thus, $\mathrm{E}[\gamma_X(x)]$ has right-hand derivative $\mathrm{E}[\gamma'_X(0)] = -\mathrm{E}[\nu_{u^\perp}(p_{u^\perp}(X))]$ at $x = 0$.

Definition 5. The *intercept distribution* of X (in the u-direction) is defined as

$$F_X(x) = \frac{\mathrm{E}[\gamma_X{}'(x)]}{\mathrm{E}[\gamma_X{}'(0)]}, \qquad x \geq 0. \tag{15}$$

Remark 6. The term 'intercept distribution' is used because $\gamma_X{}'(x)/\gamma_X{}'(0)$ is the probability distribution of the length of the intersection of X with lines, uniformly distributed among those hitting X, having direction u; see [24].

Proposition 8. *Let M be a dead leaves model associated with a random closed set, X, which is convex with intercept distribution F_X a.s., and let $\mathbb{P}^0_{N_\ell}$ and y_1 be defined as above. Then, for all $x \geq 0$,*

$$\int_x^\infty \mathbb{P}^0_{N_\ell}(y_1 > t)\,\mathrm{d}t = \frac{1}{2}(1 + Kx)^{-1} \int_x^\infty F_X(t)\,\mathrm{d}t, \tag{16}$$

where $K = -\mathrm{E}[\gamma'_X(0)]/\mathrm{E}[\gamma_X(0)]$.

Proof. It can be shown that, when X is convex, $\nu(X \ominus [0, -xu]) = \gamma_X(x)$ and

$$\nu(X \oplus [0, -xu]) = \gamma_X(0) + x\nu_{u^\perp}(p_{u^\perp}(X)).$$

Since $\nu_{u^\perp}(p_{u^\perp}(X)) = -\mathrm{E}[\gamma'_X(0)]$, (13) yields

$$L(x) = \frac{\mathrm{E}[\gamma_X(x)]}{\mathrm{E}[\gamma_X(0)] - x\,\mathrm{E}[\gamma'_X(0)]},$$

and the result then follows easily from (14) and (15).

Let us finally note that $\mathbb{P}^0_{N_\ell}(y_1 > x)$ may be seen (as in Section 2.2) as the length distribution of the 'typical cell' of the tessellation $D \cap M := \sum_i \mathbf{1}(V_i \cap D \neq \varnothing)\delta_{V_i \cap D}$, and thus as the intercept distribution of the typical cell of M (which is not convex). Note also that, by setting $x = 0$ in (16), we obtain

$$\mathbb{E}^0_{N_\ell}[y_1] = \frac{1}{2}\int_0^\infty F_X(t)\,\mathrm{d}t,$$

which show (see Remark 6) that, for X convex, the mean intercept in any direction is divided by two as a result of occlusion.

5. Conclusion

Various generalizations of this model are possible. Inhomogeneous point processes could be considered, or the independence assumption between time and objects could be broken (see [12]), enabling perspective laws to be taken into account. In the homogeneous and independent case, many open problems remain, in particular to do with computing typical cell properties, given the distribution of the leaf X. The computation of the mean perimeter and area of typical cells, as performed in [5] for the connected components of visible parts, is an interesting direction for further work.

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