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Held by Prof. D. Loiacono at Politecnico di Milano 2023/2024

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1. Disclaimers and preface

These notes were taken during AY 2023/2024 using older material, your mileage may vary. They're meant to accompany the lectures and in no way aim to substitute a professor yapping away at an iPad 30m away.

These notes are in part based on material by **Salvatore Buono**

For any questions/mistakes you can reach me [here](#).

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2. Overview of supervised learning

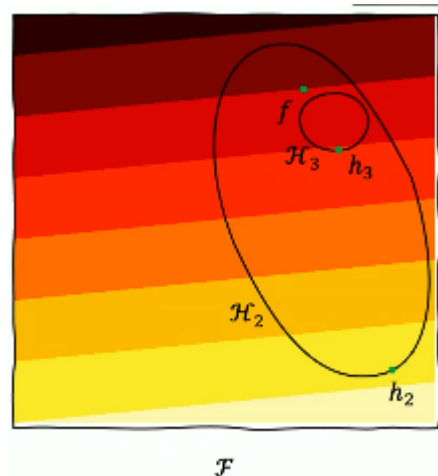
2.1. What kind of problem are we trying to solve ?

We're trying to find/approximate a function f , which is "generating" our dataset D .

The steps are:

- Define a loss function L
- Choose the hypothesis space \mathcal{H}
- Find in \mathcal{H} an approximation h of f that minimizes L .

We might be tempted to enlarge the hypothesis space, but if we do that we can have a larger risk of approximating f very poorly as we don't know if we'd be expanding \mathcal{H} in the correct direction.



2.2. The elements of a supervised learning algorithm

- The representation meaning the hypothesis space \mathcal{H} and how it's designed.
- The evaluation, so how the loss function is designed.
- The optimization algorithm, how you're looking for candidate solutions.

Some examples of representations are:

- Linear models.
- Neural networks.

Evaluation:

- Accuracy, the % of time you're correctly classifying a datapoint.
- If you have a regression problem you could use MSE.

Finally, the optimization technique really depends on your loss function:

- Gradient descent.
- Greedy search.
- Linear programming (in cases where you have some sort of constraint)

3. A supervised learning taxonomy

There are many ways you can try to classify/organize/compare, some more general paradigms other than the previous three.

3.1. Parametric vs Non-parametric

There are some learning algorithms for which the design of the hypothesis space is such that when you decide to apply a learning algorithm to a problem you have to define a set of parameters that are fixed and won't change with the data these are called **Parametric**, while non parametric models scale the number of parameters with the training set.

3.2. Frequentist vs Bayesian

No meaningful definition given by the prof here.

3.3. Direct, discriminative or generative

They're different ways to see the same problem from different perspectives, in the **direct** approach you don't really care about the probabilistic interpretation, you're just optimizing the loss function. In the **discriminative** you interpret your problem as a conditional density $p(t|x)$ you essentially try to learn the distribution and then compute the expected mean. In the **generative** tries to model the *joint* density $p(x, t)$, this allows to then infer the conditional density and generate novel samples by computing the conditional mean.

4. Linear regression

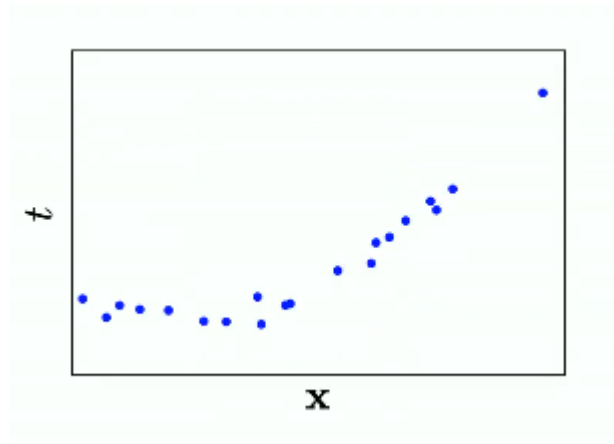
We'll focus on regression as our first type of problem, more specifically on linear models to solve.

References:

- *Pattern recognition and Machine learning*, Bishop

4.1. The model

We want to learn an approximation function $f(x)$:

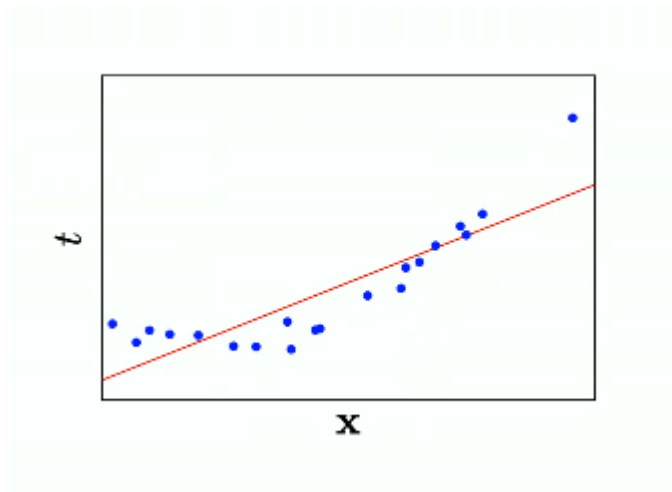


We start from a dataset

$$\mathcal{D} = \langle x, t \rangle \implies t = f(x)$$

How do we model f ? How do we evaluate our approximation ? How do we optimize our approximation ?

In linear regression we model $f(x)$ with linear functions.



A model like this clearly has room for improvement but we can veery easily explain what the model is doing. Another important property is that we can solve this analitically, this is better since models like NN are hard to "debug", in this case nothing can go wrong, we have algorithims with guaranteed convergence etc...

Another thing is that these models can capture non linear interactions.

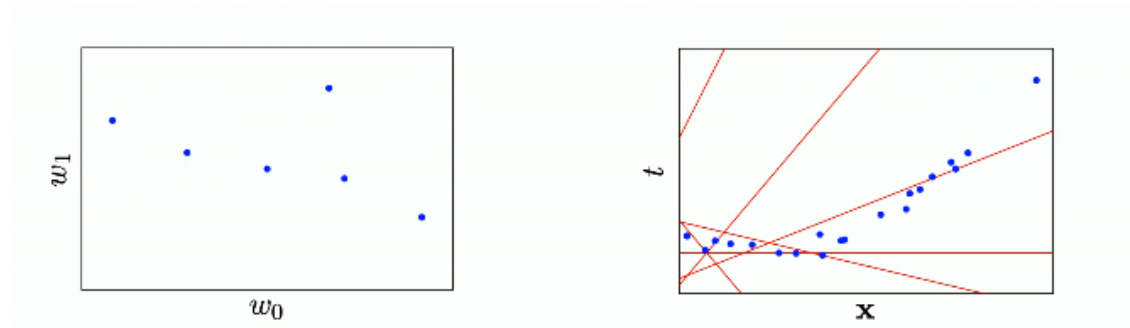
The simplest linear model can be defined as:

$$y(x, w) = w_0 + \sum_{j=1}^{D-1} w_j x_j = w^T x$$

Where:

- $x = (1, x_1, \dots, x_{D-1})$ is our input variable expressed as a vector to which we prepend 1.
- w_0 is called the bias parameter

My hypothesis space is 2D space with the two weights as dimensions.



Each point in the left image is a "solution" to our problem (a red line). We now need to *evaluate* these solutions, we need to define an error.

4.2. Loss function and optimization

A convenient error loss function is the sum of squared errors (SSE):

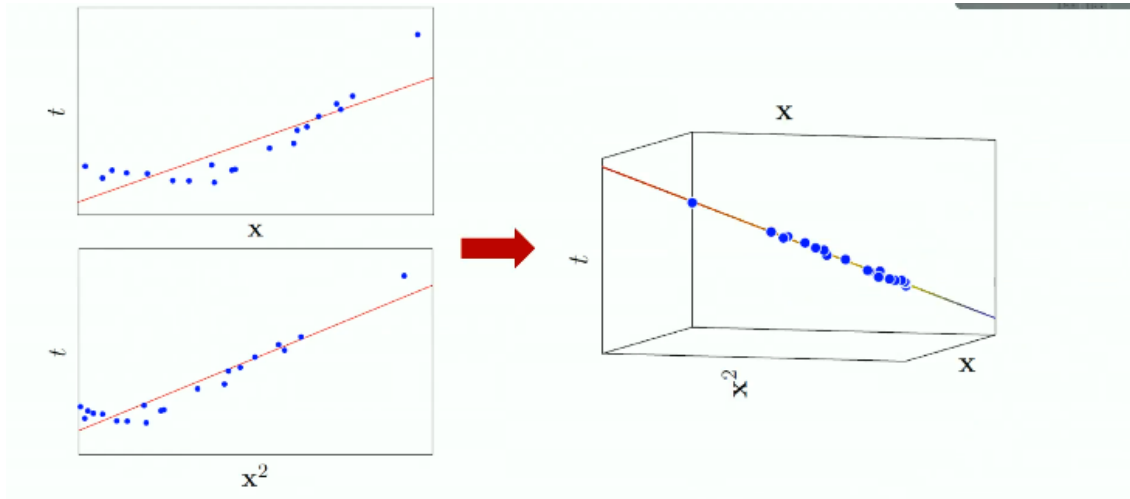
$$L(w) = \frac{1}{2} \sum_{n=1}^N (y(x_n, w) - t_n)^2$$

The reason we take the square instead of the absolute value is that ???

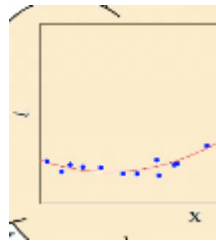
5. Linear models and basis functions

How do we capture non-linear relationships ? We can use basis functions. Instead of using x_1, x_2, \dots, x_n we can use a set of non-linear functions ϕ_i (as many as we want), my linear model will be *linear* with respect to the basis functions (we're learning the weights).

In this example i generate 2 weights for each sample, one for x and for x^2 .



If we now plot this approximation into the original space:



If we have problem specific knowledge we can pinpoint some sort of specific basis functions but generally there's no rule.

There are *families* of basis functions such as :

- Polynomial :

$$\phi_j(x) = x^j$$

- Gaussian :

$$\phi_j(x) = \exp\left(-\frac{(x - \mu_j)^2}{2\sigma^2}\right)$$

- Sigmoidal:

$$\phi_j(x) = \frac{1}{1 + \exp\left(\frac{\mu_j - x}{\sigma}\right)}$$

This can be hyper-parameterized (idk how to spell it), more on this later in the course.

6. Least squares

From now on we'll assume that our problem is solved in the feature space and not in the input space (in the basis function space in the previous case).

Prof emphasizes how important linear algebra is for this course...

$$L(w) = \frac{1}{2}RSS(w) - \frac{1}{2}(t - \phi w)^T(t - \phi w)$$

Where t is a vector with all the targets and ϕ is a matrix where on each row you have all the samples in the feature space (so on the cols you'd have features).

If you have to optimize this function with respect to a w . Let's assume it's a scalar value, how do we find the optimal value of w ?

Get the derivative and put the derivative equal to 0, right ?

Ordinary Least Squares

- For linear models, a closed-form optimization of the RSS, known as **least squares**, starting from the matrix form of the loss function:

$$L(\mathbf{w}) = \frac{1}{2}RSS(\mathbf{w}) = \frac{1}{2}(\mathbf{t} - \Phi\mathbf{w})^T(\mathbf{t} - \Phi\mathbf{w})$$

► where $\Phi = (\phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_N))^T$ and $\mathbf{t} = (t_1, \dots, t_N)^T$

- We can compute first a second derivative of $L(w)$ to find the optimal w

$$\frac{\partial L(\mathbf{w})}{\partial \mathbf{w}} = -\Phi^T(\mathbf{t} - \Phi\mathbf{w}) \qquad \frac{\partial^2 L(\mathbf{w})}{\partial \mathbf{w} \partial \mathbf{w}^T} = \Phi^T \Phi$$

$$\Rightarrow \hat{\mathbf{w}}_{OLS} = \left(\Phi^T \Phi\right)^{-1} \Phi^T \mathbf{t}$$

This is an analytical solution, the caveat is i need the matrix to not be singular cause then i'd have to invert it. It's also slow as fuck for large datasets.

You can instead apply **Gradient descent** (or SGD).

Sequential Learning

- ❑ Closed-form optimization (OLS) is not feasible with large dataset
- ❑ Instead, a **stochastic** (or **sequential**) gradient descent is possible
- ❑ **Least Mean Square** (LMS) algorithm:

$$L(\mathbf{x}) = \sum_n L(x_n)$$

$$\Rightarrow \mathbf{w}^{(n+1)} = \mathbf{w}^{(n)} - \alpha^{(n)} \nabla L(x_n)$$

$$\Rightarrow \mathbf{w}^{(n+1)} = \mathbf{w}^{(n)} - \alpha^{(n)} \left(\mathbf{w}^{(n)T} \phi(\mathbf{x}_n) - t_n \right) \phi(\mathbf{x}_n)$$

► α is called learning rate and to guarantee convergence:

$$\sum_{n=0}^{\infty} \alpha^{(n)} = +\infty \qquad \sum_{n=0}^{\infty} \alpha^{(n)^2} < +\infty$$

Prof then talks about geometric interpretation of OLS (Ordinary least squares).

7. Multiple outputs

What if my target is a multitude regression targets ?

In practice it's just like running multiple regression problems in parallel, the main optimization is using the same basis functions.

8. Regularization

What happens if i use too many features ? (p.22 to 24 of 3).

We basically add a regularizing term to the loss function that takes into account how many features we added. We can extend the loss function to take into account the complexity of our model

$$L(\mathbf{w}) = L_D(\mathbf{w}) + \lambda L_W(\mathbf{w})$$

where $L_W(\mathbf{w})$ accounts for model complexity and λ is the **regularization** coefficient and w is the vector of model coefficients (or weights).

We can design $L_W(\mathbf{w})$ in many ways, we'll look at two in particular

8.1. Ridge regression (L2 regularization)

We define $L_W(w)$ as

$$L_W(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{w} = \frac{1}{2} \|\mathbf{w}\|_2^2$$

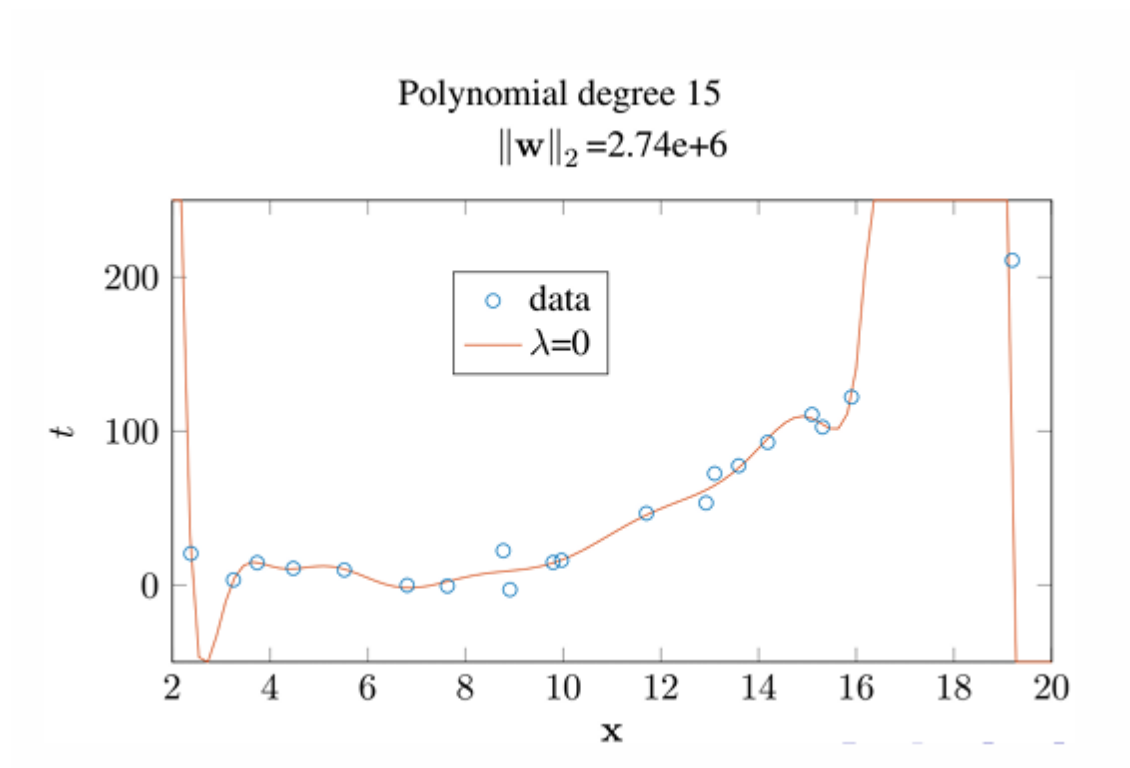
where the loss function $L(w)$ ends up being

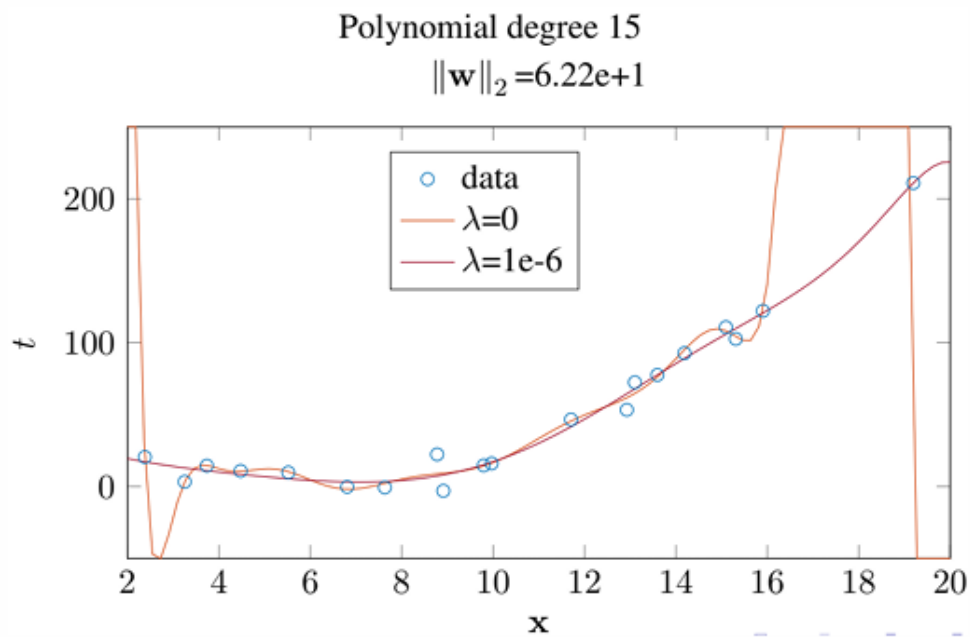
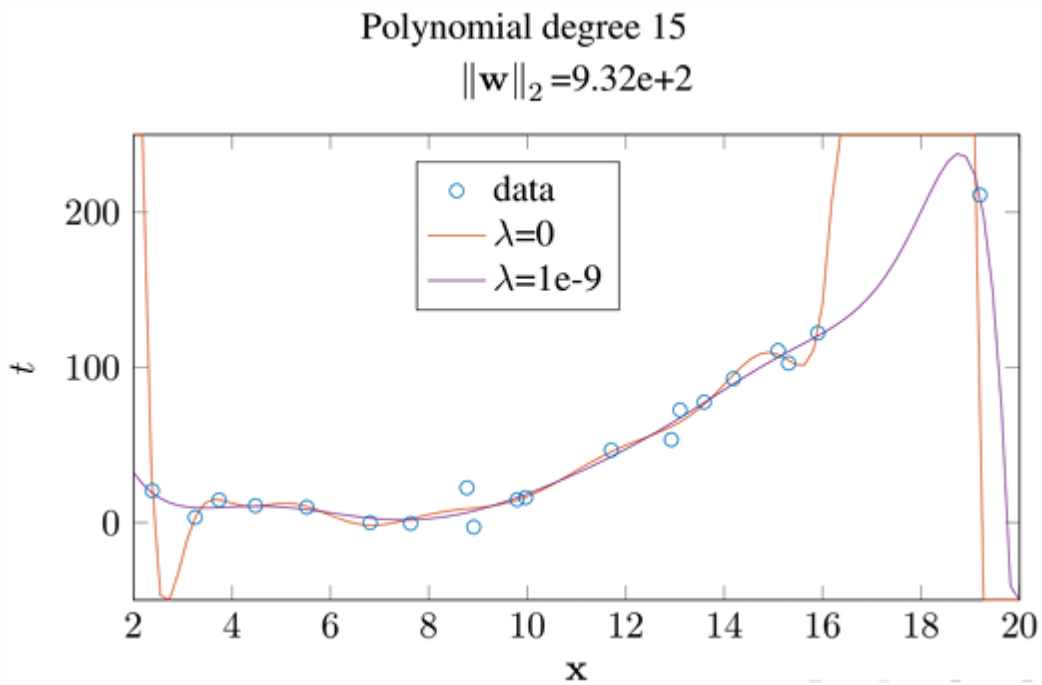
$$L(\mathbf{w}) = \frac{1}{2} \sum_{i=1}^N (t_i - \mathbf{w}^T \phi(\mathbf{x}_i))^2 + \frac{\lambda}{2} \|\mathbf{w}\|_2^2$$

the closed form (estimator) is

$$\hat{\mathbf{w}}_{ridge} = (\lambda \mathbf{I} + \Phi^T \Phi)^{-1} \Phi^T \mathbf{t}$$

We're essentially an L^2 penalty





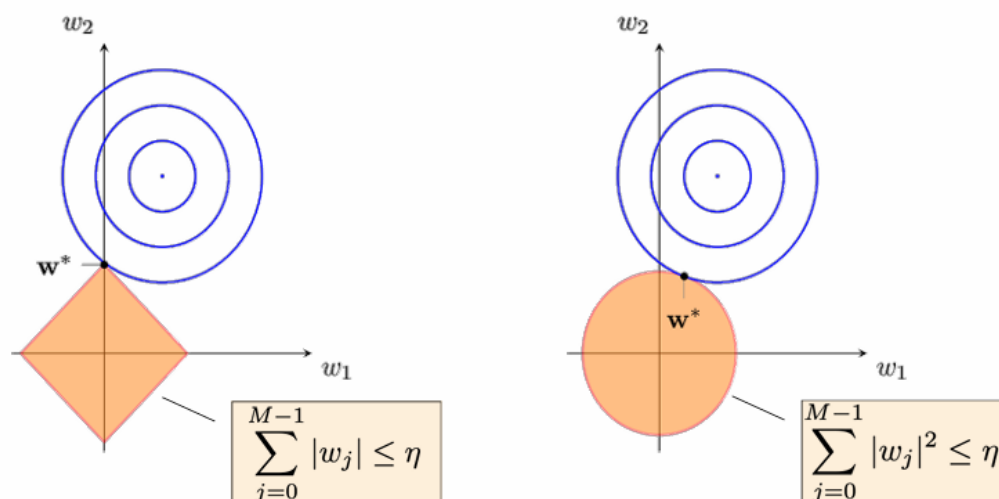
8.2. Lasso regression (L1 regularization)

Lasso stands for *least absolute shrinkage and selection operator*

$$L_W(\mathbf{w}) = \frac{1}{2} \|\mathbf{w}\|_1 = \frac{1}{2} \sum_{j=0}^{M-1} |w_j|$$

$$L(\mathbf{w}) = \frac{1}{2} \sum_{i=1}^N (t_i - \mathbf{w}^T \phi(\mathbf{x}_i))^2 + \frac{\lambda}{2} \|\mathbf{w}\|_1$$

When λ is large enough some weights might be **equal to zero**. That's because we're basically creating a constraint region in the shape of a diamond (or a cross-polytope) in the weight space. When the loss function's contours (ellipsoids) are minimized under this constraint, the solution often occurs at the corners of the diamond.



9. Least squares and Maximum likelihood

We can also approach regression in a probabilistic way by defining a probabilistic model that maps input x to outputs t using some unknown parameters w

$$y(x, w)$$

we then model the **likelihood** i.e. the probability that observed data \mathcal{D} is generated by a given set of parameters w

$$p(\mathcal{D}|w)$$

we then estimate those parameters by maximizing the likelihood that those w generated our observed sample

$$w_{ML} = \arg \max_w p(\mathcal{D}|w)$$

In the case of linear regression our model can be defined as:

$$t = y(x, w) + \epsilon = w^T \phi(x) + \epsilon$$

where $y(x, w)$ is taken to be a linear model disrupted by Gaussian noise $\epsilon \sim \mathcal{N}(0, \sigma^2)$

Given a dataset \mathcal{D} of N samples with inputs $X = \{x_1, \dots, x_N\}$ and outputs $T = \{t_1, \dots, t_N\}^T$

$$p(\mathcal{D}|w) = p(\mathbf{t}|\mathbf{X}, w, \sigma^2) = \prod_{n=1}^N \mathcal{N}(t_n|w^T \phi(\mathbf{x}_n), \sigma^2)$$

Since we're assuming the datapoints to be i.i.d. the likelihood function is the product of individual Gaussian distributions for each data points.

To find w_{ML} we usually prefer to deal with the **log**-likelihood since it is a strictly increasing map

$$\ell(w) = \ln p(\mathbf{t}|\mathbf{X}, w, \sigma^2) = \sum_{n=1}^N \ln p(t_n|\mathbf{x}_n, w, \sigma^2) = -\frac{N}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} RSS(w)$$

we then set the gradient of this object to zero and find the solution to that equation

$$\nabla \ell(w) = \sum_{n=1}^N t_n \phi(\mathbf{x}_n)^T - w^T \left(\sum_{n=1}^N \phi(\mathbf{x}_n) \phi(\mathbf{x}_n)^T \right) = 0$$

$$w_{ML} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{t}$$

10. Bayesian Linear Regression

We first define the model given our current knowledge, we capture our assumptions by wrapping them into our a priori distribution (in teal) over some unknown parameters before seeing the data, we then observe the data and compute the posterior probability distribution (in red) for the parameters given the observed data.

$$p(\textcolor{red}{parameters}|\textcolor{red}{data}) = \frac{p(\textcolor{teal}{data}|\textcolor{teal}{parameters})p(\textcolor{teal}{parameters})}{p(\textcolor{red}{data})}$$

we then use the posterior distribution to

- Make predictions
- Account for uncertainty over the parameters

Example

The way this works is basically the concept of Bayesian learning, it allows us to update our beliefs after observing data.

Say we have a prior belief that a certain coin's bias p follows a β distribution with parameters $\alpha = 2$ and $\beta = 2$, this means we think the coin is fairly balanced with a bias centered around $p = 0.5$.

Now let's say we flip the coin 10 times and observe that it lands heads 6 times and tails 4 times, using Bayes' theorem we compute the posterior distribution of the coin bias p given the observed data which will also be a β distribution (prove this hehe) with different parameters.

We've successfully employed Bayes' theorem to update our prior belief system, we can now use this posterior distribution to

- Predict the outcome of future coin flips by averaging over the posterior distribution of p
- Quantify our uncertainty about the coin bias p using summary statistics of the posterior distribution, such as the mean, median, or credible intervals
- Use the posterior distribution to calculate the expected loss under different decisions and choose the decision with the lowest expected loss.

The general formula is:

$$p(w|\mathcal{D}) = \frac{p(\mathcal{D}|w)p(w)}{p(\mathcal{D})}$$

where $p(w)$ is the prior probability over the parameter or what we know before observing the data, $p(\mathcal{D}|w)$ is the **likelihood** or the probability of observing the data (\mathcal{D}) given some value of the parameters (w).

We're essentially computing a probabilistic mean so we have to normalize the product on the numerator, that's where $p(\mathcal{D})$ comes

$$p(\mathcal{D}) = \int p(\mathcal{D}|w)p(w)dw$$

We usually model the prior distribution using a Gaussian likelihood

$$p(w) = \mathcal{N}(w|w_0, \mathbf{S}_0)$$

this means that the posterior distribution result from the Bayes theorem is still a Gaussian distribution (known property of Gaussian distributions)

$$p(\mathbf{w}|\mathbf{t}, \Phi, \sigma^2) \propto \mathcal{N}(\mathbf{w}|\mathbf{w}_0, \mathbf{S}_0)\mathcal{N}(\mathbf{t}|\Phi\mathbf{w}, \sigma^2\mathbf{I})$$

$$p(\mathbf{w}|\mathbf{t}, \Phi, \sigma^2) = \mathcal{N}(\mathbf{w}|\mathbf{w}_N, \mathbf{S}_N)$$

$$\mathbf{w}_N = \mathbf{S}_N \left(\mathbf{S}_0^{-1} \mathbf{w}_0 + \frac{\Phi^T \mathbf{t}}{\sigma^2} \right)$$

$$\mathbf{S}_N^{-1} = \mathbf{S}_0^{-1} + \frac{\Phi^T \Phi}{\sigma^2}$$

Note that:

- The posterior acts as prior for the next iteration in the case of sequential data.
- If the prior has infinite variance w_N converges to the **maximum likelihood** estimator
- If $\mathcal{S} \rightarrow \infty$ it converges to **ordinary least squares** instead.

MISSING SOME STUFF HERE p.6 of Salvatore Buono study the part where OLS and ML

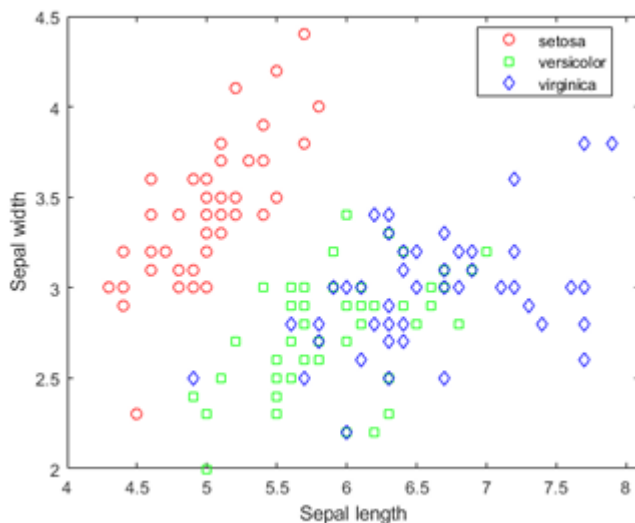
11. Linear classification

Given a dataset \mathcal{D} we want to learn an approximation function $f(x)$ that maps input x to a discrete class C_k where $K = 1, \dots, k$.

$$\mathcal{D} = \langle x, C_k \rangle \longrightarrow C_k = f(x)$$

We need to predict discrete class labels, more specifically the (linear) decision boundaries that divide the different class labels.

This type of classification is linear despite employing non-linear activation functions, the name comes from the linear decision boundaries



There are 3 main approaches:

- **Discriminant function:** we model a parametric *function* that directly maps the input to a class.
- **Probabilistic discriminative approach:** where we model $p(C_k|x)$ with respect to certain parameters and learn them from the training dataset (logistic regression).
- **Probabilistic generative approach (Bayesian):** where we model $p(C_k|x)$ and the prior $p(C_k)$ and infer the posterior distribution using Bayes' theorem

In linear classification we'll use **generalized linear models**:

$$f(\mathbf{x}, \mathbf{w}) = f\left(w_0 + \sum_{j=1}^{D-1} w_j x_j\right) = f(\mathbf{x}^T \mathbf{w} + w_0)$$

where $f(\cdot)$ is **not** linear in \mathbf{w} because its output is a discrete value or alternatively a probability value. The $f(\cdot)$ effectively partitions the input space into **decision boundaries** (**decision surfaces** in higher dimensions) which are linear functions of \mathbf{x} and \mathbf{w} as they satisfy

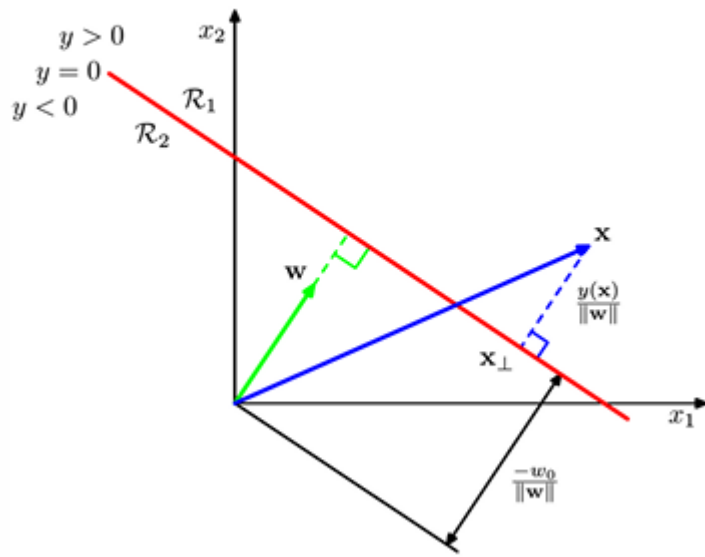
$$\mathbf{x}^T \mathbf{w} + w_0 = \text{const}$$

The most common choice for encoding for the general k -classes problem is **1-of- k** (or 1 hot encoding) encoding where $t \in B^k$ with B representing the set of 0 and 1.

With this encoding t and $f(\cdot)$ represent the density probability over the classes. In the case of two-class problems we might opt for the natural $t \in B^0$, $t \in \{-1, 1\}$ is sometimes used for some algorithms.

The discriminant linear function for a two-class problem would be

$$f(\mathbf{x}, \mathbf{w}) = \begin{cases} C_1, & \text{if } \mathbf{x}^T \mathbf{w} + w_0 \geq 0 \\ C_2, & \text{otherwise} \end{cases}$$



where the the decision surface has the following properties:

- It's equal to

$$y(\cdot) = \mathbf{x}^T \mathbf{w} + w_0 = 0$$

- It's orthogonal to \mathbf{w}
- The distance between the decision surface and the origin is

$$-\frac{w_0}{\|\mathbf{w}\|_2}$$

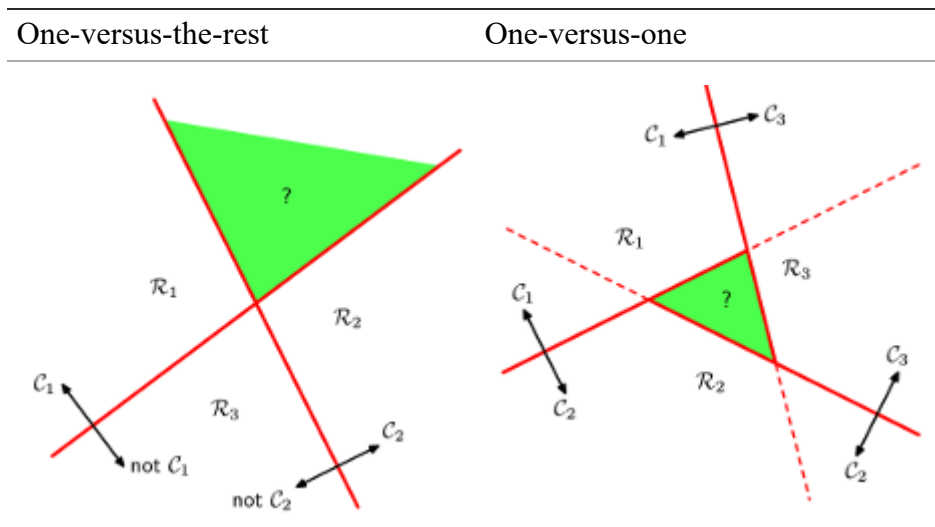
- And the distance between the \mathbf{ds} and x is

$$\frac{y(\mathbf{x})}{\|\mathbf{w}\|_2}$$

But what if we have multiple classes ? We generally denote the number of classes with K .

We have two main approaches

- **One-versus-the-rest:** where we use $K - 1$ binary classifiers where each classifier i discriminates between C_i and not C_i .
- **One-versus-one:** where we use $\frac{K(K-1)}{2}$ binary classifiers to discriminate between C_i and C_j



These approaches are not a good idea if one region is mapped to more than one class, in that case we might elect to use K linear discriminant functions:

$$y_k(\mathbf{x}) = \mathbf{x}^T \mathbf{w}_k + w_{k0}, \text{ where } k = 1, \dots, K$$

where we're basically mapping \mathbf{x} to C_k if $y_k > y_j \quad \forall j \neq k$.

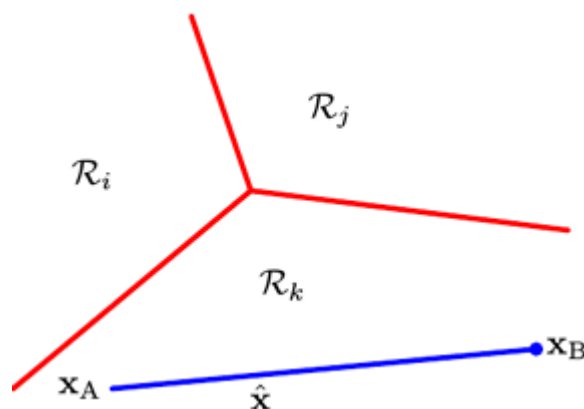
This approach presents no ambiguity for classes belonging to the same regions.

Theorem

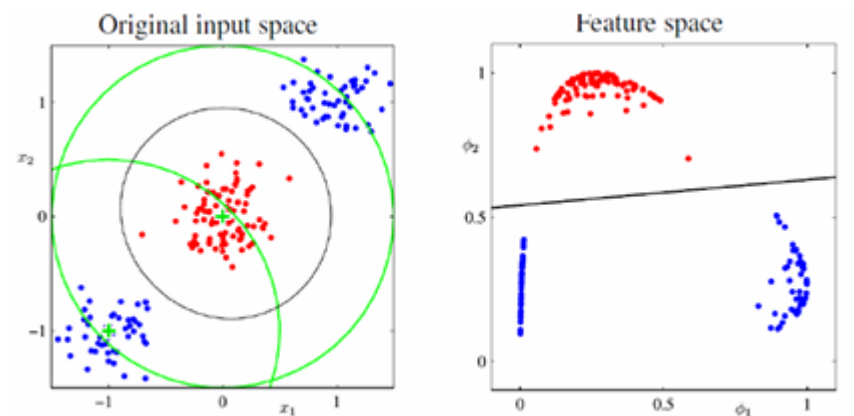
Let $\mathbf{x}_A, \mathbf{x}_B \in \mathcal{R}_k$

Thus $y_k(\mathbf{x}_A) > y_j(\mathbf{x}_A)$ and $y_k(\mathbf{x}_B) > y_j(\mathbf{x}_B)$, therefore $\forall \alpha$ such that $0 < \alpha < 1$:

$$y_k(\alpha \mathbf{x}_A + (1 - \alpha) \mathbf{x}_B) > y_j(\alpha \mathbf{x}_A + (1 - \alpha) \mathbf{x}_B)$$



Keep in mind that all of these models are applied to the problem input space, we can extend these models by using basis functions $\varphi(\mathbf{x})$, this has the effect of potentially admitting decision boundaries that are linear in the **feature space** and non linear in the input space.



11.1. Least Squares for Classification

We consider a K -class problem with a 1-of- K encoding where each class is modeled with a linear function

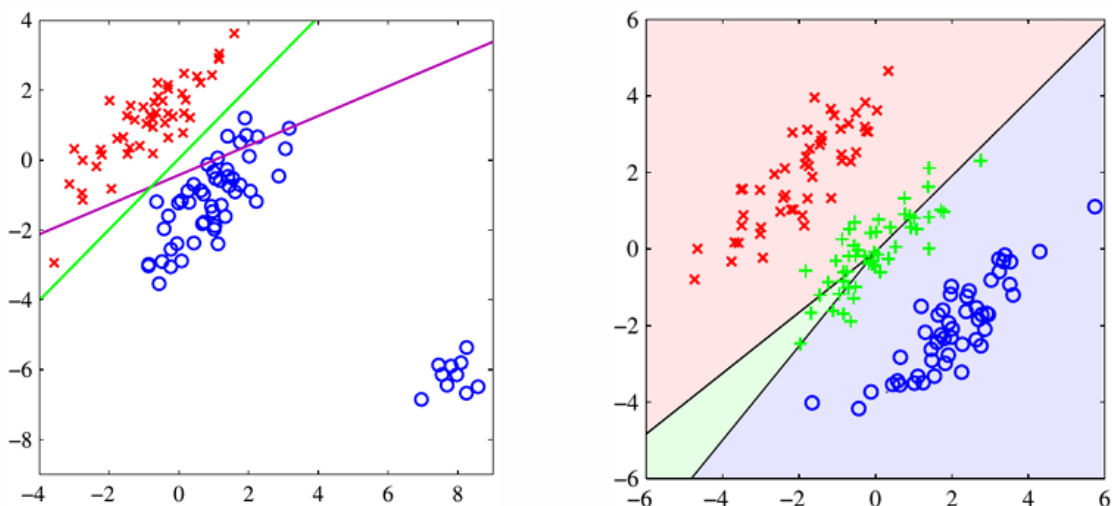
$$y_k(\mathbf{x}) = \mathbf{x}^T \mathbf{w}_k + w_{k0}, \text{ where } k = 1, \dots, K$$

or

$$\mathbf{y}(\mathbf{x}) = \tilde{\mathbf{W}}^T \tilde{\mathbf{x}}$$

where $\tilde{\mathbf{W}}$ has size $(D+1) \cdot K$ and the k -th column of $\tilde{\mathbf{W}}$ is $\tilde{\mathbf{w}}_k = (w_{k0}, \mathbf{w}_k^T)^T$.

The main problem with this approach is its sensibility to outliers, as the error for these elements will move the decision boundaries far more than other elements.



11.2. Perceptron

The perceptron is an **online** (computes and updates one sample at a time), **linear** discriminant model. The algorithm tries to find the decision boundary

by minimizing the distance of misclassified points to the decision boundary:

- Correct classifications get 0 error
- Misclassified points x_n get error $w^T \phi(x_n) t_n$

The algorithm effectively optimizes this function to be minimal:

$$L_P(\mathbf{w}) = - \sum_{n \in \mathcal{M}} \mathbf{w}^T \phi(\mathbf{x}_n) t_n$$

the optimization (minimization) can be performed by using stochastic gradient descent, note how correctly classified samples do not contribute to the loss. We can update one sample at a time using the following where the learning rate α is usually set to 1

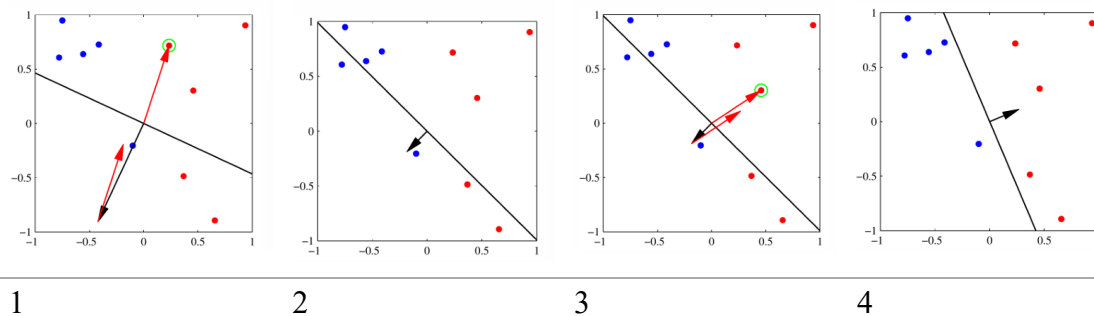
$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \alpha \nabla L_P(\mathbf{w}) = \mathbf{w}^{(k)} + \alpha \phi(\mathbf{x}_n) t_n$$

```

k = 0
repeat
  k = k+1
  n = k mod N
  if model_output != target then
    w(k+1) = w(k) + phi(x_n)*target
  end if
until convergence

```

where the dataset $\mathcal{D} = \{x_i, t_i\} \forall i = 1, \dots, N$.



it's important to note that while a single iteration will reduce the error for the single misclassified sample, this does **not** guarantee that the entire loss is reduced after each update.

Perceptron convergence theorem

If the training dataset is linearly separable in the feature space ϕ , then the perceptron learning algorithm is guaranteed to find an **exact solution** in a finite number of steps.

There's no guarantee on the number of steps or the uniqueness of the solution.

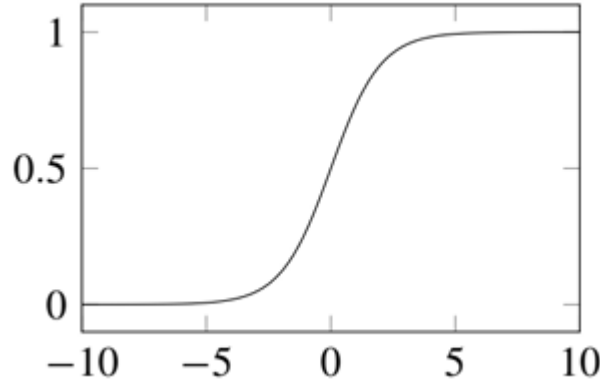
11.3. Logistic regression (two-class)

In this scheme we aim to model the conditional probability $p(C_k|\phi)$ directly

$$p(C_1 | \phi) = \frac{1}{1 + \exp(-\mathbf{w}^T \phi)} = \sigma(\mathbf{w}^T \phi)$$

$$p(C_2 | \phi) = 1 - p(C_1 | \phi)$$

where $\sigma(a)$ is known as the sigmoidal function.



Given a dataset $\mathcal{D} = \{\mathbf{x}_i, t_i\}$ where $i = 1, \dots, N$ and $t_i \in \{0, 1\}$ we model the likelihood of a single sample using a **Bernoulli** distribution, using the logistic regression model for conditioned class probability

$$p(t_n | \mathbf{x}_n, \mathbf{w}) = y_n^{t_n} (1 - y_n)^{1-t_n} \quad \text{where} \quad y_n = p(t_n = 1 | \mathbf{x}_n, \mathbf{w}) = \sigma(\mathbf{w}^T \phi_n)$$

we use maximum likelihood to maximize the probability of correctly classifying the samples:

$$p(\mathbf{t} | \mathbf{X}, \mathbf{w}) = \prod_{n=1}^N y_n^{t_n} (1 - y_n)^{1-t_n} \quad , \quad y_n = \sigma(\mathbf{w}^T \phi_n)$$

where

- y_n is the value predicted
- t_n is the real value (sometimes referred to as target or reals)

At this point we can opt to use the negative log-likelihood as a loss function to minimize (this is also known as the **cross-entropy error function**)

$$\begin{aligned} L(\mathbf{w}) &= -\ln\left(p(\mathbf{t}|\mathbf{X}, \mathbf{w})\right) \\ &= -\sum_{n=1}^N \left(t_n \ln(y_n) + (1 - t_n) \ln(1 - y_n) \right) \\ &= +\sum_{n=1}^N L_n \end{aligned}$$

The gradient for this loss function can then be derived as follows:

$$\begin{aligned} \frac{\partial L_n}{\partial y_n} &= -\left(-\frac{t_n}{y_n} - \frac{(1 - t_n)}{1 - y_n} \right) = \frac{y_n - t_n}{y_n(1 - y_n)} \\ \text{where } \frac{\partial y_n}{\partial \mathbf{w}} &= y_n(1 - y_n)\phi_n \\ \frac{\partial L_n}{\partial \mathbf{w}} &= \frac{\partial L_n}{\partial y_n} \frac{\partial y_n}{\partial \mathbf{w}} = (y_n - t_n)\phi_n \implies \nabla L(\mathbf{w}) = \sum_{n=1}^N (y_n - t_n)\phi_n \end{aligned}$$

Note that it has the same form as the sum of squared errors (SSE) for linear regression.

Unfortunately there's no closed form to be derived. You can optimize it using gradient-based techniques.

11.4. Logistic regression (multi-class)

In the multi-class we represent the posterior probabilities using a **softmax** transformation of linear functions of feature variables:

$$p(C_k|\phi) = y_k(\phi) = \frac{\exp(\mathbf{w}_k^T \phi)}{\sum_j \exp(\mathbf{w}_j^T \phi)}$$

Take a moment to understand what's happening here,

- \mathbf{w}_k is the weight vector for class C_k . It is a column vector of the same dimension as the feature vector ϕ
- ϕ is the feature vector of an input sample. It is also a column vector

The dot product $\mathbf{w}_k^T \phi$ represents a linear combination of the input features ϕ , where each feature ϕ_i is weighed by its corresponding weight w_k^i . We can think of this geometrically as a projection, we're measuring how close the feature space is to the weight space (what we're trying to learn and consequentially predict the feature space from).

This score is also known as the **logit** score for K .

At this point we're ready to compute the likelihood assuming a 1-of- K encoding

$$p(\mathbf{T}|\Phi, \mathbf{w}_1, \dots, \mathbf{w}_K) = \prod_{n=1}^N \underbrace{\left(\prod_{k=1}^K p(C_k|\phi_n)^{t_{nk}} \right)}_{\text{One term corresponding to correct class}} = \prod_{n=1}^N \left(\prod_{k=1}^K y_{nk}^{t_{nk}} \right)$$

which we can minimize using the cross-entropy error function for which we can then compute the gradient for each weight

$$L(\mathbf{w}_1, \dots, \mathbf{w}_K) = -\ln p(\mathbf{T}|\Phi, \mathbf{w}_1, \dots, \mathbf{w}_K) = -\sum_{n=1}^N \left(\sum_{k=1}^K t_{nk} \ln g_{nk} \right)$$

$$\nabla L_{\mathbf{w}_j}(\mathbf{w}_1, \dots, \mathbf{w}_K) = \sum_{n=1}^N (y_{nj} - t_{nj}) \phi_n$$

There's no closed form solution due to the non-linearity, the error function is convex and can be optimized by standard gradient-based optimization techniques.

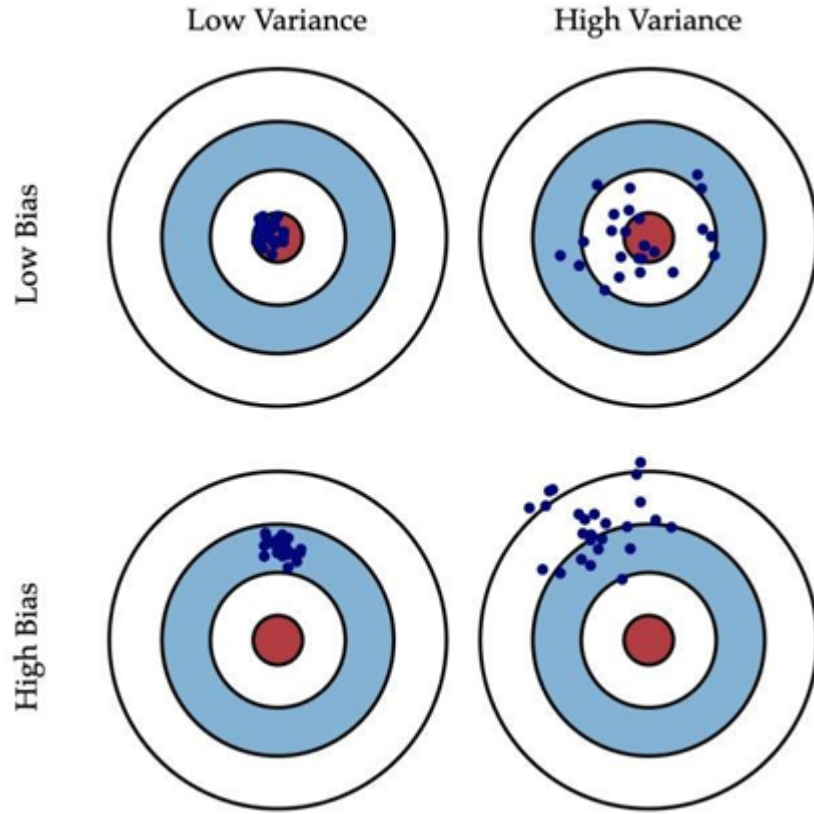
One might think that a step function would better represent a sample belonging to a single class. The step function, however, is non-differentiable making it unsuitable for gradient-based optimization, instead it would require an alternative optimization technique such as a combinatorial or evolutionary algorithm.

12. Model evaluation and selection

12.1. Bias-variance decomposition

Let $t_i = f(x_i) + \epsilon_i$ be our data points assuming $\mathbb{E}[\epsilon] = 0$ and $\text{Var}[\epsilon] = \sigma^2$, we'll denote our model with $\hat{t}_i = y(x_i)$ learned from *dataset* $\mathcal{D} = \{x_i, t_i\}$. Finally we'll take $\mathbb{E}[(t - y(\mathbf{x}))^2]$ to be our performance measure. We can decompose the **expected square error** as

$$\begin{aligned} \mathbb{E}[(t - y(\mathbf{x}))^2] &= \mathbb{E}[t^2 + y(\mathbf{x})^2 - 2ty(\mathbf{x})] \\ &= \mathbb{E}[t^2] + \mathbb{E}[y(\mathbf{x})^2] - \mathbb{E}[2ty(\mathbf{x})] \\ &= \mathbb{E}[t^2] \pm \mathbb{E}[t]^2 + \mathbb{E}[y(\mathbf{x})^2] \pm \mathbb{E}[y(\mathbf{x})]^2 - 2f(\mathbf{x})\mathbb{E}[y(\mathbf{x})] \\ &= \text{Var}[t] + \mathbb{E}[t]^2 + \text{Var}[y(\mathbf{x})] + \mathbb{E}[y(\mathbf{x})]^2 - 2f(\mathbf{x})\mathbb{E}[y(\mathbf{x})] \\ &= \text{Var}[t] + \text{Var}[y(\mathbf{x})] + (f(\mathbf{x}) - \mathbb{E}[y(\mathbf{x})])^2 \\ &= \underbrace{\text{Var}[t]}_{\sigma^2} + \underbrace{\text{Var}[y(\mathbf{x})]}_{\text{Variance}} + \underbrace{\mathbb{E}[f(\mathbf{x}) - y(\mathbf{x})]^2}_{\text{Bias}^2} \end{aligned}$$



In essence the model **variance** measures the inherent difference between models learned from different datasets. For the more experienced ML student it can be thought of as the over/under of how underfit or overfit your model is. It naturally decreases with simpler models which will require fewer samples to saturate, consequentially it will also decrease given more samples.

$$\text{variance} = \int \mathbb{E} [(y(\mathbf{x}) - \bar{y}(\mathbf{x}))^2] p(\mathbf{x}) d\mathbf{x}$$

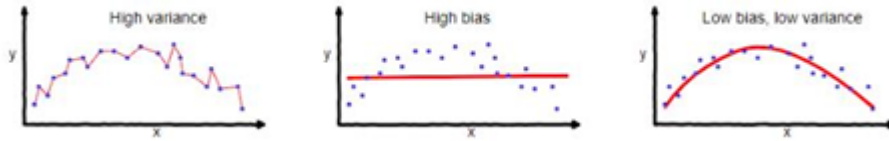
where

$$\bar{y}(\mathbf{x}) = \mathbb{E}[y(\mathbf{x})]$$

the model **bias** on the other hand measures the difference between the ground truth and what you can expect your model to learn, you can sort of assimilate this definition to that of *estimator bias*, where, given the estimator analytical form, we can compute the expected value and compare it to the analytical form of what we're trying to estimate. Think of the mean estimator for the beta distribution and how you have to correct it to get the actual mean.

Bias happens when one of our learning hypotheses is wrong, it decreases with more complex models, you can think of it as a sort of measure of how correct your assumptions are (trying to fit a linear regressor on data from a quadratic function)

$$\text{bias}^2 = \int (f(\mathbf{x}) - \bar{y}(\mathbf{x}))^2 p(\mathbf{x}) d\mathbf{x}$$



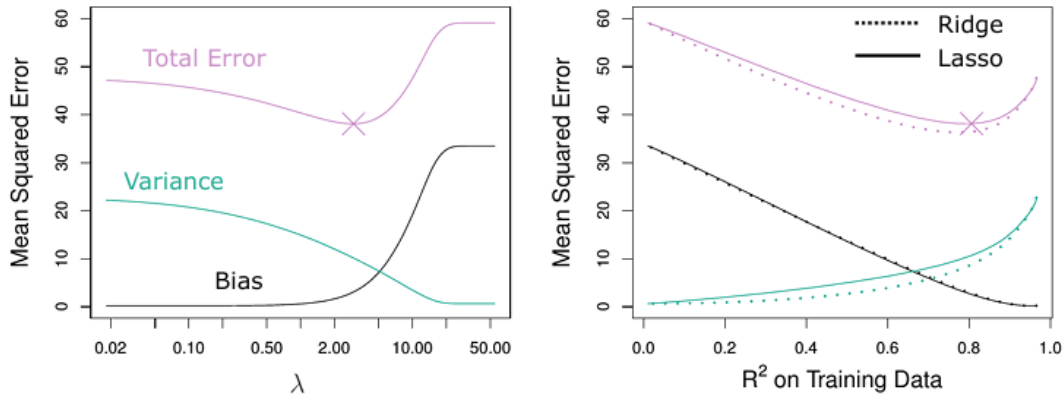
Finally **data noise** (σ^2) is the variance of data itself and is *irreducible*.

The professor proposes a case-study on K -NN, if we analyse the **MSE** we get

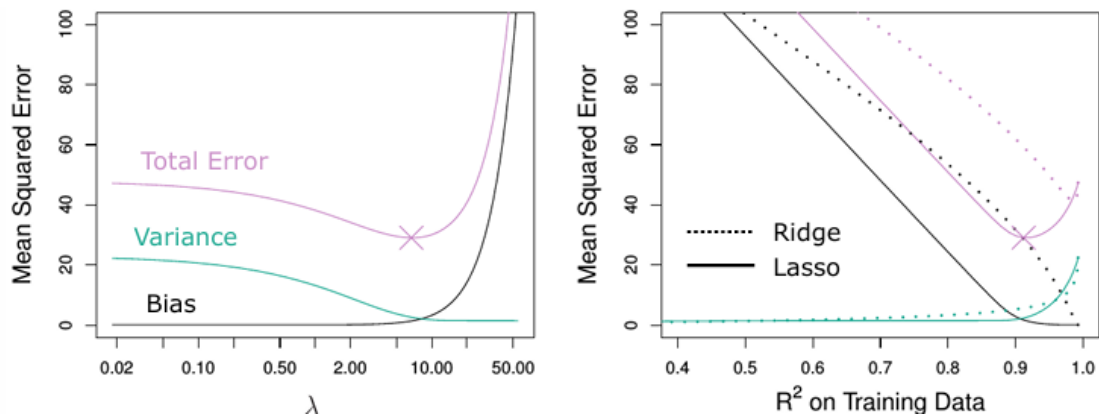
$$\mathbb{E}[(t^* - y(\mathbf{x}^*))^2] = \sigma^2 + \frac{\sigma^2}{K} + \left(f(\mathbf{x}^*) - \frac{1}{K} \sum_{i=1}^K f(\mathbf{x}_i) \right)^2$$

where the data noise σ^2 is the *irreducible* error we talked about earlier, the model **variance** is $\frac{\sigma^2}{K}$ decreases as K increases and finally the **bias** increases as K increases. This last part, while not as intuitive, is a direct effect of the inherent smoothness of high K value models. This smoothing effect means that the model gives more weight to a larger number of neighbours, effectively averaging their labels. While this reduces the model's variance, it also makes the model less sensitive to the nuances in the data. The larger the K , the more the model averages out the local variations, leading to a higher bias because it oversimplifies the true patterns in the data. Another reason for it is what we previously discussed as a possible reason for high bias in our models i.e. a wrong hypothesis, K -NN relies on the assumption that nearby points in the feature space are likely to belong to the same class. When K is small, the model focuses on very local structures, capturing the fine details of the data. As K increases, the model starts considering neighbours that are further away, which may belong to different classes. This causes the model to generalize more, thus missing local patterns and increasing bias.

Keep in mind that not all bias is bad, this is where regularization kicks in. We add bias as a way to avoid overfitting our model on unseen data.



45 of 45 features correlated to output



2 of 45 features correlated to output

Generally speaking Lasso regularization outperforms Ridge when few features are related to the output.

12.2. Model selection and assessment in practice

Given $\mathcal{D} = \{\mathbf{x}_i, t_i\}$ with $i = 1, \dots, N$ we can select the best model based on the loss function L computed on \mathcal{D} , this is called the **training error** and is not a good measure of the **prediction error** which is the real world performance of our model on unseen data

- Regression

$$L_{true} = \int \int (t - y(\mathbf{x}))^2 p(\mathbf{x}, t) d\mathbf{x} dt$$

- Classification

$$L_{true} = \int \int I(t \neq y(\mathbf{x})) p(\mathbf{x}, t) d\mathbf{x} dt$$

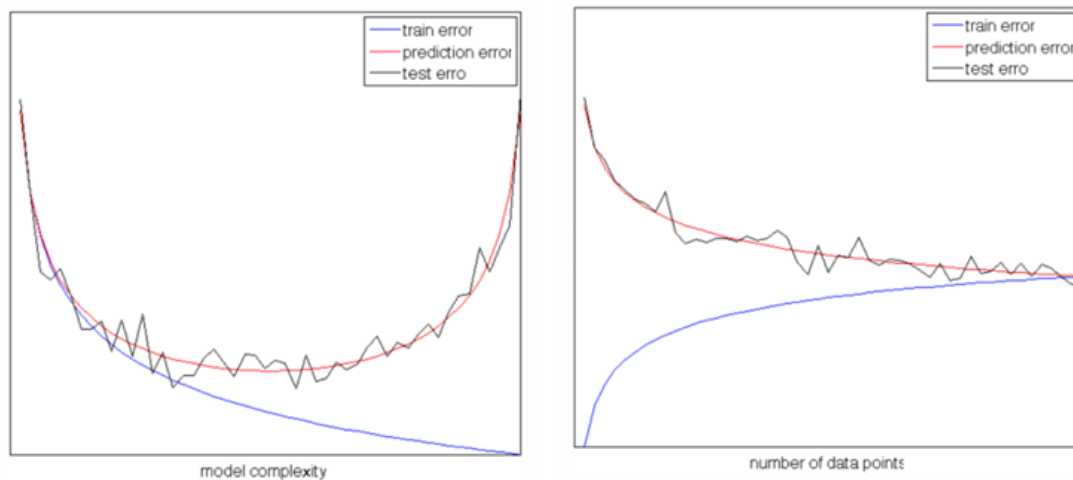
Unfortunately we usually don't have a good model for $p(\mathbf{x}, t)$ therefore we introduce the concept of a subset of \mathcal{D} reserved for testing our model on unseen data, a **test set** on which we compute the **test error**

- Regression

$$L_{test} = \frac{1}{N_{test}} \sum_{n=1}^{N_{test}} (t_n - y(\mathbf{x}_n))^2$$

- Classification

$$L_{test} = \frac{1}{N_{test}} \sum_{n=1}^{N_{test}} (I(t_n \neq y(\mathbf{x}_n)))$$



The test set error as an estimator for prediction error

Generally speaking, train-test error analysis can help us identify

- **High-bias:** where the training and test error are close but higher than expected.
- **High-variance**

Validation set

In order to avoid over-fitting the test set we introduce the validation set, the steps are as follows:

- Use **training data** to learn model parameters
- For each model learned use **validation data** to compute the validation error

- We select the model with the lowest **validation error** and finally use **test data** to estimate the prediction error

This, of course can lead to unreliable model selection whenever the validation data is not plentiful enough, on the other hand we might end up over-fitting the validation set and not choose the best model.

In order to avoid validation over-fitting we can opt to use one of the following schemes to "shuffle" the validation/training partitions

Leave-One-Out cross validation (LOO)

For each sample $\{\mathbf{x}_i, t_i\} \in \mathcal{D}$ we train the model once on $\mathcal{D} \setminus \{\mathbf{x}_i, t_i\} \in \mathcal{D}$ and then compute the error of the resulting model on $\{\mathbf{x}_i, t_i\} \in \mathcal{D}$

The prediction error can then be estimated as the average of all the error computed using a single sample

$$L_{LOO} = \frac{1}{N} \sum_{i=1}^N (t_i - y_{\mathcal{D}_i}(\mathbf{x}_i))^2$$

where $t_{\mathcal{D}_i}$ is the model trained on $\mathcal{D} \setminus \{\mathbf{x}_i, t_i\} \in \mathcal{D}$

This scheme provides an **almost unbiased** estimate of the prediction error, it is however extremely expensive especially on larger datasets

K-Fold cross validation

We randomly split the training data \mathcal{D} into k folds: $\mathcal{D}_1, \dots, \mathcal{D}_k$.

For each fold \mathcal{D}_i we train model on $\mathcal{D} - \{\mathcal{D}_i\}$ and then compute the error on \mathcal{D}_i

$$L_{\mathcal{D}_i} = \frac{k}{N} \sum_{(\mathbf{x}_n, t_n) \in \mathcal{D}_i} (t_n - y_{\mathcal{D} \setminus \mathcal{D}_i}(\mathbf{x}_n))^2$$

finally we estimate the prediction error as the average error computed

$$L_{k-fold} = \frac{1}{k} \sum_{i=1}^k L_{\mathcal{D}_i}$$

L_{k-fold} provides a **slightly biased** estimate of the prediction error but it is much cheaper with low k values (around 10)

Other metrics can also be used to evaluate models based on their complexity:

- Mallows's

$$C_p : C_p = \frac{1}{N} (RSS + 2M\hat{\sigma}^2)$$

- Akaike Information Criteria:

$$AIC = -2\ln L + 2M$$

- Bayesian Information Criteria:

$$BIC = -2\ln L + M\ln(N)$$

- Adjusted R2:

$$AdjustedR^2 = 1 - \frac{RSS/(N - M - 1)}{TSS/(N - 1)}$$

where M is the number of parameters, N is the number of samples, $\hat{\sigma}^2$ is the estimate of noise variance, RSS is the residual sum of squares and finally TSS is the total sum of squares.

AIC and BIC are generally used when maximizing the log-likelihood while BIC will generally penalize more than AIC model complexity.

12.2.1. Dealing with model complexity

A common pitfall is to think that using more features is always better, in fact one might argue that trying to fit

$$y = w_0 + w_1x + w_2x^2$$

is better than

$$y = w_0 + w_1x$$

since we can always set $w_2 = 0$ but this actually increases the probability of over fitting the data and implies larger model variance since we might not have enough data to infer that $w_2 \approx 0$. With this in mind we aim to reduce the model variance in one (or more) of three common ways:

- **Feature selection:** we should design the feature space by selecting the most effective subset of all the possible features
- **Dimensionality reduction:** The input space can be mapped to lower dimensional space
- **Regularization:** Shrink parameter values towards 0

Feature selection

One approach for **feature selection** might be to brute force all possible feature combinations but this is obviously overly expensive in terms of compute. We generally can elect to use one of 3 common practices:

- **Filtering** where features are ranked on some evaluation metrics (e.g. correlation, variance etc..) and select the top k , this can be very fast but fails to capture any subset of mutually dependent features (it might make our model worse !)
- **Embedded:** we select (read turn off) features as we train using Lasso or Decision trees, this naturally captures feature relationships and is guaranteed not to make our model worse but can be very computationally expensive
- **Wrapper:** a search algorithm is used to find a subset of features that are evaluated by training a model with them and assessing its performance, can be implemented in one of two ways
 - Forward selection: starting from an empty model and adding features one at a time
 - Backwards elimination: where we start with all the features and remove them one at a time.

Dimensionality reduction

The main difference between feature selection and dimensionality reduction is that the latter uses **all** the features and maps them (through combinations) into a lower-dimensionality space, there are a lot of ways to implement this but we'll focus on two main ones.

Principal component analysis aims to find an orthonormal base of input which accounts for most of the variance

1. We standardize the data in order to allow each feature to contribute equally
2. Compute the covariance matrix on the standardized data in order to capture how much each pair of features vary together

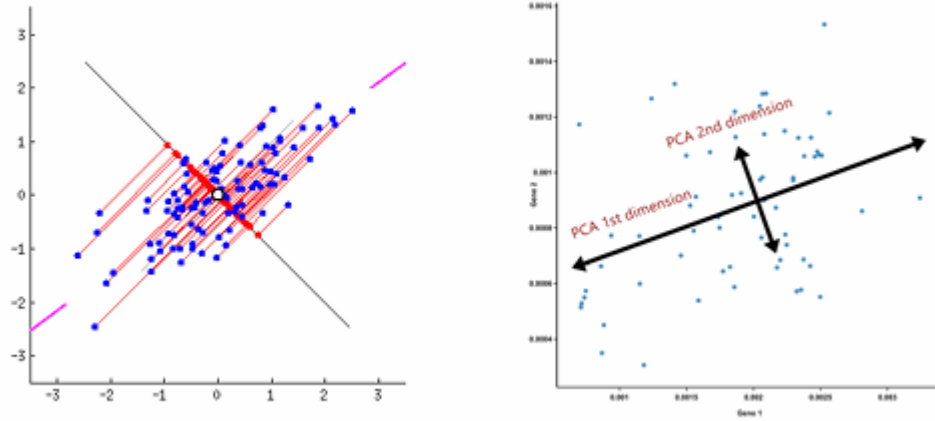
$$S = \frac{1}{N-1} \sum_{n=1}^N (x_n - \bar{x})(x_n - \bar{x})^T$$

3. Compute the eigenvalues and eigenvectors for S , then the eigenvector with the largest eigenvalue will be the first principal component (**PC**) and so on.

The k -th largest eigen value will have a proportion of variance captured:

$$\frac{\lambda_k}{\sum_i \lambda_i}$$

At this point we've built a new **covariance-ordered** orthonormal basis for the feature space where ALL the features are taken into account, allowing us to select the first k PCs to build a reduced dimensional representation of the data. One can think of this approach as a way to build a basis of linearly independent vectors that represent a certain space.



In Fig.1 we find the highest variance and set it as PC, in Fig.2 we then find the second dimension