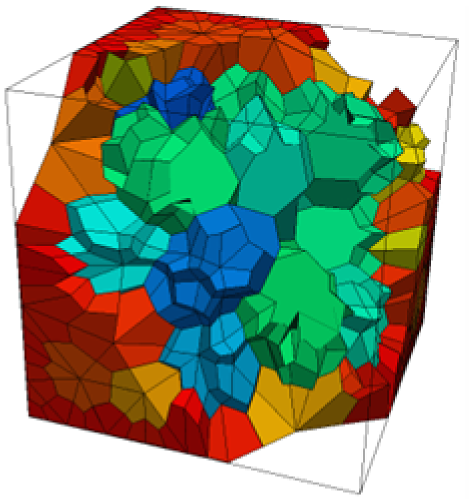
Exam preparation

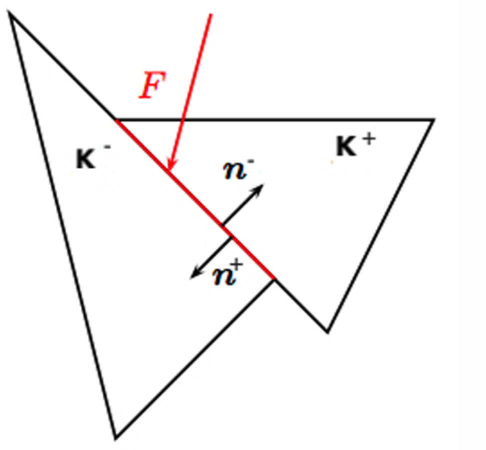
1. Discontinuous Galerkin F.E.M.

We focus on the approximation of Poisson problem, stability and convergence properties with respect to the **DG** norm (no proofs).
We start from the model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

we consider the approximation as done on triangles
K like





Triangulation of the domain

The weak formulation is obtained by multiplying both sides of the partial differential equation by a smooth test function v on each triangle K

$$\int_K -\Delta u v = \int_K f v$$

we integrate by parts and sum over all K

$$\sum_{K\in T_h}\int_K \nabla u\cdot\nabla v - \boxed{\sum_{K\in T_h}\int_{\partial K}\nabla u\cdot \mathbf{n}_K v} = \int_\Omega fv$$

to deal with the directional derivative some notation is introduced, after some steps we end up with the following **DG** formulation.
Find $u_h \in V_h^r$ such that

$$\mathcal{A}(u_h, v_h) = \int_\Omega f v_h \quad \forall v_h \in V_h^r$$

where

$$\begin{aligned}\mathcal{A}(w,v) = &\underbrace{\sum_{K\in T_h}\int_K \nabla w\cdot\nabla v}_{\text{Volume integral}} - \underbrace{\sum_{F\in\mathcal{F}_h}\int_F \{\nabla_h w\}\cdot[v]}_{\text{Flux term on faces}} \\ &- \underbrace{\sum_{F\in\mathcal{F}_h}\int_F [w]\cdot\{\{\nabla_h v\}\}}_{\text{Symmetric term}} + \underbrace{\sum_{F\in\mathcal{F}_h}\int_F \gamma[w]\cdot[v]}_{\text{Stabilization term}}\end{aligned}$$

and where

- $\mathcal{T}_h$ is the triangulation of the domain Ω into finite elements K
- The average operator $\{\!\{\cdot\}\!\}$ computes the vector average between the two sides for all the faces
- The jump operator $[\![\cdot]\!]$ computes the discontinuity across faces.
- ∇_h is the discrete gradient operator
- V_h^r is the finite element space consisting of piecewise polynomial functions of degree r, which are allowed to be discontinuous across element boundaries.

finally we have the interior penalty forms

$$\begin{aligned}\mathcal{A}(w,v) = &\sum_{K\in T_h}\int_K \nabla w\cdot\nabla v - \sum_{F\in\mathcal{F}_h}\int_F \{\{\nabla_h w\}\}\cdot[v] \\ &- \theta\sum_{F\in\mathcal{F}_h}\int_F [w]\cdot\{\{\nabla_h v\}\} + \sum_{F\in\mathcal{F}_h}\int_F \gamma[w]\cdot[v]\end{aligned}$$

where

- $\theta = 1$: **Symmetric** interior penalty
- $\theta = -1$: **Non-Symmetric** interior penalty
- $\theta = 0$: **Incomplete** interior penalty

Non standard boundary conditions

In the case of **Non-homogeneous Dirichlet** boundary conditions we modify the right hand side in order to apply the boundary condition

$$u = g_D \quad \text{on } \partial\Omega$$

the interior penalty formulation of the bilinear form becomes

$$\mathcal{A}(w,v) = \int_\Omega fv - \theta \sum_{F\in\mathcal{F}_h^B} \int_F g_D \nabla_h v \cdot \mathbf{n} + \sum_{F\in\mathcal{F}_h^B} \int_F \gamma g_D v$$

In the case of **Neumann** boundary conditions of the form

$$\nabla u \cdot \mathbf{n} = g_N \quad \text{on } \partial\Omega$$

the bilinear form has to be modified as

$$\begin{aligned}\mathcal{A}(w,v) = &\sum_{K\in T_h} \int_K \nabla w \cdot \nabla v - \sum_{F\in\mathcal{F}_h'} \int_F \{\nabla_h w\} \cdot [v] \\ &- \theta \sum_{F\in\mathcal{F}_h'} \int_F [w] \cdot \{\{\nabla_h v\}\} + \sum_{F\in\mathcal{F}_h'} \int_F \gamma [w] \cdot [v]\end{aligned}$$

and the r.h.s has to be modifies as

$$\int_\Omega fv + \sum_{F\in\mathcal{F}_h^B} \int_F g_N v$$

2. Spectral element methods

Given f find $u{:}\,\Omega \subset \mathbb{R}^d$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in weak form this becomes, given $f \in L^2(\Omega)$ find $u \in V = H_0^1(\Omega)$ such that

$$a(u,v) = F(v) \quad \forall v \in V$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v, \quad F(v) = \int_\Omega fv$$

2.1. Galerkin s.e.m.

Let $\hat{K}$ be the reference element $\hat{K} = (-1,1)^d$, for any element $K \in \mathcal{T}_h$ (the mesh) , there exists a (bijective and differentiable) mapping

$$F_k : \hat{K} \to K$$

The problem becomes

$$\text{Find } u_h \in V_h^p : a(u_h, \nu_h) = F(\nu_h) \quad \forall \nu_h \in V_h^p$$

where

- $p \geq 1$ integer and $\mathbb{Q}^p$ is the space of polynomials of degree $\leq p$ w.r.t each variable
- $X_h^p = \left\{\nu_h \in C^0(\overline{\Omega}) : \nu_h|_K = \widehat{\nu} \circ F_K^{-1} \text{ with } \hat{\nu} \in \mathbb{Q}^p(\widehat{K}) \forall K \in \mathcal{T}_h\right\}$

- $V_h^p = X_h^p \cap V$ (To account for the homogeneous boundary conditions)

2.2. Interpolation estimates

Let $v \in H^{s+1}(\Omega)$, $s \geq 0$ then there exists an interpolant $\prod_h^p v$ such that

$$\|\nu - \Pi_h^p\|_{H^1(\Omega)} \leq C_s \frac{h^{\min(s,p)}}{p^s} \|\nu\|_{H^{s+1}(\Omega)}$$
$$\|\nu - \Pi_h^p\|_{L^2(\Omega)} \leq C_s \frac{h^{\min(s,p)+1}}{p^{s+1}} \|\nu\|_{H^{s+1}(\Omega)}$$

2.3. Error estimates

Let u be the solution of the weak formulation and let u_h be the approximate solution with the SEM. Assume that $u \in H^{s+1}(\Omega)$, then

$$\|u - u_h\|_V \leq C_s \frac{h^{\min(s,p)}}{p^s} \|u\|_{H^{s+1}(\Omega)}$$

Moreover, if u is in analytic form

$$\|u - u_h\|_V \lesssim \exp(-\gamma p)$$

where γ depends on u

2.4. SEM-NI

The idea is to use GLL quadrature to switch out the integrals for numerical integration, the mass matrix then becomes **diagonal** which simplifies the implementation and reduces computation cost.
Let u be the solution of the weak formulation and let u_h be the approximate solution with the $SEM - NI$ if we assume $u \in H^{s+1}(\Omega)$ then

$$\|u - u_h\|_V \leq C_s \frac{h^{\min(s,p)}}{p^s} (\|f\|_{H^s(\Omega)} + \|u\|_{H^{s+1}(\Omega)})$$

3. Heat equation

Consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f, & \mathbf{x} \in \Omega, t > 0, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega \\ \text{Boundary condition} \end{cases}$$

where the boundary condition can take one of two forms

- **Dirichlet**:

$$u(\mathbf{x}, t) = g_D(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma_D \text{ and } t > 0,$$

- **Neumann**:

$$\frac{\partial u(\mathbf{x},t)}{\partial n} = g_N(\mathbf{x},t), \quad \mathbf{x} \in \Gamma_N \text{ and } t > 0,$$

Stability

Suppose that the data are regular enough. Then, the following a priori estimates hold for the exact solution

$$\|u(t)\|^2_{L^2(\Omega)} + \alpha \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)} ds \leq \|u_0\|^2_{L^2(\Omega)} + \frac{C_\Omega^2}{\alpha} \int_0^t \|f(s)\|^2_{L^2(\Omega)} ds$$

where C_Ω is the Poincare inequality constant and α is the coercivity constant of $a(\cdot,\cdot)$

Proof

Let us consider problem (4); since the corresponding equations must hold for each $v \in V$, it will be legitimate to set $v = u(t)$ (t being given), solution of the problem itself, yielding

$$\int_\Omega \frac{\partial u(t)}{\partial t} u(t)\, d\Omega + a(u(t), u(t)) = \int_\Omega f(t) u(t)\, d\Omega \quad \forall t > 0.$$

Considering the individual terms, we have

$$\int_\Omega \frac{\partial u(t)}{\partial t} u(t)\, d\Omega = \frac{1}{2}\frac{\partial}{\partial t} \int_\Omega |u(t)|^2 d\Omega = \frac{1}{2}\frac{\partial}{\partial t} \|u(t)\|^2_{L^2(\Omega)}.$$

The bilinear form is coercive, then we obtain

$$a(u(t), u(t)) \geq \alpha \|u(t)\|^2_V.$$

Thanks to the Cauchy-Schwarz inequality, we find

$$(f(t), u(t)) \leq \|f(t)\|_{\mathcal{L}^2(\Omega)} \|u(t)\|_{\mathcal{L}^2(\Omega)}.$$

3.1. Semi-discrete form

Let $V = H^1_{\Gamma_D}(\Omega)$ the weak formulation for the **heat** equation becomes

$$\int_\Omega \frac{\partial u(t)}{\partial t} v\, d\Omega + a(u(t), v) = \int_\Omega f(t) v\, d\Omega \quad \forall v \in V$$

where $a(\cdot,\cdot) = \int_\Omega \nabla u \nabla v$, the initial condition is preserved as $u(0) = u_0$ and the boundary conditions are assumed to be homogeneous.
The semi-discrete form can then be obtained by

$$\int_\Omega \frac{\partial u_h(t)}{\partial t} v_h\, d\Omega + a(u_h(t), v_h) = \int_\Omega f(t) v_h\, d\Omega \quad \forall v_h \in V_h$$

where $V_h \subset V$.

3.2. Time discretization and the θ-method

Proper care must be taken in order to discretize the time component, one way of handling it is to introduce the θ-method by approximating the temporal derivative with a simple difference quotient and replacing **all** the other terms with a linear combination of the values at time t^k and t^{k1} in the following manner

$$\int_\Omega \frac{u_h^{k+1} - u_h^k}{\Delta t} d\Omega + \int_\Omega \left(\theta \nabla u_h^{k+1} + (1-\theta)\nabla u_h^k\right)\nabla v_h = \int_\Omega \left(\theta f^{k+1} + (1-\theta) f^k\right) v_h \, d\Omega \quad \forall v_h \in V_i$$

this yields

- **Forward Euler**: when $\theta = 0$ accurate to order one with respect to Δt
- **Backwards Euler**: when $\theta = 1$ accurate to order one with respect to Δt
- **Crank-Nicolson**: when $\theta = \frac{1}{2}$ accurate to order two with respect to Δt

Moreover, order two is achieved only $\theta = \frac{1}{2}$.

Stability

In the case where $\theta = 0$ we have the following condition for stability

$$\exists c > 0 \quad : \quad \Delta t \leq ch^2 \quad \forall h > 0$$