

# APDE - Advanced Partial Differential Equations - F. Gazzola

**HELD BY PROF. F. GAZZOLA AT POLITECNICO DI MILANO**

These notes were taken during AY 2023/2024 using older material, your mileage may vary. They're meant to accompany the lectures and in no way aim to substitute a professor yapping away at an iPad 30m away.

Any reference of form  $(x.x.x)$  where  $x \in \mathbb{N}$  is a reference to a Theorem/Definition/etc... in the professors own book:

\*\*[Elements of Advanced Mathematical Analysis for Physics and Engineering](#)

I'm using the September 2013 version.

These notes are based on the work done by students Ravizza and Mescolini, you can access their notes by logging on the [AIM website](#) and checking out **Portale appunti**. This would NOT have been possible without their notes..

For any questions/mistakes you can reach me [here](#).

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## 1. Sobolev spaces and initial derivation for discrete domains

$C^*$  are banach spaces but not *Hilbert* spaces (i.e. we cannot use Lax-Milgram).

### Example

Second ordinary differential equation:

$$\begin{cases} -u'' + u = f \\ u(a) = u(b) = 0 \end{cases}$$

What happens if  $f$  is not continuous ? Let's try a weak solution.

$$-u''\varphi + u\varphi = f\varphi \quad \forall \varphi \in \mathcal{D}(a, b)$$

$$\mathcal{D}(a, b) = C_c^\infty(a, b)$$

We then integrate

$$\begin{aligned} -\int_a^b u''\varphi + \int_a^b u\varphi &= \int_a^b f\varphi \\ \int_a^b (-u''\varphi + u\varphi) &= \int_a^b f\varphi \end{aligned}$$

This integral obviously has sense if and only if the integrands are  $L^1$ :

- To have  $u\varphi \in L^1$  we need both of those functions to be  $L^2$
- To have  $u'\varphi' \in L^1$  we need both of those derivatives to be in  $L^1$
- We also need  $f \in L^2$

In order to be talking sense (i.e. having a legal formulation) this is enough. We will end up with a weak formulation satisfying these properties.

- $(u, u', f) \in L^2$

Keep in mind that  $L^2$  contains discontinuous functions.

### Weak derivative

We say that  $v = u'$  in the weak sense (distributional sense) if:

$$\int_I u\varphi' = - \int_I u'\varphi \quad \forall \varphi \in \mathcal{D}(I)$$

### The simplest Sobolev space

We define the  $H^1$  space as the following  
 $H^1(I) = \{u \in L^2(I), u' \in L^2(I) \text{ in a weak sense}\}$

$H^1$  is the simplest one dimensional Sobolev space. We are going to change the dimensionality of the space, the order of the derivative and the summability of the space.

Can we strengthen the assumptions on the hypotheses on our solution if we strengthen those on  $f \in L^2$ ? Answer is yes but we'll get there.

## Examples of weak derivatives

**First example:**

$C^1(\bar{I}) \subset H^1(I)$ : The weak derivative coincides with the classic one.

**Second example:**

Let  $f(x) = |x|$  and  $I = (-1, 1)$

Conjecture: We want to prove that the derivative  $f'(x)$ , defined as below, belongs to  $L^2$ .

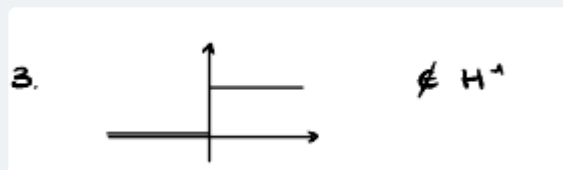
$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases}$$

Let's try it using the definition of weak derivative:

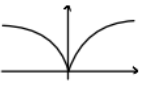
$$\begin{aligned} \int_{-1}^1 |x| \varphi'(x) dx &= \int_{-1}^0 |x| \varphi'(x) dx + \int_0^1 |x| \varphi'(x) dx \\ &\stackrel{\text{to check if it is the weak derivative}}{=} \int_{-1}^0 -x \varphi'(x) dx + \int_0^1 x \varphi'(x) dx \\ &\stackrel{\text{IPP}}{=} \int_{-1}^0 \varphi(x) dx - [x \varphi(x)]_{-1}^0 - \int_0^1 \varphi(x) dx + [x \varphi(x)]_0^1 \\ &= \int_{-1}^0 \varphi(x) dx - \int_0^1 \varphi(x) dx \end{aligned} \quad \forall \varphi \in \mathcal{D}(-1, 1)$$

We used the definition of weak derivative, written our initial integral in it's form, therefore proven that it is the weak derivative.

**Third example:**



**Fourth example:**

6.   $|x|^\alpha$  For which  $\alpha$  the function is in  $H^1$ ?

$\nearrow$  problem of integrability if  $\alpha < 0$

$$D \approx |x|^{\alpha-1} \quad \text{no} \quad \int \frac{1}{|x|^{1-\alpha}} dx < \infty \quad 2(1-\alpha) < 1 \Rightarrow \alpha > \frac{1}{2}$$

So if  $\alpha > 1$  this is a  $H^1$  function

## Separability of $H^1$ (6.2.3)

$H^1(I)$  is a separable Hilbert space when endowed with the following scalar product.

$$(u, v)_{H^1} = \int_I (u'v' + uv)$$

### Proof

We start by proving that it is indeed a scalar product, i.e. the following properties:

- Symmetry (obvious)
- It is a norm:

$$\|u\|_{H^1}^2 = \int_I [(u')^2 + u^2]$$

We then need to show that it is a Banach space with the following scheme:

Take a Cauchy sequence  $\{u_n\}$  s.t.  $\forall \epsilon > 0 \exists N_\epsilon, n, m > N_\epsilon \Rightarrow \|u_n - u_m\|_{H^1}^2 < \epsilon$   
 $\Rightarrow \|u_n - u_m\|_{L^2}^2 + \|u'_n - u'_m\|_{L^2}^2 < \epsilon$   
 $\swarrow \quad \searrow$   
 2 Cauchy sequences  
 $L^2$  Banach space  $\Rightarrow \exists u \in L^2(I)$  s.t.  $u_n \rightarrow u$  in  $L^2$   
 $\exists v \in L^2(I)$  s.t.  $u'_n \rightarrow v$  in  $L^2$   
 The point is:  $u' = v$ ?  $\rightarrow$  because the limit must  $\in H^1(I) \rightarrow$  if  $u' = v \Rightarrow u' \in L^2 \Rightarrow u \in H^1$   
 $\int u v' = \lim_{n \rightarrow \infty} \int u_n v' = \lim_{n \rightarrow \infty} \int u'_n v = - \int v v$   $\forall v \in \mathcal{D}(I) \subset L^2(I)$   
 $\hookrightarrow$  dense in  $L^2(I)$   $\square$

All we need now is proving that the Hilbert space we constructed is indeed **separable**, we can use  $H^1$  own isomorphism to  $L^2$  which is separable in its own right, then construct a linear map between the two:

$$Lu = (u, u')$$

### Map between Sobolev and Continuous spaces (6.2.5)

Every function in  $H^1$  can be represented by a continuous function, formally:

$$H^1(I) \subset C^0(\bar{I})$$

This is interesting because  $H^1$  also contains discontinuous functions, then how is it a subset of  $C^0$  ??

This is of course not what the theorem is saying, more specifically

### Proof

"Prof"  $u \in H^1(I)$  then  $u \in \overset{I \cap \mathbb{R}}{C^2(I)} \subset \overset{I \cap \mathbb{R}}{C^1(I)}$   
 $\int_a^x u'(t) dt$  well defined  $\forall x$



$u(x) = u(a) + \int_a^x \underbrace{u'(t)}_{\in L^1} dt$  Fundamental theorem of calculus  
 $u' \in L^1 \Rightarrow u \in AC$

$u(x_n) - u(x) = \int_x^{x_n} u'(t) dt \rightarrow 0$  if  $x_n \rightarrow x \Rightarrow u$  is continuous  $\square$

### Definition (6.2.7)

We define the closure of  $\mathcal{D}(I)$  with respect to the  $H^1(I)$  norm as the following:

$$H_0^1(I) = \overline{\mathcal{D}(I)}^{H^1(I)}$$

Recall that  $\mathcal{D}(\Omega)$  is the space of *smooth* functions with compact support over whatever  $\Omega$  is.

Essentially we're defining the space of  $H^1(I)$  functions that vanish at the boundary of  $I$ .

Note that the closure with respect to the  $L^2$  norm has special properties, specifically:

$$\overline{\mathcal{D}(I)}^{L^2(I)} = L^2(I)$$

This is because  $\mathcal{D}(I)$  is dense in  $L^2(I)$ , meaning that the space of *smooth enough* functions with a compact support within the boundaries of  $I$  is dense in  $L^2$ . If you think about it this makes sense since these functions would be  $L^\infty$  (i think...).

### Remark (6.2.7)

If and only if  $I \neq \mathbb{R}$

$$H_0^1(I) \subsetneq H^1(I)$$

### Definition

A weak solution satisfies

$$u \in H_0^1(a, b)$$

$$\int_a^b (u' \varphi' + u \varphi) = \int_a^b f \varphi \quad \forall \varphi \in H_0^1(a, b)$$

We can rewrite this as

$$(u, \varphi)_{H^1} = (f, \varphi)_{L^2}$$

You can then apply **Lax-Milgram** (1.7.4), prove the hypotheses on the bilinear form and conclude that  $\exists! u$ .

### Poincaré inequality (6.2.9)

Let  $I = (a, b)$  bounded then:

$$\|u\|_{L^2(I)} \leq (b-a) \|u'\|_{L^2(a,b)} \quad \forall u \in H_0^1(I)$$

As a consequence the map  $u \rightarrow \|u'\|_{L^2}$  defines a norm in  $H_0^1(I)$  which is equivalent to the norm of  $H^1(I)$

#### Proof

It's better if you use the books proof.

Proof It suffices to prove (PI)  $\forall u \in \mathcal{D}(I)$  (by density)

$\forall x \in (a, b) \quad u(x) = \int_a^x u'(t) dt \quad \text{fundamental theorem of calculus}$   
(u(a) = 0)

$\Rightarrow |u(x)| \leq \int_a^x |u'(t)| dt \leq \int_a^b |u'(t)| dt \leq \sqrt{b-a} \|u'\|_{L^2(a,b)} \quad \text{Hölder}$

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$\Rightarrow \|u\|_{L^\infty(a,b)} \leq \|u'\|_{L^2(a,b)} \leq \sqrt{b-a} \|u'\|_{L^2(a,b)}$

$\|u\|_{L^2(a,b)}^2 = \int_a^b u(t)^2 dt \leq (b-a) \|u\|_{L^\infty(a,b)}^2 \leq (b-a)^2 \|u'\|_{L^2(a,b)}^2$   
Hölder inequality

$\Rightarrow \|u\|_{L^2(a,b)} \leq (b-a) \|u'\|_{L^2(a,b)} \quad \bullet \quad \text{Poincaré inequality}$

$\|u'\|_{L^2(a,b)}^2 \leq \|u'\|_{L^2(a,b)}^2 + \|u\|_{L^2(a,b)}^2 = \|u\|_{H^1(a,b)}^2 \leq \|u'\|_{L^2(a,b)}^2 + (b-a)^2 \|u'\|_{L^2(a,b)}^2$   
 $= [1 + (b-a)^2] \|u'\|_{L^2(a,b)}^2$

The two norms are equivalent

$\begin{cases} u' \equiv 0 \\ u(a) = u(b) = 0 \end{cases} \Rightarrow u = 0 \rightarrow \text{the annihilation property is satisfied} \\ \text{so it is a norm}$

On  $H_0^1(I)$  we may take  $\|u\|_{H_0^1} = \|u'\|_{L^2}$  as a norm and  $(u, v) \mapsto \int_a^b u'v'$  as a scalar product  $\square$

Let  $H_1$  and  $H_2$  be Hilbert spaces such that:

$$H_1 \subset H_2 \quad (H^1 \subset L^2, H_0^1 \subset L^2)$$

Any Hilbert space is the dual of itself:

$$H_1' \approx H_1$$

$$H_2' \approx H_2$$

Therefore if  $H_1 \subset H_2$  and  $H_2' \subset H_1'$  (since a smaller Hilbert space will have a larger number of linear and continuous functionals, this is taken straight from notes I don't condone this behaviour):

$$H_2 \subset H_1$$

But this is absurd, look at the hypothesis !

There's clearly a mistake somewhere and that mistake lies in the assumption that we can use both isomorphisms at the same time, which isn't true due to the **Riesz representation theorem** (1.5.5).

### Hilbert (Gelfand) triple (Bottom of 6.2.18)

The following is true:

$$H_1 \subset H_2 \underbrace{\approx}_{(1)} H_2' \subset H_1'$$

$$H_0^1(I) \subset L^2(I) \underbrace{\subset}_{(2)} (H_0^1(I))' = H^{-1}(I)$$

Where:

1. Is the Riesz representation theorem (1.5.5)
2. The pivot space is the space in which we make use of the dual (in this case  $L^2$ )

Moreover if we define:

$$Lu(v) = \int_I u v \quad \forall u \in L^2(I)$$

Then the map  $v \rightarrow Lu(v)$  is linear and continuous  $\forall v \in H_0^1(I)$

### Proposition 6.2.10

Let  $F \in H^{-1}(I)$  then:

$$\exists f_0, f_1 \in L^2(I)$$

Such that:

$$\langle F, v \rangle = \int_I f_0 v + \int_I f_1 v \quad \forall v \in H_0^1$$

and

$$\|F\|_{H^{-1}(I)} = \max\{\|f_0\|_{L^2(I)}, \|f_1\|_{L^2(I)}\}$$

In the case where  $I = \mathbb{R}$  then substitute  $H_0^1(I)$  with  $H^1(I)$ .

In the case where  $I$  is unbounded we can take  $f_0 = 0$ .

This is a sort of representation for linear and continuous functionals over  $H_0^1(I)$ .

Now suppose that  $f_0 = 0$  :

$$\langle F, v \rangle = \int_I f_1 v' \underbrace{=}_{IBP} - \int_I f_1' v$$

but this basically means  $F = -f_1'$ .

...

Not so fast bucko, this is an illegal move since  $f_1 \in L^2$  so we don't know its derivative necessarily. We, on the other hand, can say (by defining it as such) that :

$$F = -f_1'$$

Implies that  $F$  has  $-1$  derivative in  $L^2$ .

### Dirac delta

Ex.  $\delta \in H^{-1}$   $\forall c \in [a, b]$   $\langle \delta_c, u \rangle = u(c)$   $\delta$  is a linear and continuous functional  
Heaviside function belongs to  $L^2(I)$ ,  $I$  bounded,  $\delta$  is its derivative

## 2. The $\mathbb{R}^n$ case

We'll take a domain  $\Omega$  in  $\mathbb{R}^n$  to be an open set, for example  $\partial\Omega$ .

### Example

$$\Omega = \{(x, y) \in [0, 1]^2, x, y \in \mathbb{Q}\}$$

$$\partial\Omega = [0, 1]^2$$

Our main assumptions will be that  $\Omega$  has to be an open set and  $\partial\Omega$  smooth. Let's say we want  $\partial\Omega \in C^1$ , it can be constructed as the union of locally  $C^1$  functions. (3.1.2)

Now we want to define  $H^1(\Omega)$  (we'll worry about the boundary later on).

### Weak derivative in $\mathbb{R}^n$

The  $i$ -th weak derivative  $w = \frac{\partial u}{\partial x_i}$  of  $u$  is such that:

$$\int_{\Omega} u \frac{\partial v}{\partial x_i} = \int_{\Omega} w v \quad \forall v \in \mathcal{D}(I)$$

### Sobolev space (6.,2.11)

Let  $\Omega \in \mathbb{R}^n$  be an open set, the Sobolev space  $H^1$  is defined as:



$$H^1(\Omega) = \left\{ u \in L^2(\Omega) ; \frac{\partial u}{\partial x_i} \in L^2(\Omega) \quad \forall i = 1, \dots, n \right\}$$

(6.2.12)

Given  $u \in H^2(\Omega)$ :

We take

$$H^1(\Omega) \approx L^2(\Omega)^{n+1}$$

Meaning we take these spaces to be isomorphic.

We define the gradient as :

$$\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x_i} \\ \vdots \\ \vdots \end{bmatrix} \in L^2(\Omega)^n$$

We define the norm to be:

$$|\nabla u|^2 = \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i} \right)^2$$

Taking derivative in the classical sense:

$$\left( u \in C^1(\Omega) \cap L^2(\Omega) \right) \wedge \left( u \frac{\partial u}{\partial x_i} \in L^2(\Omega) \right) \implies u \in H^1(\Omega)$$

Moreover the classical partial derivatives coincides with the weak ones, additionally if  $\Omega$  is bounded:

$$\implies C^1(\overline{\Omega}) \subset H^1(\Omega)$$

We finally define the bilinear form on  $H^1$  to be:

$$(u, v)_{H^1} = \int_{\Omega} (u \cdot v + \nabla u \times \nabla v)$$

Where the first is a product between two scalar functions and the second is a scalar product (sum of  $n + 1$  terms).

The former will be sometimes omitted.

We can define a norm (6.2.13) therefore

### $H^1$ separability in $\mathbb{R}^n$ (6.2.14)

$H^1$  is separable since we can:

- Define a scalar product such as:

$$(u, v)_{H^1}$$

- We can define a norm
- The space is complete

Refer to the book for an actual proof.

## Closure to $H^1$ (6.2.15)

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} \subset H^1(\Omega)$$

The book goes in depth on how actually it is defined..

## Example

Ex.  $f(x) = \frac{1}{|x|^2}$  in  $\mathbb{R}^m \setminus B_1$   $f \in L^p(\mathbb{R}^m \setminus B_1) \iff \int_{\mathbb{R}^m \setminus B_1} |f(x)|^p dx < +\infty$

We can take advantage of polar coordinates

$$\int_{\mathbb{R}^m \setminus B_1} |f(x)|^p dx = c_m \int_1^\infty \frac{r^{m-1}}{r^{2p}} dr \quad m-1-2p < -1 \Rightarrow p > \frac{m}{2}$$

## Example (6.2.16)

Basically just show that the function and its gradient are  $L^2(\Omega)$  and you're good to go. This is also interesting because it's a counter example to the notion that (in  $n \geq 2$ ) a function only need be limited to be in the respective Sobolev space.

Ex.  $m=2$   $B_{1/2} = \{ |x| < \frac{1}{2} \}$   $f(x) = \log |\log |x|| \notin L^\infty$   $B_{1/2} \subset \mathbb{R}^2$   
 $\notin C^0$

Show that  $f \in H^1(B_{1/2})$

$|x| = \rho$   $\int_0^{1/2} \rho \log^2 |\log \rho| d\rho < +\infty \iff f \in L^2(B_{1/2})$

$|\nabla f(x)| = |f'(\rho)|$   $f(\rho) = \log |\log \rho|$   $f'(\rho) = \frac{1}{\rho \log \rho}$

$\nabla f \in L^2(B_{1/2}) \iff \int_0^{1/2} \frac{\rho}{\rho^2 \log^2 \rho} d\rho < +\infty$

$\Rightarrow f \in H^1(B_{1/2})$  but  $f \notin L^\infty(B_{1/2})$   $H^1(B_{1/2}) \not\subset L^\infty(B_{1/2})$  contrary example if  $m \geq 2$

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$$



No

I take a function that is 0 at the boundary

If  $u \in C^0(\bar{\Omega})$  then  $u \in H_0^1(\Omega) \iff u = 0$  on  $\partial\Omega$



If the function has compact support

## Poincaré inequality (6.2.18)

Assume that  $\Omega$  is bounded. Then there exists a constant  $C(\Omega)$  such that:

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2} \quad \forall u \in H_0^1(\Omega)$$


Moreover the map  $u \rightarrow \|u\|_{L^2}$  is a norm in  $H_0^1(\Omega)$  which is equivalent to the norm  $\|u\|_{H^1}$ .

**Poincaré inequality**

**Rem.** Holds in dimension 1 if the domain is bounded.

Let  $\Omega \subset \mathbb{R}^n$  be open and bounded in one direction.  
 Then  $\exists c > 0$  s.t.  $\|u\|_{L^2} \leq c \|\nabla u\|_{L^2} \quad \forall u \in H_0^1(\Omega)$   
 In particular,  $u \mapsto \|\nabla u\|_{L^2}$  is an equivalent norm in  $H_0^1(\Omega)$ .

**Proof.** By density we may assume that  $u \in \mathcal{D}(\Omega)$ .




$x_1$  all directions in  $\mathbb{R}^n$  excluded  $x_1$ .  
 $\Omega \subset (a, b) \times \mathbb{R}^{n-1}$

$\forall x \in \Omega \quad u(x) = \int_a^{x_1} u_{x_1}(t, x') dt$   
 (FTC in 1 dimension with partial derivative)

$|u(x)| \leq \int_a^{x_1} |u_{x_1}(t, x')| dt \leq \sqrt{b-a} \left[ \int_a^{x_1} |u_{x_1}(t, x')|^2 dt \right]^{1/2}$

$|u(x)|^2 \leq (b-a) \int_a^{x_1} |u_{x_1}(t, x')|^2 dt$

$\int_{\mathbb{R}^{n-1}} |u(x)|^2 dx' \leq (b-a) \int_{\mathbb{R}^{n-1}} \int_a^{x_1} |u_{x_1}(t, x')|^2 dt dx'$

$\int_a^b \int_{\mathbb{R}^{n-1}} |u(x)|^2 dx' dx_1 \leq (b-a)^2 \int_{\mathbb{R}^{n-1}} \int_a^b |u_{x_1}(x)|^2 dx_1 dx' \rightarrow$  

$\|u\|_{L^2(\Omega)}^2 = \int_{\Omega} |u(x)|^2 dx \leq (b-a)^2 \int_{\Omega} |\nabla u(x)|^2 dx$

Professor then goes on to define the relevant Hilbert triple

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$$

Where the second continuous embedding is to be meant as follows:

Each function  $u \in L^2(\Omega)$  is identified with the linear functional  $I_u \in H^{-1}(\Omega)$  defined as :

$$\langle I_u, v \rangle := \int_{\Omega} u v \quad \forall v \in H_0^1(\Omega)$$

Essentially we can map each  $L^2$  function to a functional from  $H^1$  to  $\mathbb{R}$ .

Note that we denote by  $H^{-1}(\Omega)$  the dual space of  $H_0^1(\Omega)$ .

### Proposition (6.2.19)

Let  $\Omega \subset \mathbb{R}^n$  be open and let  $F \in H^{-1}(\Omega)$ , then  $\exists \{f_i\}_{i=0}^n \in L^2(\Omega)$  such that:

$$\langle F, v \rangle = \int_{\Omega} f_0 v + \sum_{i=1}^n f_i \frac{\partial u}{\partial x_i} \quad \forall v \in H_0^1(\Omega)$$

And:

$$\|F\|_{H^{-1}} = \max_{i=0, \dots, n} \|f_i\|_L^2$$

Moreover, if  $\Omega$  is bounded, we can take  $f_0 = 0$

### Remark

Note that in this case  $v \in L^2(\Omega)$  because  $v \in H_0^1(\Omega)$

### Proof

There's a proof in the book but here's what the prof wrote:

Proof. If  $f_0 = 0$ ,  $\langle F, v \rangle = \int_{\Omega} f \cdot \nabla v$ ,  $f = (f_1, f_2, \dots, f_m) \in L^2(\Omega)^m$

For smooth  $f$ ,  $\int_{\Omega} f \cdot \nabla v = - \int_{\Omega} v \operatorname{div} f + \int_{\partial\Omega} v (f \cdot \nu)$   $\nu \rightarrow$  normal

$\Rightarrow$  "  $F = - \operatorname{div} f$  " the linear continuous functional in  $H^{-1}$  is the divergent of a function in  $L^2$

$H^1(\Omega) \not\subset L^\infty(\Omega)$  if  $n \geq 2$   
 $H^1(\Omega) \subset L^2(\Omega) \quad \forall n$

### Sobolev embedding theorem (6.2.20 - 6.2.21)

Let  $\Omega \subset \mathbb{R}^n$  be an open domain with  $\partial\Omega \in Lip$  and  $n \geq 2$ . Then:

$$H^1(\Omega) \subset L^p(\Omega) \quad \begin{cases} \forall 2 \leq p < \infty & \text{if } n = 2 \\ \forall 2 \leq p < \frac{2n}{n-2} & \text{if } n = 3 \end{cases}$$

In addition to the above, if  $\Omega$  is bounded, then the embeddings become compact:

*the following continuous embeddings:*

- (i) if  $n = 2$  then  $H^1(\Omega) \subset\subset L^q(\Omega)$ , for every  $1 \leq q < \infty$ ;
- (ii) if  $n \geq 3$  then  $H^1(\Omega) \subset\subset L^q(\Omega)$ , for every  $1 \leq q < 2^* = \frac{2n}{n-2}$ .

*Proof.* See [1, Theorem 6.2]. □

### Remark

The following is called the **Critical Sobolev exponent**:

$$2^* = \frac{2n}{n-2}$$

A sequence that converges in the  $H^1(\Omega)$  sense implies convergence in a certain  $L^p$  sense, formally:

$$u_n \xrightarrow{H^1(\Omega)} u \implies u_n \xrightarrow{L^p} u$$

### Remark

The functions inside  $H^1(\Omega)$  are defined up to a negligible set.

### 3. The $H^k$ spaces

We basically take 6.2.11 and apply it a bunch of times to define higher order weak derivatives.

Note that in the image below the prof *formally* sets  $H^0 = L^2$  (6.3.4)



#### Multi-index notation

We call a multi-index a vector:

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$$

With norm:

$$|\alpha| = \sum_{i=1}^n \alpha_i$$

We'll use this to define partial derivatives kinda like this:

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

#### Definition 6.3.1

Let  $u \in L^1_{\text{loc}}(\Omega)$  and  $\Omega \subseteq \mathbb{R}^n$  be an open set. Given a multi-index  $\alpha$  we'll say that  $u$  admits a weak derivative  $D^\alpha u$  if there exists  $g \in L^1_{\text{loc}}(\Omega)$  such that:

$$\int_{\Omega} u D^\alpha \varphi = (-1)^{|\alpha|} \int_{\Omega} g \varphi \quad \forall \varphi \in \mathcal{D}(\Omega)$$

#### Definition and Separability Theorem (6.3.2 - 6.3.5)

Let  $\Omega$  be as above and  $k \in \mathbb{N}$ . The  $H^k(\Omega)$  is defined by:

$$H^k(\Omega) = \{u \in L^2(\Omega), \quad D^\alpha u \in L^2(\Omega) \quad \forall |\alpha| \leq k\} \quad \forall k \in \mathbb{N}$$

We take  $H^k(\Omega)$  to be a [separable](#) Hilbert space with scalar product:

$$(u, v)_{H^k(\Omega)} = \int_{\Omega} \left( uv + \sum_{1 \leq |\alpha| \leq k} D^{\alpha} u D^{\alpha} v \right)$$

With induced norm:

$$\|u\|_{H^k(\Omega)} = \left( \int_{\Omega} \sum_{0 \leq |\alpha| \leq k} |D^{\alpha} u|^2 \right)^{\frac{1}{2}}$$

### Proof

Check the book for a more detailed proof, here's what the prof wrote,

$H^k$  is a separable Hilbert space with scalar product

$$(u, v)_{H^k} = \int_{\Omega} \left( uv + \sum_{1 \leq |\alpha| \leq k} D^{\alpha} u D^{\alpha} v \right)$$

- it has a scalar product
- it is complete (use a Cauchy sequence)
- it's separable because its  $L^2$

$$\|u\|_{H^k} = \left( \int_{\Omega} \sum_{0 \leq |\alpha| \leq k} |D^{\alpha} u|^2 \right)^{1/2} \quad \text{induced norm}$$

### Separable Banach spaces with no scalar product. (6.3.8)

$$H^{k+1}(\Omega) = \{u \in H^k(\Omega), \forall u \in H^k(\Omega)\}$$

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega); D^{\alpha} u \in L^p(\Omega) \forall |\alpha| \leq k\} \quad \forall k \in \mathbb{N}$$

↳ separable Banach spaces, with no scalar product

$$\text{If } u \in L^p(\Omega) \text{ and } D^{\alpha} u \in L^p(\Omega) \forall |\alpha| = k \Rightarrow u \in W^{k,p}(\Omega)$$

$H^k$  defined inductively

### Remark (6.3.4)

We define  $H^k(\Omega)$  inductively. Check the book.

### Fourier definition for higher order Sobolev spaces(6.3.5 - 6.3.6)

We can define  $H^k(\mathbb{R}^n, \mathbb{C})$  in a more straightforward way using the Fourier transform.

$$H^k(\mathbb{R}^n, \mathbb{C}) = \{u \in L^2(\mathbb{R}^n, \mathbb{C}); (1 + |\xi|^2)^{\frac{k}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n, \mathbb{C})\}$$

In this construction  $H^0 = L^2$  because  $(1 + |\xi|^2)^{\frac{k}{2}} = 1$  if  $k = 0$ , we need, however, to define a scalar product to go further.

$$(u, v)_{H^k} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^k \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi \quad \rightarrow \text{norm}$$

$$(1 + |\xi|^2)^{k/2} \rightarrow \text{weight function}$$

Note that  $k$  doesn't *have* to be an integer in this construction, it just has to be  $k = s \geq 1$ . We then end up with what are called **Non Local PDE's**

### Remark

This only works on  $\mathbb{R}^n$ , not on bounded domains (Duh!)

## 4. Non-integer $H^k$

As previously mentioned, we'll take  $s \geq 0$ .

$$H^s(\mathbb{R}^n, \mathbb{C}) = \left\{ u \in L^2(\mathbb{R}^n, \mathbb{C}), \quad (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n, \mathbb{C}) \right\}$$

Endowed with the scalar product

$$(u, v)_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi$$

for every  $u, v \in H^s(\mathbb{R}^n, \mathbb{C})$ .

(6.4.1)

We can extend this construction to a general domain  $\Omega$ .

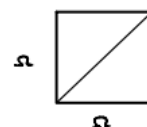
### Theorem (6.4.5)

**Def** let  $\Omega \subseteq \mathbb{R}^n$  be an open set and let  $s > 0$ ; if  $s \in (k, k+1)$  we set  $s = k + t$  with  $t \in (0, 1)$  and we define

$$H^s(\Omega) = \left\{ u \in H^k(\Omega); \int_{\Omega} \int_{\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{n+2t}} dx dy < \infty \quad \forall |\alpha| = k \right\}$$

$$s = \frac{1}{2} \Rightarrow \alpha = 0 \quad \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+1}} dx dy < \infty$$

↳ it's a property of the incremental ratio



$$(u, v)_{H^s(\Omega)} = (u, v)_{H^k(\Omega)} + \sum_{|\alpha|=k} \int_{\Omega} \int_{\Omega} \frac{(\partial^\alpha u(x) - \partial^\alpha u(y))(\partial^\alpha v(x) - \partial^\alpha v(y))}{|x - y|^{n+2t}} dx dy$$

$H^s(\Omega)$  is an Hilbert space, because it becomes a Banach space if we define the norm from the scalar product  
 $H^s(\Omega)$  is separable because it is the cartesian product of separable Hilbert spaces

## 5. $H_0^s$ spaces and Trace operators

### Definition (6.5.1)

For every  $s \geq 0$ :

$$H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$$

A special case is the following:

$$\Omega = \mathbb{R}^n \implies H_0^s(\Omega) = H^s(\Omega)$$

Otherwise

$$\text{If } \Omega \neq \mathbb{R}^n \begin{cases} L^2(\Omega) = H^0(\Omega) \\ H^1(\Omega) \end{cases} \quad \begin{cases} \overline{\mathcal{D}(\Omega)}^{L^2(\Omega)} = L^2(\Omega) \\ \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega) \subsetneq H^1(\Omega) \end{cases}$$

What if  $s \in [0, 1]$  ?

Let's put ourselves in the case where  $\Omega \neq \mathbb{R}^n$

$$\begin{array}{lll} \text{What happens with } H^s(\Omega) \text{ } s \in [0, 1] ? & & \\ H_0^s(\Omega) = H^s(\Omega) & \forall s \in [0, \frac{1}{2}] & \Omega \neq \mathbb{R}^n \\ H_0^s(\Omega) \subsetneq H^s(\Omega) & \forall s > \frac{1}{2} & \end{array}$$

Read the book for a better understanding of what's going on in the background.

Essentially  $\mathcal{D}(\Omega)$  is dense in  $H^s(\Omega)$  if and only if  $s \leq \frac{1}{2}$ . Meaning the "closure" and the space are the same. (What's written in the pic).

The  $\mathbb{R}^n$  case is also explained.

A problem now arises, how do we define the value of a function  $u \in H^s(\Omega)$  on the boundary ? Remember these functions are defined up to a negligible set.

How to define  $u|_{\partial\Omega} = f \neq 0$  ?

Which is the restriction on the boundary ?

If  $s \in [0, \frac{1}{2}]$  it's impossible to define the restriction even if  $f=0$   
 If  $s > \frac{1}{2}$  we can try with an approximation



We'll solve it this way

### Trace operator (6.5.2 - 6.5.4)



Assume  $\partial\Omega \in C^\infty$ , now note that  $C^\infty(\bar{\Omega})$  is dense in  $H^s(\Omega) \forall s \geq 0$  so the idea is to define a sequence that approximates  $u$  and evaluate it at the boundary in this fashion:

Take  $u \in H^s(\Omega)$ ,  $s > \frac{1}{2}$ , then construct a sequence:

$$\{u_m\} \subset C^\infty(\bar{\Omega})$$

Such that:

$$u_m \xrightarrow{H^s(\Omega)} u$$

Now define the **Trace operator** as:

$$\gamma_0 u = \lim_{m \rightarrow \infty} u_m|_{\partial\Omega}$$

This  $\gamma_0 u$  is an approximation and in general this definition fails to actually define what happens at the boundary, or even define a way to evaluate a *type* of boundary condition. The book goes *much* more in depth.

### Redefine the boundary points

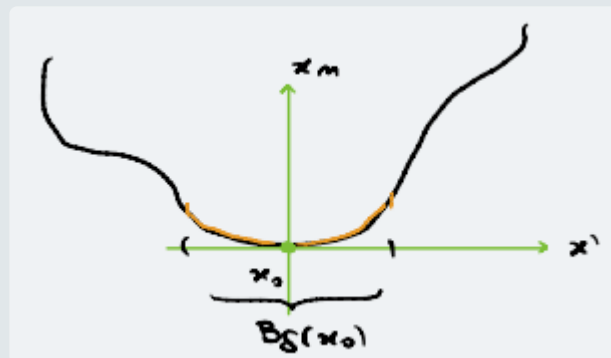
We can see any point  $x_0 \in \partial\Omega$  as:

$$x_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

At this point there exists a veery tiny  $\delta > 0$  and a function  $\varphi \in C^\infty(B_\delta(x_0))$  (locally smooth) such that:

$$x \in \partial\Omega \iff x_n = \varphi(x') \quad \forall x' \in B_\delta(x'_0)$$

Where  $x'$  are as below:



The relevant Sobolev space is

$$H^s(B_\delta(x'_0))$$

Note that it's of dimension  $n - 1$

We say that  $\gamma_0 u \in H^s(\partial\Omega)$  if

$$u_0(x') = u(x', \varphi(x')) \in H^s(B_\delta(x'_0)) \quad \forall x_0 \in \partial\Omega$$

The following is called the *restriction* of  $u$  to the boundary:

$$u(x', \varphi(x'))$$

### Theorem (6.5.3)

Let  $s > \frac{1}{2}$ , the trace operator is such that:

$$\gamma_0: H^s(\Omega) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega)$$

We essentially lose half a degree of derivative.

### Remark

$\gamma_0 u$  is only surjective, not injective. We *can* get injectivity if we add more constraints but we can't conclude that with our current formulation (we get it in PDEs, using the lifting operator)

Some examples follow.

Ex. Heat equation  

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$
 has a unique solution

If  $\partial\Omega \in C^\infty$ ,  $\exists \gamma_0: H^s(\Omega) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega)$  linear and continuous that extends  
 $u \mapsto u|_{\partial\Omega}$  for  $C^\infty(\bar{\Omega})$  functions

N.B.  $H^{s-\frac{1}{2}}$  cannot be improved, it is the optimum space  
 When we go to the boundary we lose half derivative

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{in } \partial\Omega \end{cases} \quad \forall f \in H^s(\partial\Omega) \quad \exists! u \in H^{s+\frac{1}{2}}(\Omega)$$

$\exists$  want to find an harmonic extension to the boundary  
 The traces are well defined if  $s > \frac{1}{2}$

$\gamma_0 \rightarrow$  trace operator (of order 0)  
 $\hookrightarrow$  number of derivatives

### Trace operator in the case of the Neumann boundary condition (6.5.7)

Neumann problem

$$\begin{cases} \Delta u = 0 & \Omega \\ \frac{\partial u}{\partial \nu} = g & \partial\Omega \end{cases} \quad \frac{\partial u}{\partial \nu} \rightarrow \text{flux across the boundary}$$

$$0 = \int_{\Omega} \Delta u = \int_{\Omega} \operatorname{div}(\nabla u) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \quad \text{we need } \int_{\partial\Omega} g = 0$$

We need to find the normal derivative of  $u$  at the boundary

We choose  $\partial\Omega \in C^\infty$



$$\delta_j u = \frac{\partial^j u}{\partial \nu^j} \Big|_{\partial\Omega} \quad \forall u \in C^\infty(\bar{\Omega})$$

$$\delta_j : H^s(\Omega) \rightarrow H^{s-j-\frac{1}{2}} \quad \forall j < s - \frac{1}{2} \quad j \text{ integer}$$

$H^{s-j-\frac{1}{2}} \rightarrow$   $\frac{1}{2}$  case  $j$  because we want the  $j$ -th derivative and  $\frac{1}{2}$  case before because  $\frac{1}{2}$  in the boundary

$$\text{If } j=0 \Rightarrow s > \frac{1}{2}$$

$$\text{If } j=1 \Rightarrow s > \frac{3}{2}$$

$\delta_j$  are linear and continuous operators



$$H_0^s(\Omega)$$

$$H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$$

### Theorem (6.5.8)

\*Thm  $\partial\Omega \in C^\infty, u \in H^s(\Omega)$

$$u \in H_0^s(\Omega) \Leftrightarrow \delta_j u = 0 \quad \forall j < s - \frac{1}{2}$$

$$H_0^s(\Omega) = \bigcap_{j=0}^{\lfloor s-\frac{1}{2} \rfloor} \ker \{\delta_j\}$$

$\lfloor s - \frac{1}{2} \rfloor \rightarrow$  integer part of  $s - \frac{1}{2}$   
 $\delta_j \rightarrow$  linear operator, it has a kernel

## 6. Embeddings theorems

In this section we will make use of [embeddings](#) in order to evaluate the regularity of  $H^s(\Omega)$  elements.

### Sobolev Theorem (6.6.1)

Let  $\Omega = \mathbb{R}^n$  or an open set with Lipschitz boundary. Let  $s \geq 0$ , then the following continuous embeddings hold:

$$H^s(\Omega) \subset \begin{cases} L^p(\Omega) & \forall 2 \leq p \leq \frac{2n}{n-2s} & \text{if } n > 2s \\ L^p(\Omega) & \forall 2 \leq p \leq \infty & \text{if } n = 2s \\ C^0(\bar{\Omega}) & & \text{if } n < 2s \end{cases}$$

So if we increase  $s$  we increase the regularity but we also have to take into account  $n$  (the number of dimensions).

Theorem 6.6.1 ensures the continuity of the elements of  $H^1(\Omega)$  when  $n = 1$ , while if  $n = 2$ , the fact that a function belongs to  $H^1(\Omega)$  is not any more sufficient to guarantee its continuity. In the case  $n = 2$ , in order to have continuity of a function  $u$  we have to assume that  $u \in H^s(\Omega)$  with  $s > 1$ . We deduce that, as the space dimension increases, the regularity of the functions of  $H^s(\Omega)$  decreases.

If  $\Omega \subseteq \mathbb{R}^n$  is bounded then all of the above embeddings are **compact** except for:

$$H^s(\Omega) \subset L^{\frac{2n}{n-2s}} \quad \text{when } n > 2s$$