APDE - Advanced Partial Differential Equations - F. Gazzola

Held by prof. F. Gazzola at Politecnico di Milano

Disclaimers

Read me! :

- These notes were taken during AY 2023/2024 using older material, your mileage may vary. They're meant to accompany the lectures and in no way aim to substitute a professor yapping away at an iPad 30m away.
- Any reference of form $(x.x.x)$ where $x \in \mathbb{N}$ is a reference to a a Theorem/Definition/etc... in the professors own book:
  Elements of Advanced Mathematical Analysis for Physics and Engineering
  By: A. Ferrero , F. Gazzola , M. Zanotti
  ISBN: 978-88-7488-645-6.
  I'm using the September 2013 version.
- These notes are based on the work done by students Ravizza and Mescolini, you can access their notes by logging on the AIM website and checking out **Portale appunti**. This would NOT have been possible if not for their notes..
- For any questions/mistakes you can reach me here.

Sobolev spaces and initial derivation for discrete domains

C^* are banach spaces but not *Hilbert* spaces (i.e. we cannot use Lax-Milgram).

Example

Second ordinary differential equation:

$$\begin{cases} -u'' + u = f \\ u(a) = u(b) = 0 \end{cases}$$

What happens if f is not continuous ? Let's try a weak solution.

$$-u''\varphi + u\varphi = f\varphi \qquad \forall \varphi \in \mathcal{D}(a,b)$$

$$\mathcal{D}(a,b) = C_c^\infty(a,b)$$

We then integrate

$$-\int_a^b u''\varphi + \int_a^b u\varphi = \int_a^b f\varphi$$

$$\int_a^b (-u''\varphi + u\varphi) = \int_a^b f\varphi$$

This integral obviously has sense if and only if the integrands are L^1:

- To have $u\varphi \in L^1$ we need both of those functions to be L^2
- To have $u'\varphi' \in L^1$ we need both of those derivatives to be in L^1
- We also need $f \in L^2$

In order to be talking sense (i.e. having a legal formulation) this is enough. We will end up with a weak formulation satisfying these properties.

- $(u, u', f) \in L^2$

Keep in mind that L^2 contains discontinuous functions.

Weak derivative

We say that $v = u'$ in the weak sense (distributional sense) if:

$$\int_I u\varphi' = -\int_I u\varphi \qquad \forall\varphi \in \mathcal{D}(I)$$

The simplest Sobolev space

We define the H^1 space as the following
$H^1(I) = \{u \in L^2(I),\ u' \in L^2(I) \text{ in a weak sense}\}$

H^1 is the simplest one dimensional Sobolev space. We are going to change the dimensionality of the space, the order of the derivative and the summability of the space. Can we strengthen the assumptions on the hypotheses on our solution if we strengthen those on $f \in L^2$? Answer is yes but we'll get there.

Examples of weak derivatives

First example:
$C^1(\bar{I}) \subset H^1(I)$: The weak derivative coincides with the classic one.

Second example:
Let $f(x) = |x|$ and $I = (-1, 1)$

Conjecture: We want to prove that the derivative $f'(x)$, defined as below, belongs to L^2.

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases}$$

Let's try it using the definition of weak derivative:

$$\int_{-1}^{1} |x| \varphi'(x)\, dx = \int_{-1}^{0} \varphi(x)\, dx - \int_{0}^{1} \varphi(x)\, dx \qquad \forall \varphi \in \mathcal{D}(-1,1)$$

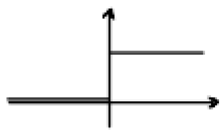
↳ to check if it is the weak derivative

$$= -\int_{-1}^{0} x\varphi'(x)\, dx + \int_{0}^{1} x\varphi'(x)\, dx \underset{IPP}{=} \int_{-1}^{0} \varphi(x)\, dx - \underbrace{[x\varphi(x)]_{-1}^{0}}_{=0} - \int_{0}^{1} \varphi(x)\, dx + \underbrace{[x\varphi(x)]_{0}^{1}}_{=0}$$

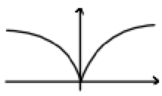
$$= \int_{-1}^{0} \varphi(x)\, dx - \int_{0}^{1} \varphi(x)\, dx$$

We used the definition of weak derivative, written our initial integral in it's form, therefore proven that it is the weak derivative.

Third example:

3.  $\notin H^1$

Fourth example:

4. 

$|x|^\alpha$ For which α the function is in H^1 ?

↗ problem of integrability if $\alpha - 1 < 0$

$D \approx |x|^{\alpha-1}$ so $\int \frac{1}{|x|^{(1-\alpha)2}} < \infty \qquad 2(1-\alpha) < 1 \Rightarrow \alpha > \frac{1}{2}$

If $\alpha > 1$ this is a H^1 function

Separability of H^1 (6.2.3)

$H^1(I)$ is a separable Hilbert space when endowed with he following scalar product.

$$(u,v)_{H^1} = \int_I (u'v' + u\, v)$$

Proof

We start by proving that it is indeed a scalar product, i.e. the following properties:

- Symmetry (obvious)
- It is a norm:

$$||u||_{H^1}^2 = \int_I [(u')^2 + u^2]$$

We then need to show that it is a Banach space with the following scheme:

Take a Cauchy sequence $\{u_n\}$ s.t. $\forall \varepsilon > 0 \ \exists N_\varepsilon$, $n, m > N_\varepsilon \Rightarrow \|u_n - u_m\|_{H^1}^2 < \varepsilon$

$\Rightarrow \underbrace{\|u_n - u_m\|_{L^2}^2 + \|u_n' + u_m'\|_{L^2}^2}_{\text{2 Cauchy sequences}} < \varepsilon$

L^2 Banach space $\Rightarrow$ $\exists\, u \in L^2(I)$ s.t. $u_n \to u$ in L^2

$\exists\, v \in L^2(I)$ s.t. $u_n' \to v$ in L^2

The point is: $u' = v$? $\to$ because the limit must $\in H^1(I) \Rightarrow$ if $u' = v \Rightarrow u' \in L^2 \Rightarrow u \in H^1$

$$\int u\varphi' = \lim_{n \to +\infty} \int u_n \varphi' = -\lim_m \int u_m' \varphi = -\int v \varphi \qquad \forall \varphi \in \mathcal{D}(I) \subset L^2(I)$$

$\hookrightarrow$ dense in $L^2(I)$

□

All we need now is proving that the Hilbert space we constructed is indeed **separable**, we can use H^1 own isomorphism to L^2 which is separable in its own right, then construct a linear map between the two:

$$Lu = (u, u')$$

Map between Sobolev and Continuous spaces (6.2.5)

Every function in H^1 can be represented by a continuous function, formally:

$$H^1(I) \subset C^0(\bar{I})$$

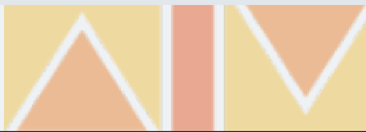
This is interesting because H^1 also contains discontinuous functions, then how is it a subset of C^0 ??
This is of course not what the theorem is saying, more specifically

Proof

"Proof" $u \in H^1(I)$ then $u' \in L^2(I) \subset L^1(I)$

$|I| < \infty$

$\int_a^x u'(t)\,dt$ well defined $\forall x$



[redacted]

$u(x) = u(a) + \int_a^x \underbrace{u'(t)}_{\in L^1} dt$ Fundamental theorem of calculus

$u' \in L^1 \Rightarrow u \in AC$

$u(x_n) - u(x) = \int_x^{x_n} u'(t)\,dt \to 0$ if $x_n \to x$ $\Rightarrow$ u is continuous

□

Definition (6.2.7)

We define the closure of $\mathcal{D}(I)$ with respect to the $H^1(I)$ norm as the following:

$$H_0^1(I) = \overline{\mathcal{D}(I)}^{H^1(I)}$$

Recall that $\mathcal{D}(\Omega)$ is the space of *smooth* functions with compact support over whatever Ω is.

Essentially we're defining the space of $H^1(I)$ functions that vanish at the boundary of I.

Note that the closure with respect to the L^2 norm has special properties, specifically:

$$\overline{\mathcal{D}(I)}^{L^2(I)} = L^2(I)$$

This is because $\mathcal{D}(I)$ is dense in $L^2(I)$, , meaning that the space of *smooth enough* functions with a compact support within the boundaries of I is dense in L^2. If you think about it this makes sense since these functions would be L^∞ (i think...).

Remark (6.2.7)

If and only if $I \neq \mathbb{R}$

$$H_0^1(I) \subsetneq H^1(I)$$

Definition

A weak solution satisfies

$$u \in H_0^1(a,b)$$

$$\int_a^b (u'\varphi' + u\,\varphi) = \int_a^b f\varphi \qquad \forall \varphi \in H_0^1(a,b)$$

We can rewrite this as

$$(u,\varphi)_{H^1} = (f,\varphi)_{L^2}$$

You can then apply **Lax-Milgram** (1.7.4), prove the hypotheses on the bilinear form and conclude that $\exists! u$.

Poincarè inequality (6.2.9)

Let $I = (a,b)$ bounded then:

$$||u||_{L^2(I)} \leq (b-a)\,||u'||_{L^2(a,b)} \qquad \forall u \in H_0^1(I)$$

As a consequence the map $u \to ||u'||_{L^2}$ defines a norm in $H_0^1(I)$ which is equivalent to the norm of $H^1(I)$

Proof

It's better if you use the books proof.

Proof It sufficies to prove (PI) $\forall u \in \mathcal{D}(I)$ (by density)

$\forall x \in [a,b] \qquad u(x) = \int_a^x u'(t)\, dt$ fundamental theorem of calculus $(u(a)=0)$

$$\Rightarrow |u(x)| \le \int_a^x |u'(t)|\, dt \le \int_a^b |u'(t)|\, dt \underset{\text{Holder}}{\le} \sqrt{b-a}\, \|u'\|_{L^2(a,b)}$$

$$\Rightarrow \|u\|_{L^\infty(a,b)} \le \|u'\|_{L^1(a,b)} \le \sqrt{b-a}\, \|u'\|_{L^2(a,b)}$$

$$\|u\|^2_{L^2(a,b)} = \int_a^b u(t)^2\, dt \underset{\text{Holder inequality}}{\le} (b-a)\, \|u\|^2_{L^\infty(a,b)} \le (b-a)^2\, \|u'\|^2_{L^2(a,b)}$$

$$\Rightarrow \|u\|_{L^2(a,b)} \le (b-a)\, \|u'\|^2_{L^2(a,b)}$$ Poincaré inequality

$$\|u'\|^2_{L^2(a,b)} \le \|u'\|^2_{L^2(a,b)} + \|u\|^2_{L^2(a,b)} = \|u\|^2_{H^1(a,b)} \le \|u'\|^2_{L^2(a,b)} + (b-a)^2\, \|u'\|^2_{L^2(a,b)}$$

$$= [1 + (b-a)^2]\, \|u'\|^2_{L^2(a,b)}$$

The two norms are equivalent

$$\begin{cases} u' \equiv 0 \\ u(a) = u(b) = 0 \end{cases} \Rightarrow u = 0 \rightarrow$$ the annichilation property is satisfied
↳ it is a norm

In $H_0^1(I)$ we may take $\|u\|_{H_0^1} = \|u'\|_2$ as a norm and $(u,v) \mapsto \int_a^b u'v'$ as a scalar product

$\square$

Let H_1 and H_2 be Hilbert spaces such that:

$$H_1 \subset H_2 \qquad (H^1 \subset L^2,\ H_0^1 \subset L^2)$$

Any Hilbert space is the dual of itself:

$$H_1' \approx H_1$$
$$H_2' \approx H_2$$

Therefore if $H_1 \subset H_2$ and $H_2' \subset H_1'$ (since a smaller Hilbert space will have a larger number of linear and continuous functionals, this is taken straight from notes I don't condone this behaviour):

$$H_2 \subset H_1$$

But this is absurd, look at the hypothesis !
There's clearly a mistake somewhere and that mistake lies in the assumption that we can use both isomorphisms at the same time, which isn't true due to the **Riesz representation theorem** (1.5.5).

Hilbert (Gelfand) triple (Bottom of 6.2.18)

The following is true:

$$H_1 \subset H_2 \underbrace{\approx}_{(1)} H_2' \subset H_1'$$

$$H_0^1(I) \subset L^2(I) \underbrace{\subset}_{(2)} (H_0^1(I))' = H^{-1}(I)$$

Where:

1. Is the Riesz representation theorem (1.5.5)
2. The pivot space is the space in which we make use of the dual (in this case L^2)

Moreover if we define:

$$Lu(v) = \int_I u\, v \qquad \forall u \in L^2(I)$$

Then the map $v \to Lu(v)$ is linear and continuous $\forall v \in H_0^1(I)$

Proposition 6.2.10

Let $F \in H^{-1}(I)$ then:

$$\exists f_0, f_1 \in L^2(I)$$

Such that:

$$< F, v >= \int_I f_0 v + \int_I f_1 v \qquad \forall v \in H_0^1$$

and

$$||F||_{H^{-1}(I)} = \max\{||f_0||_{L^2}(I)\ ,\ ||f_1||_{L^2}(I)\}$$

In the case where $I = \mathbb{R}$ then substitute $H_0^1(I)$ with $H^1(I)$.

In the case where I is unbounded we can take $f_0 = 0$.

This a sort of representation for linear and continuous functionals over $H_0^1(I)$.

Now suppose that $f_0 = 0$:

$$< F, v >= \int_I f_1 v' \underbrace{=}_{IBP} - \int_I f_1' v$$

but this basically means $F = -f_1'$.

...

Not so fast bucko, this is an illegal move since $f_1 \in L^2$ so we don't know its derivative necessarily.

We, on the other hand, can say (by defining it as such) that :

$$F = -f_1'$$

Implies that F has -1 derivative in L^2.

Dirac delta

8o. $\delta \in H^{-1}$ $\quad \forall c \in [a,b]$ $\quad \langle \delta_c , u \rangle = u(c)$ $\quad \delta$ is a linear and continuous functional
Heaviside function belongs to $L^2(I)$, I bounded, δ is its derivative

The $\mathbb{R}^n$ case

We'll take a domain Ω in $\mathbb{R}^n$ to be an open set, for example $\partial\Omega$.

Example

$$\Omega = \{(x,y) \in [0,1]^2 \ , \ x,y \in \mathbb{Q}\}$$

$$\partial\Omega = [0,1]^2$$

Our main assumptions will be that Ω has to be an open set and $\partial\Omega$ smooth. Let's say we want $\partial\Omega \in C^1$, it can be constructed as the union of locally C^1 functions. (3.1.2)

Now we want to define $H^1(\Omega)$ (we'll worry about the boundary later on).

Weak derivative in $\mathbb{R}^n$

The i-th weak derivative $w = \frac{\partial u}{\partial x_i}$ of u is such that:

$$\int_\Omega u \ \frac{\partial v}{\partial x_i} = \int_\Omega w \ v \qquad \forall v \in \mathbb{D}(I)$$

Sobolev space (6.,2.11)

Let $\Omega \in \mathbb{R}^n$ be an open set, the Sobolev space H^1 is defined as:

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \ ; \ \frac{\partial u}{\partial x_i} \in L^2(\Omega) \qquad \forall i = 1, .., n \right\}$$

(6.2.12)
Given $u \in H^2(\Omega)$:

We take

$$H^1(\Omega) \approx L^2(\Omega)^{n+1}$$

Meaning we take these spaces to be isomorphic.

We define the gradient as :

$$\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x_i} \\ .. \\ .. \end{bmatrix} \in L^2(\Omega)^n$$

We define the norm to be:

$$|\nabla u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)$$

Taking derivative in the classical sense:

$$\left(u \in C^1(\Omega) \bigcap L^2(\Omega)\right) \wedge \left(u \frac{\partial u}{\partial x_i} \in L^2(\Omega)\right) \implies u \in H^1(\Omega)$$

Moreover the classical partial derivatives coincides with the weak ones, additionally if Ω is bounded:

$$\implies C^1(\overline{\Omega}) \subset H^1(\Omega)$$

We finally define the bilinear form on H^1 to be:

$$(u, v)_{H^1} = \int_\Omega (u \cdot v + \nabla u \times \nabla v)$$

Where the first is a product between two scalar functions and the second is a scalar product (sum of $n+1$ terms).
The former will be sometimes omitted.

We can define a norm (6.2.13) therefore

H^1 separability in $\mathbb{R}^n$ (6.2.14)

H^1 is separable since we can:

- Define a scalar product such as:

$$(u, v)_{H^1}$$

- We can define a norm
- The space is complete

Refer to the book for an actual proof.

Closure to H^1 (6.2.15)

$$H^1_0(\Omega) = \overline{\mathbb{D}(\Omega)}^{H^1(\Omega)} \subset H^1(\Omega)$$

The book goes in depth on how actually it is defined..

Example

Es. $f(x) = \frac{1}{|x|^2}$ in $\mathbb{R}^n \setminus B_1$ $\quad f \in L^p(\mathbb{R}^n \setminus B_1) \iff \int_{\mathbb{R}^n \setminus B_1} |f(x)|^p \, dx < +\infty$

We can take advantage of polar coordinates

$$\int_{\mathbb{R}^n \setminus B_1} |f(x)|^p \, dx = c_n \int_1^\infty \frac{\rho^{n-1}}{\rho^{2p}} \, d\rho \qquad n-1-2p < -1 \implies p > \frac{n}{2}$$

Example (6.2.16)

Basically just show that the function and its gradient are $L^2(\Omega)$ and you're good to go. This is also interesting because it's a counter example to the notion that (in $n \geq 2$) a function only need be limited to be in the respective Sobolev space.

Ex. $n = 2$ $\quad B_{1/2} = \{|x| < \frac{1}{2}\}$ $\quad f(x) = \log|\log|x|| \notin L^\infty$, $\notin C^0$ $\quad B_{1/2} \subset \mathbb{R}^2$

Show that $f \in H^1(B_{1/2})$

$|x| = \rho \qquad \int_0^{1/2} \rho \log^2|\log|\rho|| \, d\rho < +\infty \iff f \in L^2(B_{1/2})$

$|\nabla f(x)| = |f'(\rho)| \qquad f(\rho) = \log|\log(\rho)| \qquad f'(\rho) = \frac{1}{\rho \log \rho}$

$\nabla f \in L^2(B_{1/2}) \iff \int_0^{1/2} \frac{\rho}{\rho^2 \log^2 \rho} \, d\rho < +\infty$

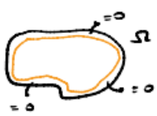
$\implies f \in H^1(B_{1/2})$ but $f \notin L^\infty(B_{1/2})$ $\qquad H^1(B_{1/2}) \not\subset L^\infty(B_{1/2})$ if $n \geq 2$ $\quad$ contrary example

$\boxed{H^1_0(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}}$

No

I take a function that is 0 at the boundary

If $u \in C^0(\bar{\Omega})$ then $u \in H^1_0(\Omega) \iff u = 0$ on $\partial\Omega$



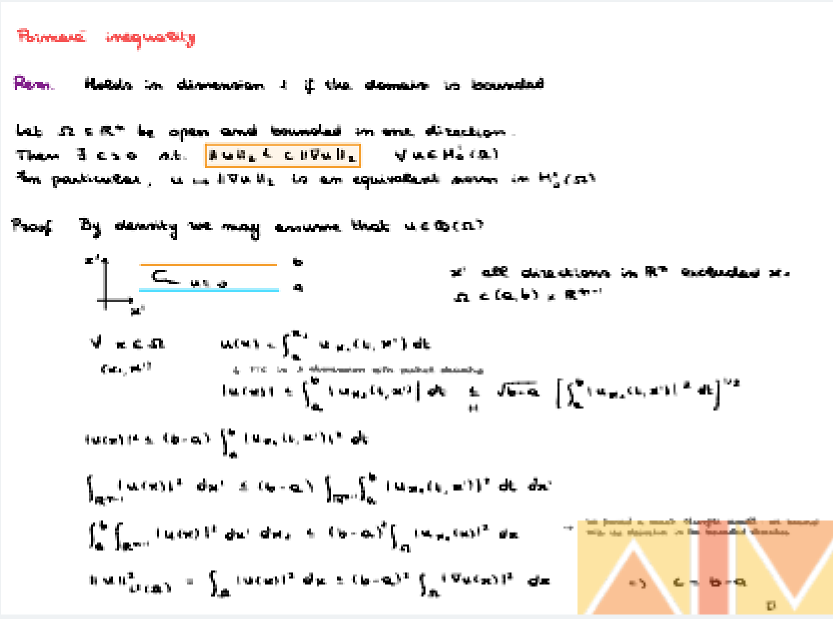
If the function has compact support

Poincarè inequality (6.2.18)

Assume that Ω is bouned. Then there exists a constant $C(\Omega)$ such that:

$$\|u\|_{L^2} \le C\,\|\nabla u\|_{L^2} \qquad \forall u \in H_0^1(\Omega)$$

Moreover the map $u \to \|u\|_{L^2}$ us a norm in $H_0^1(\Omega)$ which is equivalent to the norm $\|u\|_{H^1}$.


Poincaré inequality

Rem. Holds in dimension 1 if the domain is bounded

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded in one direction.
Then $\exists\, c > 0$ s.t. $\boxed{\|u\|_2 \le c\,\|\nabla u\|_2}$ $\forall u \in H_0^1(\Omega)$
In particular, $u \mapsto \|\nabla u\|_2$ is an equivalent norm in $H_0^1(\Omega)$

Proof By density we may assume that $u \in \mathcal{D}(\Omega)$

Professor then goes on to define the relevant Hilibert triple

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$$

Where the second continuous embedding is to be meant as follows:
Each function $u \in L^2(\Omega)$ is identified with the linear functional $I_u \in H^{-1}(\Omega)$ defined as :

$$< I_u, v > \;\; := \int_\Omega u\, v \qquad \forall v \in H_0^1(\Omega)$$

Essentially we can map each L^2 function to a functional from H^1 to $\mathbb{R}$.

Note that we denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$.

Proposition (6.2.19)

Let $\Omega \subset \mathbb{R}^n$ be open and let $F \in H^{-1}(\Omega)$, then $\exists \{f_i\}_{i=0}^n \in L''(\Omega)$ such that:

$$< F, v > \;\; = \int_\Omega f_0\, v + \sum_{i=1}^n f_i \frac{\partial u}{\partial x_i} \qquad \forall v \in H_0^1(\Omega)$$

And:

$$\|F\|_{H^{-1}} = \max_{i=0,\dots,n} \|f_i\|_L^2$$

Moreover, if Ω is bounded, we can take $f_0 = 0$

Remark

Note that in this case $v \in L^2(\Omega)$ because $v \in H_0^1(\Omega)$

Proof

There's a proof in the book but here's what the prof wrote:

Proof If $f_0 = 0$ $\quad \langle F, v \rangle = \int_\Omega f \cdot \nabla v \qquad f = (f_1, f_2, \dots, f_n) \in L^2(\Omega)^n$

For smooth f, $\quad \int_\Omega f \cdot \nabla v = -\int_\Omega v \operatorname{div} f + \cancel{\int_{\partial\Omega} v (f \cdot \nu)} \qquad \nu \to$ normal

$\Rightarrow$ "$F = -\operatorname{div} f$" $\quad$ the linear continuous functional in H^{-1} is the divergent of a function in L^2

$\square$

$H^1(\Omega) \not\subset L^\infty(\Omega) \quad$ if $n \geq 2$
$H^1(\Omega) \subset L^2(\Omega) \quad \forall n$

Sobolev embedding theorem (6.2.20 - 6.2.21)

Let $\Omega \subset \mathbb{R}^n$ be an open domain with $\partial\Omega \in Lip$ and $n \geq 2$. Then:

$$H^1(\Omega) \subset L^p(\Omega) \qquad \begin{cases} \forall 2 \leq p < \infty & if\ n = 2 \\ \forall 2 \leq p < \frac{2n}{n-2} & if\ n = 3 \end{cases}$$

In addition to the above, if Ω is bounded, then the embeddings become compact:

the following continuous embeddings:
(i) if $n = 2$ then $H^1(\Omega) \subset\subset L^q(\Omega)$, for every $1 \leq q < \infty$;
(ii) if $n \geq 3$ then $H^1(\Omega) \subset\subset L^q(\Omega)$, for every $1 \leq q < 2^* = \frac{2n}{n-2}$.

Proof. See [1, Theorem 6.2]. $\square$

Remark

The following is called the **Critical Sobolev exponent**:

$$2^* = \frac{2n}{n-2}$$

A sequence that converges in the $H^1(\Omega)$ sense implies convergence in a certain L^p sense, formally:

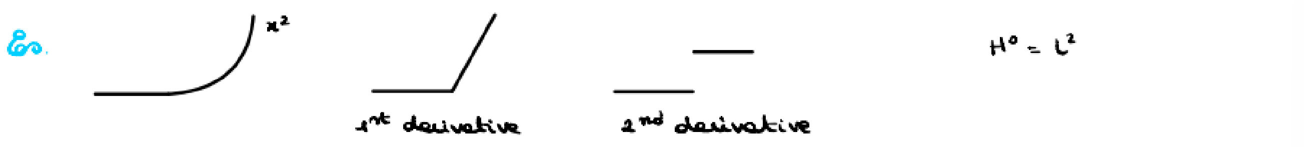
$$u_n \xrightarrow{H^1(\Omega)} u \implies u_n \xrightarrow{L^p} u$$

> **Remark**
>
> The functions inside $H^1(\Omega)$ are defined up to a negligible set.

The H^k spaces

We basically take 6.2.11 and apply it a bunch of times to define higher order weak derivatives.

Note that in the image below the prof *formally* sets $H^0 = L^2$ (6.3.4)



> **Multi-index notation**
>
> We call a multi-index a vector:
>
> $$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$$
>
> With norm:
>
> $$|\alpha| = \sum_{i=1}^{m} \alpha_i$$
>
> We'll use this to define partial derivatives kinda like this:
>
> $$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1}, \dots, \partial x_n^{\alpha_n}}$$

> **Definition 6.3.1**
>
> Let $u \in L^1_{\text{loc}}(\Omega)$ and $\Omega \subseteq \mathbb{R}^n$ be an open set. Given a multi-index α we'll say that u admits a weak derivative $D^\alpha u$ if there exists $g \in L^1_{\text{loc}}(\Omega)$ such that:
>
> $$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g\varphi \qquad \forall \varphi \in \mathcal{D}(\Omega)$$

> **Definition and Separability Theorem (6.3.2 - 6.3.5)**
>
> Let Ω be as above and $k \in \mathbb{N}$. The $H^k(\Omega)$ is defined by:
>
> $$H^k(\Omega) = \{u \in L^2(\Omega), \qquad D^\alpha u \in L^2(\Omega) \qquad \forall |\alpha| \le k\} \qquad \forall k \in \mathbb{N}$$

We take $H^k(\Omega)$ to be a separable Hilbert space with scalar product:

$$(u,v)_{H^k(\Omega)} = \int_\Omega \left(uv + \sum_{1\le|\alpha|\le k} D^\alpha u\, D^\alpha v \right)$$

With induced norm:

$$\|u\|_{H^k(\Omega)} = \left(\int_{\Omega \wedge 0\le|\alpha|\le k} \sum |D^\alpha u|^2 \right)^{\frac{1}{2}}$$

Proof

Check the book for a more detailed proof, here's what the prof wrote,

H^k is a separable Hilbert space with scalar product

$$(u,v)_{H^k} = \int_\Omega (uv + \sum_{1\le|\alpha|\le k} D^\alpha u\, D^\alpha v)$$

- it has a scalar product
- it is complete (use a Cauchy sequence)
- it's separable because its L^2

$$\|u\|_{H^k} = \left(\int_\Omega \sum_{0\le|\alpha|\le k} |D^\alpha u|^2\right)^{1/2}$$ induced norm

Separable Banach spaces with no scalar product. (6.3.8)

$$\boxed{H^{k+1}(\Omega) = \{u \in H^k(\Omega),\ \nabla u \in H^k(\Omega)\}}$$

$$\boxed{W^{k,p}(\Omega) = \{u \in L^p(\Omega);\ D^\alpha u \in L^p(\Omega)\ \ \forall |\alpha|\le k\} \quad \forall k \in \mathbb{N}}$$

separable Banach spaces, with no scalar product

If $u \in L^p(\Omega)$ and $D^\alpha u \in L^p(\Omega)$ $\forall |\alpha| = k$ $\Rightarrow$ $u \in W^{k,p}(\Omega)$

H^k defined inductively

Remark (6.3.4)

We define $H^k(\Omega)$ inductively. Check the book.

Fourier definition for higher order Sobolev spaces(6.3.5 - 6.3.6)

We can define $H^k(\mathbb{R}^n, \mathbb{C})$ in a more straightforward way using the Fourier transform.

$$H^k(\mathbb{R}^n, \mathbb{C}) = \{u \in L^2(\mathbb{R}^n, \mathbb{C}); \quad (1+|\xi|^2)^{\frac{k}{2}}\, \hat{u}(\xi) \in L^2(\mathbb{R}^n, \mathbb{C})\}$$

In this construction $H^0 = L^2$ because $(1+|\xi|^2)^{\frac{k}{2}} = 1$ if $k = 0$, we need, however, to define a scalar product to go further.

$$(u,v)_{H^k} = \int_{\mathbb{R}^n} (1+|\xi|^2)^k\, \hat{u}(\xi)\, \hat{v}(\xi)\ d\xi \quad \to \text{norm}$$

$$(1+|\xi|^2)^{k/2} \quad \to \text{weight function}$$

Note that k doesn't *have* to be an integer in this construction, it just has to be $k = s \geq 1$. We then end up with what are called **Non Local PDE's**

Non-integer H^k

As previously mentioned, we'll take $s \geq 0$.

$$H^s(\mathbb{R}^n, \mathbb{C}) = \left\{u \in L^2(\mathbb{R}^n, \mathbb{C})\ , \quad (1+|\xi|^2)^{\frac{k}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n, \mathbb{C})\right\}$$

Endowed with the scalar product

$$(u,v)_{H^s} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s\ \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\ d\xi$$

for every $u, v \in H^s(\mathbb{R}^n, \mathbb{C})$.

(6.4.1)

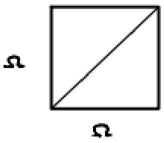
We can extend this construction to a general domain Ω.

Theorem (6.4.5)

Def let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $s>0$; if $s \in (k, k+1)$ we set $s = k+t$ with $t \in (0,1)$ and we define

$$\boxed{H^s(\Omega) = \left\{ u \in H^k(\Omega);\ \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2t}}\, dx\, dy < \infty \quad \forall\, |\alpha| = k \right\}}$$

$s = \frac{1}{2} \implies \alpha = 0 \qquad \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+1}}\, dx\, dy < +\infty$



↳ it's a property of the incremental ratio

$$(u,v)_{H^s(\Omega)} = (u,v)_{H^k(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha v(x) - D^\alpha v(y))}{|x-y|^{n+2t}}\, dx\, dy$$

$H^s(\Omega)$ is an Hilbert space, because it becomes a Banach space if we define the norm from the scalar product
$H^s(\Omega)$ is separable because it is the cartesian product of separable Hilbert spaces

H_0^s spaces and Trace operators

Definition (6.5.1)

For every $s \geq 0$:

$$H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$$

A special case is the following:

$$\Omega = \mathbb{R}^n \implies H_0^s(\Omega) = H^s(\Omega)$$

Otherwise

If $\Omega \neq \mathbb{R}^n$ $\to L^2(\Omega) = H^0(\Omega) \qquad \overline{\mathcal{D}(\Omega)}^{L^2(\Omega)} = L^2(\Omega)$
$\searrow H^1(\Omega) \qquad \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega) \subsetneq H^1(\Omega)$

What if $s \in [0,1]$?
Let's put ourselves in the case where $\Omega \neq \mathbb{R}^n$

What happens with $H^s(\Omega)$ $s \in [0,1]$?
$H_0^s(\Omega) = H^s(\Omega) \qquad \forall s \in [0, \frac{1}{2}] \qquad \Omega \neq \mathbb{R}^n$
$H_0^s(\Omega) \subsetneq H^s(\Omega) \qquad \forall s > \frac{1}{2}$

Read the book for a better understanding of what's going on in the background.
Essentially $\mathcal{D}(\Omega)$ is dense in $H^s(\Omega)$ if and only if $s \leq \frac{1}{2}$. Meaning the "closure" and the space are the same. (What's written in the pic).
The $\mathbb{R}^n$ case is also explained.

A problem now arises, how do we define the value of a function $u \in H^s(\Omega)$ on the boundary ? Remember these functions are defined up to a negligible set.

How to define $u|_{\partial\Omega} = g \neq 0$?   Which is the restriction on the boundary ?

If $s \in [0, \frac{1}{2}]$ it's impossible to define the restriction even if $g = 0$
If $s > \frac{1}{2}$ we can try with an approximation

We'll solve it this way

Trace operator (6.5.2 - 6.5.4)

Assume $\partial\Omega \in C^\infty$, now note that $C^\infty(\overline{\Omega})$ is dense in $H^s(\Omega)$ $\forall s \geq 0$