NAPDE - Numerical Analysis for Partial Differential Equations - A. Paola

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1. Disclaimer and preface

These notes were taken during AY 2023-2024, the lecturer is prof. Antonietti Paola.
For any questions or errors you can reach out to me here.

Part 1

1. Boundary-value problems

In general, these types of problems are written as: ”Some operator applied to some function u set to some value”.

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ \text{BC} & \text{in } \partial\Omega \end{cases}$$

Some specific examples include the diffusion problem, where:

$$\mathcal{L}u = -\text{div}(\mu(x)\nabla u)$$

Where $\mu(x)$ is the diffusion coefficient which is positive almost everywhere (from here on out a.e.).
There's also the ADR problem where:

$$\mathcal{L}u = -\text{div}(\mu(x)\nabla u) + \underline{b} \cdot \nabla u + \sigma u$$

Where $\mu(x)$ is as above, $\underline{b} \in \mathbb{R}^d$ and $\sigma = \sigma(x) \geq 0$.
In PDE courses you're usually taught to think of these as at least $L^2(\Omega)$ functions but for the purposes of this course
we'll relax the constraints and assume them to be $L^\infty(\Omega)$ to simplify the analysis.
Here, have some notation, this is what we're going to assume:

$$\begin{cases} \mu(x) \in L^\infty(\Omega) \\ \sigma(x) \in L^\infty(\Omega) \\ \underline{b} \in [L^\infty(\Omega)]^d \\ f \in L^2(\Omega) \end{cases}$$

1.1. Weak formulation

This is a very general form which isn’t very useful for numerical analysis, it’s better to use some form of weak formulation, this will help us find an integral representation of the problem and derive numerical models from that.
Let's start quick and dirty, ignore the legality of steps, the regularity of v and just try to come up with an integral form, we'll worry about conditions later !

$$\int_\Omega [-\text{div}(\mu(x)\nabla u) + \underline{b} \cdot \nabla u + \sigma u \cdot v] = \int_\Omega [f \cdot v]$$

Now, integrating by parts we get:

$$\int_\Omega \mu(x)\nabla u \cdot nv - \underbrace{\int_{\partial\Omega} \mu\nabla u \cdot nv}_{\text{notice the dominion}} + \int_\Omega \underbrace{b}\nabla uv + \int_\Omega \sigma uv = \int_\Omega fv \qquad \forall v$$