31^{st} August, 2020

1. Exercise 1

Consider the 1D heat equation in the domain $(0, d)$

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), & 0 < x < d, 0 < t \leq T, \\ u(t,0) = 0, & 0 < t \leq T, \\ \frac{\partial u(t,d)}{\partial x} = g(t), & 0 < t \leq T, \\ u(0,x) = u_0(x), & 0 < x < d, \end{cases} \tag{1}$$

for suitable functions f, g, and u_0

1.1.

Question

Introduce the **semi-discrete Galerkin** Finite Element formulation of (1), suitably defining the introduced function space(s).

Solution

We start by multiplying for each $t > 0$ the differential equation by a test function $v = v(\mathbf{x})$ and integrating on $(0, d)$. We set the set to which the test function belongs as $V = H^1_{\Gamma_D}(\Omega)$ and for each $t > 0$ we aim to find $u(t) \in V$ such that

$$\int_\Omega \frac{\partial u(t)}{\partial t} v \, d\Omega + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f(t) v d\Omega \quad \forall v \in V$$

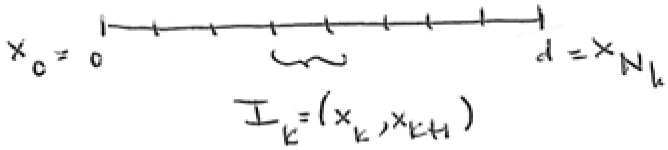
where $\Omega = (0, d)$.
At this point we discretize the equation into (note how the rightmost term appears as a result of integration by parts)

$$\int_0^d \frac{\partial u_h(t)}{\partial t} v_h \text{ dx} + \int_0^d \frac{\partial u_h(t)}{\partial x} \frac{\partial v_h(t)}{\partial x} \text{ dx} = \int_0^d f \, v_h \, \text{dx} + \\ + g \, v_h \quad \forall v_h \in V_h \tag{2}$$

where we've set $u_h(t = 0) = u_{0_h} \in V_h$ and V_h is the space of continuous on $(0, d)$ piece-wise polynomials of degree $\leq r$, in short we seek a solution $u_h(t) \in V_h$ where

$$V_h = \left\{ v_h \in C^0(0, d) \colon v_h(0) = 0 \, , \, v_{h|I_k} \in \mathbb{P}^r \quad \forall k = 0, \ldots, N_h - 1 \right\}$$

where each I_k is a piece of the interval



Discretization of the interval

1.2.

Question

Write the θ-method for the full discretization of (1)

Solution

The θ-method generalizes numerical approximation schemes for time discretization of space-discretized schemes where $0 \leq \theta \leq 1$ taking (2) we discretize the time derivative as a simple difference normalize by the timestep, in essence

$$\frac{1}{\Delta t}\int_0^d (u_h^{n+1} - u_h^n)v_h \,\mathrm{dx} + \int_0^d \left(\theta u_{h_x}^{n+1} + (1-\theta)u_{h_x}^n\right) v_h \,\mathrm{dx} = \int_0^d (\theta f^{n+1} + (1-\theta)f^n)v_h \,\mathrm{dx} +$$

$$+(\theta g^{n+1} + (1-\theta)g^n)v_h \qquad \forall v_h \in V_h \;\wedge\; n = 0, 1, \ldots$$

where each timestep is $n\Delta t$ long.
For $\theta = 0$ and $\theta = 1$ we obtain respectively **forward** and **backward Euler** methods which are accurate to order one with respect Δt, while for $\theta = \frac{1}{2}$ we obtain the **Crank-Nicolson** which is of second order in Δt, more precisely $\theta = \frac{1}{2}$ is the only value for which we obtain a second-order method,

1.3.

Question

Discuss the numerical stability of the scheme, depending upon the value of θ (no proof needed).

Solution

Given that $f \neq 0$.

- If $\theta < \frac{1}{2}$ it can be shown that the θ-method is absolutely stable only for

  $$\Delta t \leq \frac{2}{(1-2\theta)\lambda_h^{N_h}}$$

  where $\lambda_h^{N_h}$ is the largest eigenvalue of the generalized eigenvalue problem $A\mathbf{x} = \lambda M\mathbf{x}$. It can be shown that $\lambda_h^{N_h} \lesssim h^{-2}$ therefore

  $$\Delta t \approxeq \frac{2h^2}{(1-2\sigma)}$$

- If $\theta \geq \frac{1}{2}$ then it is unconditionally absolutely stable.

1.4.

Question

Prove the stability properties of the backward Euler discretization (assume for simplicity $g = 0$).

Solution

Backward Euler is equivalent to applying $\theta = 1$, therefore we take (2), apply $g = 0$ and ,finally, impose the test function $v_h = u_h^{n+1}$, then we get

$$\underbrace{\frac{1}{\Delta t}\int_0^d \left(u_h^{n+1} - u_h^n\right)u_h^{n+1}dx}_{(1)} + \underbrace{\int_0^d \left(u_{h_x}^{n+1}\right)^2 dx}_{(2)} = \underbrace{\int_0^d f^{n+1}u_h^{n+1}}_{(3)}$$

where $\Omega = (0, d)$.

1. We apply $(a - b, a) \geq \frac{1}{2}(\|a\|^2 - \|b\|^2) \quad \forall a, b$ to get

$$(1) \geq \frac{1}{2\Delta t}\left(\|u_h^{k+1}\|_{L^2}^2 - \|u_h^k\|_{L^2}^2\right)$$

2. We use the coercivity of $a(\cdot, \cdot)$ to get

$$(2) \geq \alpha\|u_h^{k+1}\|_{L^2}^2$$

3. We apply Cauchy-Schwartz $\rightarrow$ Poincare $\rightarrow$ Young

$$\begin{aligned}\left|\int_0^d f^{n+1}u_h^{n+1}\right| &\underbrace{\leq}_{\text{C-S}} \left\|f^{n+1}\right\|_{L^2}\left\|u_h^{n+1}\right\|_{L^2} \\ &\underbrace{\leq}_{\text{Poincare}} \left\|f^{n+1}\right\|_{L^2} C_p \left\|u_{h_x}^{n+1}\right\|_{L^2} \\ &\underbrace{\leq}_{\text{Young}} \frac{1}{2}\left(\left\|f^{n+1}\right\|C_p\right)^2 + \frac{1}{2}\left\|u_{h_x}^{n+1}\right\|_{L^2}^2\end{aligned}$$

16th June, 2023

1. Problem 1

This problem isn't in the course material for AA 2023/2024, won't be in future exams.

2. Problem 2

Consider the following problem for $\Omega = (0, 1)^2$

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u) + \frac{\partial u}{\partial y} = f & \text{in } \Omega, \\ \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\epsilon > 0$, and $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ are two *suitable* functions.

2.1.

Question

Introduce the variational (weak) formulation of (2) and explain why a standard Galerkin Finite Element approximation may not be suitable for (2) when $\epsilon \ll 1$

Solution

Problem (2) is the non-conservative form of the **Advection-Diffusion-Reaction** equation where $\mathbf{b} = 1$ and $\sigma = 0$, we multiply both sides of the equation by a test function and integrate over the domain.
The problem then becomes finding $u \in V = H_0^1(\Omega)$ such that

$$a(u, v) = F(v) \quad \forall v \in V$$

where the bilinear form and the linear forms are defined as

$$a(u, v) = \int_\Omega \varepsilon \nabla u \cdot \nabla v d\Omega + \int_\Omega \frac{\partial u}{\partial y} v d\Omega,$$

$$F(v) = \int_\Omega f v$$

In cases where $\epsilon \ll 1$ the Peclet number becomes too big and therefore **GLS** or **SUPG** methods are needed to stabilize the method.

2.2.

Question

Introduce a Finite Element **Galerkin-Least-Squares (GLS)** approximation of (2) for $g = 0$

Solution