

APDE - Advanced Partial Differential Equations - F. Gazzola

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1. Disclaimers and preface

These notes were taken during AY 2023/2024 using older material, your mileage may vary. They're meant to accompany the lectures and in no way aim to substitute a professor yapping away at an iPad 30m away.

Any reference of form $(x.x.x)$ where $x \in \mathbb{N}$ is a reference to a Theorem/Definition/etc... in the professors own book:

****[Elements of Advanced Mathematical Analysis for Physics and Engineering](#)**

I'm using the September 2013 version.

These notes are based on the work done by students Ravizza and Mescolini, you can access their notes by logging on the [AIM website](#) and checking out [Portale appunti](#). This would NOT have been possible without their notes..

For any questions/mistakes you can reach me [here](#).

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2. Sobolev spaces and initial derivation for discrete domains

C^* are banach spaces but not *Hilbert* spaces (i.e. we cannot use Lax-Milgram).

Example

Second ordinary differential equation:

$$\begin{cases} -u'' + u = f \\ u(a) = u(b) = 0 \end{cases}$$

What happens if f is not continuous ? Let's try a weak solution.

$$\begin{aligned} -u''\varphi + u\varphi &= f\varphi \quad \forall \varphi \in \mathcal{D}(a,b) \\ \mathcal{D}(a,b) &= C_c^\infty(a,b) \end{aligned}$$

We then integrate

$$\begin{aligned} -\int_a^b u''\varphi + \int_a^b u\varphi &= \int_a^b f\varphi \\ \int_a^b (-u''\varphi + u\varphi) &= \int_a^b f\varphi \end{aligned}$$

This integral obviously has sense if and only if the integrands are L^1 :

- To have $u\varphi \in L^1$ we need both of those functions to be L^2
- To have $u'\varphi' \in L^1$ we need both of those derivatives to be in L^1
- We also need $f \in L^2$

In order to be talking sense (i.e. having a legal formulation) this is enough. We will end up with a weak formulation satisfying these properties.

- $(u, u', f) \in L^2$

Keep in mind that L^2 contains discontinuous functions.

Weak derivative

We say that $v = u'$ in the weak sense (distributional sense) if:

$$\int_I u\varphi' = - \int_I u\varphi \quad \forall \varphi \in \mathcal{D}(I)$$

The simplest Sobolev space

We define the H^1 space as the following

$$H^1(I) = \{u \in L^2(I), u' \in L^2(I) \text{ in a weak sense}\}$$

H^1 is the simplest one dimensional *sobolev* space. We are going to change the dimensionality of the space, the order of the derivative and the summability of the space.

Can we strengthen the assumptions on the hypotheses on our solution if we strengthen those on $f \in L^2$? Answer is yes but we'll get there.

Examples of weak derivatives

First example:

$C^1(\bar{I}) \subset H^1(I)$: The weak derivative coincides with the classic one.

Second example:

Let $f(x) = |x|$ and $I = (-1, 1)$

Conjecture: We want to prove that the derivative $f'(x)$, defined as below, belongs to L^2 .

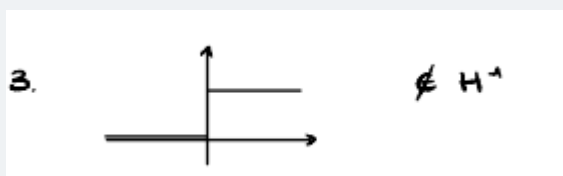
$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases}$$

Let's try it using the definition of weak derivative:

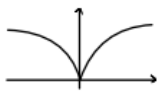
$$\begin{aligned} \int_{-1}^1 |x| \varphi'(x) dx &= \int_{-1}^0 \varphi(x) dx - \int_0^1 \varphi(x) dx \quad \forall \varphi \in \mathcal{D}(-1, 1) \\ &\quad \hookrightarrow \text{to check if it is the weak derivative} \\ - \int_{-1}^0 x \varphi'(x) dx + \int_0^1 x \varphi'(x) dx &= \int_{-1}^0 \varphi(x) dx - [x \varphi(x)]_{-1}^0 - \int_0^1 \varphi(x) dx + [x \varphi(x)]_0^1 \\ &= \int_{-1}^0 \varphi(x) dx - \int_0^1 \varphi(x) dx \end{aligned}$$

We used the definition of weak derivative, written our initial integral in it's form, therefore proven that it is the weak derivative.

Third example:



Fourth example:

4. 

$|x|^\alpha$ For which α the function is in H^1 ?
 \hookrightarrow problem of integrability if $\alpha-1 < 0$
 $D \approx |x|^{\alpha-1} \Rightarrow \int \frac{1}{|x|^{1-\alpha/2}} < \infty \quad 2(1-\alpha) < 1 \Rightarrow \alpha > \frac{1}{2}$
 If $\alpha > 1$ this is a H^1 function

Separability of H^1 (6.2.3)

$H^1(I)$ is a **separable** Hilbert space when endowed with the following scalar product.

$$(u, v)_{H^1} = \int_I (u'v' + uv)$$

Proof

We start by proving that it is indeed a scalar product, i.e. the following properties:

- Symmetry (obvious)
- It is a norm:

$$\|u\|_{H^1}^2 = \int_I [(u')^2 + u^2]$$

We then need to show that it is a Banach space with the following scheme:

Take a Cauchy sequence $\{u_n\}$ s.t. $\forall \varepsilon > 0 \exists N_\varepsilon, n, m > N_\varepsilon \Rightarrow \|u_n - u_m\|_{H^1}^2 < \varepsilon$
 $\Rightarrow \|u_n - u_m\|_{L^2}^2 + \|u'_n - u'_m\|_{L^2}^2 < \varepsilon$
 \swarrow 2 Cauchy sequences \checkmark
 L^2 Banach space $\Rightarrow \exists u \in L^2(I)$ s.t. $u_n \rightarrow u$ in L^2
 $\exists v \in L^2(I)$ s.t. $u'_n \rightarrow v$ in L^2
The point is: $u' = v$? \rightarrow because the limit must $\in H^1(I) \rightarrow$ if $u' = v \Rightarrow u' \in L^2 \Rightarrow u \in H^1$
 $\int u \varphi' = \lim_{n \rightarrow \infty} \int u_n \varphi' = - \lim_{n \rightarrow \infty} \int u'_n \varphi = - \int v \varphi \quad \forall \varphi \in \mathcal{D}(I) \subset L^2(I)$
 \hookrightarrow dense in $L^2(I)$ \square

All we need now is proving that the Hilbert space we constructed is indeed **separable**, we can use H^1 own isomorphism to L^2 which is separable in its own right, then construct a linear map between the two:

$$Lu = (u, u')$$

Map between Sobolev and Continuous spaces (6.2.5)

Every function in H^1 can be represented by a continuous function, formally:

$$H^1(I) \subset C^0(\bar{I})$$

This is interesting because H^1 also contains discontinuous functions, then how is it a subset of C^0 ??
This is of course not what the theorem is saying, more specifically

Proof

"Proof" $u \in H^1(I)$ then $u' \in L^2(I) \subset L^1(I)$
 \uparrow
 $|I| < \infty$
 $\int_a^x u'(t) dt$ well defined $\forall x$

 $u(x) = u(a) + \int_a^x u'(t) dt$ Fundamental theorem of calculus
 $\underbrace{u' \in L^1}_{\in L^1} \Rightarrow u \in AC$
 $u(x_m) - u(x) = \int_x^{x_m} u'(t) dt \rightarrow 0$ if $x_m \rightarrow x \Rightarrow u$ is continuous \square

Definition (6.2.7)

We define the closure of $\mathcal{D}(I)$ with respect to the $H^1(I)$ norm as the following:

$$H_0^1(I) = \overline{\mathcal{D}(I)}^{H^1(I)}$$

Recall that $\mathcal{D}(\Omega)$ is the space of **smooth** functions with compact support over whatever Ω is.

Essentially we're defining the space of $H^1(I)$ functions that vanish at the boundary of I .

Note that the closure with respect to the L^2 norm has special properties, specifically:

$$\overline{\mathcal{D}(I)}^{L^2(I)} = L^2(I)$$

This is because $\mathcal{D}(I)$ is dense in $L^2(I)$, meaning that the space of *smooth enough* functions with a compact support within the boundaries of I is dense in L^2 . If you think about it this makes sense since these functions would be L^∞ (i think...).

Remark (6.2.7)

If and only if $I \neq \mathbb{R}$

$$H_0^1(I) \subsetneq H^1(I)$$

Definition

A weak solution satisfies

$$u \in H_0^1(a, b)$$

$$\int_a^b (u' \varphi' + u \varphi) = \int_a^b f \varphi \quad \forall \varphi \in H_0^1(a, b)$$

We can rewrite this as

$$(u, \varphi)_{H^1} = (f, \varphi)_{L^2}$$

You can then apply **Lax-Milgram** (1.7.4), prove the hypotheses on the bilinear form and conclude that $\exists! u$.

Poincarè inequality (6.2.9)

Let $I = (a, b)$ bounded then:

$$\|u\|_{L^2(I)} \leq (b-a) \|u'\|_{L^2(a,b)} \quad \forall u \in H_0^1(I)$$

As a consequence the map $u \rightarrow \|u'\|_{L^2}$ defines a norm in $H_0^1(I)$ which is equivalent to the norm of $H^1(I)$

Proof

It's better if you use the books proof.

Proof It suffices to prove (PI) $\forall u \in \mathcal{D}(I)$ (by density)

$$\forall x \in [a, b] \quad u(x) = \int_a^x u'(t) dt \quad \text{fundamental theorem of calculus}$$

$$\Rightarrow |u(x)| \leq \int_a^x |u'(t)| dt \leq \int_a^b |u'(t)| dt \stackrel{\text{Hölder}}{\leq} \sqrt{b-a} \|u'\|_{L^2(a,b)}$$

$$\Rightarrow \|u\|_{L^\infty(a,b)} \leq \|u'\|_{L^2(a,b)} \leq \sqrt{b-a} \|u'\|_{L^2(a,b)}$$

$$\|u\|_{L^2(a,b)}^2 = \int_a^b u(t)^2 dt \stackrel{\text{Hölder inequality}}{\leq} (b-a) \|u\|_{L^\infty(a,b)}^2 \leq (b-a)^2 \|u'\|_{L^2(a,b)}^2$$

$$\Rightarrow \|u\|_{L^2(a,b)} \leq (b-a) \|u'\|_{L^2(a,b)} \quad \bullet \quad \text{Poincaré inequality}$$

$$\begin{aligned} \|u'\|_{L^2(a,b)}^2 &\leq \|u'\|_{L^2(a,b)}^2 + \|u\|_{L^2(a,b)}^2 = \|u\|_{H^1(a,b)}^2 \leq \|u'\|_{L^2(a,b)}^2 + (b-a)^2 \|u'\|_{L^2(a,b)}^2 \\ &= [1 + (b-a)^2] \|u'\|_{L^2(a,b)}^2 \end{aligned}$$

The two norms are equivalent

$$\begin{cases} u' \equiv 0 \\ u(a) = u(b) = 0 \end{cases} \Rightarrow u = 0 \rightarrow \text{the annihilation property is satisfied} \\ \text{so it is a norm}$$

On $H_0^1(I)$ we may take $\|u\|_{H_0^1} = \|u'\|_2$ as a norm and $(u, v) \mapsto \int_a^b u' v'$ as a scalar product \square

Let H_1 and H_2 be Hilbert spaces such that:

$$H_1 \subset H_2 \quad (H^1 \subset L^2, H_0^1 \subset L^2)$$

Any Hilbert space is the dual of itself:

$$H_1' \approx H_1$$

$$H_2' \approx H_2$$

Therefore if $H_1 \subset H_2$ and $H_2' \subset H_1'$ (since a smaller Hilbert space will have a larger number of linear and continuous functionals, this is taken straight from notes I don't condone this behaviour):

$$H_2 \subset H_1$$

But this is absurd, look at the hypothesis !

There's clearly a mistake somewhere and that mistake lies in the assumption that we can use both isomorphisms at the same time, which isn't true due to the **Riesz representation theorem** (1.5.5).

Hilbert (Gelfand) triple (Bottom of 6.2.18)

The following is true:

$$H_1 \subset H_2 \underbrace{\approx}_{(1)} H_2' \subset H_1'$$

$$H_0^1(I) \subset L^2(I) \underbrace{\subset}_{(2)} (H_0^1(I))' = H^{-1}(I)$$

Where:

1. Is the Riesz representation theorem (1.5.5)
2. The pivot space is the space in which we make use of the dual (in this case L^2)

Moreover if we define:

$$Lu(v) = \int_I u v \quad \forall u \in L^2(I)$$

Then the map $v \rightarrow Lu(v)$ is linear and continuous $\forall v \in H_0^1(I)$

Proposition 6.2.10

Let $F \in H^{-1}(I)$ then:

$$\exists f_0, f_1 \in L^2(I)$$

Such that:

$$\langle F, v \rangle = \int_I f_0 v + \int_I f_1 v \quad \forall v \in H_0^1$$

and

$$\|F\|_{H^{-1}(I)} = \max\{\|f_0\|_{L^2(I)}, \|f_1\|_{L^2(I)}\}$$

In the case where $I = \mathbb{R}$ then substitute $H_0^1(I)$ with $H^1(I)$.

In the case where I is unbounded we can take $f_0 = 0$.

This is a sort of representation for linear and continuous functionals over $H_0^1(I)$.

Now suppose that $f_0 = 0$:

$$\langle F, v \rangle = \int_I f_1 v' \stackrel{IBP}{=} - \int_I f_1' v$$

but this basically means $F = -f_1'$.

...

Not so fast bucko, this is an illegal move since $f_1 \in L^2$ so we don't know its derivative necessarily.

We, on the other hand, can say (by defining it as such) that :

$$F = -f_1'$$

Implies that F has -1 derivative in L^2 .

Dirac delta

Ex. $\delta \in H^{-1} \quad \forall c \in [a, b] \quad \langle \delta_c, u \rangle = u(c)$ δ is a linear and continuous functional
Heaviside function belongs to $L^2(I)$, I bounded, δ is its derivative

2.1. Derivation in higher dimensions

We'll take a domain Ω in \mathbb{R}^n to be an open set, for example $\partial\Omega$.

Example

$$\Omega = \{(x, y) \in [0, 1]^2, x, y \in \mathbb{Q}\}$$

$$\partial\Omega = [0, 1]^2$$

Our main assumptions will be that Ω has to be an open set and $\partial\Omega$ smooth. Let's say we want $\partial\Omega \in C^1$, it can be constructed as the union of locally C^1 functions. (3.1.2)

Now we want to define $H^1(\Omega)$ (we'll worry about the boundary later on).

Weak derivative in \mathbb{R}^n

The i -th weak derivative $w = \frac{\partial u}{\partial x_i}$ of u is such that:

$$\int_{\Omega} u \frac{\partial v}{\partial x_i} = \int_{\Omega} w v \quad \forall v \in \mathbb{D}(I)$$

Sobolev space (6.,2.11)

Let $\Omega \in \mathbb{R}^n$ be an open set, the Sobolev space H^1 is defined as:

$$H^1(\Omega) = \left\{ u \in L^2(\Omega); \frac{\partial u}{\partial x_i} \in L^2(\Omega) \quad \forall i = 1, \dots, n \right\}$$

(6.2.12)

Given $u \in H^2(\Omega)$:

We take

$$H^1(\Omega) \approx L^2(\Omega)^{n+1}$$

Meaning we take these spaces to be isomorphic.

We define the gradient as :

$$\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x_i} \\ \vdots \\ \vdots \end{bmatrix} \in L^2(\Omega)^n$$

We define the norm to be:

$$|\nabla u|^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2$$

Taking derivative in the classical sense:

$$\left(u \in C^1(\Omega) \cap L^2(\Omega) \right) \wedge \left(u \frac{\partial u}{\partial x_i} \in L^2(\Omega) \right) \implies u \in H^1(\Omega)$$

Moreover the classical partial derivatives coincides with the weak ones, additionally if Ω is bounded:

$$\implies C^1(\overline{\Omega}) \subset H^1(\Omega)$$

We finally define the bilinear form on H^1 to be:

$$(u, v)_{H^1} = \int_{\Omega} (u \cdot v + \nabla u \times \nabla v)$$

Where the first is a product between two scalar functions and the second is a scalar product (sum of $n + 1$ terms).

The former will be sometimes omitted.

We can define a norm (6.2.13) therefore

H^1 separability in \mathbb{R}^n (6.2.14)

H^1 is separable since we can:

- Define a scalar product such as:

$$(u, v)_{H^1}$$

- We can define a norm
- The space is complete

Refer to the book for an actual proof.

Closure to H^1 (6.2.15)

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} \subset H^1(\Omega)$$

The book goes in depth on how actually it is defined..

Example

Ex. $f(x) = \frac{1}{|x|^2}$ in $\mathbb{R}^n \setminus B_1$ $f \in L^p(\mathbb{R}^n \setminus B_1) \Leftrightarrow \int_{\mathbb{R}^n \setminus B_1} |f(x)|^p dx < +\infty$

We can take advantage of polar coordinates

$$\int_{\mathbb{R}^n \setminus B_1} |f(x)|^p dx = c_n \int_1^\infty \frac{\rho^{n-1}}{\rho^{2p}} d\rho \quad n-1-2p < -1 \Rightarrow p > \frac{n}{2}$$

Example (6.2.16)

Basically just show that the function and its gradient are $L^2(\Omega)$ and you're good to go. This is also interesting because it's a counter example to the notion that (in $n \geq 2$) a function only need be limited to be in the respective Sobolev space.

Ex. $m=2$ $B_{1/2} = \{ |x| < \frac{1}{2} \}$ $f(x) = \log |\log |x|| \notin L^\infty$ $B_{1/2} \subset \mathbb{R}^2$
 $\notin C^0$

Show that $f \in H^1(B_{1/2})$

$$|x| = \rho \quad \int_0^{1/2} \rho \log^2 |\log \rho| d\rho < +\infty \quad \Leftrightarrow \quad f \in L^2(B_{1/2})$$

$$|\nabla f(x)| = |f'(\rho)| \quad f(\rho) = \log |\log \rho| \quad f'(\rho) = \frac{1}{\rho \log \rho}$$

$$\nabla f \in L^2(B_{1/2}) \quad \Leftrightarrow \quad \int_0^{1/2} \frac{\rho}{\rho^2 \log^2 \rho} d\rho < +\infty$$

$\Rightarrow f \in H^1(B_{1/2})$ but $f \notin L^\infty(B_{1/2})$ $H^1(B_{1/2}) \not\subset L^\infty(B_{1/2})$ contrary example if $m \geq 2$

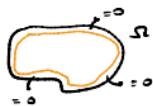
$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$$



No

\exists take a function that is 0 at the boundary

$\exists f \quad u \in C^0(\bar{\Omega})$ then $u \in H_0^1(\Omega) \Leftrightarrow u = 0$ on $\partial\Omega$



$\exists f$ the function has compact support

Poincaré inequality (6.2.18)

Assume that Ω is bounded. Then there exists a constant $C(\Omega)$ such that:

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2} \quad \forall u \in H_0^1(\Omega)$$

Moreover the map $u \rightarrow \|u\|_{L^2}$ is a norm in $H_0^1(\Omega)$ which is equivalent to the norm $\|u\|_{H^1}$.

Poincaré inequality

Rem. Holds in dimension 1 if the domain is bounded

Let $\Omega \subset \mathbb{R}^n$ be open and bounded in one direction.

Then $\exists c > 0$ s.t. $\|u\|_{L^2} \leq c \|\nabla u\|_{L^2} \quad \forall u \in H_0^1(\Omega)$

In particular, $u \mapsto \|\nabla u\|_{L^2}$ is an equivalent norm in $H_0^1(\Omega)$

Proof. By density we may assume that $u \in \mathcal{D}(\Omega)$



x_1 all directions in \mathbb{R}^n excluded x_2
 $\Omega \subset (a, b) \times \mathbb{R}^{n-1}$

$$\forall x \in \Omega \quad u(x) = \int_a^x u_{x_1}(t, x') dt$$

by FTC in 1 dimension after partial derivation

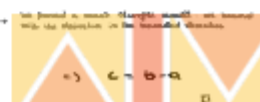
$$|u(x)| \leq \int_a^b |u_{x_1}(t, x')| dt \leq \sqrt{b-a} \left[\int_a^b |u_{x_1}(t, x')|^2 dt \right]^{1/2}$$

$$|u(x)|^2 \leq (b-a) \int_a^b |u_{x_1}(t, x')|^2 dt$$

$$\int_{\mathbb{R}^{n-1}} |u(x)|^2 dx' \leq (b-a) \int_a^b \int_{\mathbb{R}^{n-1}} |u_{x_1}(t, x')|^2 dx' dt$$

$$\int_a^b \int_{\mathbb{R}^{n-1}} |u(x)|^2 dx' dx \leq (b-a)^2 \int_a^b \int_{\mathbb{R}^{n-1}} |u_{x_1}(x)|^2 dx' dx$$

$$\|u\|_{L^2(\Omega)}^2 = \int_{\Omega} |u(x)|^2 dx \leq (b-a)^2 \int_{\Omega} |\nabla u(x)|^2 dx$$



Professor then goes on to define the relevant Hilbert triple

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$$

Where the second continuous embedding is to be meant as follows:

Each function $u \in L^2(\Omega)$ is identified with the linear functional $I_u \in H^{-1}(\Omega)$ defined as :

$$\langle I_u, v \rangle := \int_{\Omega} u v \quad \forall v \in H_0^1(\Omega)$$

Essentially we can map each L^2 function to a functional from H^1 to \mathbb{R} .

Note that we denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$.

Proposition (6.2.19)

Let $\Omega \subset \mathbb{R}^n$ be open and let $F \in H^{-1}(\Omega)$, then $\exists \{f_i\}_{i=0}^n \in L^2(\Omega)$ such that:

$$\langle F, v \rangle = \int_{\Omega} f_0 v + \sum_{i=1}^n f_i \frac{\partial v}{\partial x_i} \quad \forall v \in H_0^1(\Omega)$$

And:

$$\|F\|_{H^{-1}} = \max_{i=0, \dots, n} \|f_i\|_L^2$$

Moreover, if Ω is bounded, we can take $f_0 = 0$

Remark

Note that in this case $v \in L^2(\Omega)$ because $v \in H_0^1(\Omega)$

Proof

There's a proof in the book but here's what the prof wrote:

Proof $\exists f, f_0 = 0 \quad \langle F, v \rangle = \int_{\Omega} f \cdot \nabla v \quad f = (f_1, f_2, \dots, f_m) \in L^2(\Omega)^m$

For smooth f , $\int_{\Omega} f \cdot \nabla v = - \int_{\Omega} v \operatorname{div} f + \int_{\partial\Omega} v (f \cdot \nu) \quad \nu = \text{normal}$

\Rightarrow " $F = - \operatorname{div} f$ " the linear continuous functional in H^{-1} is the divergent of a function in L^2

$H^1(\Omega) \not\subset L^\infty(\Omega) \quad \text{if } n \geq 2$
 $H^1(\Omega) \subset L^2(\Omega) \quad \forall n$

Sobolev embedding theorem (6.2.20 - 6.2.21)

Let $\Omega \subset \mathbb{R}^n$ be an open domain with $\partial\Omega \in Lip$ and $n \geq 2$. Then:

$$H^1(\Omega) \subset L^p(\Omega) \quad \begin{cases} \forall 2 \leq p < \infty & \text{if } n = 2 \\ \forall 2 \leq p < \frac{2n}{n-2} & \text{if } n = 3 \end{cases}$$

In addition to the above, if Ω is bounded, then the embeddings become compact:

the following continuous embeddings:

(i) if $n = 2$ then $H^1(\Omega) \subset\subset L^q(\Omega)$, for every $1 \leq q < \infty$;

(ii) if $n \geq 3$ then $H^1(\Omega) \subset\subset L^q(\Omega)$, for every $1 \leq q < 2^* = \frac{2n}{n-2}$.

Proof. See [1, Theorem 6.2]. □

Remark

The following is called the **Critical Sobolev exponent**:

$$2^* = \frac{2n}{n-2}$$

A sequence that converges in the $H^1(\Omega)$ sense implies convergence in a certain L^p sense, formally:

$$u_n \xrightarrow{H^1(\Omega)} u \implies u_n \xrightarrow{L^p} u$$

Remark

The functions inside $H^1(\Omega)$ are defined up to a negligible set.

2.2. Higher order Sobolev spaces

We basically take 6.2.11 and apply it a bunch of times to define higher order weak derivatives.

Note that in the image below the prof *formally* sets $H^0 = L^2$ (6.3.4)



Multi-index notation

We call a multi-index a vector:

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$$

With norm:

$$|\alpha| = \sum_{i=1}^m \alpha_i$$

We'll use this to define partial derivatives kinda like this:

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

Definition 6.3.1

Let $u \in L^1_{\text{loc}}(\Omega)$ and $\Omega \subseteq \mathbb{R}^n$ be an open set. Given a multi-index α we'll say that u admits a weak derivative $D^\alpha u$ if there exists $g \in L^1_{\text{loc}}(\Omega)$ such that:

$$\int_{\Omega} u D^\alpha \varphi = (-1)^{|\alpha|} \int_{\Omega} g \varphi \quad \forall \varphi \in \mathcal{D}(\Omega)$$

Definition and Separability Theorem (6.3.2 - 6.3.5)

Let Ω be as above and $k \in \mathbb{N}$. The $H^k(\Omega)$ is defined by:

$$H^k(\Omega) = \left\{ u \in L^2(\Omega), \quad D^\alpha u \in L^2(\Omega) \quad \forall |\alpha| \leq k \right\} \quad \forall k \in \mathbb{N}$$

We take $H^k(\Omega)$ to be a **separable** Hilbert space with scalar product:

$$(u, v)_{H^k(\Omega)} = \int_{\Omega} \left(uv + \sum_{1 \leq |\alpha| \leq k} D^\alpha u D^\alpha v \right)$$

With induced norm:

$$\|u\|_{H^k(\Omega)} = \left(\int_{\Omega} \sum_{0 \leq |\alpha| \leq k} |D^\alpha u|^2 \right)^{\frac{1}{2}}$$

Proof

Check the book for a more detailed proof, here's what the prof wrote,

H^k is a separable Hilbert space with scalar product

$$(u, v)_{H^k} = \int_{\Omega} \left(uv + \sum_{1 \leq |\alpha| \leq k} D^\alpha u D^\alpha v \right)$$

- it has a scalar product
- it is complete (use a Cauchy sequence)
- it's separable because its L^2

$$\|u\|_{H^k} = \left(\int_{\Omega} \sum_{0 \leq |\alpha| \leq k} |D^\alpha u|^2 \right)^{1/2} \quad \text{induced norm}$$

Separable Banach spaces with no scalar product. (6.3.8)

$$H^{k+1}(\Omega) = \{u \in H^k(\Omega), \nabla u \in H^k(\Omega)\}$$

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega); D^\alpha u \in L^p(\Omega) \forall |\alpha| \leq k\} \quad \forall k \in \mathbb{N}$$

↳ separable Banach spaces, with no scalar product

$$\text{If } u \in L^p(\Omega) \text{ and } D^\alpha u \in L^p(\Omega) \forall |\alpha| = k \Rightarrow u \in W^{k,p}(\Omega)$$

H^k defined inductively

Remark (6.3.4)

We define $H^k(\Omega)$ inductively. Check the book.

Fourier definition for higher order Sobolev spaces(6.3.5 - 6.3.6)

We can define $H^k(\mathbb{R}^n, \mathbb{C})$ in a more straightforward way using the Fourier transform.

$$H^k(\mathbb{R}^n, \mathbb{C}) = \{u \in L^2(\mathbb{R}^n, \mathbb{C}); (1 + |\xi|^2)^{\frac{k}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n, \mathbb{C})\}$$

In this construction $H^0 = L^2$ because $(1 + |\xi|^2)^{\frac{k}{2}} = 1$ if $k = 0$, we need, however, to define a scalar product to go further.

$$(u, v)_{H^k} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^k \hat{u}(\xi) \hat{v}(\xi) d\xi \quad \rightarrow \text{norm}$$

$$(1 + |\xi|^2)^{k/2} \rightarrow \text{weight function}$$

Note that k doesn't *have* to be an integer in this construction, it just has to be $k = s \geq 1$. We then end up with what are called **Non Local PDE's**

Remark

This only works on \mathbb{R}^n , not on bounded domains (Duh!)

2.3. Non-integer Sobolev spaces

As previously mentioned, we'll take $s \geq 0$.

$$H^s(\mathbb{R}^n, \mathbb{C}) = \left\{ u \in L^2(\mathbb{R}^n, \mathbb{C}), (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n, \mathbb{C}) \right\}$$

Endowed with the scalar product

$$(u, v)_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi$$

for every $u, v \in H^s(\mathbb{R}^n, \mathbb{C})$.

(6.4.1)

We can extend this construction to a general domain Ω .

Theorem (6.4.5)

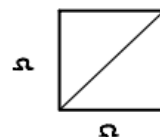
Def Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $s > 0$; if $s \in (k, k+1)$ we set $s = k + t$ with $t \in (0, 1)$ and we define

$$H^s(\Omega) = \left\{ u \in H^k(\Omega); \int_{\Omega} \int_{\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{n+2t}} dx dy < \infty \quad \forall |\alpha| = k \right\}$$

$$s = \frac{1}{2} \Rightarrow \alpha = 0 \quad \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+1}} dx dy < \infty$$

↳ it's a property of the incremental ratio

$$(u, v)_{H^s(\Omega)} = (u, v)_{H^k(\Omega)} + \sum_{|\alpha|=k} \int_{\Omega} \int_{\Omega} \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha v(x) - D^\alpha v(y))}{|x - y|^{n+2t}} dx dy$$



$H^s(\Omega)$ is an Hilbert space, because it becomes a Banach space if we define the norm from the scalar product

$H^s(\Omega)$ is separable because it is the cartesian product of separable Hilbert spaces

2.4. The trace operator

Definition (6.5.1)

For every $s \geq 0$:

$$H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$$

A special case is the following:

$$\Omega = \mathbb{R}^n \implies H_0^s(\Omega) = H^s(\Omega)$$

Otherwise

$$\begin{aligned} \text{If } \Omega \neq \mathbb{R}^n \quad & \begin{aligned} & \xrightarrow{\quad} L^2(\Omega) = H^0(\Omega) \\ & \searrow \quad \quad H^1(\Omega) \end{aligned} \quad \begin{aligned} & \overline{\mathcal{D}(\Omega)}^{L^2(\Omega)} = L^2(\Omega) \\ & \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega) \subsetneq H^1(\Omega) \end{aligned} \end{aligned}$$

What if $s \in [0, 1]$?

Let's put ourselves in the case where $\Omega \neq \mathbb{R}^n$

What happens with $H^s(\Omega)$ $s \in [0, 1]$?

$$H^s_0(\Omega) = H^s(\Omega) \quad \forall s \in [0, \frac{1}{2}]$$

$$\Omega \neq \mathbb{R}^n$$

$$H^s_0(\Omega) \subsetneq H^s(\Omega) \quad \forall s > \frac{1}{2}$$

Read the book for a better understanding of what's going on in the background.

Essentially $\mathcal{D}(\Omega)$ is dense in $H^s(\Omega)$ if and only if $s \leq \frac{1}{2}$. Meaning the "closure" and the space are the same. (What's written in the pic).

The \mathbb{R}^n case is also explained.

A problem now arises, how do we define the value of a function $u \in H^s(\Omega)$ on the boundary? Remember these functions are defined up to a negligible set.

How to define $u|_{\partial\Omega} = g \neq 0$?

Which is the restriction on the boundary?

If $s \in [0, \frac{1}{2}]$ it's impossible to define the restriction even if $g=0$

If $s > \frac{1}{2}$ we can try with an approximation



We'll solve it this way

Trace operator (6.5.2 - 6.5.4)

Assume $\partial\Omega \in C^\infty$, now note that $C^\infty(\bar{\Omega})$ is **dense** in $H^s(\Omega) \forall s \geq 0$ so the idea is to define a sequence that approximates u and evaluate it at the boundary in this fashion:

Take $u \in H^s(\Omega)$, $s > \frac{1}{2}$, then construct a sequence:

$$\{u_m\} \subset C^\infty(\bar{\Omega})$$

Such that:

$$u_m \xrightarrow{H^s(\Omega)} u$$

Now define the **Trace operator** as:

$$\gamma_0 u = \lim_{m \rightarrow \infty} u_m|_{\partial\Omega}$$

This $\gamma_0 u$ is an approximation and in general this definition fails to actually define what happens at the boundary, or even define a way to evaluate **a type** of boundary condition. The book goes **much** more in depth.

Redefine the boundary points

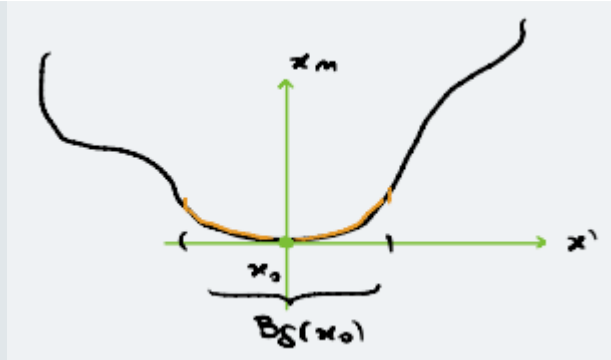
We can see any point $x_0 \in \partial\Omega$ as:

$$x_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

At this point there exists a veery tiny $\delta > 0$ and a function $\varphi \in C^\infty(B_\delta(x_0))$ (locally smooth) such that:

$$x \in \partial\Omega \iff x_n = \varphi(x') \quad \forall x' \in B_\delta(x'_0)$$

Where x' are as below:



The relevant Sobolev space is

$$H^s(B_\delta(x'_0))$$

Note that it's of dimension $n - 1$

We say that $\gamma_0 u \in H^s(\partial\Omega)$ if

$$u_0(x') = u(x', \varphi(x')) \in H^s(B_\delta(x'_0)) \quad \forall x_0 \in \partial\Omega$$

The following is called the *restriction* of u to the boundary:

$$u(x', \varphi(x'))$$

Theorem (6.5.3)

Let $s > \frac{1}{2}$, the trace operator is such that:

$$\gamma_0: H^s(\Omega) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega)$$

We essentially lose half a degree of derivative.

Remark

$\gamma_0 u$ is only surjective, not injective. We *can* get injectivity if we add more constraints but we can't conclude that with our current formulation (we get it in PDEs, using the lifting operator)

Some examples follow.

8. Heat equation

$$\begin{cases} \Delta u = 0 & \Omega \\ g = 0 & \partial\Omega \end{cases}$$
 has a unique solution

If $\partial\Omega \in C^\infty$, $\exists \gamma_0: H^0(\Omega) \rightarrow H^{\sigma-\frac{1}{2}}(\partial\Omega)$ linear and continuous that extends $u \mapsto u|_{\partial\Omega}$ for $C^\infty(\bar{\Omega})$ functions

N.B. $H^{\sigma-\frac{1}{2}}$ cannot be improved, it is the optimum space
 When we go to the boundary we lose half derivative

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases} \quad \forall g \in H^\sigma(\partial\Omega) \quad \exists! u \in H^{\sigma+\frac{1}{2}}(\Omega)$$

\exists want to find an harmonic extension to the boundary
 The traces are well defined if $s > \frac{1}{2}$

$\gamma_0 \rightarrow$ trace operator (of order 0)
 ↳ number of derivatives

Trace operator in the case of the Neumann boundary condition (6.5.7)

Neumann problem

$$\begin{cases} \Delta u = 0 & \Omega \\ \frac{\partial u}{\partial \nu} = g & \partial\Omega \end{cases} \quad \frac{\partial u}{\partial \nu} \rightarrow \text{flux across the boundary}$$

$$0 = \int_{\Omega} \Delta u = \int_{\Omega} \operatorname{div}(\nabla u) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \quad \text{we need } \int_{\partial\Omega} g = 0$$

We need to find the normal derivative of u at the boundary

We choose $\partial\Omega \in C^\infty$



$$\gamma_j u = \frac{\partial^j u}{\partial \nu^j} \Big|_{\partial\Omega} \quad \forall u \in C^\infty(\bar{\Omega})$$

$$\gamma_j : H^\sigma(\Omega) \rightarrow H^{\sigma-j-\frac{1}{2}} \quad \forall j \leq \sigma - \frac{1}{2} \quad j \text{ integer}$$

$H^{\sigma-j-\frac{1}{2}} \rightarrow \exists$ lose j because we want the j -th derivative and \exists lose $\frac{1}{2}$ like before because \exists in on the boundary

$$\text{If } j=0 \Rightarrow \sigma > \frac{1}{2}$$

$$\text{If } j=1 \Rightarrow \sigma > \frac{3}{2}$$

γ_j are linear and continuous operators



$$H_0^s(\Omega)$$

$$H_0^s(\Omega) = \overline{D(\Omega)}^{H^s(\Omega)}$$

Theorem (6.5.8)

*Thm $\partial\Omega \in C^\infty, u \in H^s(\Omega)$

$$u \in H^s_0(\Omega) \iff \partial_j u = 0 \quad \forall j \leq n - \frac{1}{2}$$

$$H^s_0(\Omega) = \bigcap_{j=0}^{[n-\frac{1}{2}]} \ker \{\partial_j\}$$

$[n - \frac{1}{2}] \rightarrow$ integer part of $n - \frac{1}{2}$
 $\partial_j \rightarrow$ linear operator, it has a kernel

2.5. Embedding theorems

In this section we will make use of **embeddings** in order to evaluate the regularity of $H^s(\Omega)$ elements.

Sobolev Theorem (6.6.1)

Let $\Omega = \mathbb{R}^n$ or an open set with **Lipschitz** boundary. Let $s \geq 0$, then the following **continuous embeddings** hold:

$$H^s(\Omega) \subset \begin{cases} L^p(\Omega) & \forall 2 \leq p \leq \frac{2n}{n-2s} & \text{if } n > 2s \\ L^p(\Omega) & \forall 2 \leq p \leq \infty & \text{if } n = 2s \\ C^0(\overline{\Omega}) & & \text{if } n < 2s \end{cases}$$

So if we increase s we increase the regularity but we also have to take into account n (the number of dimensions).

Theorem 6.6.1 ensures the continuity of the elements of $H^1(\Omega)$ when $n = 1$, while if $n = 2$, the fact that a function belongs to $H^1(\Omega)$ is not any more sufficient to guarantee its continuity. In the case $n = 2$, in order to have continuity of a function u we have to assume that $u \in H^s(\Omega)$ with $s > 1$. We deduce that, as the space dimension increases, the regularity of the functions of $H^s(\Omega)$ decreases.

If $\Omega \subseteq \mathbb{R}^n$ is bounded then all of the above embeddings are **compact** except for:

$$H^s(\Omega) \subset L^{\frac{2n}{n-2s}} \quad \text{when } n > 2s$$

Here's a demonstration (example 6.6.5) of this theorem in action, assume the following:

- $n \geq 3$
- Ω unbounded
- $\{u_m\} \subset H^1(\Omega)$ such that $\|u_m\|_{H^1(\Omega)} \leq c < \infty$

Thanks to theorem 1.4.13, we know that:

$$\{u_{m_k}\} \quad s.t. \quad u_{m_k} \xrightarrow[\text{weakly}]{H^1(\Omega)} u \in H^1(\Omega)$$

We can therefore conclude that the above sequence converges to $L^p \forall 2 \leq p \leq \frac{2n}{n-2s}$, formally:

$$u_k, u \in L^p\Omega \quad \text{and} \quad u_k \xrightarrow{L^p(\Omega)} u$$

2.6. Negative order Sobolev spaces

These spaces are defined extending definition 6.4.1.

Negative non-integer Sobolev spaces (6.7)

Let $s < 0$

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}$$

Where \mathcal{S}' is the complex space of **tempered distributions** (the dual for the space of fast decreasing functions).

If we let k be an integer then we can define the following duality:

$$H^{-k}(\Omega) = [H_0^k(\Omega)]'$$

We can also define the following Hilbert triple:

$$H_0^k(\Omega) \subset L^2(\Omega) \subset H^{-k}(\Omega)$$

At this point there's a lot of boilerplate that can honestly be studied far better in the book anyway, I won't bother writing it down since it's really just stuff that is gonna be referenced later. I'll include references to the book whenever needed.

3. Weak solutions of partial differential equations

In this section we aim to derive existence and uniqueness results for weak PDE solutions.

3.1. Homogeneous Dirichlet problem

Let Ω be an open, bounded subset of \mathbb{R}^n and let its boundary $\partial\Omega \in C^1$. Take $\alpha \in \mathbb{R}$, $f \in L^2(\Omega)$.

We define the Homogeneous Dirichlet Problem (7.1) as:

$$\begin{cases} -\Delta u + \alpha u &= f & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

If $f \in C^0(\Omega)$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ the equation is satisfied in the classical sense, meaning u is a classical solution.

A **weak solution** on the other hand satisfies (7.1.1) $u \in H_0^1(\Omega)$ and:

$$\int_{\Omega} \nabla u \nabla v + \alpha \int_{\Omega} uv = \int_{\Omega} fv \quad \forall v \in H_0^1(\Omega)$$

If u weak solution is $C^2(\overline{\Omega})$ then you can easily conclude that it is also a classical solution.

Important counter example

If $n > 1$ then:

$$\Delta u \in C^0(\Omega) \not\Rightarrow u \in C^2(\Omega)$$

In higher dimensional spaces it's wise to avoid formulating your problems using such regular spaces, prefer instead Sobolev spaces

Remark (7.1.2)

If u is a classical solution of (7.1) then necessarily $f \in C^0(\overline{\Omega})$. Therefore if f is $L^2(\Omega)$ we cannot expect a weak solution to also be classical.

Theorem (7.1.3)

Let $\alpha \geq 0$ and $f \in L^2(\Omega)$ then there exists a **unique** solution $u \in H_0^1(\Omega)$ of 7.1 (the Homogeneous Dirichlet problem).

Moreover the unique solution is also the unique minimizer of $J(v)$:

$$J(v) = \frac{1}{2} \int_{\Omega} (|\nabla v|^2 + \alpha v^2) - \int_{\Omega} f v \quad v \in H_0^1(\Omega)$$

The intuition for the "minimum" being unique is that this is basically a parabola.

Proof

We endow the Hilbert space $H_0^1(\Omega)$ with the following scalar product (6.2.18):

$$(u, v) = \int_{\Omega} \nabla u \nabla v \quad \forall u, v \in H_0^1(\Omega).$$

Given the assumption $\alpha \geq 0$, the bilinear form:

$$a(u, v) = \int_{\Omega} (\nabla u \nabla v + \alpha uv) \quad \forall u, v \in H_0^1(\Omega)$$

is coercive over $H_0^1(\Omega)$.

Moreover the linear form $v \rightarrow \int_{\Omega} f v$ is continuous over $H_0^1(\Omega)$.

Then the assertion follows from the Lax-Milgram theorem.

It's interesting to note that what we wrote above doesn't necessarily break for $\alpha < 0$, in fact, it *can* work. By the Poincaré inequality there exists a constant $\lambda_1 > 0$ such that:

$$\lambda_1 \|u\|_{L^2(\Omega)}^2 \leq \|\nabla u\|_{L^2(\Omega)}^2 \quad \forall u \in H_0^1(\Omega)$$

Now....

$$\begin{aligned} \int_{\Omega} (|\nabla u|^2 + \alpha u^2) &= \int_{\Omega} |\nabla u|^2 + \alpha \int_{\Omega} u^2 \\ \text{(using Poincaré)} &\geq \int_{\Omega} |\nabla u|^2 - \frac{|\alpha|}{\lambda_1} \int_{\Omega} |\nabla u|^2 \\ &= \frac{\lambda_1 - |\alpha|}{\lambda_1} \int_{\Omega} |\nabla u|^2 \end{aligned}$$

In fact if $\alpha < 0$:

$$\alpha \lambda_1 \|u\|_{L^2(\Omega)}^2 \geq \alpha \|\nabla u\|_{L^2(\Omega)}^2 \Rightarrow \alpha \|u\|_{L^2(\Omega)}^2 \geq -\frac{|\alpha|}{\lambda_1} \|\nabla u\|_{L^2(\Omega)}^2$$

Meaning α just needs to be bigger than $-\lambda_1$ to retain coercivity.

Exercise - Alternative characterization of poincaré constant (7.1.5)

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with $\partial\Omega \in C^1$. Find $\lambda \in \mathbb{R}$ such that $u = 0$ is not the only solution to the HDP:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Generally speaking $u = 0$ is always a solution to Eigenvalue problems, let's assume $u \neq 0$ as a sort of 3rd constraint to our problem:

$$\int_{\Omega} |\nabla u|^2 = \lambda \int_{\Omega} u^2 \implies \lambda > 0$$

Note that if $\lambda = 0$ then $u = 0$ is the only solution.

The spectrum of $-\Delta$, which contains the values of λ for which the problem has a non trivial solution:

$$\sigma(-\Delta) \subset (0, +\infty)$$

In fact thanks to Theorem 1.8.10 we can order the eigenvalues of $-\Delta$ as a sequence of strictly increasing values.

$$\sigma(-\Delta) = \{0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots\}$$

Therefore we conclude that

$$\lambda_1 = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}$$

There's a small tangent on Fréchet derivatives, there's no mention of it in the book so I'll avoid it, you can find references to it at page 17 of Ravizza's notes.

3.2. Non Homogeneous Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1 , we look for a function $u: \bar{\Omega} \rightarrow \mathbb{R}$ satisfying:

$$\begin{cases} -\Delta u + \alpha u &= f & \text{in } \Omega \\ u &= g & \text{on } \partial\Omega \end{cases}$$

with f defined over $H^{-1}(\Omega)$, g on $H^{1/2}(\partial\Omega)$ and $\alpha \in \mathbb{R}$. If $g \in H^{1/2}(\partial\Omega)$ we can use Theorem 6.5.3 to infer that:

$$\exists u_0 \in H^1(\Omega) \text{ s.t. } \gamma_0 u_0 = g$$

We then define the following set which will aid in later constructions:

$$K = \{v \in H^1(\Omega); v - u_0 \in H_0^1(\Omega)\}$$

Classical solution for NHDP (7.1.7)

A classical solution for the NHDP is a function $u \in C^2(\bar{\Omega})$ satisfying the NHDP.

Weak solution for the NHDP (7.1.7)

A weak solution for the NHDP is a function $u \in K$ such that:

$$\int_{\Omega} (\nabla u \nabla v + \alpha uv) = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega)$$

Where $\langle f, v \rangle$ is equal to $\int_{\Omega} f v$

The same reasoning as HDP applies, therefore any classical solution is automatically also a weak solution, conversely any weak solution $u \in C^2(\bar{\Omega})$ is also a classical solution.

We'll now analyse the existence of weak solutions

Dirichlet Principle (7.1.8)

Let $\alpha > -\lambda_1$ (where the eigenvalue is to be meant as the smallest eigenvalue under homogeneous Dirichlet conditions), $f \in H^{-1}(\Omega)$, $g \in H^{1/2}(\partial\Omega)$.

Then the NHDP admits a unique *weak* solution $\bar{u} \in K$.

Moreover \bar{u} is a weak solution if and only if it minimizes the Dirichlet *energy* over K

$$J(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + \alpha u^2) - \langle f, u \rangle$$

Proof

The proof was given by the prof but it is way too long, it's far better to study it from the book, if need be you can also make use of Ravizza p.20

3.3. Non Homogeneous Neumann problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1 , we look for a function u satisfying:

$$\begin{cases} -\Delta u + \alpha u &= f & \Omega \\ \frac{\partial u}{\partial \nu} &= g & \partial\Omega \end{cases}$$

with $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$ and $(\alpha > 0) \in \mathbb{R}$.

Classical solution for the NHNP (7.1.11)

We define a classical solution one where the boundary conditions are to be meant $f \in C^0, g \in C^0$

$$u \in C^2(\Omega) \cap C^1(\bar{\Omega})$$

Weak solution for the NHNP (7.1.11)

Recalling that the trace operator "takes" half an order in terms of Sobolev regularity, take $g \in H^{\frac{1}{2}}(\partial\Omega)$. A weak solution is $u \in H^1(\Omega)$ such that:

$$\int_{\Omega} (\nabla u \cdot \nabla v + \alpha uv) = \int_{\Omega} f v + \langle g, \gamma_0 v \rangle \quad \forall v \in H^1(\Omega)$$

Where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ by way of theorem 6.5.3 (it's really important and mentioned a lot, make sure you understand it)

Let's now work on the regularity conditions, when is a regular weak solution also a classical one? In order to do this let's assume $u \in C^2(\bar{\Omega})$ is a weak solution for the NHNP, using the Gauss-Green formula (really we're just integrating by parts) we infer that

$$\int_{\Omega} (-\Delta u + \alpha u) v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v = \int_{\Omega} f v + \int_{\partial\Omega} g v \quad \forall v \in C^1(\bar{\Omega}).$$

If we take $v \in \mathcal{D}(\Omega)$, meaning it has compact support

$$\int_{\Omega} (-\Delta u + \alpha u) v = \int_{\Omega} f v \quad \forall v \in \mathcal{D}(\Omega)$$

Which, as a consequence of 2.4.22, implies that

$$-\Delta u + \alpha u = f \quad \text{a. e. in } \Omega$$

Inserting this equation into the one we got from the Gauss-Green formula we get

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v = \int_{\partial\Omega} g v \quad \forall v \in C^1(\bar{\Omega}).$$

The previous identity holds true for every $v \in H^1(\Omega)$, since $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$ and $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is a continuous surjective function. Hence we may conclude that $\frac{\partial u}{\partial \nu} = g$ on $\partial\Omega$

We now want to prove the existence and uniqueness of a weak solution

Existence and uniqueness of a NHNP weak solution (7.1.12)

Let $f \in L^2(\Omega)$, $g \in H^{-1/2}(\partial\Omega)$ and $\alpha > 0$. Then there exists a unique weak solution $u \in H^1(\Omega)$. Moreover, u can be obtained as the solution of the following minimum problem

$$\min_{v \in H^1(\Omega)} \left\{ \frac{1}{2} \int_{\Omega} (|\nabla v|^2 + \alpha v^2) - \int_{\Omega} f v - \langle g, \gamma_0 v \rangle \right\}.$$

Proof

You can prove this by applying the Lax-Milgram theorem 1.7.4

Just like for homogeneous we want to explore what happens when we relax the $a > 0$ constraint. Let's set $a = 0$, the problem becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

Assume that the problem admits a solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, then by the divergence theorem it holds:

$$\int_{\Omega} f = - \int_{\Omega} \Delta u = - \int_{\Omega} \operatorname{div}(\nabla u) = - \int_{\Omega} \frac{\partial u}{\partial \nu} = - \int_{\partial\Omega} g = - \langle g, 1 \rangle$$

If \exists a solution u to this problem then the following holds:

$$\int_{\Omega} f = - \int_{\partial\Omega} g$$

But this same exact relation is satisfied by $\hat{u} = u + k$ where k is a constant. More on this can be found in Theorem 3.6.7. (btw I think the book got the second property wrong, it's supposed to be an equal !)
Moreover if $\int_{\Omega} f \neq - \int_{\partial\Omega} g$ then there are **no** solutions.

Alternatively one could study the variational version of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

if we multiply by u and integrate by parts as we're so used to do by now we get:

$$\int_{\Omega} |\nabla u|^2 = \lambda \int_{\Omega} u^2$$

Therefore if $\lambda = 0$ we get $\nabla u = 0$ implying that u is a constant.

Existence theorem for NHNP

Assume that $f \in L^2(\Omega)$ and $g \in H^{-\frac{1}{2}}(\partial\Omega)$

- If $\alpha > 0$ then $\exists!$ a weak solution $\bar{u} \in H^1(\Omega)$, moreover \bar{u} is the unique minimizer of the energy function mentioned in the previous theorem.
- If $\alpha < 0$ and $-\alpha \notin \sigma_N(-\Delta) \implies$



Danger

This isn't well defined look it up better p.23 of Ravizza

We have the same conditions for existence and uniqueness of Neumann and Dirichlet solutions.

- Dirichlet problems: $\alpha > -\lambda_1$
- Neumann problems: $\alpha > 0$

4. Stokes Equations

Here u will be a **field**, p will be the pressure.

These two will be our unknown variables, while f will be a given **force**

$$\begin{cases} -\eta\Delta u + \nabla p = f & \text{in } \Omega \subset \mathbb{R}^n \\ \nabla \cdot u = 0 & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We'll assume the fluid to be incompressible, with no velocity at the boundary.

We mean $u \in L^2(\Omega)$

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_2 \end{pmatrix} \wedge u_i \in L^2(\Omega) \quad \forall i$$

alternatively

$$u \in [L^2(\Omega)]^n$$

Here's some more notation to familiar yourself:

- $E(\Omega) = \{v \in L^2(\Omega) : \operatorname{div}(v) \in L^2(\Omega)\}$
- $u \in H^1(\Omega) : \frac{\partial u}{\partial x_i} \quad \forall i, j \in 1, \dots, m$
- $\nabla u = \left(\frac{\partial u_i}{\partial x_j} \right) \quad n \times n \quad \text{matrix}$
- $\nabla u : \nabla v = \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}$
- $H_0^*(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$

In Hilbert spaces:

- $H^1(\Omega) \quad (u, v)_{H^1} = \int_{\Omega} (\nabla u; \nabla v + uv)$
- $H_0^1(\Omega) \quad (u, v)_{H_0^1} = \int_{\Omega} \nabla u : \nabla v$
- $E(\Omega) \quad (u, v)_{E(\Omega)} = \int_{\Omega} u \cdot v + \int_{\Omega} (\nabla \cdot u)(\nabla \cdot v)$
- $V(\Omega) = \{u \in H_0^1(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega\} = \overline{\{u \in \mathcal{D}(\Omega) : \nabla \cdot u = 0\}}^{H^1(\Omega)}$

$V(\Omega)$

- Is a closed subspace of $H_0^1(\Omega)$, because it it's basically the same with an additional constraint
- It's a Hilbert space with product:

$$(u, v)_V = (u, v)_{H_0^1(\Omega)} = \int_{\Omega} \nabla u : \nabla v \quad \forall u, v \in V$$

4.1. Helmholtz-Weyl decomposition

Let Ω be an open bounded (conncted) set of \mathbb{R}^n with boundary $\partial\Omega \in C^2$.

Capital bold letters ($\mathbf{L}^2, \mathbf{H}^1, \dots$) denote spaces of vector valued functions (with values in \mathbb{R}^n), while italic small letters (L^2, H^1, \dots) denote spaces of scalar functions: for brevity, we set $L^2 := L^2(\Omega), \dots$ and we will precise the set only when it is different from Ω .

We'll denote the divergence of some field u with $\nabla \cdot u$.

Keep in mind that $\nabla \cdot f = 0$ is meant in the weak sense since $f \in L^2$ (we can't directly compute it).

Recall that \mathbf{E} is defined as:

$$\mathbf{E} := \{u \in (L^2(\Omega))^n; \operatorname{div} u \in L^2(\Omega)\},$$

where $\operatorname{div} u \in L^2(\Omega)$ if and only if there exists $w \in L^2(\Omega)$ such that

$$\int_{\Omega} w \varphi = - \int_{\Omega} u \cdot \nabla \varphi \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Consider the following spaces:

$$\begin{aligned} \mathbf{G}_1 &:= \{f \in \mathbf{L}^2; \nabla \cdot f = 0, \gamma_{\nu} f = 0\}, \\ \mathbf{G}_2 &:= \{f \in \mathbf{L}^2; \nabla \cdot f = 0, \exists g \in H^1, f = \nabla g\}, \\ \mathbf{G}_3 &:= \{f \in \mathbf{L}^2; \exists g \in H_0^1, f = \nabla g\}. \end{aligned}$$

$f \in L^2(\Omega)$ therefore it's in a subspace of $\mathbf{E}(\Omega)$, f is a vector and g a scalar.

In $n = 1$ this is what these spaces become:

- \mathbf{G}_1 : functions in $L^2(\Omega)$ with derivative equal to zero that vanish at the boundary (it's just the element 0)
- \mathbf{G}_2 : functions in $L^2(\Omega)$ with derivative equal to 0, that are constant and have a potential (\mathbb{R})
- \mathbf{G}_3 : functions in $L^2(\Omega)$ which have the potential in $H_0^1(\Omega)$ and are the derivative of a function that vanishes at the boundary

$$\mathbf{G}_3 = \left\{ f \in L^2, \int_{\Omega} f = 0 \right\}$$

Helmholtz - Weyl theorem (7.1.16)

The spaces \mathbf{G}_i ($i = 1, 2, 3$) are mutually orthogonal in \mathbf{L}^2 and $\mathbf{L}^2 = \mathbf{G}_1 \oplus \mathbf{G}_2 \oplus \mathbf{G}_3$ (this means that every $f \in \mathbf{L}^2$ can be uniquely written as the sum of three functions $f_i \in \mathbf{G}_i$ for $i = 1, 2, 3$).

Proof

In the book

4.2. Strong form of Stokes problem

Let $f \in V'$ dual space of V ,

$V \subset H_0^1(\Omega) \subset V'$ (V' is larger than H^{-1})

$$\begin{cases} -\eta \Delta u + \nabla p = f & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

where $u = 0$ is called the **non-slip boundary condition**.

As usual we multiply both sides by a test function $v \in V$ and applying (6.7.6 - 6.16) we get:

$$\eta \int_{\Omega} \nabla u : \nabla v + \int_{\Omega} \nabla p \cdot v = \int_{\Omega} f v$$

where $\nabla p \in \mathbb{G}_2 \oplus \mathbb{G}_3$ and $\mathbf{V} \subset \mathbb{G}_1$ therefore

$$\int_{\Omega} \nabla p \cdot v = 0 \quad \forall v \in \mathbf{V}$$

from which we derive

$$\eta \int_{\Omega} \nabla u : \nabla v = \int_{\Omega} f v \quad \forall v \in \mathbf{V}$$

where $\nabla u : \nabla v$ is the "Euclidean scalar product" of the Jacobian matrices of u and v .

Next, given that $\forall f \in \mathbf{L}^2$ the right hand side of the equation represents a linear and continuous function over \mathbf{V} , we infer (using Lax-Milgram 1.7.4):

$$\forall f \in \mathbf{L}^2 \quad \exists! u \in \mathbf{V} \quad \text{s. t.} \quad \eta \int_{\Omega} \nabla u : \nabla v = \int_{\Omega} f v \quad \forall v \in \mathbf{V}$$

This is just a uniqueness result for the weak solution, can we get a strong solution now ? We'll need to make the weak solution more regular!

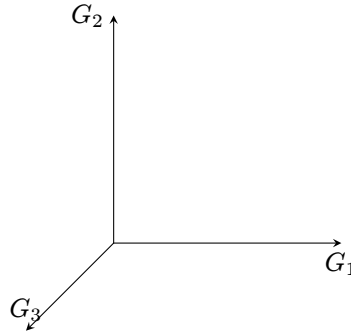
We're going to use the following result of Elliptic regularity

$$\text{if } \Delta u \in H^k(\Omega) \text{ and } \gamma_0 u \in H^{k+3/2}(\Omega) \implies u \in H^{k+2}(\Omega)$$

thus we rewrite the existence condition as

$$\forall f \in \mathbf{L}^2 \quad \exists! u \in \mathbf{H}^2 \cap \mathbf{V} \quad \text{s. t.} \quad \int_{\Omega} (\eta \Delta u + f) v = 0 \quad \forall v \in \mathbf{G}_1.$$

where we used the density of \mathbf{V} in \mathbf{G}_1 to replace the condition on v .



In view of 7.1.16, we infer that $(\eta \Delta u + f) \in (\mathbb{G}_2 \oplus \mathbb{G}_3)$. You can intuitively get to this conclusion given that you're multiplying something by $v \in \mathbf{G}_1$ and getting zero.

With this information we can say that

$$\exists p \in H^1(\Omega) \quad \text{s. t.} \quad -(\eta \Delta u + f) = \nabla p$$

we have essentially proven theorem 7.1.19 which guarantees a unique (u, p) solution to our Stokes problem in the sense of traces.

Corollary - projection (7.1.20)

For every $f \in L^2$, there exists a unique $v \in \mathbf{H}^2 \cap \mathbf{V}$ such that

$$P_1(\Delta v) = P_1 f$$

in particular if $P_1 f = 0$ then $v = 0$.

In conclusion

$\forall f \in L^2(\Omega) \quad \exists! u \in V \cap \mathbf{H}^2(\Omega) , \exists! p \in H^1(\Omega)$ with $\int_{\Omega} p = p_0$ satisfying the Stokes problem

4.3. Eigenvalue problem (Stokes)

The Eigenvalue problem associated with the Stokes equation is:

$$\begin{cases} -\eta\Delta u + \nabla p = \lambda u & \Omega \\ \nabla \cdot u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

where $u \in V \subset \mathbf{G}_1$, $\eta\Delta u \in \mathbf{G}_1 \oplus \mathbf{G}_2$ and $\nabla p \in \mathbf{G}_2$

Métivier Theorem (7.1.21)

The eigenvalue problem associated with the stokes equation admits a sequence of positive eigenvalues $\{\lambda_m\}$ which diverges to infinity. Each eigenvalue has finite multiplicity.

More over, the eigenfunctions can be chosen so that they form an orthogonal basis in \mathbf{G}_1 and \mathbf{V}

4.4. Semi-linear elliptic equations

Let Ω be bounded with $\partial\Omega \in C^1$, consider the original Dirichlet problem:

$$\begin{cases} -\Delta u &= -\alpha u + f(x) \\ u &= 0 \end{cases}$$

we take f to be independent of the position, we then obtain:

$$\begin{cases} -\Delta u &= F(u) & \Omega \\ u &= 0 & \partial\Omega \end{cases}$$