APDE - Advanced Partial Differential Equations - F. Gazzola

Held by prof. F. Gazzola at Politecnico di Milano

(i) Disclaimers

Read me!:

- These notes were taken during AY 2023/2024 using older material, your mileage may vary. They're meant to accompany the lectures and in no way aim to substitute a professor yapping away at an iPad 30m away.
- Any reference of form $(x.\,x.\,x)$ where $x\in\mathbb{N}$ is a reference to a a Theorem/Definition/etc... in the professors own book:

Elements of Advanced Mathematical Analysis for Physics and Engineering

By: A. Ferrero , F. Gazzola , M. Zanotti ISBN: 978-88-7488-645-6.

I'm using the September 2013 version.

- These notes are based on the work done by students Ravizza and Mescolini, you
 can access their notes by logging on the <u>AIM website</u> and checking out **Portale**appunti. This would NOT have been possible if not for their notes..
- For any questions/mistakes you can reach me <u>here</u>.

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Sobolev spaces and initial derivation for discrete domains

 C^* are banach spaces but not *Hilbert* spaces (i.e. we cannot use Lax-Milgram).

△ Example

Second ordinary differential equation:

$$egin{cases} -u''+u=f\ u(a)=u(b)=0 \end{cases}$$

What happens if f is not continuous? Let's try a weak solution.

$$egin{aligned} -u''arphi + uarphi &= farphi & orall arphi \in \mathcal{D}(a,b) \ \mathcal{D}(a,b) &= C_c^\infty(a,b) \end{aligned}$$

We then integrate

$$-\int_a^b u''\varphi + \int_a^b u\varphi = \int_a^b f\varphi$$

$$\int_a^b (-u''arphi+uarphi) = \int_a^b farphi$$

This integral obviously has sense if and only if the integrands are L^1 :

- ullet To have $uarphi\in L^1$ we need both of those functions to be L^2
- To have $u'\varphi'\in L^1$ we need both of those derivatives to be in L^1
- ullet We also need $f\in L^2$

In order to be talking sense (i.e. having a legal formulation) this is enough. We will end up with a weak formulation satisfying these properties.

$$ullet (u,u',f)\in L^2$$

Keep in mind that L^2 contains discontinuous functions.

We say that v = u' in the weak sense (distributional sense) if:

$$\int_I uarphi' = -\int_I uarphi \qquad orall arphi \in \mathcal{D}(I)$$

We define the H^1 space as the following $H^1(I)=\{u\in L^2(I),\ u'\in L^2(I)\ \text{in a weak sense}\}$

 H^1 is the simplest one dimensional Sobolev space. We are going to change the dimensionality of the space, the order of the derivative and the summability of the space. Can we strengthen the assumptions on the hypotheses on our solution if we strengthen those on $f \in L^2$? Answer is yes but we'll get there.

Examples of weak derivatives

First example:

 $C^1(\overline{I})\subset H^1(I)$: The weak derivative coincides with the classic one.

Second example:

Let f(x) = |x| and I = (-1, 1)

Conjecture: We want to prove that the derivative f'(x), defined as below, belongs to L^2 .

$$f'(x) = egin{cases} -1 & ext{if } x < 0 \ +1 & ext{if } x > 0 \end{cases}$$

Let's try it using the definition of weak derivative:

$$= \int_{-\pi}^{\pi} A(x) \, dx - \int_{\pi}^{\pi} A(x) \, dx$$

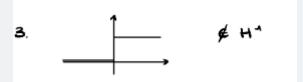
$$= \int_{-\pi}^{\pi} x \, A_{1}(x) \, dx + \int_{\pi}^{\pi} x \, A_{1}(x) \, dx - \int_{\pi}^{\pi} A(x) \, dx - \left[x \, A(x)\right]_{\pi}^{\pi} - \int_{\pi}^{\pi} A(x) \, dx + \left[x \, A(x)\right]_{\pi}^{\pi}$$

$$= \int_{\pi}^{\pi} x \, A_{1}(x) \, dx + \int_{\pi}^{\pi} x \, A_{1}(x) \, dx - \int_{\pi}^{\pi} A(x) \, dx - \left[x \, A(x)\right]_{\pi}^{\pi} - \int_{\pi}^{\pi} A(x) \, dx + \left[x \, A(x)\right]_{\pi}^{\pi}$$

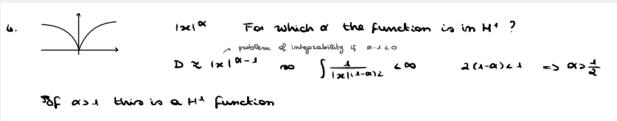
$$= \int_{\pi}^{\pi} x \, A_{1}(x) \, dx - \int_{\pi}^{\pi} A(x) \, dx - \int_{\pi}^{\pi} A(x) \, dx - \int_{\pi}^{\pi} A(x) \, dx + \left[x \, A(x)\right]_{\pi}^{\pi}$$

We used the definition of weak derivative, written our initial integral in it's form, therefore proven that it is the weak derivative.

Third example:



Fourth example:



riangle Separability of H^1 (6.2.3)

 $H^1(I)$ is a separable Hilbert space when endowed with he following scalar product.

$$(u,v)_{H^1}=\int_I (u'v'+u\ v)$$

△ Proof

We start by proving that it is indeed a scalar product, i.e. the following properties:

- Symmetry (obvious)
- It is a norm:

$$||u||_{H^1}^2 = \int_I [(u')^2 + u^2]$$

We then need to show that it is a Banach space with the following scheme:

Take a cauchy requerce funt at
$$A = 0$$
 $A = 0$ $A = 0$

All we need now is proving that the Hilbert space we constructed is indeed **separable**, we can use H^1 own isomorphism to L^2 which is separable in its own right, then construct a linear map between the two:

$$Lu=(u,u')$$

Every function in H^1 can be represented by a continuous function, formally:

$$H^1(I)\subset C^0(\overline{I})$$

This is interesting because H^1 also contains discontinuous functions, then how is it a subset of C^0 ??

This is of course not what the theorem is saying, more specifically

Proof

"Prof"
$$u \in H^1(\Sigma)$$
 than $u' \in L^2(\Sigma)$ $c \mid l' \mid (\Sigma)$

$$\int_{0}^{\infty} u'(t) dt \quad \text{well defined } V \times$$

$$u(\pi) = u(\alpha) + \int_{0}^{\infty} u'(t) dt \quad \text{Fundamental theorem of calculus}$$

$$u(\pi_0) = u(\pi_0) + \int_{0}^{\infty} u'(t) dt \quad \Rightarrow u \in AC$$

$$u(\pi_0) = u(\pi_0) = \int_{0}^{\pi_0} u'(t) dt \quad \Rightarrow 0 \quad \text{if } \pi_0 \to \infty \implies u \text{ is continuous}$$

△ Definition (6.2.7)

We define the closure of $\mathcal{D}(I)$ with respect to the $H^1(I)$ norm as the following:

$$H^1_0(I) = \overline{\mathcal{D}(I)}^{H^1(I)}$$

Recall that $\mathcal{D}(\Omega)$ is the space of *smooth* functions with compact support over whatever Ω is.

Essentially we're defining the space of $H^1(I)$ functions that vanish at the boundary of I.

Note that the closure with respect to the L^2 norm has special properties, specifically:

$$\overline{\mathcal{D}(I)}^{L^2(I)} = L^2(I)$$

This is because $\mathcal{D}(I)$ is dense in $L^2(I)$, , meaning that the space of *smooth enough* functions with a compact support within the boundaries of I is dense in L^2 . If you think about it this makes sense since these functions would be L^{∞} (i think...).

A Remark (6.2.7)

If and only if $I
eq \mathbb{R}$

$$H_0^1(I) \subseteq H^1(I)$$

Definition

A weak solution satisfies

$$u\in H^1_0(a,b)$$

$$\int_a^b (u'arphi'+uarphi) = \int_a^b farphi \qquad orall arphi \in H^1_0(a,b)$$

We can rewrite this as

$$(u,arphi)_{H^1}=(f,arphi)_{L^2}$$

You can then apply **Lax-Milgram** (1.7.4), prove the hypotheses on the bilinear form and conclude that $\exists ! u$.

△ Poincarè inequality (6.2.9)

Let I = (a, b) bounded then:

$$||u||_{L^2(I)} \le (b-a) \ ||u'||_{L^2(a,b)} \qquad orall u \in H^1_0(I)$$

As a consequence the map $u o ||u'||_{L^2}$ defines a norm in $H^1_0(I)$ which is equivalent to the norm of $H^1(I)$

△ Proof

It's better if you use the books proof.

Proof St sufficien to prove (PI) 4
$$u \in O(I)$$
 (by density)

$$4 \times e(a,b) \quad u(x) = \int_{a}^{x} u'(b) db \quad \text{fundamental theorem of collines}$$

$$=> |u(x)| \le \int_{a}^{x} |u'(b)| db \le \int_{a}^{b} |u'(b)| db \stackrel{\triangle}{=} \sqrt{b-a} ||u'||_{L^{2}(a,b)}$$

=) If
$$u$$
 if u (a , b) \in if u' if u (a , b) \in $\sqrt{b-a}$ if u' if u (a , b) \in (a - a) \in if a' if a (a , b) \in if a' if a (a , b) \in if a' if a (a , b) \in if a' if a (a , b) \in if a' if a (a , b) \in if a' if a (a , b) \in if a' if a (a , b) \in if a' if a (a , b) \in if a' if a (a , b) \in if a' if a if a

Let H_1 and H_2 be Hilbert spaces such that:

$$H_1\subset H_2 \qquad (H^1\subset L^2,\ H_0^1\subset L^2)$$

Any Hilbert space is the dual of itself:

$$H_1'pprox H_1$$

$$H_2'pprox H_2$$

Therefore if $H_1 \subset H_2$ and $H_2' \subset H_1'$ (since a smaller Hilbert space will have a larger number of linear and continuous functionals, this is taken straight from notes I don't condone this behaviour):

$$H_2\subset H_1$$

But this is absurd, look at the hypothesis!

There's clearly a mistake somewhere and that mistake lies in the assumption that we can use both isomorphisms at the same time, which isn't true due to the **Riesz representation theorem** (1.5.5).

A Hilbert (Gelfand) triple (Bottom of 6.2.18)

The following is true:

$$H_1\subset H_2\underbrace{pprox}_{(1)}H_2'\subset H_1'$$

$$H^1_0(I) \subset L^2(I)$$
 \subset $(H^1_0(I))' = H^{-1}(I)$

Where:

- 1. Is the Riesz representation theorem (1.5.5)
- 2. The pivot space is the space in which we make use of the dual (in this case L^2)

Moreover if we define:

$$Lu(v) = \int_I u \, v \qquad orall u \in L^2(I)$$

Then the map v o Lu(v) is linear and continuous $orall v \in H^1_0(I)$

△ Proposition 6.2.10

Let $F \in H^{-1}(I)$ then:

$$\exists f_0, f_1 \in L^2(I)$$

Such that:

$$< F, v> = \int_I f_0 v + \int_I f_1 v \qquad orall v \in H^1_0$$

and

$$||F||_{H^{-1}(I)} = \max\{||f_0||_{L^2}(I) \ , \ ||f_1||_{L^2}(I)\}$$

In the case where $I=\mathbb{R}$ then substitute $H^1_0(I)$ with $H^1(I)$.

In the case where I is unbounded we can take $f_0 = 0$.

This a sort of representation for linear and continuous functionals over $H_0^1(I)$.

Now suppose that $f_0 = 0$:

$$< F, v> = \int_I f_1 v' \underbrace{=}_{IRP} - \int_I f'_1 v$$

but this basically means $F = -f_1'$.

...

Not so fast bucko, this is an illegal move since $f_1 \in L^2$ so we don't know its derivative necessarily.

We, on the other hand, can say (by defining it as such) that :

$$F=-f_1'$$

Implies that F has -1 derivative in L^2 .

△ Dirac delta

&. $G \in H^{-1}$ $\forall c \in (a,b]$ $cG_c, u > = u(c)$ G is a linear and continuous functional Heaviside function belongs to $L^2(I)$, I bounded, G is its derivative

The \mathbb{R}^n case

We'll take a domain Ω in \mathbb{R}^n to be an open set, for example $\partial\Omega$.

A Example

$$\Omega = \{(x,y) \in [0,1]^2 \; , \; x,y \in \mathbb{Q} \}$$
 $\partial \Omega = [0,1]^2$

Our main assumptions will be that Ω has to be an open set and $\partial\Omega$ smooth. Let's say we want $\partial\Omega\in C^1$, it can be constructed as the union of locally C^1 functions. (3.1.2)

Now we want to define $H^1(\Omega)$ (we'll worry about the boundary later on).

lacktriangle Weak derivative in \mathbb{R}^n

The *i*-th weak derivative $w=\frac{\partial u}{\partial x_i}$ of u is such that:

$$\int_{\Omega} u \; rac{\partial v}{\partial x_i} = \int_{\Omega} w \, v \qquad orall v \in \mathbb{D}(I)$$

△ Sobolev space (6.,2.11)

Let $\Omega \in \mathbb{R}^n$ be an open set, the Sobolev space H^1 is defined as:

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \; ; \; rac{\partial u}{\partial x_i} \in L^2(\Omega) \qquad orall i = 1, \ldots, n
ight\}$$

(6.2.12)

Given $u \in H^2(\Omega)$:

We take

$$H^1(\Omega)pprox L^2(\Omega)^{n+1}$$

Meaning we take these spaces to be isomorphic.

We define the gradient as:

$$abla u = egin{bmatrix} rac{\partial u}{\partial x_i} \ dots \ dots \end{bmatrix} \in L^2(\Omega)^n$$

We define the norm to be:

$$|
abla u|^2 = \sum_{i=1}^n \left(rac{\partial u}{\partial x_i}
ight)$$

Taking derivative in the classical sense:

$$\Big(u\in C^1(\Omega)igcap L^2(\Omega)\Big)\wedge \Big(urac{\partial u}{\partial x_i}\in L^2(\Omega)\Big)\Longrightarrow u\in H^1(\Omega)$$

Moreover the classical partial derivatives coincides with the weak ones, additionally if Ω is bounded:

$$\Longrightarrow C^1(\overline{\Omega})\subset H^1(\Omega)$$

We finally define the bilinear form on H^1 to be:

$$(u,v)_{H^1} = \int_\Omega \left(u \cdot v +
abla u imes
abla v
ight)$$

Where the first is a product between two scalar functions and the second is a scalar product (sum of n+1 terms).

The former will be sometimes omitted.

We can define a norm (6.2.13) therefore

$riangleq H^1$ separability in \mathbb{R}^n (6.2.14)

 H^1 is separable since we can:

Define a scalar product such as:

$$(u,v)_{H^1}$$

- · We can define a norm
- The space is complete

Refer to the book for an actual proof.

\triangle Closure to H^1 (6.2.15)

$$H^1_0(\Omega) = \overline{\mathbb{D}(\Omega)}^{H^1(\Omega)} \subset H^1(\Omega)$$

The book goes in depth on how actually it is defined..

Example

80.
$$f(x) = \frac{1}{|x|^2}$$
 in $\mathbb{R}^m \setminus B_1$ $f \in L^p(\mathbb{R}^m \setminus B_1)$ (=) $\int_{\mathbb{R}^m \setminus B_1} |f(x)|^p dx \ c + \infty$

We can take adventage of poeces coordinates

$$\int_{\mathbb{R}^{n}/B_{1}} |f(x)|^{p} dx = cm \int_{0}^{x} \frac{b_{x}b}{b_{x}y} dy \qquad m-1-2b < -\gamma = 2b > \frac{4}{m}$$

△ Example (6.2.16)

Basically just show that the function and its gradient are $L^2(\Omega)$ and you're good to go. This is also interesting because it's a counter example to the notion that (in $n \geq 2$) a function only need be limited to be in the respective Sobolev space.

Show that
$$f \in H_1(B)/r$$

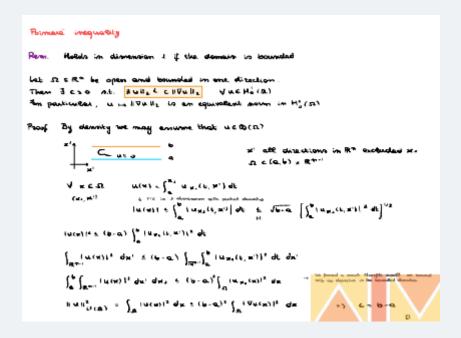
$$|\Delta f(x)| = |f_1(f_1)|$$

$$|\Delta f(x)| = |f_1(f_$$

Assume that Ω is bouned. Then there exists a constant $C(\Omega)$ such that:

$$\|u\|_{L^2} \leq C\, \|
abla u\|_{L^2} \qquad orall u \in H^1_0(\Omega)$$

Moreover the map $u \to \|u\|_{L^2}$ us a norm in $H^1_0(\Omega)$ which is equivalent to the norm $\|u\|_{H^1}$.



Professor then goes on to define the relevant Hilibert triple

$$H^1_0(\Omega)\subset L^2(\Omega)\subset H^{-1}(\Omega)$$

Where the second continuous embedding is to be meant as follows:

Each function $u \in L^2(\Omega)$ is identified with the linear functional $I_u \in H^{-1}(\Omega)$ defined as :

$$0 < I_u, v > \ := \int_\Omega u \ v \qquad orall v \in H^1_0(\Omega)$$

Essentially we can map each L^2 function to a functional from H^1 to \mathbb{R} .

Note that we denote by $H^{-1}(\Omega)$ the dual space of $H^1_0(\Omega)$.

A Proposition (6.2.19)

Let $\Omega\subset\mathbb{R}^n$ be open and let $F\in H^{-1}(\Omega)$, then $\exists\{f_i\}_{i=0}^n\in L''(\Omega)$ such that:

$$0 < F, v > \ = \int_{\Omega} f_0 \ v + \sum_{i=1}^n f_i rac{\partial u}{\partial x_i} \qquad orall v \in H^1_0(\Omega)$$

And:

$$\|F\|_{H^{-1}} = \max_{i=0,...,n} \|f_i\|_L^2$$

Moreover, if Ω is bounded, we can take $f_0=0$

△ Remark

Note that in this case $v\in L^2(\Omega)$ because $v\in H^1_0(\Omega)$

△ Proof

There's a proof in the book but here's what the prof wrote:

Proof of fo =0
$$(F, v) = \int_{\Omega} f \cdot \nabla v \qquad f = (f_{1}, f_{2}, ..., f_{m}) \in L^{2}(\Omega)^{m}$$

For smooth f , $\int_{\Omega} f \cdot \nabla v = -\int_{\Omega} v \operatorname{div} f + \int_{\partial \Omega} v (f_{1}v) \qquad v_{1} \text{ normal}$

=) " $F = -\operatorname{div} f$ " the linear continuous functioned in H^{-1} is the divergent of a function in L^{2}

H¹(Ω) $\not\subset L^{\infty}(\Omega)$ if $m \ge 1$

H²(Ω) $\not\subset L^{2}(\Omega)$ $\forall m$

△ Sobolev embedding theorem (6.2.20 - 6.2.21)

Let $\Omega \subset \mathbb{R}^n$ be an open domain with $\partial \Omega \in Lip$ and $n \geq 2$. Then:

$$H^1(\Omega) \subset L^p(\Omega) \qquad egin{cases} orall 2 \leq p < \infty & if \ n=2 \ orall 2 \leq p < rac{2n}{n-2} & if \ n=3 \end{cases}$$

In addition to the above, if Ω is bounded, then the embeddings become compact:

the following continuous embeddings:

- (i) if n = 2 then $H^1(\Omega) \subset L^q(\Omega)$, for every $1 \le q < \infty$;
- (ii) if $n \ge 3$ then $H^1(\Omega) \subset \subset L^q(\Omega)$, for every $1 \le q < 2^* = \frac{2n}{n-2}$.

Proof. See [1, Theorem 6.2].

△ Remark

The following is called the **Critical Sobolev exponent**:

$$2^* = \frac{2n}{n-2}$$

A sequence that converges in the $H^1(\Omega)$ sense implies convergence in a certain L^p sense, formally:

$$u_n \stackrel{H^1(\Omega)}{\longrightarrow} u \implies u_n \stackrel{L^p}{\longrightarrow} u$$

The H^k spaces

We basically take 6.2.11 and apply it a bunch of times to define higher order weak derivatives.

Note that in the image below the prof *formally* sets $H^0=L^2$ (6.3.4)



We call a multi-index a vector:

$$lpha=(lpha_1,\ldots,lpha_n)\in\mathbb{N}^n$$

With norm:

$$|lpha|=\sum_{i=1}^mlpha_i$$

We'll use this to define partial derivatives kinda like this:

$$D^lpha v = rac{\partial^{|lpha|} v}{\partial x_1^{lpha_1}, \ldots, \partial x_n^{lpha_n}}$$

△ Definition 6.3.1

Let $u \in L^1_{loc}(\Omega)$ and $\Omega \subseteq \mathbb{R}^n$ be an open set. Given a multi-index α we'll say that u admits a weak derivative $D^{\alpha}u$ if there exists $g \in L^1_{loc}(\Omega)$ such that:

$$\int_{\Omega} u D^{lpha} arphi = (-1)^{|lpha|} \int_{\Omega} g arphi \qquad orall arphi \in \mathcal{D}(\Omega)$$

△ Definition and Separability Theorem (6.3.2 - 6.3.5)

Let Ω be as above and $k \in \mathbb{N}$. The $H^k(\Omega)$ is defined by:

$$H^k(\Omega) = \{u \in L^2(\Omega), \qquad D^lpha u \in L^2(\Omega) \qquad orall |lpha| < k \ \} \qquad orall k \in \mathbb{N}$$

We take $H^k(\Omega)$ to be a separable Hilbert space with scalar product:

$$(u,v)_{H^k(\Omega)} = \int_\Omega \left(uv + \sum_{1 \leq |lpha| \leq k} D^lpha u \ D^lpha v
ight)$$

With induced norm:

$$\|u\|_{H^k(\Omega)} = \left(\int_{\Omega \wedge 0 \leq |lpha| \leq k} \sum |D^lpha u|^2
ight)^{rac{1}{2}}$$

△ Proof

Check the book for a more detailed proof, here's what the prof wrote,

H" is a neperable Hilbert space with scalar product

- · it has a scalar product
- · it is complete (use a couchy reguence)
- · it's separable because its L2

△ Separable Banach spaces with no scalar product. (6.3.8)

He defined industries?

At
$$n \in \Gamma_b(\mathfrak{D})$$
 and $D_K n \in \Gamma_b(\mathfrak{D})$ A $|\alpha| = K = \mathcal{I}$ and $n \in \Gamma_b(\mathfrak{D})$ and $n \in \Gamma_b(\mathfrak{D})$

△ Remark (6.3.4)

We define $H^k(\Omega)$ inductively. Check the book.

△ Fourier definition for higher order Sobolev spaces(6.3.5 - 6.3.6)

We can define $H^k(\mathbb{R}^n,\mathbb{C})$ in a more straightforward way using the Fourier transform.

$$H^k(\mathbb{R}^n,\mathbb{C})=\{u\in L^2(\mathbb{R}^n,\mathbb{C});\quad (1+|\xi|^2)^{rac{k}{2}}\ \hat{u}(\xi)\in L^2(\mathbb{R}^n,\mathbb{C})\}$$

In this construction $H^0=L^2$ because $(1+|\xi|^2)^{\frac{k}{2}}=1$ if k=0, we need, however, to define a scalar product to go further.

$$(u,v)_{HK} = \int_{\mathbb{R}^m} (u+|\xi|^2)^K \hat{u}(\xi) \hat{v}(\xi) d\xi \rightarrow move$$

 $(u+|\xi|^2)^{K/2} \rightarrow weight function$

Note that k doesn't *have* to be an integer in this construction, it just has to be $k = s \ge 1$. We then end up with what are called **Non Local PDE's**

Non-integer H^k

As previously mentioned, we'll take $s \ge 0$.

$$H^s(\mathbb{R}^n,\mathbb{C})=\left\{u\in L^2(\mathbb{R}^n,\mathbb{C})\ ,\quad (1+|\xi|^2)^{rac{k}{2}}\hat{u}(\xi)\in L^2(\mathbb{R}^n,\mathbb{C})
ight\}$$

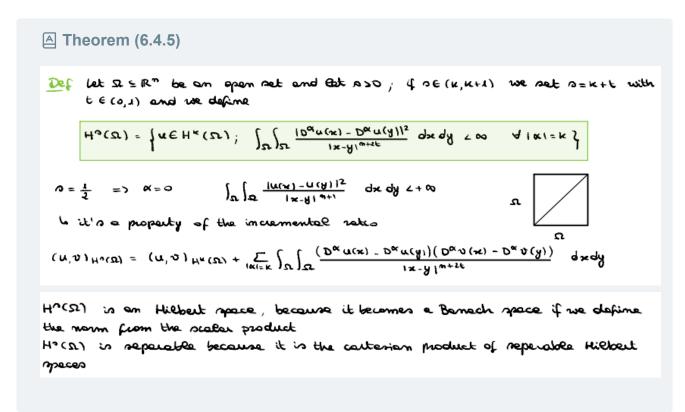
Endowed with the scalar product

$$(u, v)_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi$$

for every $u, v \in H^s(\mathbb{R}^n, \mathbb{C})$.

(6.4.1)

We can extend this construction to a general domain Ω .



H_0^s spaces and Trace operators

□ Definition (6.5.1)

For every $s \geq 0$:

$$H^s_0(\Omega)=\overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$$

A special case is the following:

$$\Omega = \mathbb{R}^n \Longrightarrow H^s_0(\Omega) = H^s(\Omega)$$

Otherwise