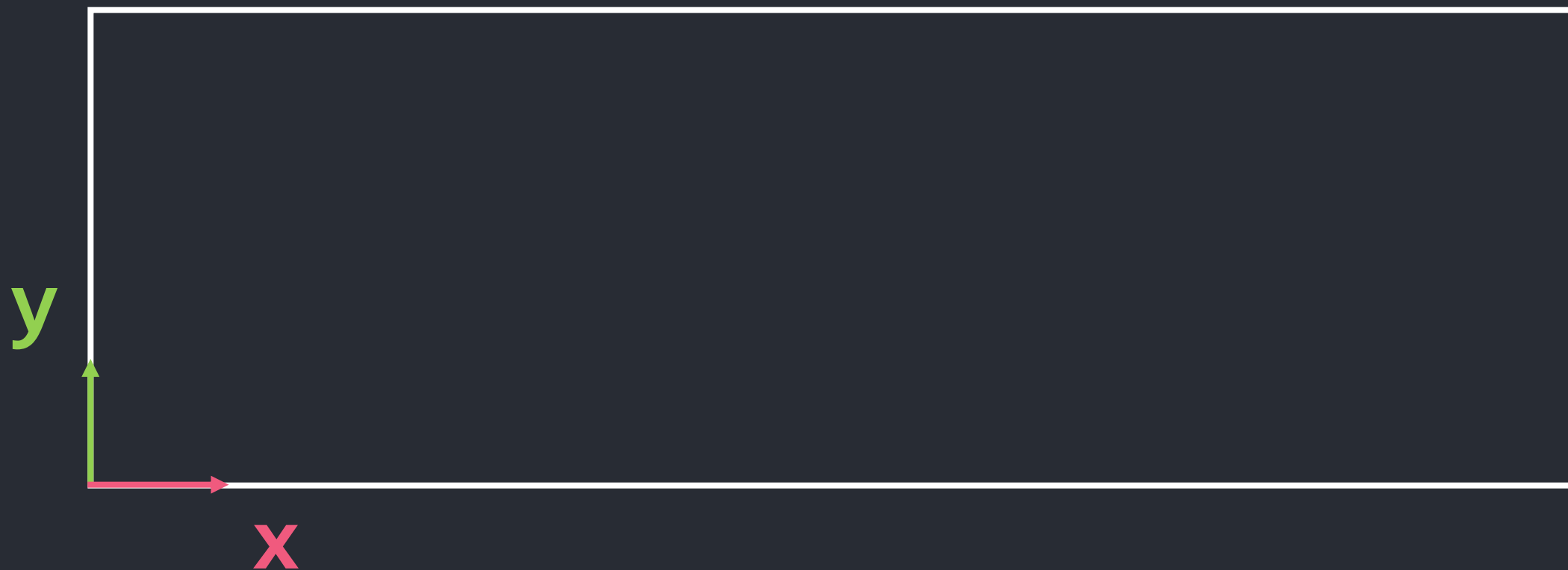
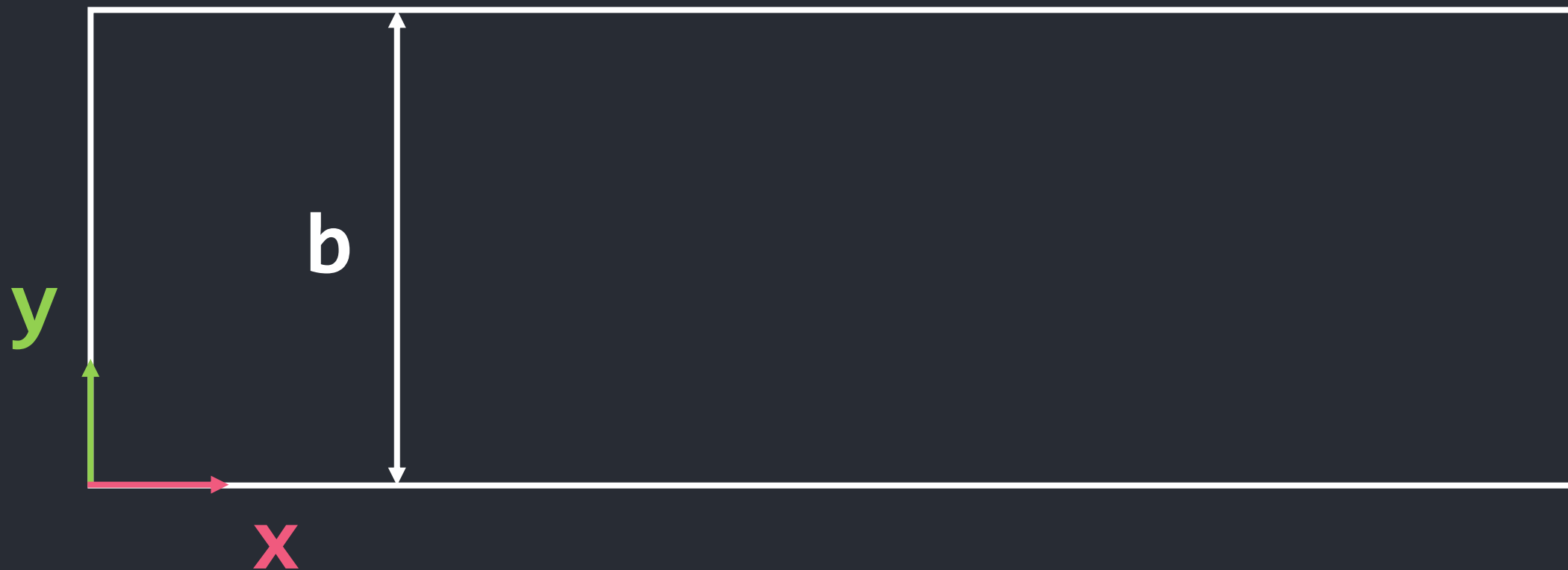
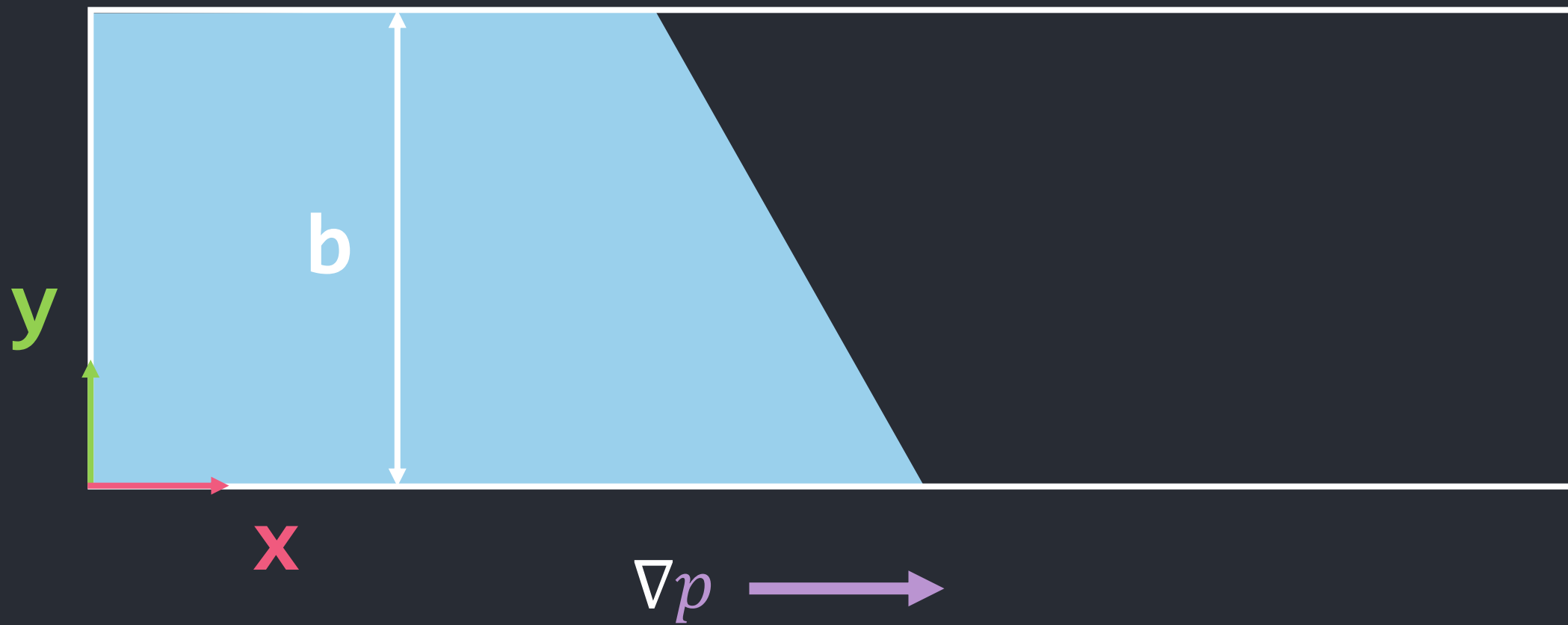


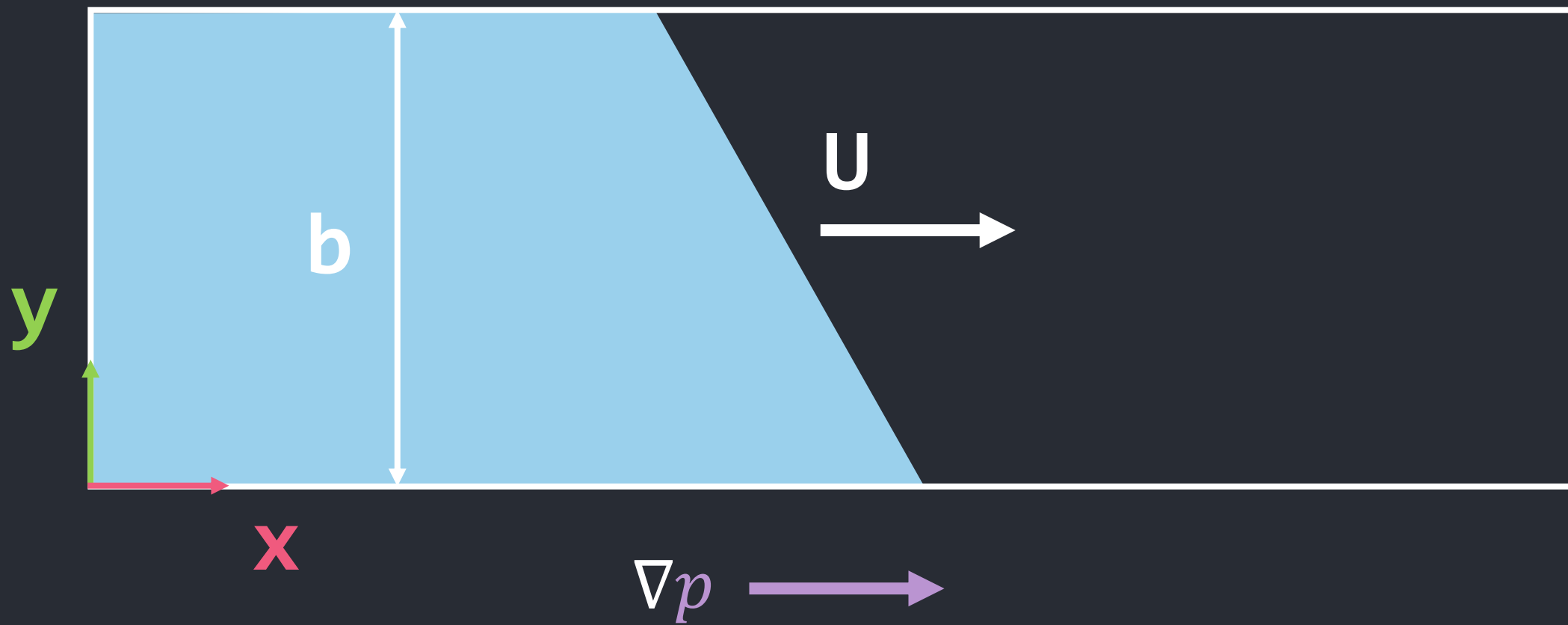
Navier-Stokes Equation

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}$$

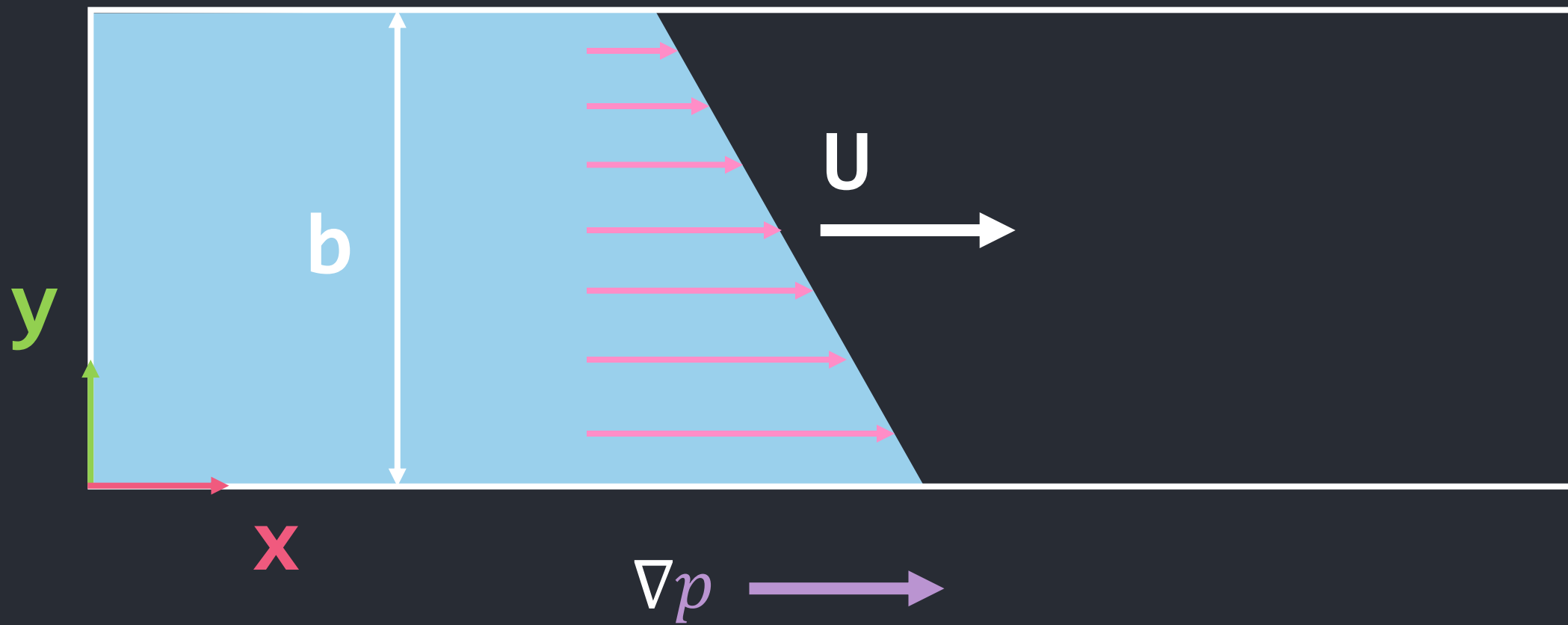




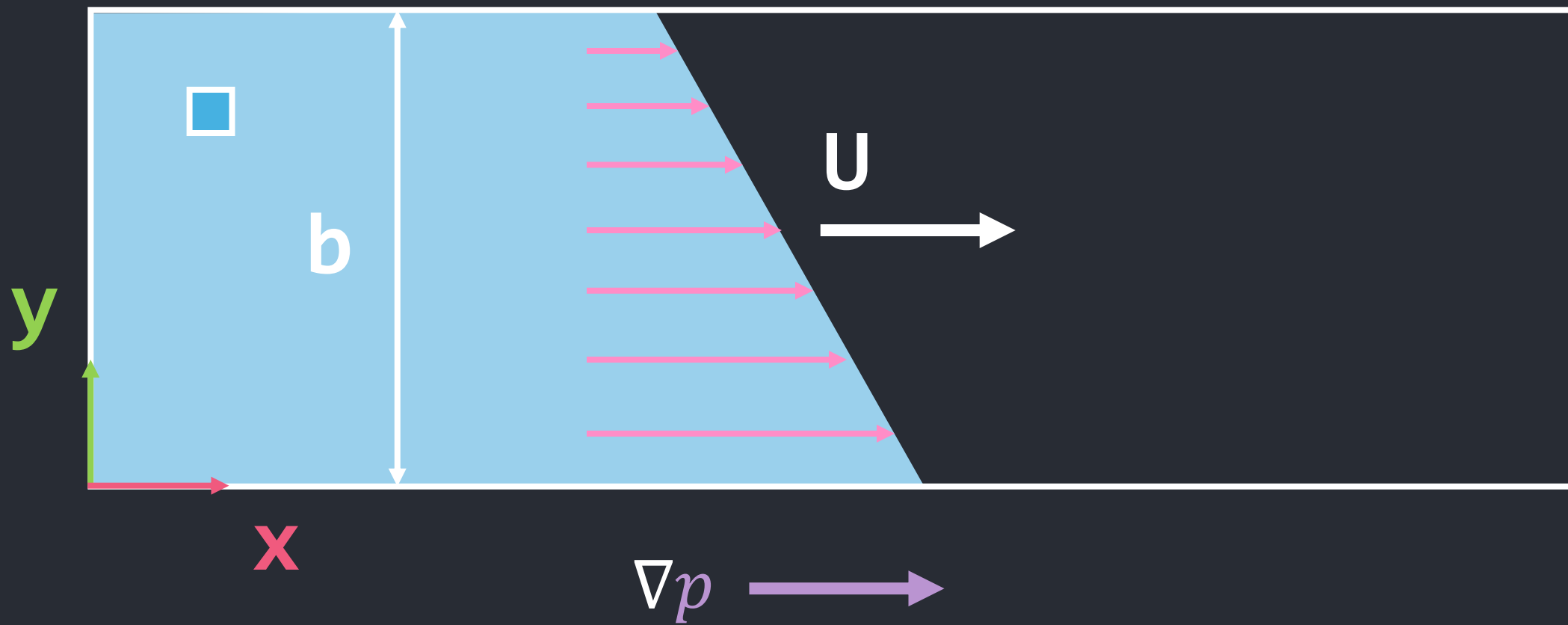




$$u = u(x, y)$$



$$u = u(x, y)$$

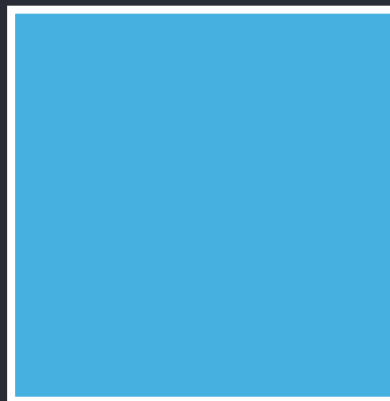


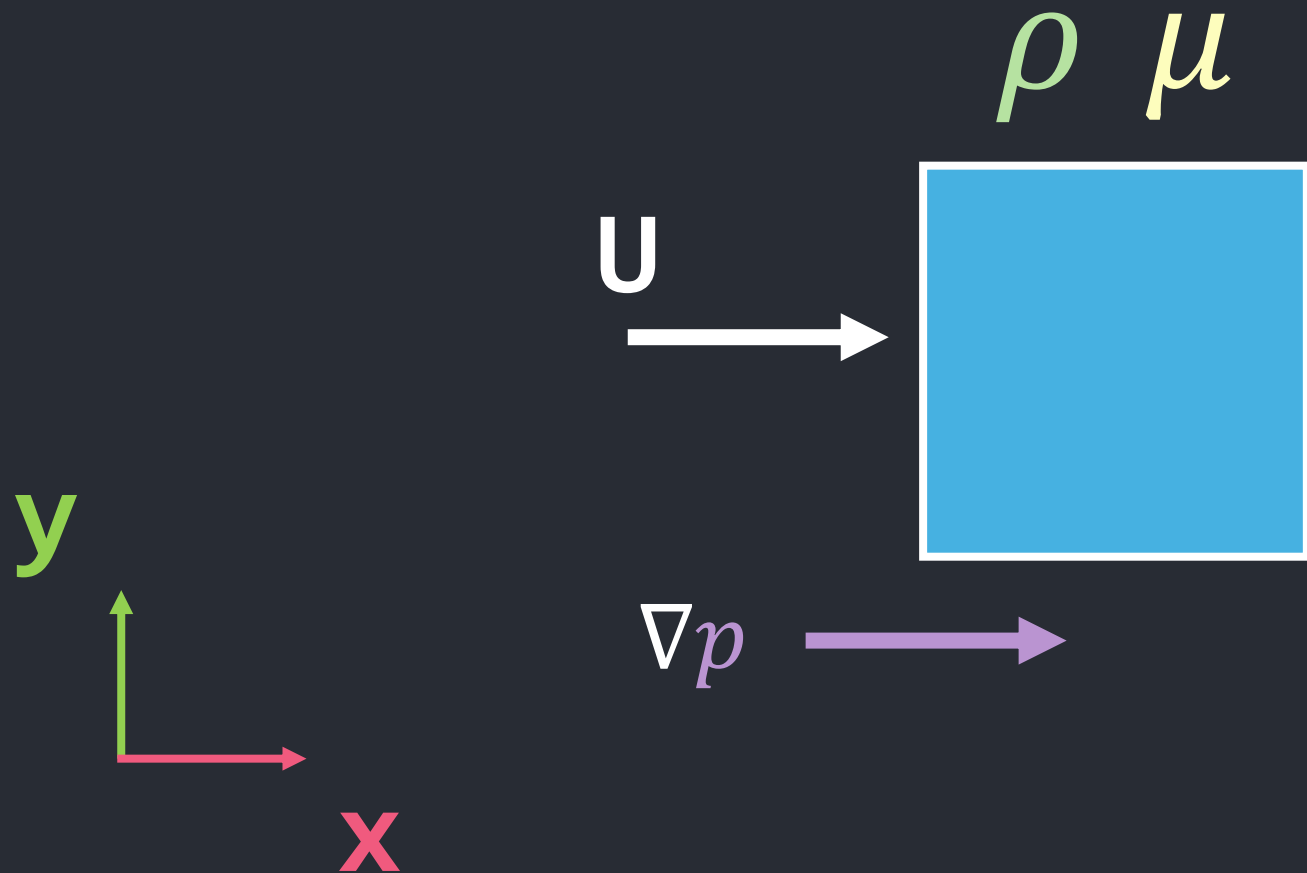
y

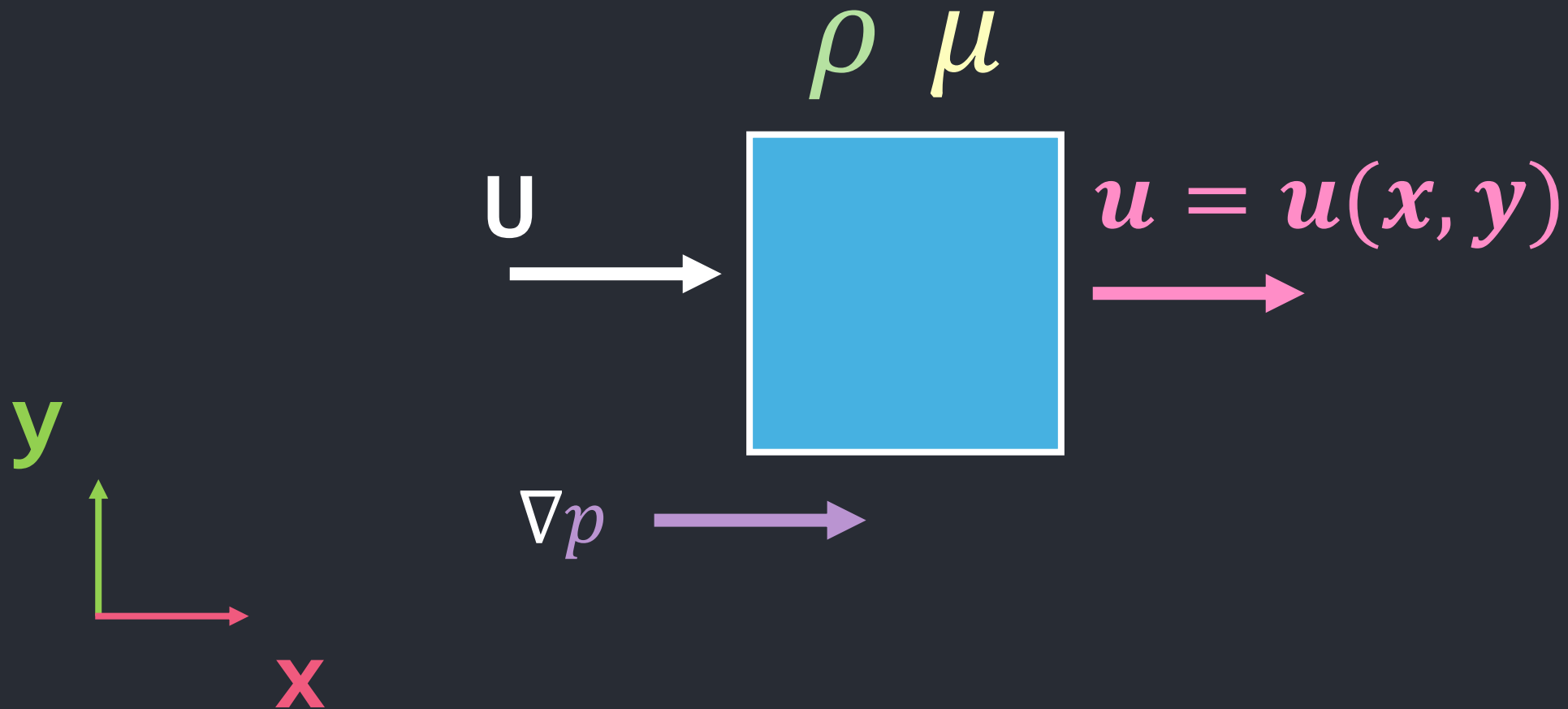


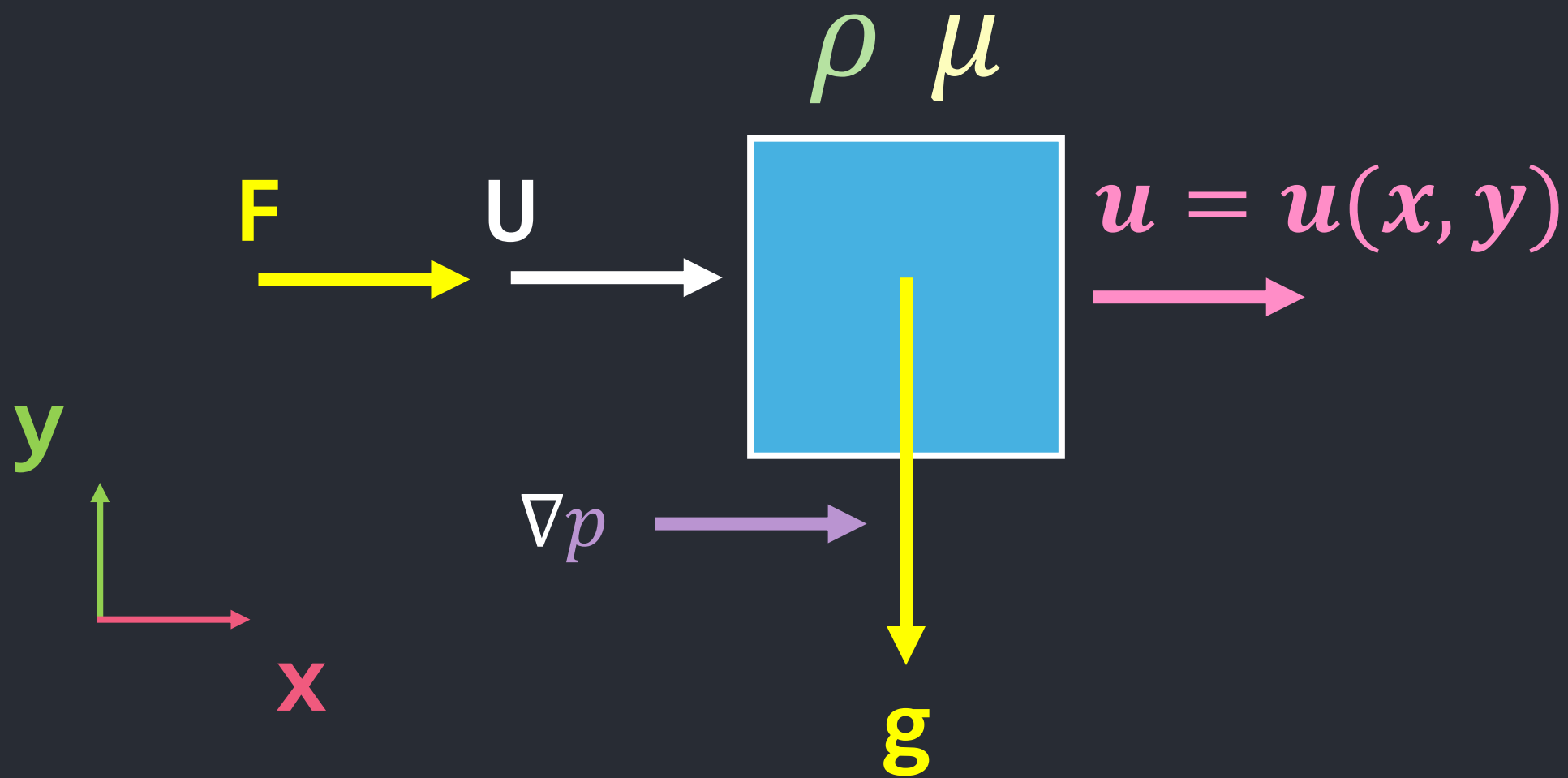
x

ρ









Navier-Stokes Equation

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}$$

$\mathbf{u} \rightarrow \text{Speed: } m/s$

$\rho \rightarrow \text{Fluid Density: } kg/m^2$

$p \rightarrow \text{Pressure: } Pa$

$\mu \rightarrow \text{Dynamic Viscosity: } Pa$

$\mathbf{F} \rightarrow \text{External Force: } N$

$t \rightarrow \text{Time: } s$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}$$

Newton's 2nd Law

$$ma = \sum F$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}$$



\mathbf{ma}

$\frac{\mathbf{ma}}{V}$

$$\underbrace{\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right)}_{\frac{ma}{V}} = \underbrace{-\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}}_{\frac{F}{\bar{V}}}$$

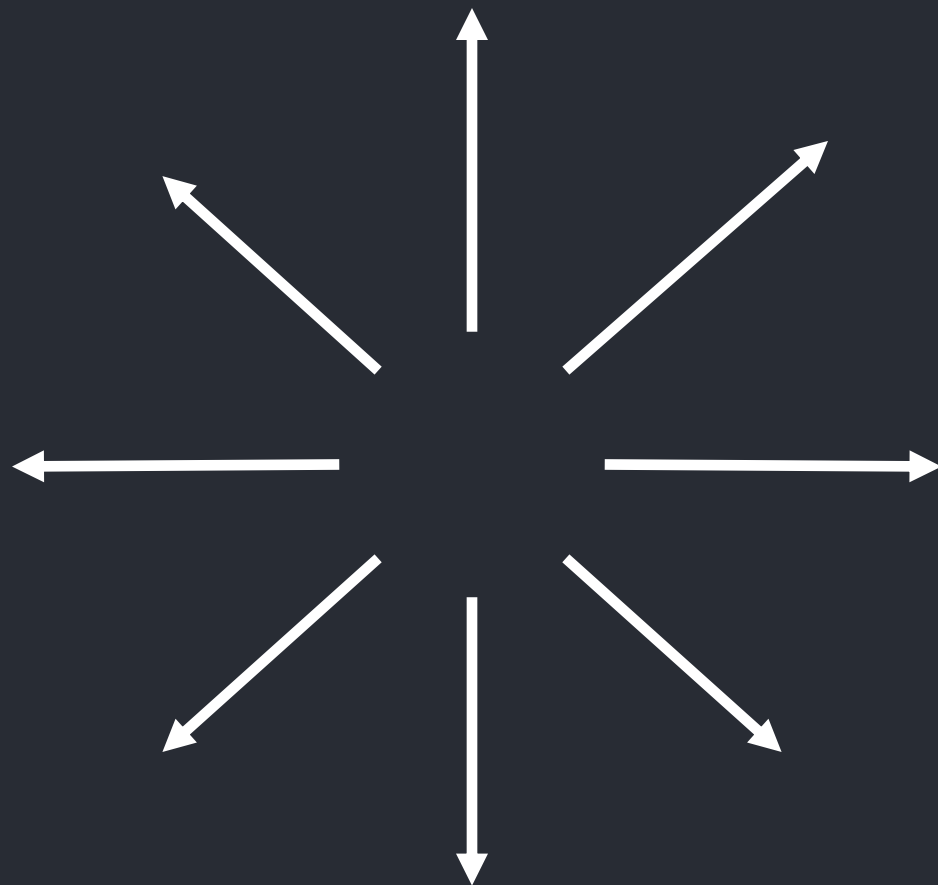
Divergence

$$\nabla \cdot \boldsymbol{u}$$

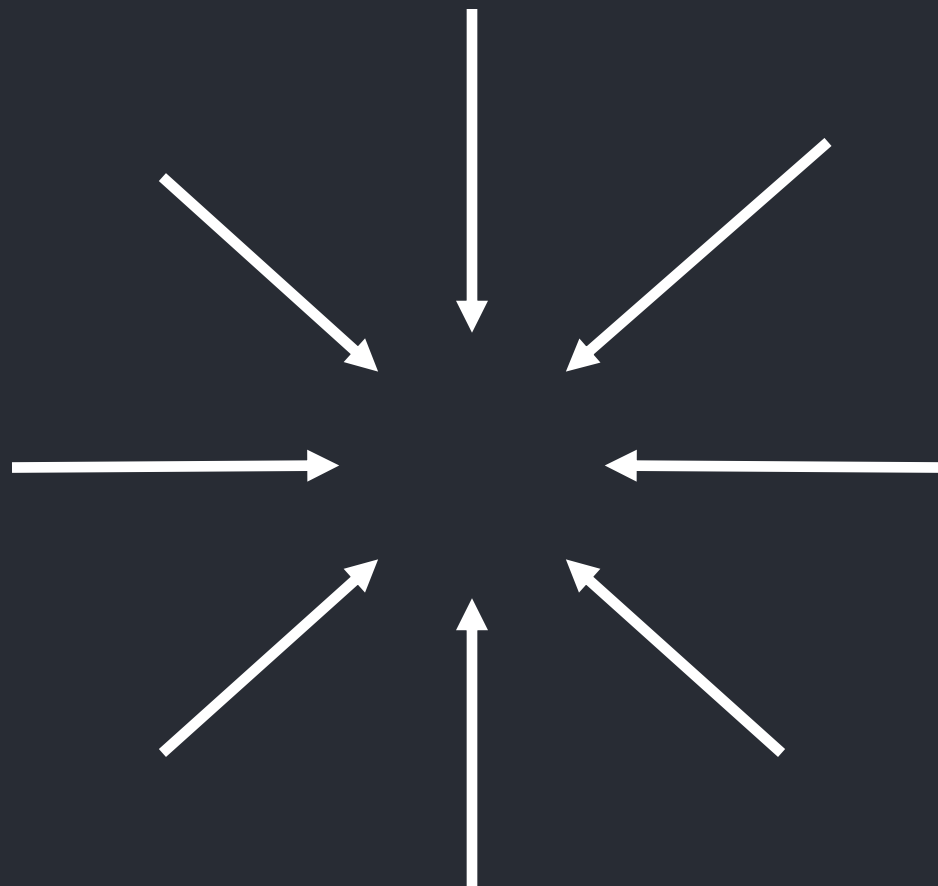
$$\boldsymbol{u} = u(x, y)$$

$$\nabla \cdot \boldsymbol{u} = \operatorname{div} \boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}$$

$$\nabla u > 0$$



$$\nabla u < 0$$



$$\nabla u = 0$$

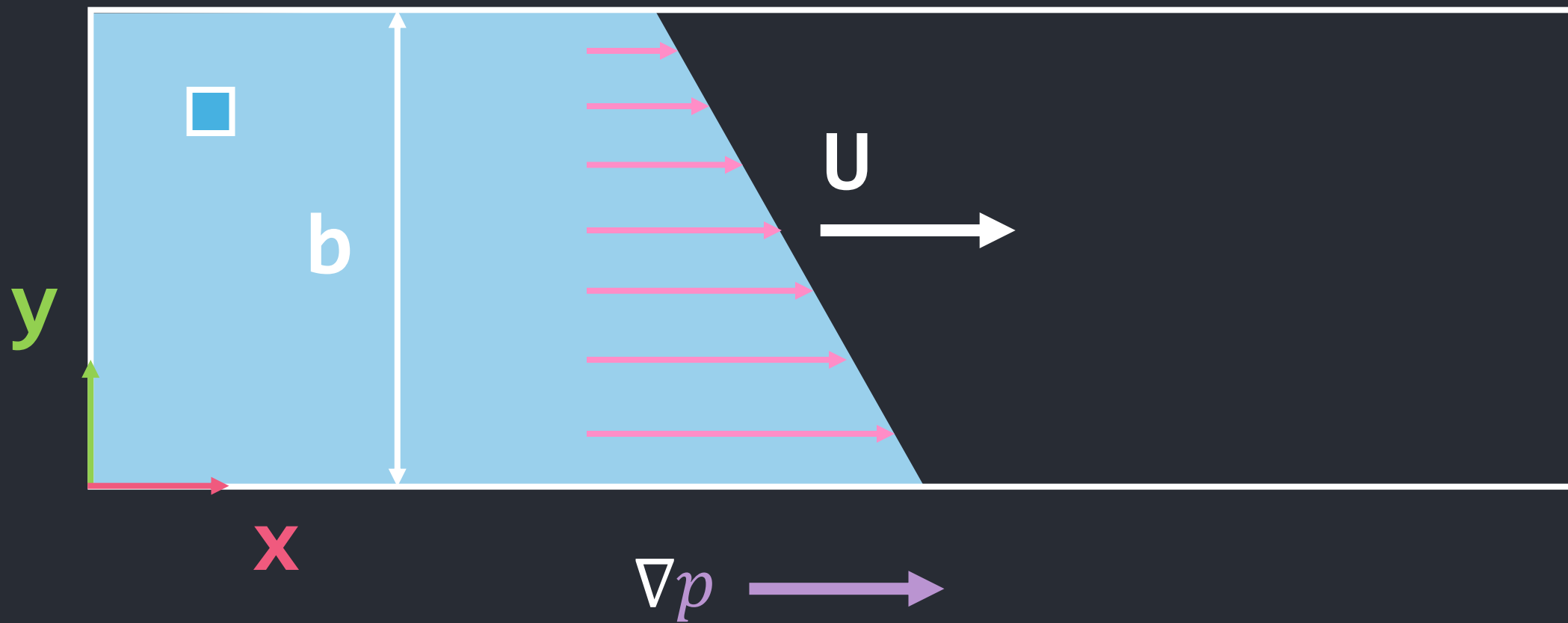


Incompressible Flow

(AKA: Continuity Equation)

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$u = u(x, y)$$



Conditions

- *No flow in vertical direction $\rightarrow u_y = 0 \frac{m}{s}$*
- *Flow driven by pressure gradient $\rightarrow \frac{\partial p}{\partial x} \neq 0 \frac{Pa}{m}$*
- *No gravity effects $\rightarrow g = 0 \frac{m}{s^2}$*
- *Boundary of the pipe $\rightarrow y = [0, b]$*
- *Incompressible Flow $\rightarrow \nabla \mathbf{u} = \mathbf{0}$*

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho g$$

Incompressible Flow

(Continuity Equation)

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

Fully Developed Flow

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

Fully Developed Flow

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

0 No flow in y

Fully Developed Flow

$$\frac{\partial u_x}{\partial x} = 0$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho g$$

Steady Flow

No flow in y

No g effects

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho g$$

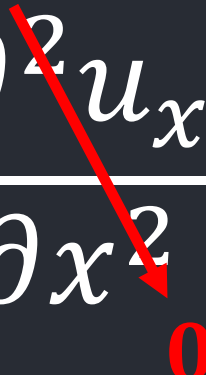
Diagram illustrating the simplification of the Navier-Stokes equation for a fully developed flow. Red arrows point from the following terms to a red '0' below them:

- $\frac{\partial \mathbf{u}}{\partial t}$ (Steady Flow)
- $u_x \frac{\partial u_x}{\partial x}$ (No flow in y)
- $u_y \frac{\partial u_x}{\partial y}$ (No flow in y)
- ρg (No g effects)

Fully Developed

$$0 = -\nabla p + \mu \nabla^2 u$$

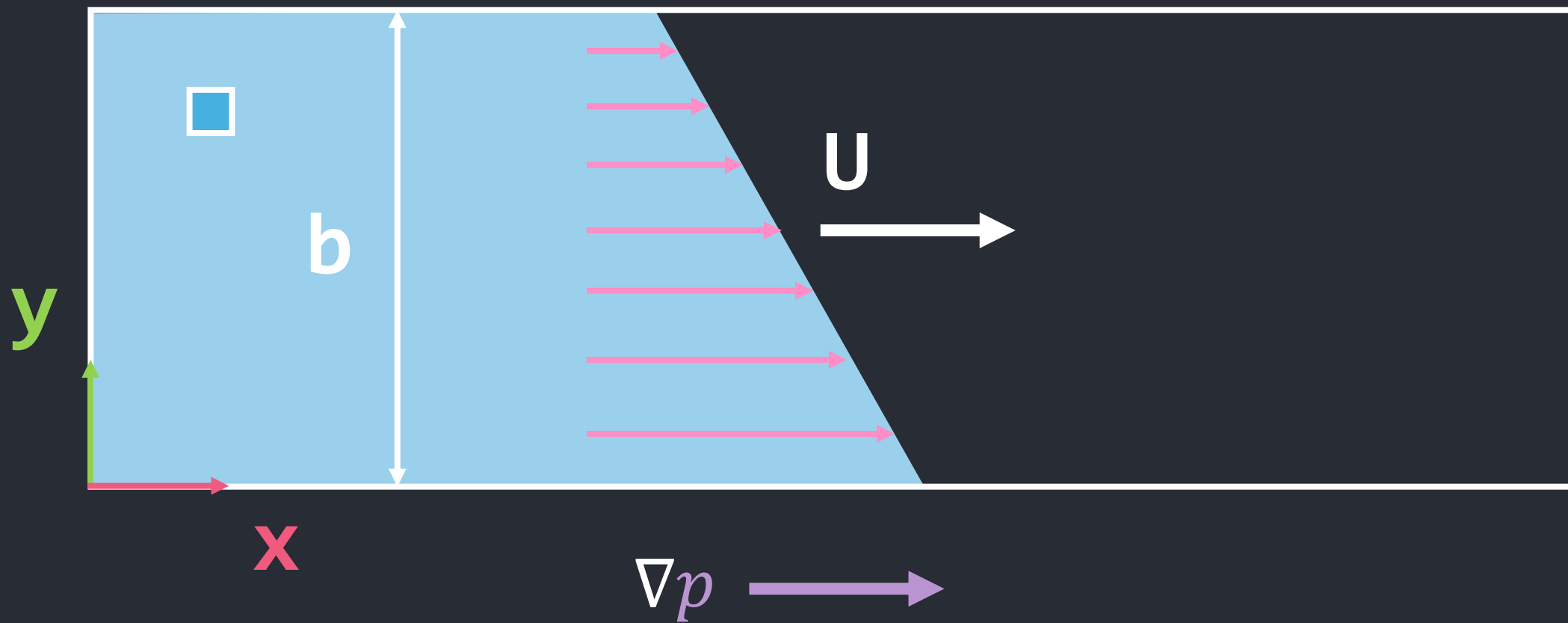
Fully Developed

$$0 = -\nabla p + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right)$$


A red arrow points from the top of the $\frac{\partial^2 u_x}{\partial x^2}$ term to a red zero located below the ∂x^2 denominator.

$$0 = -\nabla p + \mu \left(\frac{\partial^2 u_x}{\partial y^2} \right)$$

$$u = u(x, y)$$



Flow Profile

$$u(y) = ???$$

$$0 = -\nabla p + \mu \left(\frac{\partial^2 u_x}{\partial y^2} \right)$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{1}{\mu} \nabla p = \textit{const}$$

Flow Profile

$$u(y) = \frac{1}{2\mu} \nabla p (b^2 - by) + U \left(1 - \frac{y}{b} \right)$$

