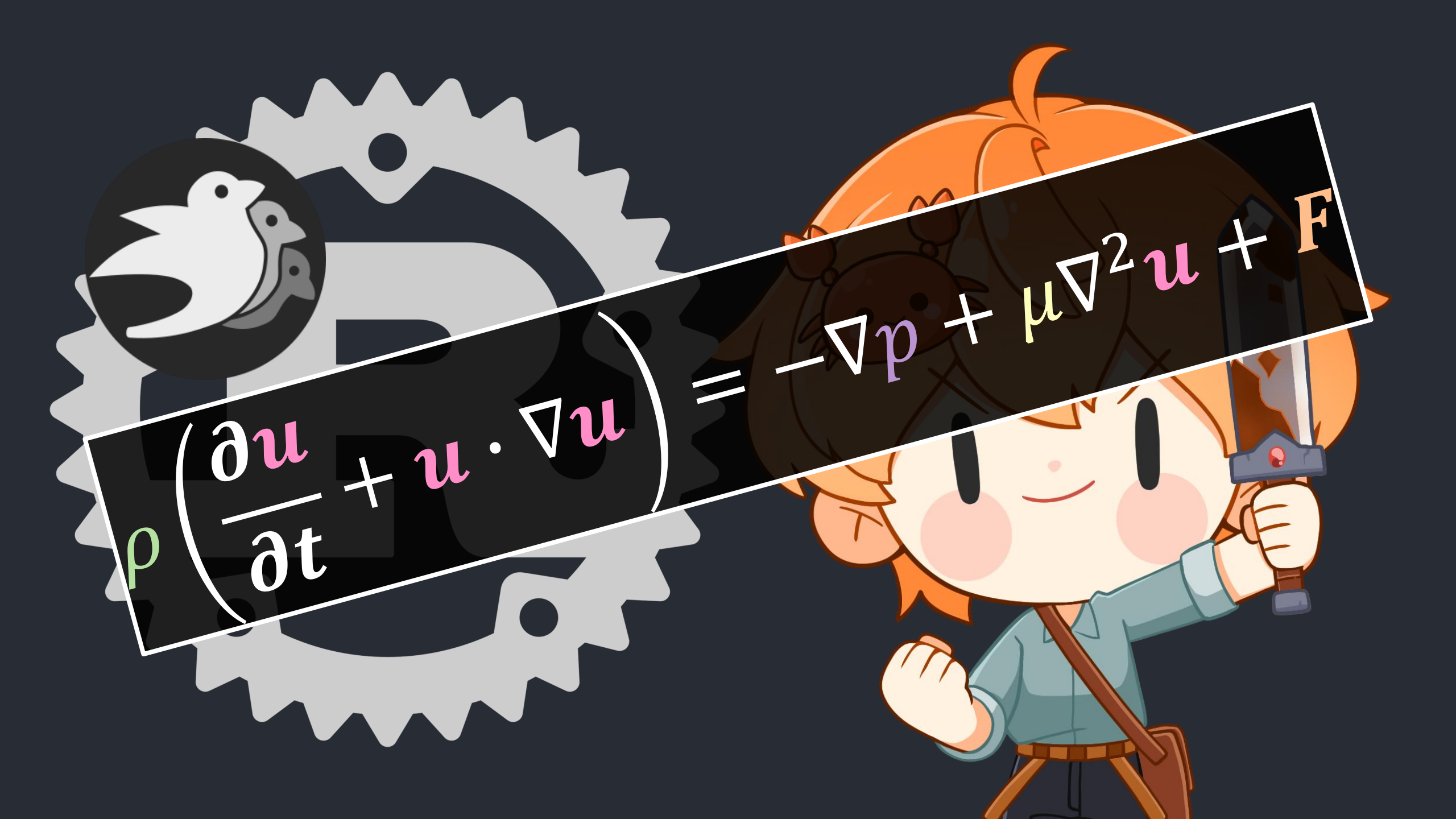
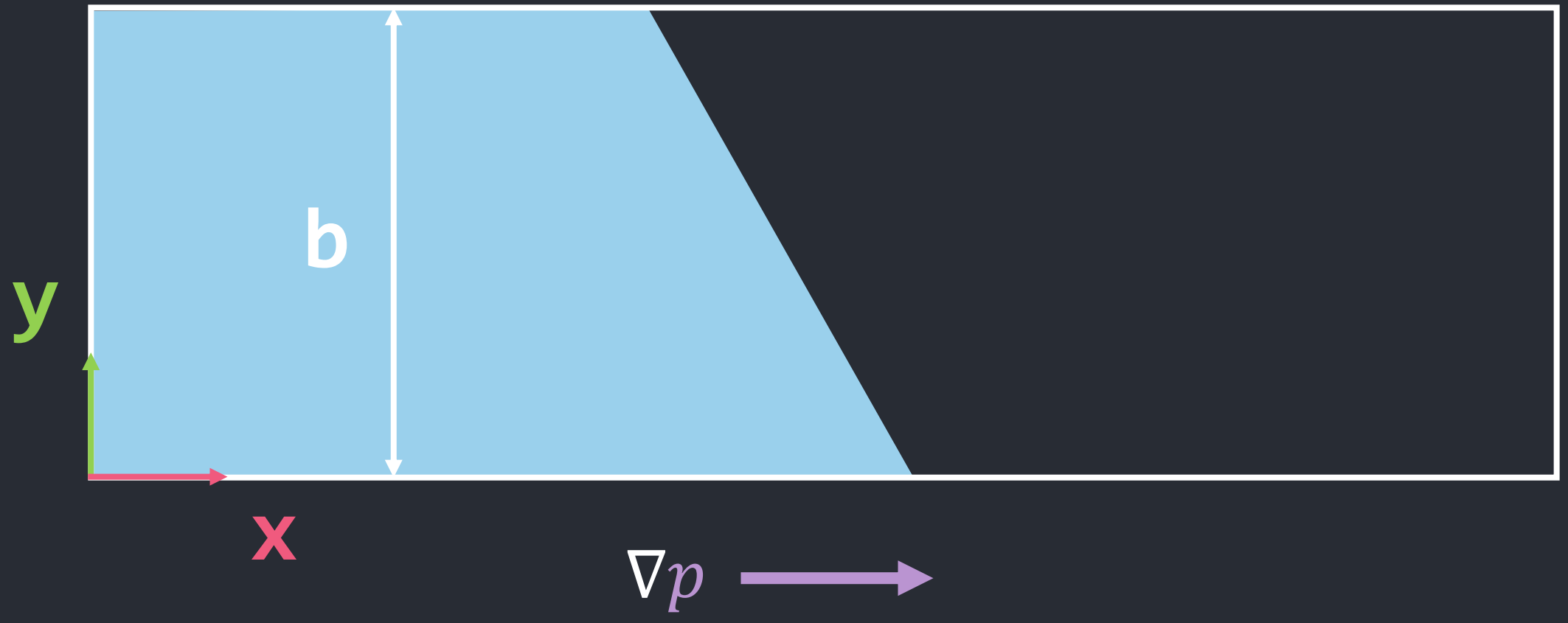

$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p + \mu \nabla^2 u + \mathbf{F}$$

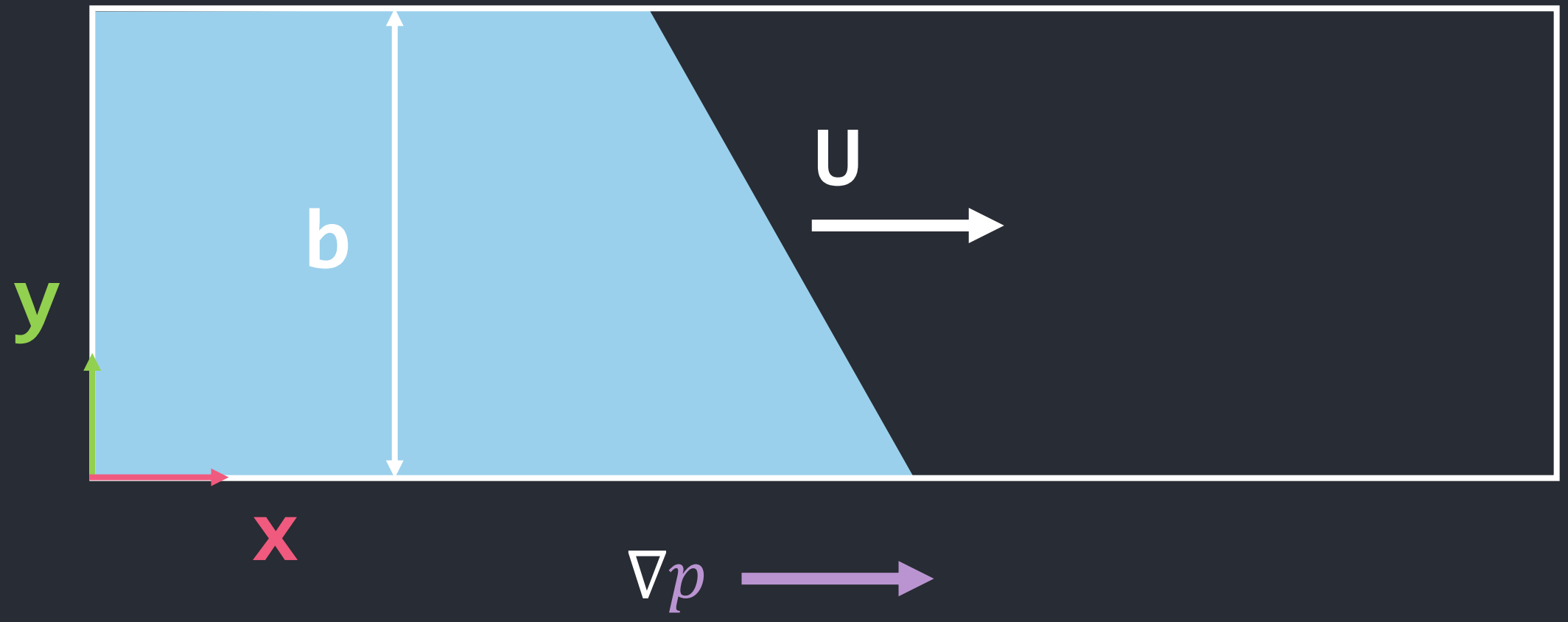
Navier-Stokes Equation

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} + \boldsymbol{F}$$

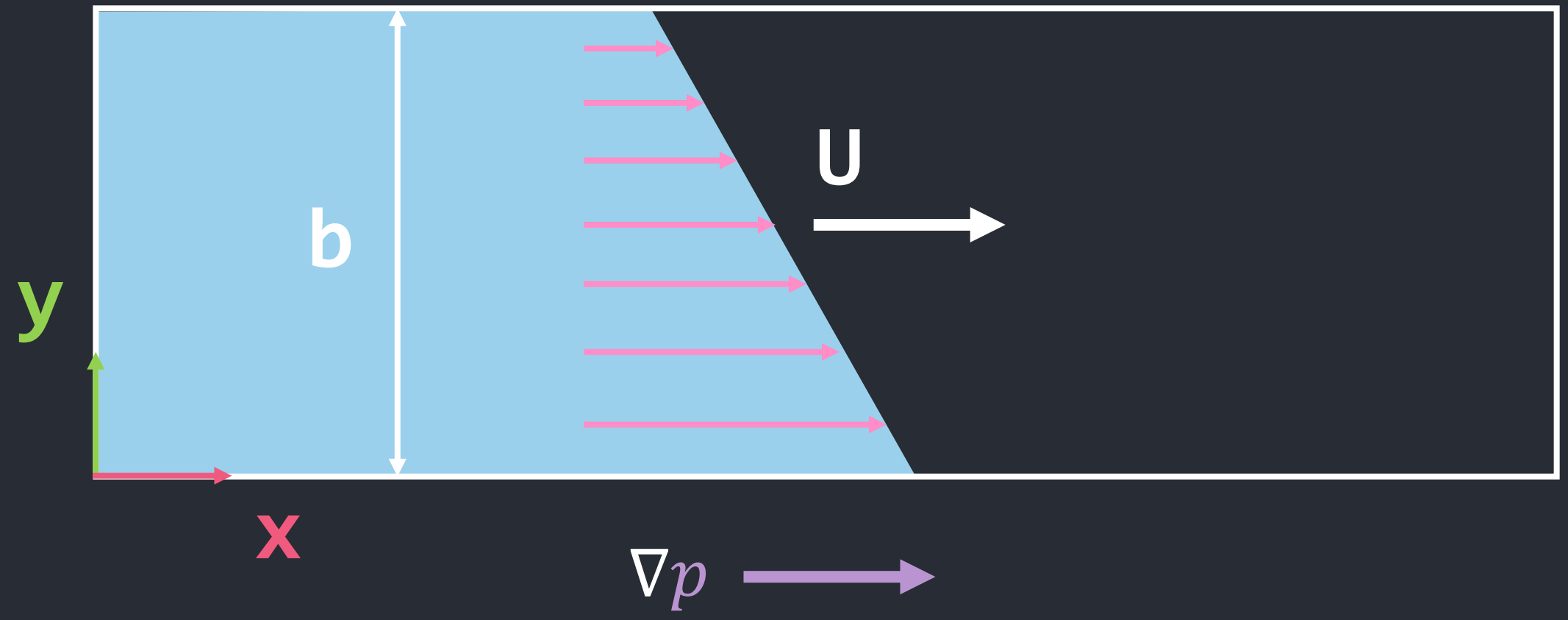
y
x

b
y
x

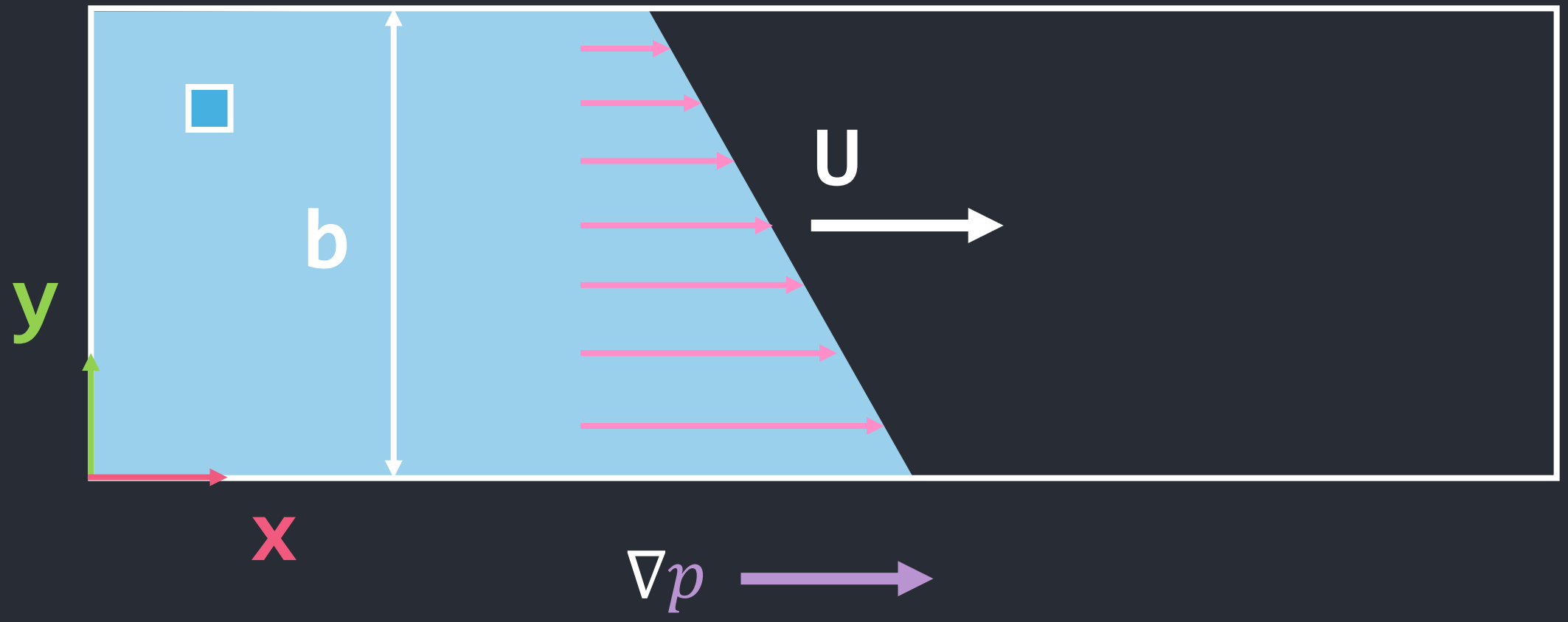

b
y
x
∇p

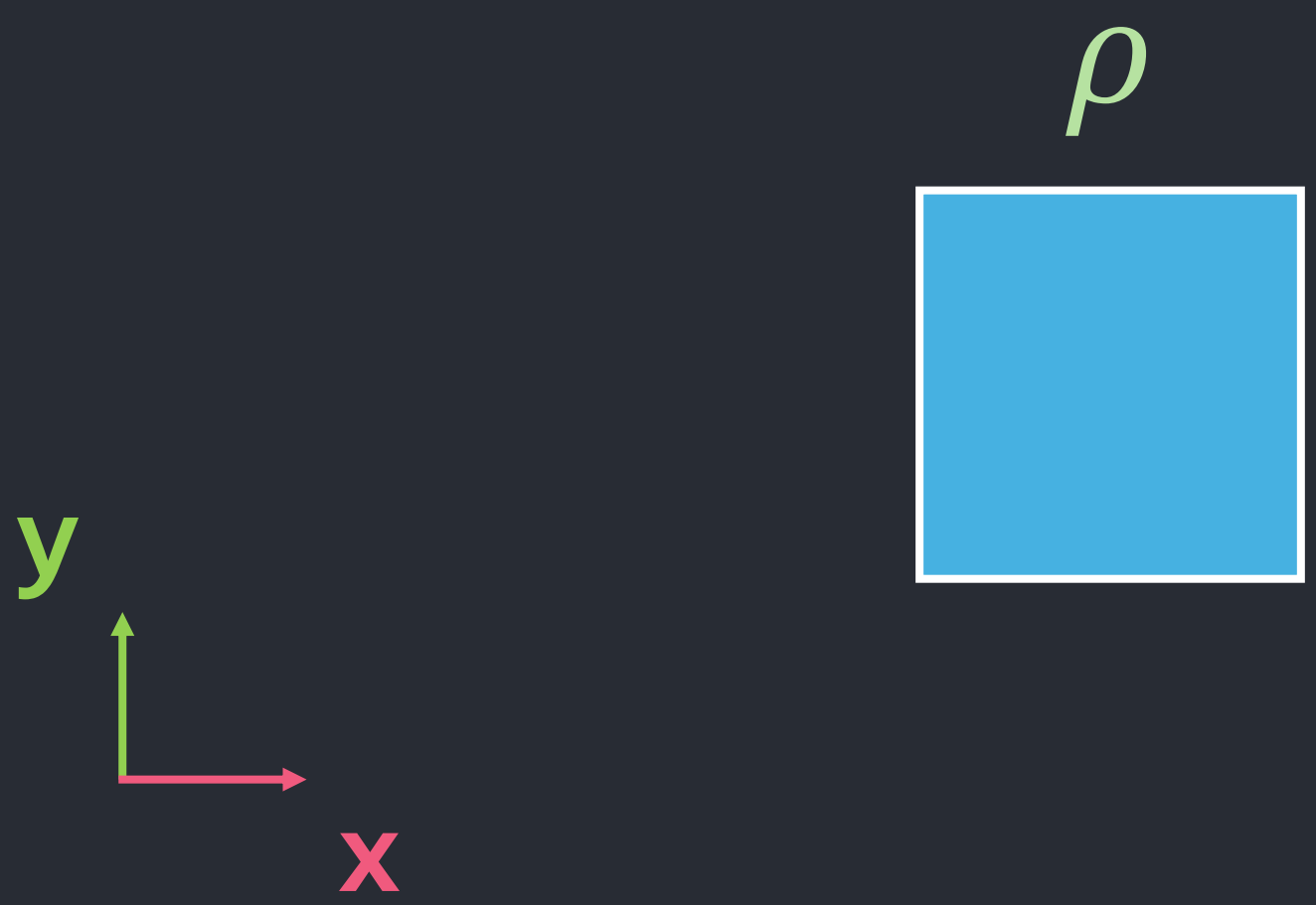

b
U
y
x
∇p

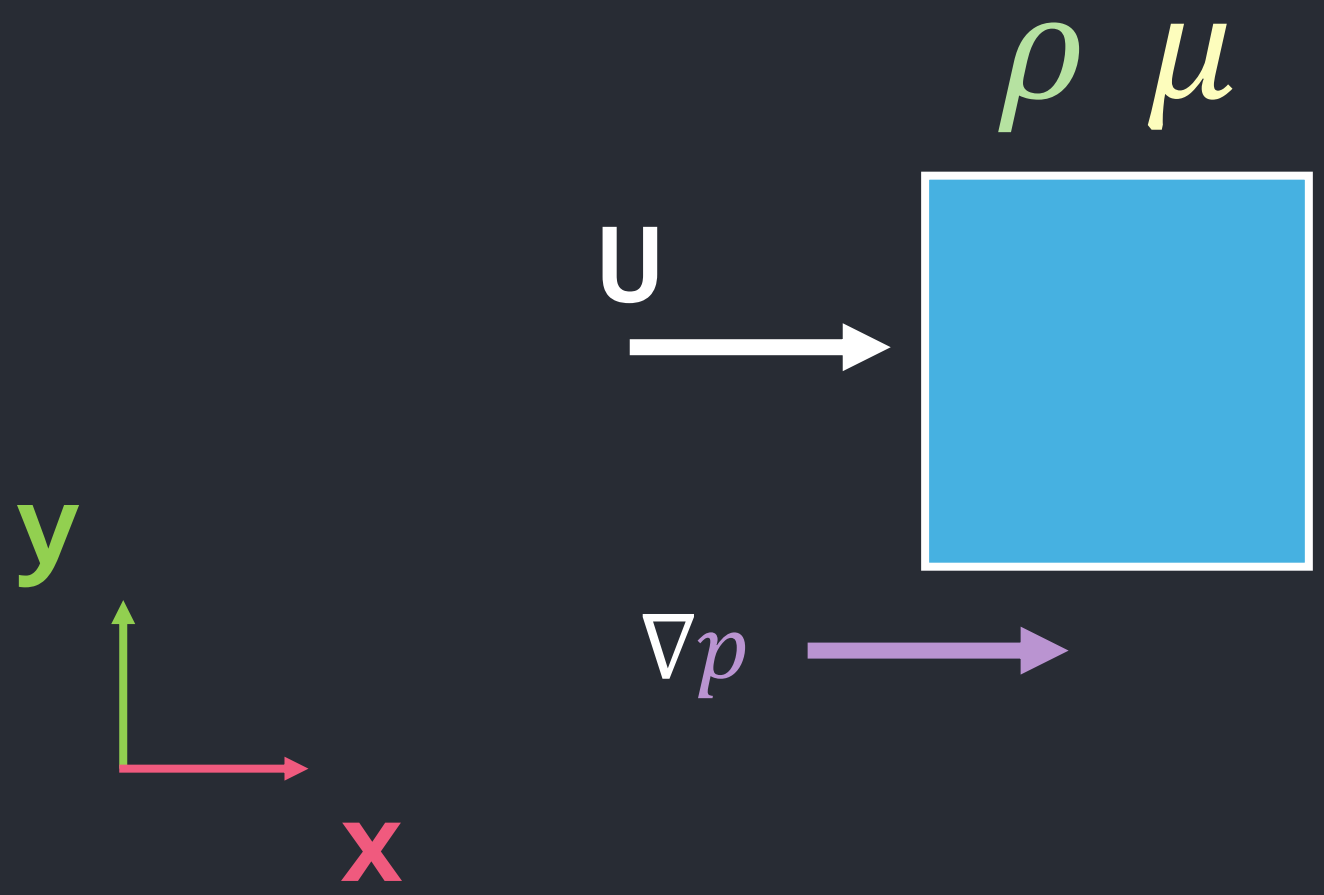
$$u = u(x, y)$$

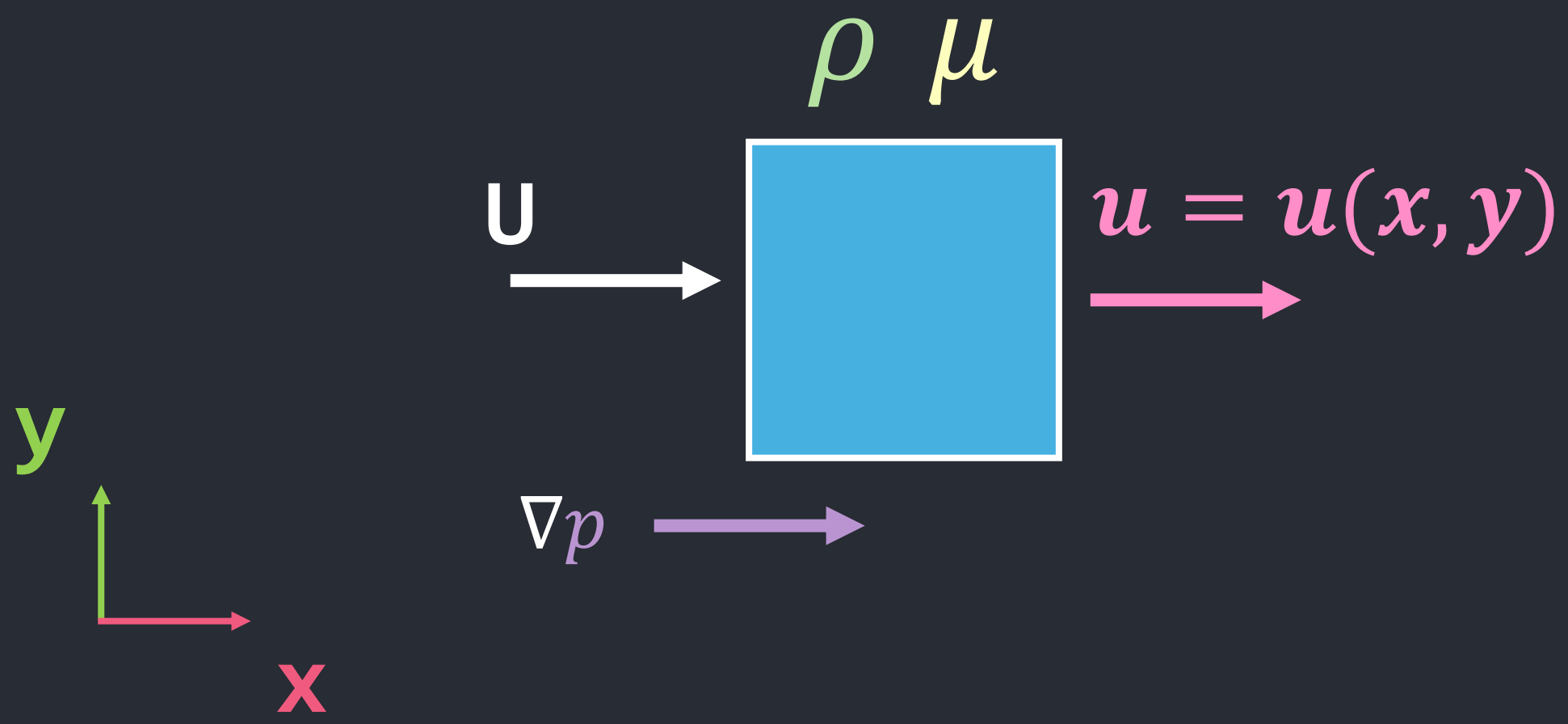


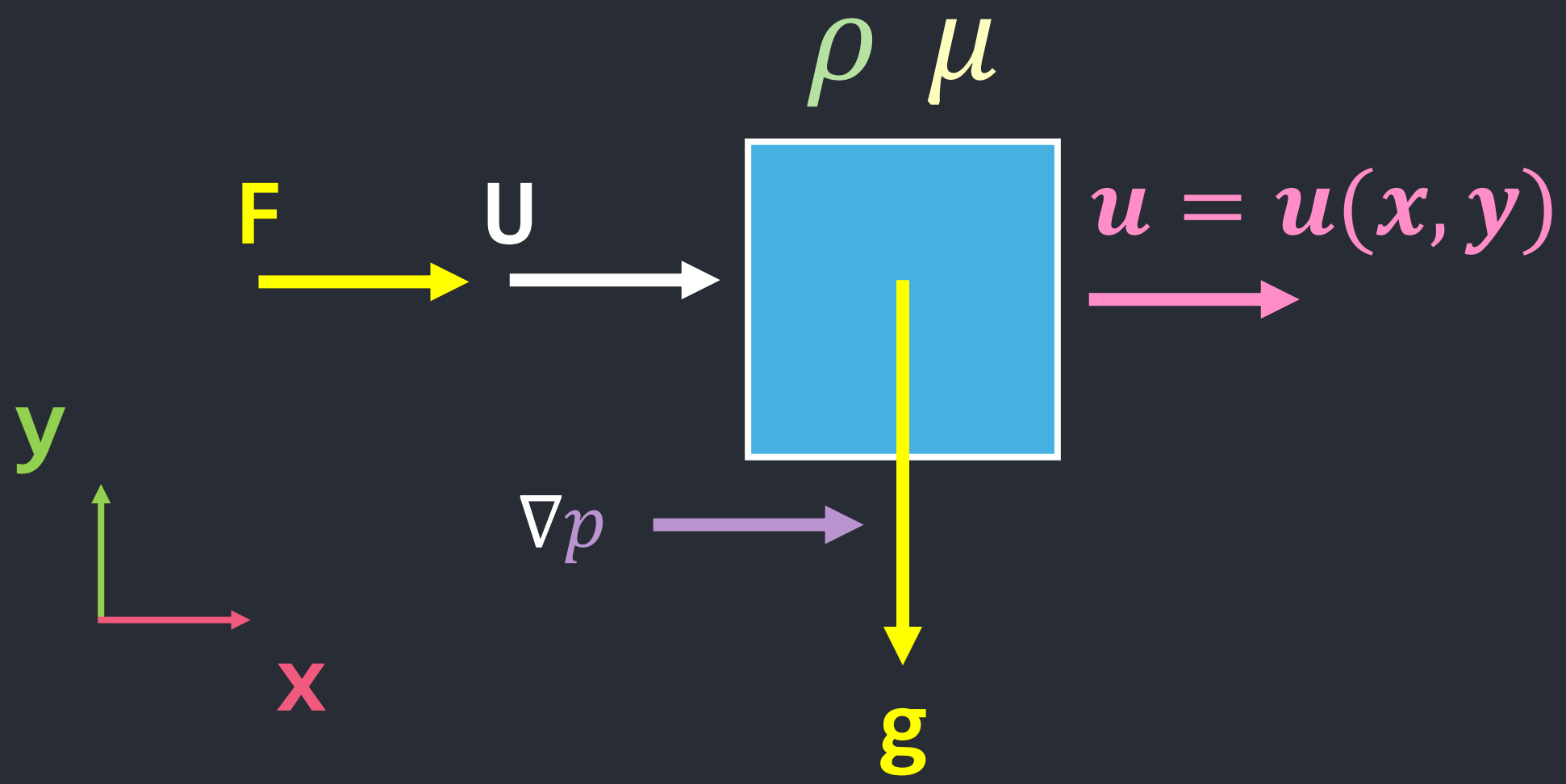
$$u = u(x, y)$$




ρ
y
x


ρ μ
u
y
x
∇p


ρ μ
U
$u = u(x, y)$
∇p
y
x


ρ μ
F
U
$u = u(x, y)$
y
x
∇p
g

Navier-Stokes Equation

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} + \boldsymbol{F}$$

$\boldsymbol{u} \rightarrow Speed: m/s$

$\rho \rightarrow Fluid\ Density: kg/m^2$

$p \rightarrow Pressure: Pa$

$\mu \rightarrow Dynamic\ Viscousity: Pa \cdot s$

$\boldsymbol{F} \rightarrow External\ Force: N$

$\boldsymbol{t} \rightarrow Time: s$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} + \boldsymbol{F}$$

Newton's 2nd Law

$$ma = \sum F$$

$$\underbrace{\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)}_{\frac{\boldsymbol{ma}}{\boldsymbol{V}}} = -\nabla p + \mu\nabla^2\boldsymbol{u} + \boldsymbol{F}$$

$$\underbrace{\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)}_{\frac{\boldsymbol{ma}}{\boldsymbol{V}}} = \underbrace{-\nabla p + \mu\nabla^2\boldsymbol{u} + \boldsymbol{F}}_{\frac{\boldsymbol{F}}{\boldsymbol{V}}}$$

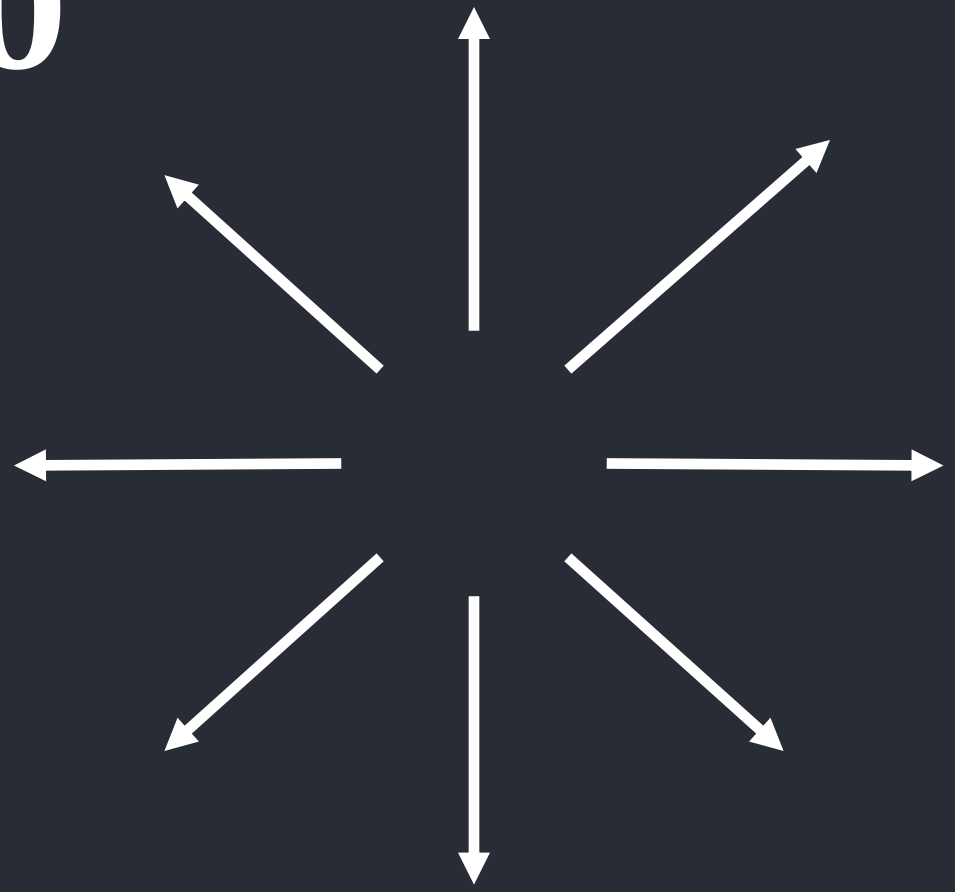
$\boldsymbol{Divergence}$

$$\nabla \cdot \boldsymbol{u}$$

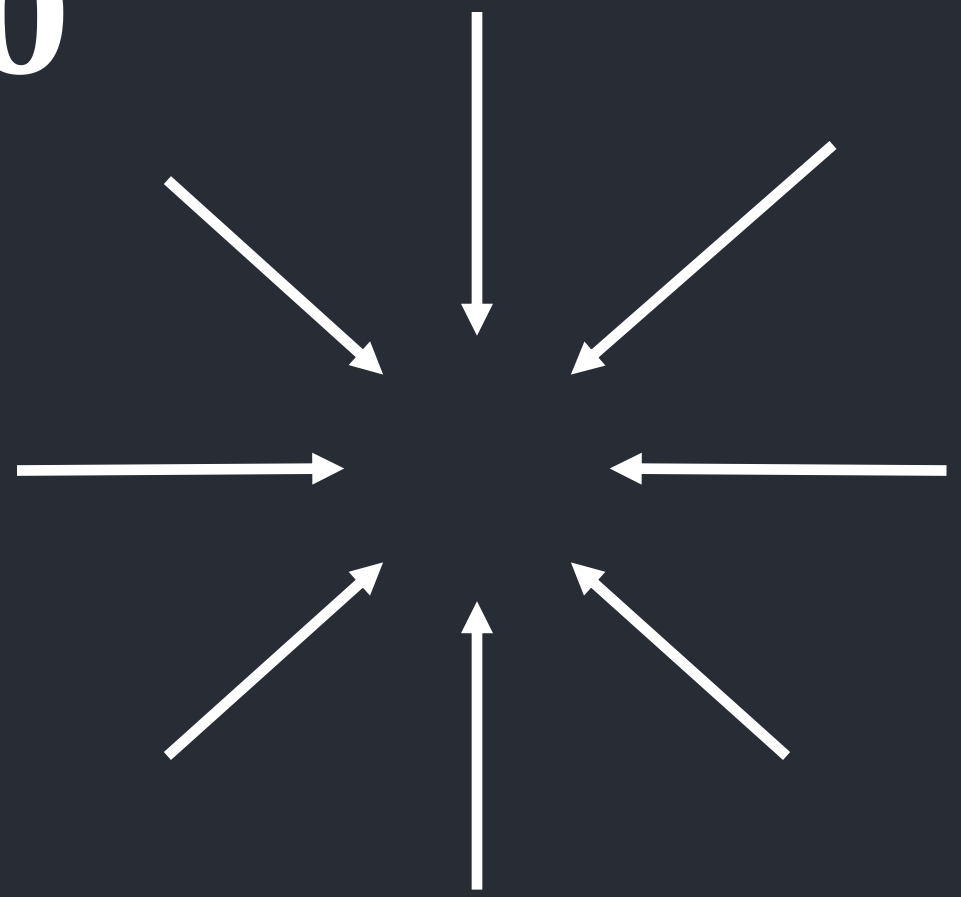
$$\boldsymbol{u} = u(x, y)$$

$$\nabla \cdot \boldsymbol{u} = \boldsymbol{div}\, \boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}$$

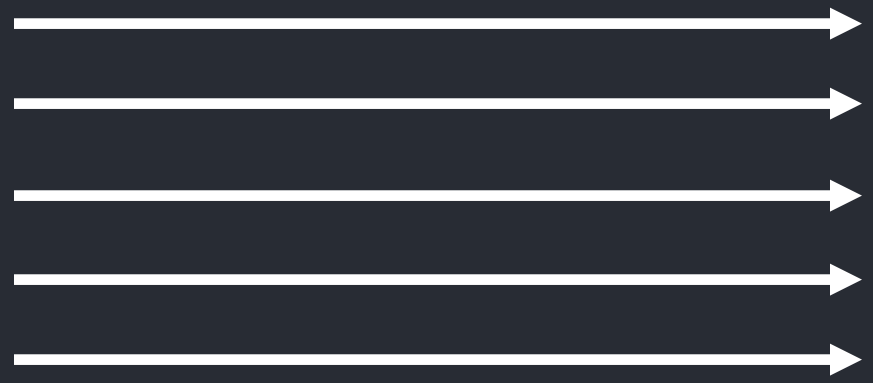
$$\nabla \cdot \boldsymbol{u} > 0$$



$$\nabla \cdot \boldsymbol{u} < \mathbf{0}$$



$$\nabla \cdot \boldsymbol{u} = \mathbf{0}$$



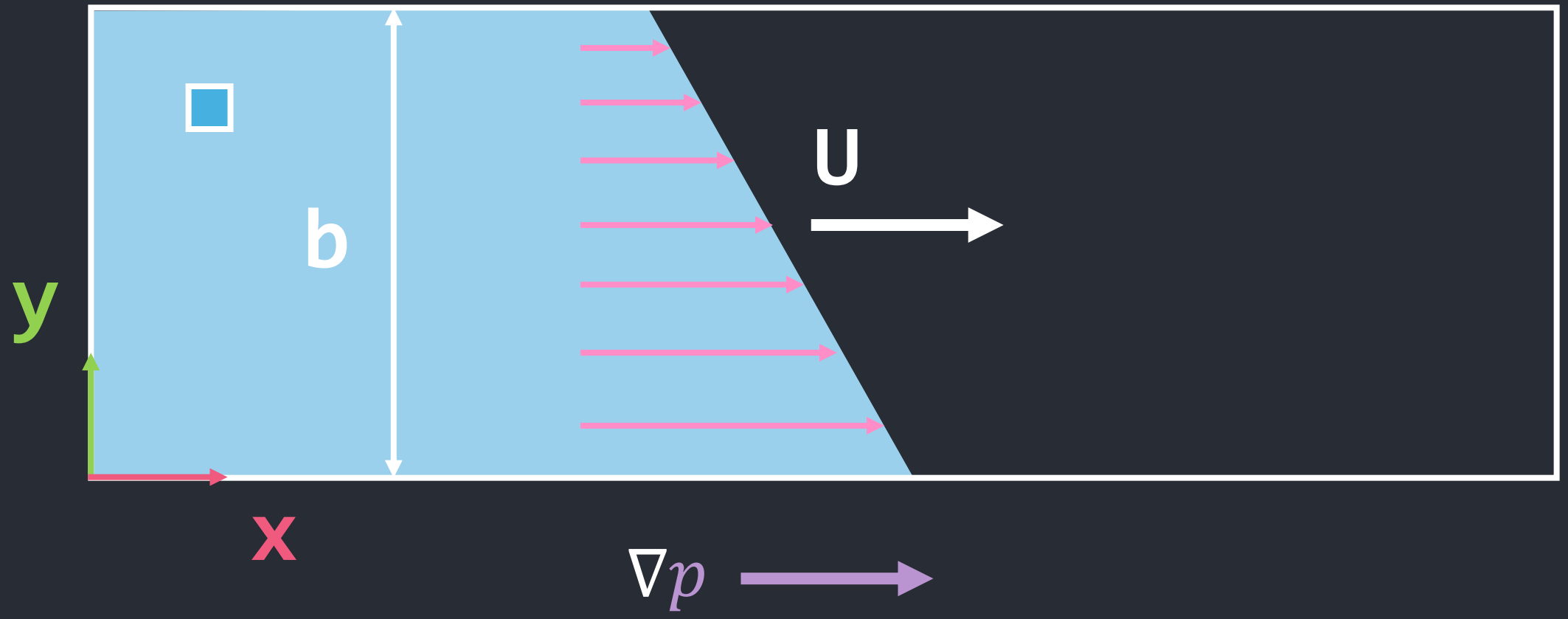
$\boldsymbol{Continuity\ Equation}$

$$\nabla \cdot \boldsymbol{u} = \mathbf{0}$$

$\boldsymbol{Incompressible\ Flow}$

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \mathbf{0}$$

$$u = u(x, y)$$



Conditions

- $No\ flow\ in\ vertical\ direction \rightarrow u_y = 0\ \frac{m}{s}$
- $Flow\ driven\ by\ pressure\ gradient \rightarrow \frac{\partial p}{\partial x} \neq 0\ \frac{Pa}{m}$
- $No\ gravity\ effects\ \rightarrow g = 0\ \frac{m}{s^2}$
- $Boundary\ of\ the\ pipe\ \rightarrow y = [0, b]$
- $Incompressible\ Flow \rightarrow \nabla \cdot \boldsymbol{u} = \mathbf{0}$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)\right)=-\nabla p+\mu\nabla^2\boldsymbol{u}+\rho g$$

$\boldsymbol{Incompressible\ Flow}$

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \mathbf{0}$$

Fully Developed Flow

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \mathbf{0}$$

Fully Developed Flow

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

0 *No flow in y*

Fully Developed Flow

$$\frac{\partial u_x}{\partial x} = \mathbf{0}$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)\right)=-\nabla p+\mu\nabla^2\boldsymbol{u}+\rho g$$

Steady Flow

No flow in y

No g effects

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)\right)=-\nabla p+\mu\nabla^2\boldsymbol{u}+\rho g$$

$\frac{\partial \boldsymbol{u}}{\partial t}$ → **0**

$u_x\frac{\partial u_x}{\partial x}$ → **0**

$u_y\frac{\partial u_x}{\partial y}$ → **0**

ρg → **0**

Fully Developed

$$0 = -\nabla p + \mu \nabla^2 \boldsymbol{u}$$

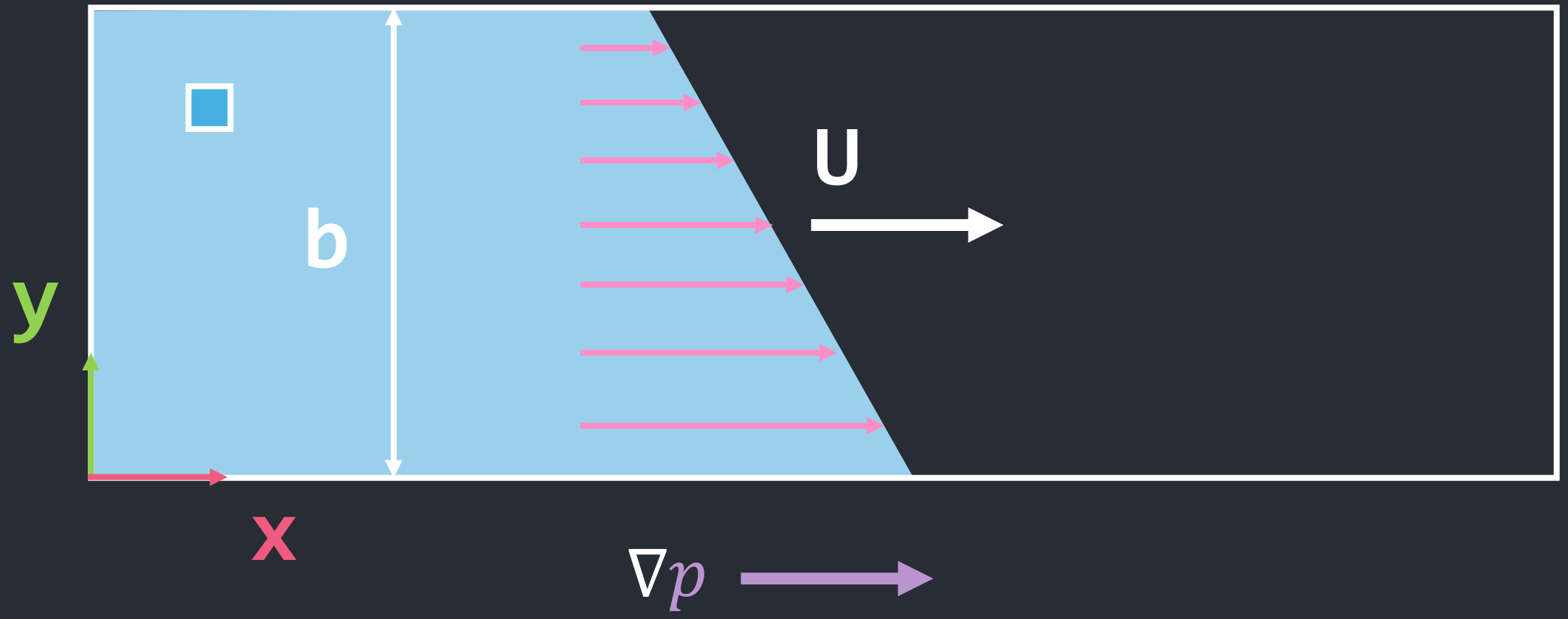
Fully Developed

$$0 = -\nabla p + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right)$$

0

$$0 = -\nabla p + \mu\left(\frac{\partial^2 u_x}{\partial y^2}\right)$$

$$u = u(x, y)$$



$\boldsymbol{Flow\ Profile}$

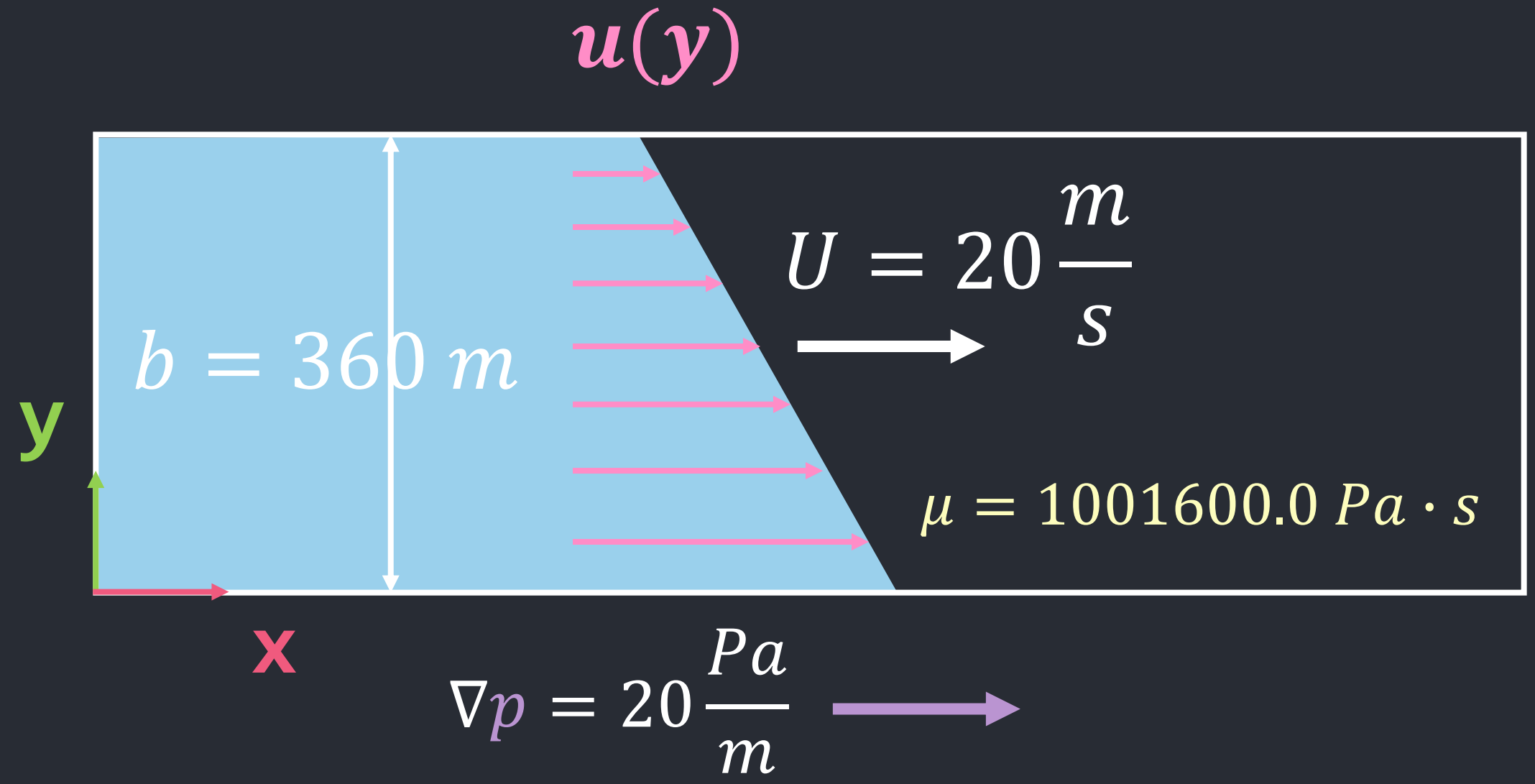
$$u(y) = ???$$

$$0 = -\nabla p + \mu\left(\frac{\partial^2 u_x}{\partial y^2}\right)$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{1}{\mu}\nabla p = const$$

Flow Profile

$$u(y) = \frac{1}{2\mu}\nabla p(b^2 - by) + U\left(1 - \frac{y}{b}\right)$$


$u(y)$
$b = 360\ m$
$U = 20\ \frac{m}{s}$
$\mu = 1001600.0\ Pa \cdot s$
y
x
$\nabla p = 20\ \frac{Pa}{m}$

Rust + Bevy