# Seminar: Computational Methods for X-ray Computed Tomography

## Session 1

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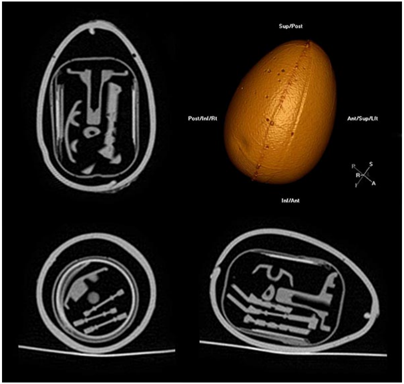
# Outline

# Surprise Egg

# The Fastest Way

# The Safer Way




# What is Tomography?

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- tomos(section or slice) + graphos(to describe)
- images of slices of an object without slicing it
- information from the outside
- measuring the transmission of waves or particles which more or less travel in straight lines but whose intensity is attenuated by the material through which they travel.

- tomographic reconstruction is what helps us find out what's inside things we can't look inside.
- the process of taking projections and putting them together to form a 3D interior model

# Important Applications of tomographic Imaging

1. Medical and biological imaging
    - Clinical X-ray CT
    - visualize damage to bones and teeth
    - image soft tissues such as lungs
    - electron tomography can be used to image very small objects such as viruses

# Important Applications of tomographic Imaging

2. Non destructive inspection and testing
    - monitor oil and gas pipes for both their content and their integrity
    - In museums, tomography is used to look inside artistic and cultural artifacts, archaeological finds, and fossils, without causing damage.

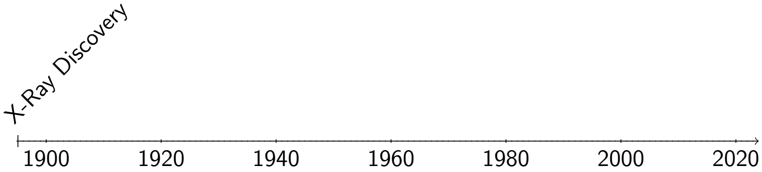
# Important Applications of tomographic Imaging

3. Materials science
   - Development of advanced materials requires understanding their properties at the micro- and nano-scale

# Important Applications of tomographic Imaging

4. Security Screening
    - inspect parcels and luggage, especially at airports
    - 3D view allows objects to be seen that would otherwise be obscured by denser objects on top of them
    - looking for bombs and weapons, tomography can be used to detect contraband

# A Little History

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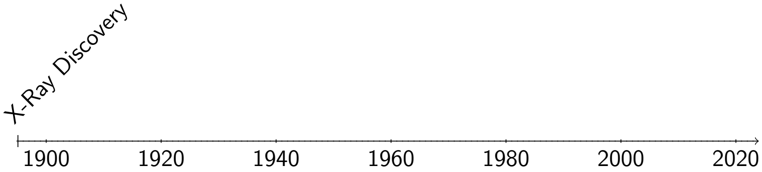


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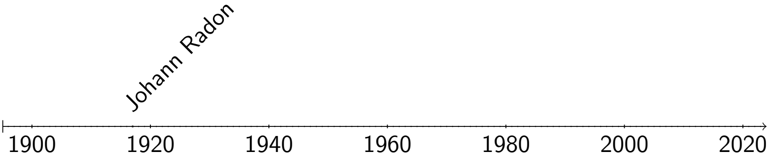
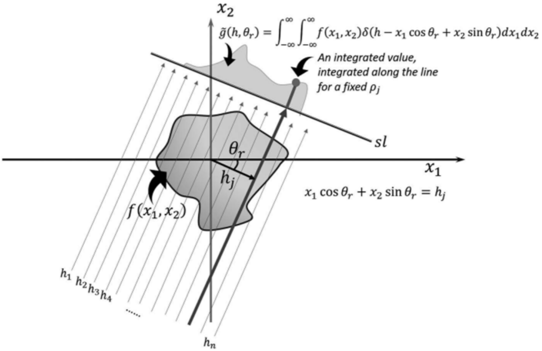
- About six weeks after his discovery, he took a picture using X-rays of his wife's hand.


Hand mit Ringen
PHYSIK. INSTITUT DER UNIVERSITÄT WÜRZBURG.




- Johann Radon: published a paper 1917 about the problem of recovering a function on the plane from its line integrals.



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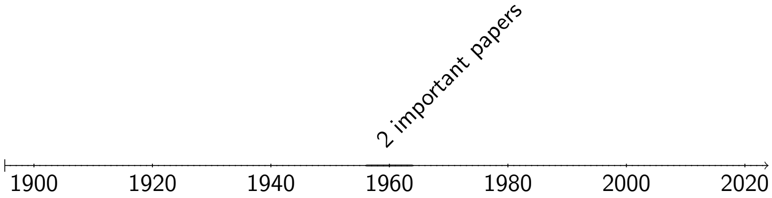


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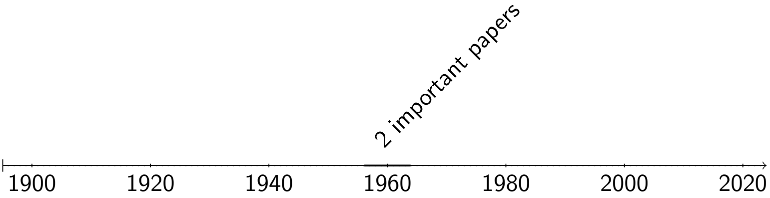


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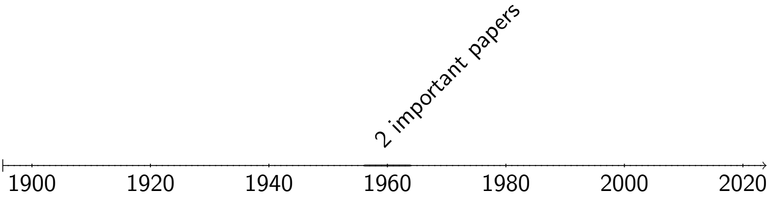


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- The rise of tomography to the 1960s, especially with Allan Cormack's 1963 paper.
- Cormack's experimental measurements: published in 1964.
- Godfrey Hounsfield experiments: applying X-ray CT to a preserved human brain, and then to animal brains from butcher shops.

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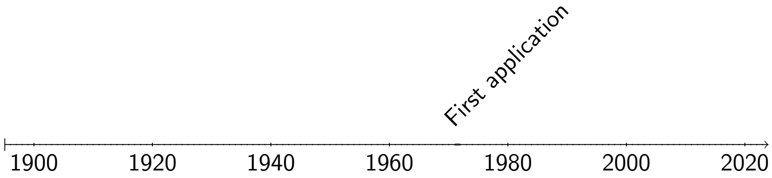


- The system was first tested on a patient in 1971, and a patent was granted in 1972

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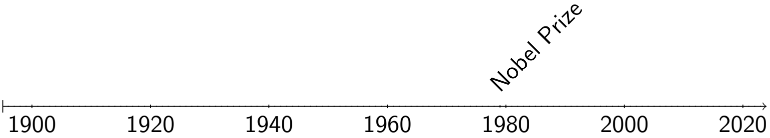


- Hounsfield and Cormack jointly won the Nobel Prize in Physiology or Medicine in 1979 for the invention of X-ray CT

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i) Cone-beam CT

- modern medical CT machines use a helical-scan cone beam
- the patient is translated on a table
- the X-ray source and the detector array rotate about a horizontal axis.

- Dental cone-beam CT: widely used in dental hospitals,
- for extensive surgical procedures
- circular-scan cone-beam geometry

ii) Positron Emission Tomogrpahy

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- PET scans use a radioactive tracer to show how an organ is functioning in real time.
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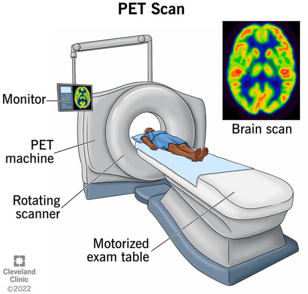
- can measure vital functions: blood flow, oxygen use, blood sugar (glucose) metabolism ...
- can be combined with CT or MRI to image both metabolism and anatomy.



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- a parallel-beam limited-angle problem.
- used, e.g., to image virus particles. (The Electron Microscopy Data Bank reported that 200 out of the 250 protein structures registered in 2020 were found using electron tomography)

- widely used in materials science to understand smaller structures than is possible using X-ray tomography.
- uses a scanning transmission electron microscope (STEM), where a pencil beam of high-energy electrons is focused into a narrow beam which is raster scanned
- or an ordinary TEM, in which the full object is illuminated by a parallel beam

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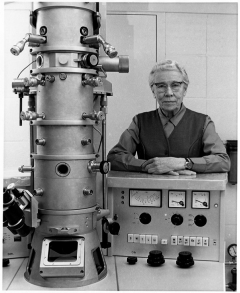

Figure: Katherine Esau next to her TEM

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- X-ray tomography at a higher resolution than possible on a laboratory system
- a high-intensity beam that is highly collimated and close to monochromatic.
- located at large national or multinational facilities such as the Diamond Light Source at Harwell in the UK or the European Synchrotron Radiation Facility (ESRF) in Grenoble, France.
- high cost of the facilities.

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- chapter 7: Limited-data problems: singular values and functions of the Radon transform.

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- chapter 12: inverse problems and regularization, a trade-off between fitting the data and imposing prior assumptions about the image.
- chapter 13: modern practical optimization methods that can be applied to tomographic imaging

# Linear Operators

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- The range is the set of all functions $K[f]$ for some valid $f$, and it is often a smaller set than the co-domain.
- most of the problems in tomography will be formulated as an operator that takes an image to some data.

# example 1: Integration

For a function $f : [-\pi, \pi] \to [-\pi, \pi]$, we can regard the operation:

$$K[f](y) \;=\; \int_{-\pi}^{y} f(x)\, dx$$

as an operator whose domain and co-domain are integrable functions on the interval $[-\pi, \pi]$
Integrating a function makes it a little smoother, so in this example the range is a smaller set than the co-domain.

# Linearity of an operator

We say an operator is linear if it has both the superposition property

$$K[f1 + f2] = K[f1] + K[f2]$$

for any two functions f1 and f2 in the domain and the scaling property

$$K[\alpha f] = \alpha K[f]$$

for any function f in the domain and any scalar $\alpha$ .

# How to combine two functions ?

Suppose I give you two different functions and I ask you to think of all the ways you might combine the two functions to get a new function.

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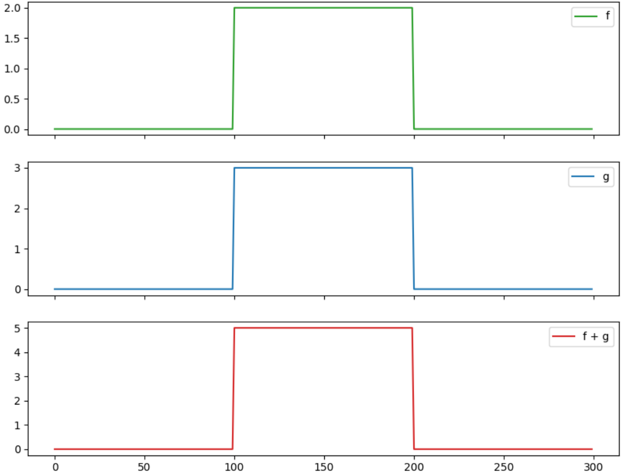

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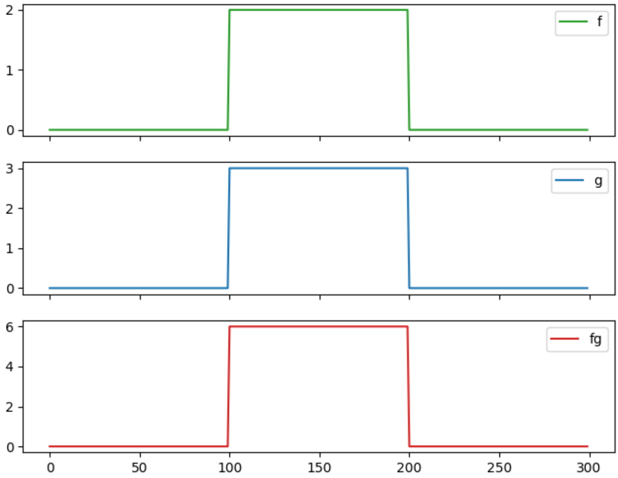

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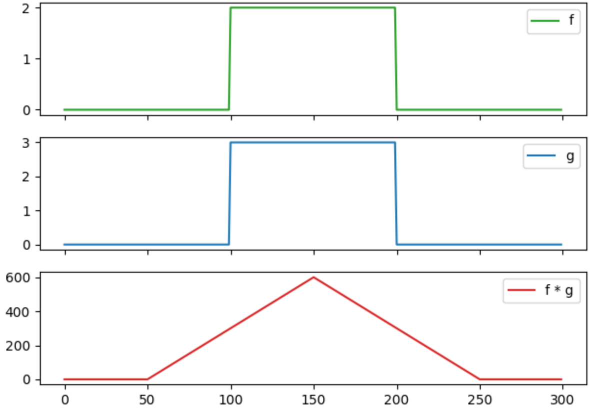

# example 2: Convolution operator for periodic functions

We consider again $2\pi$-periodic functions on the interval $[-\pi, \pi]$, and we define the convolution operator as:

$$g(x) = K[f](x) = \int_{-\pi}^{\pi} h(y-x) f(y)\, dy$$

or

$$g = h * f$$

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$$g = h * f$$

- $h$ defines the system (periodic)
- $y - x$ wraps around when it goes outside the interval $[-\pi, \pi]$
- we refer to convolution of periodic functions as circular convolution

# General form of convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

To convolve a kernel with an input signal : flip the signal move to the desired time and accumulate every interaction with the kernel

# Visualization

# example 3: Integration as Convolution

We can formulate the integration operator in Example 1 as a convolution by choosing:

$$h(x) = \begin{cases} 1, & x \in [-\pi, 0], \\ 0, & x \in ]0, \pi], \end{cases}$$

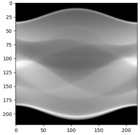
so that

$$h * f(x) = \int_{-\pi}^{x} f(y)\, dy$$

We will now discuss the inverse process of computing or reconstructing the function $f$ from the data $g$, which we refer to as an inverse problem.

# What is an inverse problem?

Direct Problem $\longrightarrow$




Direct problem: given object $f$ determime data $y$

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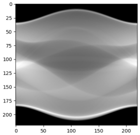
Inverse Problem

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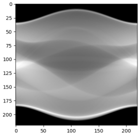


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Solving the inverse problems means reconstructing the object from the measured data

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- The forward problem is a model for how the observations arise
- the inverse problem is a model for how, from a set of observations, we compute the causal factors that produced them.
- an inverse problem typically takes a "forward problem," such as "given the properties of the interior of the object, how much of an X-ray beam penetrates the object,"
- attempts to invert it: deducing interior information from these exterior measurements

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  3. What do you already know?

# Sensitivity to noise

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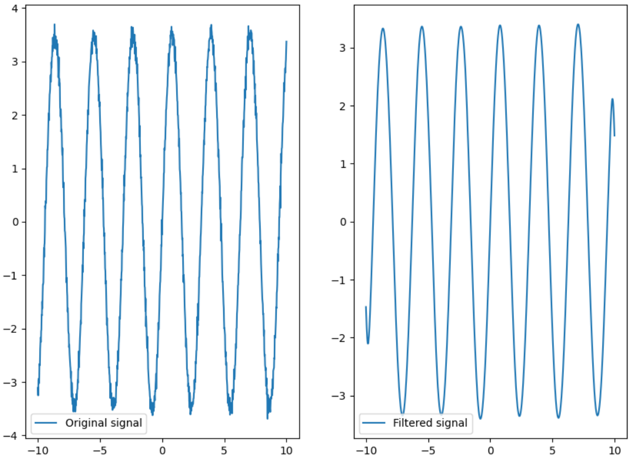
- $\rightarrow$artifacts in the reconstructed image, such as streaks or blurring
- necessity to minimize the noise in CT to improve image quality and diagnostic accuracy.
- Filtering: a technique that involves modifying pixel values in an image to reduce noise while preserving image details.

# Applying a Gaussian Filter to a noisy signal



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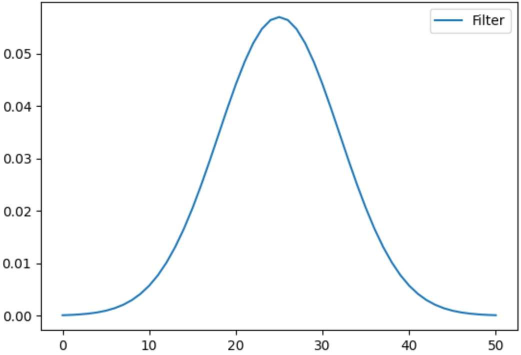

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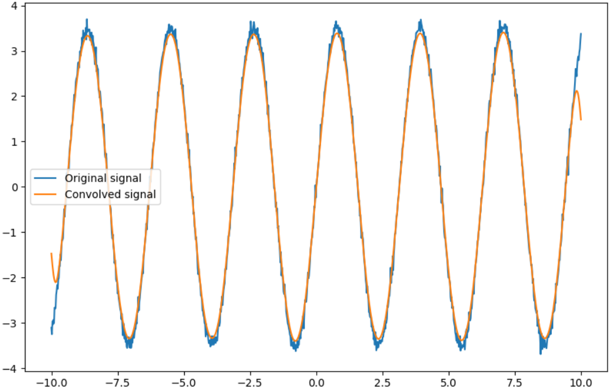

Inverse problems can be quite difficult to solve. This is because many inverse problems are ill posed, according to the following definition.

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If a problem violates any of these criteria, we say that it is ill posed.

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- Often a solution might not exist due to errors such that the measured data g has a component outside the range
- A unique solution means that $K$ has an inverse $K^{-1}$ so that $K^{-1}[K[f]] = f$

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- The solution is stable if any perturbation of the data g produces a bounded perturbation of the solution f
- instability: a small perturbation in the data can make an arbitrarily large change in the solution f

Solution?

# Stabilization by regularization

The important principle is that you can't beat the analysis!
As you measure more data and try to get a more accurate approximation to the solution, you approach the ill-posed continuum problem with its inherent instability

# Stabilization by regularization

- Suppose the operator $K : U \to V$ has a bounded inverse
- but, the noisy data $g$ lives in a co-domain larger than the range
- $\to$ There is no $f$ such that $K[f] = g$ the first Hadamard criterion is violated

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- $\to$ There is no $f$ such that $K[f] = g$ the first Hadamard criterion is violated

To overcome this, it is common to instead consider a solution that minimizes a norm of the residual $K[f] - g$

$$f_{min} = argmin_f ||K[f] - g||_V^2$$

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- regularizing the solution by introducing filters in the expansion (filtering techniques in Chapters 6, 7, 10, 11 and 12)
- terminate an iterative solver before the noise starts to dominate the reconstruction (this approach is discussed in Chapter 11)

# The Fourier Transform

There are various conventions in defining the Fourier transform. Here, we define the Fourier transform for a function $f$ of one variable $x$ by

$$f(\hat{\omega}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\omega} \, dx$$

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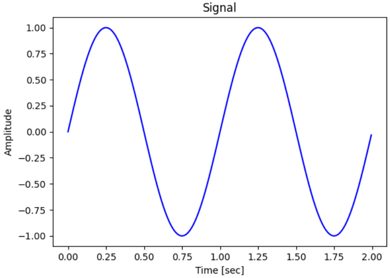
The inverse Fourier transform of a function f is then defined as

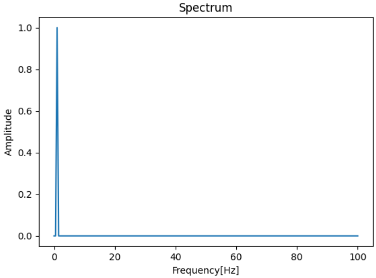
$$f(\check{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{ix\omega} \, d\omega$$

# The Fourier Transform

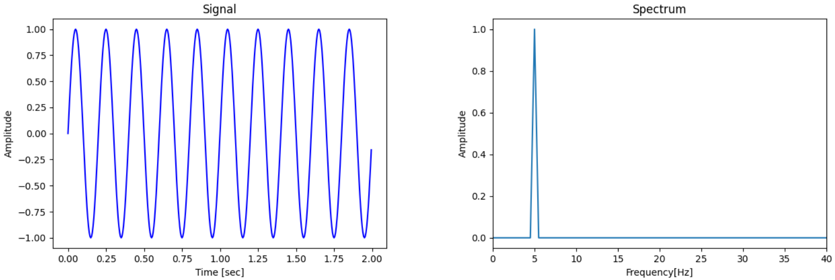
- The Fourier transform gives the frequency components of a function
- The Fourier transform of a real function is complex, encoding both the magnitude and the phase of the frequency components
- We say “in frequency space” or “in Fourier space” to mean considering the Fourier transform of functions with respect to the frequency variable.

# Examples

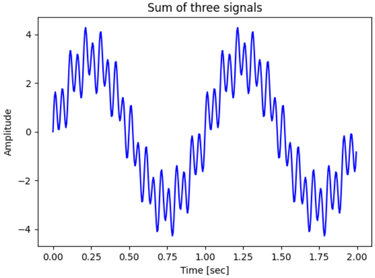

Signal
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
Amplitude
0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75 2.00
Time [sec]

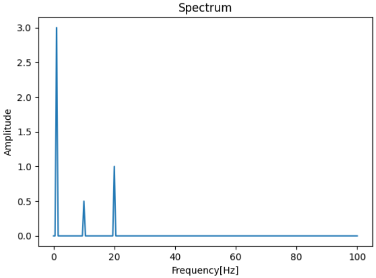

Spectrum
1.0
0.8
0.6
0.4
0.2
0.0
Amplitude
0 20 40 60 80 100
Frequency[Hz]

# Examples


Signal
Amplitude
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75 2.00
Time [sec]
Spectrum
Amplitude
1.0
0.8
0.6
0.4
0.2
0.0
0 5 10 15 20 25 30 35 40
Frequency[Hz]

# Examples


Sum of three signals
Amplitude
Time [sec]
4
2
0
−2
−4
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00


Spectrum
Amplitude
Frequency[Hz]
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
20
40
60
80
100

# The 2D Fourier Transform

Let $f$ be a function of $x = (x_1, x_2)$. The the 2D Fourier Transform is a function of the angular frequency vector $\omega = (\omega_1, \omega_2)$ given by

$$f(\hat{\omega}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-ix\omega} \, dx_1 dx_2$$

where $x\omega = x_1\omega_1 + x_2\omega_2$

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The 2D inverse Fourier Transform is

$$\check{f(x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\omega) e^{ix\omega} \, d\omega_1 d\omega_2$$

$\rightarrow$ The 2D Fourier transform is just the 1D Fourier transform with respect to each of the variables in turn.

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- $f(x)h(x) \xrightarrow{\text{Fourier Transform}} F(\omega)H(\omega)$

# Quiz

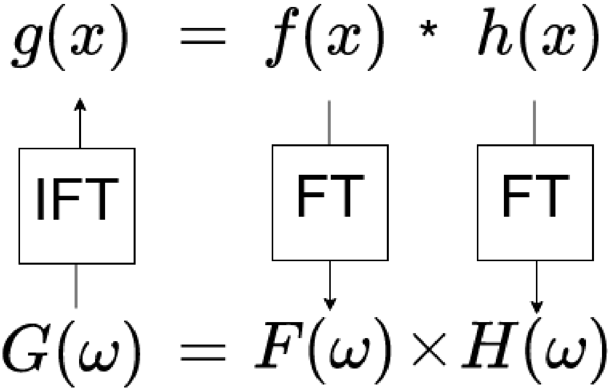
Which of the following propreties hold?

- $\alpha f_1(x) + \beta f_2(x) \xrightarrow{\text{Fourier Transform}} \alpha F_1(\omega) + \beta F_2(\omega)$: Linearity
- $f(ax) \xrightarrow{\text{Fourier Transform}} \frac{1}{|a|} F(\frac{\omega}{a})$: Scaling
- $f(x)h(x) \xrightarrow{\text{Fourier Transform}} F(\omega)H(\omega)$

# Convolution and Fourier Transform

| **Spatial Domain** | **Frequency Domain** |
|---|---|
| $g(x) = f(x) * h(x)$ | $G(\omega) = F(\omega)H(\omega)$ |
| $\downarrow$ | $\downarrow$ |
| Convolution | Multiplication |

# Convolution using Fourier Transform



# Systems of Linear Equations

- discretize a CT reconstruction Problem
- we arrive at a system of linear equations:

$$Ax = b$$

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- we arrive at a system of linear equations:

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- A represents a linear operator
- b vector holding the measured data
- x the unkown solution: the reconstruction we want to compute

# Notation

column vector: $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ , row vector: $v^\top = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$

# Notation

column vector: $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, row vector: $v^\top = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$

If the matrix $A$ has dimensions $m \times n$ (m rows and n columns), then we write it as :

$$A = \begin{pmatrix} | & | & \dots & | \\ \mathbf{c_1} & \mathbf{c_2} & \dots & \mathbf{c_n} \\ | & | & \dots & | \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{pmatrix}$$

# example

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Two linear equations with unkowns x1 and x2 :

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad a_{21}x_1 + a_{22}x_2 = b_2$$

# The first Problem

- There may not always exist an $x$ such that the equation $Ax = b$ holds.
- This depends on the size and the rank of the matrix $A$.

# Rank

- The rank $r$ of an $m \times n$ matrix $A$ is the number of linearly independent rows of the matrix (it is also equal to the number of linearly independent columns), and for a nonzero matrix the rank satisfies $1 \leq r \leq min(m, n)$

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- example :

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow$$

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So this matrix has the rank $r = 2$

# Range

- The range, or column space, Range($A$), of an $m \times n$ matrix $A$ is the linear subspace spanned by the columns of the matrix:

$$Range(A) \equiv \{u \in \mathbb{R}^m | u = \alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_n c_n, arbitrary \alpha_j\}$$

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So this Matrix has Rank $= 2$ and $Range(A) = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

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So this Matrix has Rank $= 2$ and $Null(A) = \alpha_1 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

- The dimensions of the range and null space are $r$ and $n - r$, respectively.
- In the last example : $n = 2$ , $r = 1$ and $n - r = 1$ is the dimension of the Null space.

# Linear Systems

Consider two linear sytems with the the same matrix $A$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{pmatrix} x = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \; and \; \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$$

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- The first system, has infinitely many solutions; any vector of the form $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ satisfies this equation because the last, arbitrary, component is in the null space of $A$.

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- The second system has no solution because the right-hand side does not belong to the range of $A$

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Since the data $b$ always comes from the forward projection of an image $x$, the system $Ax = b$ will always have a solution.

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Since the data $b$ always comes from the forward projection of an image $x$, the system $Ax = b$ will always have a solution.

Is that true?

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- The right-hand side that we encounter when computing a tomographic reconstruction from measured data is contaminated with noise

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- The right-hand side that we encounter when computing a tomographic reconstruction from measured data is contaminated with noise
- The key issue: whether the system is consistent
- whether we can find a vector $x$ that satisfies the equation
- whether the right-hand side $b$ lies in the range $\mathrm{Range}(A)$

# Linear Least Squares Problem

- formulation that seeks to handle inconsistency of a linear problem
- define a meaningful “solution” to an inconsistent system

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- define a meaningful "solution" to an inconsistent system
- Idea : find an $x$ such that $Ax$ approximates the right-hand side $b$ in some optimal way, here by minimizing the 2-norm of the residual vector $b - Ax$

# Formally:

Assume that the right-hand side has the form

$$b = A\overline{x} + e$$

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Then the optimal estimate of $x$ is obtained by solving the least-squares problem :

$$x_{LS} = argmin_x \ \frac{1}{2}||b - Ax||_2^2$$

# Recipe: Compute a least-squares solution

Let $A$ be an $m \times n$ Matrix and $b$ a vector in $\mathbb{R}^n$

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1. Compute the Matrix $A^T A$ and the vector $A^T b$
2. Form the augmented matrix for the matrix equation $A^T Ax = A^T b$ and row reduce
3. This equation is always consistent, and any solution $x$ is a least-squares solution.

# Example

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Ensuring the system is inconsistent:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 5 \\ 2 & 3 & 5 & 7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 1 \\ 2 & 3 & 5 & 7 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

$$A^T A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 15 \\ 9 & 14 & 24 \\ 15 & 24 & 42 \end{pmatrix}$$

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$$A^T b = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 23 \\ 35 \\ 59 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 15 \\ 9 & 14 & 24 \\ 15 & 24 & 42 \end{pmatrix}$$

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If we form the augmented matrix equation $A^TAx = A^Tb$ and solve it we find that all least-squares solutions have the form :

$$\begin{pmatrix} \frac{7}{3} \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

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Among all possible solutions, the minimum-norm least-squares solution $x^0_{LS}$ is obtained for $\alpha = 0$

# Discussion

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- the least-squares solution is unique if and only if the null space of $A$ is trivial (the case when r = n)
- If $r \neq n$, then the least-squares solution has an arbitrary, undetermined component in the null space $Null(A)$.
- from a computational point of view it should generally not be used to compute $x_{LS}$ due to the influence of rounding errors; a better approach is to use a $QR$ factorization (or $SVD$) of $A$ .

Thank you!